Exercise 1

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$$\int_a^b y_1^*(x)\, \mathcal{L}[y_2(x)]\, dx = \int_a^b y_1^*(x) \left( \left(r(x)\, [y_2(x)]'\right)' + q(x)\, [y_2(x)] \right) dx$$

$$= \int_a^b y_1^*(x) \left[ \left( (r(x))' (y_2(x))' + r(x) [y_2(x)]'' \right) + q(x) [y_2(x)] \right] dx$$

$$= \int_a^b y_1^* \left[ r y_2'' + r' y_2' + q y_2 \right] dx$$

$$= \left[ (y_1^* r) y_2' + (y_1^* r') y_2 \right]_a^b - \int_a^b (y_1^* r)' y_2' + (y_1^* r')' y_2 - (y_1^* q) y_2 \, dx$$

$$= \left[ y_1^* (r y_2' + r' y_2) - (y_1^* r)' y_2 \right]_a^b + \int_a^b \left( (y_1 r)'' - (y_1 r')' + (y_1 q) \right)^* y_2 \, dx$$

$$= \left[ y_1^* (r y_2' + (r' - r') y_2) - (y_1^*)' r y_2 \right]_a^b + \langle \mathcal{L} y_1^*, y_2 \rangle$$

$$= \left[ y_1^* r y_2' - (y_1^*)' r y_2 \right]_a^b + \langle \mathcal{L} y_1^*, y_2 \rangle$$

for $\left[ r( y_1^* y_2' - (y_1^*)' y_2 \right]_a^b = 0$, $\langle y_1, \mathcal{L} y_2 \rangle = \langle \mathcal{L} y_1, y_2 \rangle$

$\Rightarrow \mathcal{L}$ is Hermitian

Exercise 2.

$$\int_a^b y_1^*(x)\, \mathcal{L}[y_2(x)]\, dx = \int_a^b \left( \mathcal{L}[y_1(x)] \right)^* y_2(x)\, dx$$

$$\mathcal{L}[y] = -\lambda p(x) y$$

$$\int_a^b y_1^*(x)\left(-\lambda p(x) y_2(x)\right) dx = \int_a^b \left(-\lambda p(x) y_1(x)\right)^* y_2(x)\, dx$$

$\Rightarrow \lambda = \lambda^* \Rightarrow \lambda$ are real.

Exercise 3.

$$\begin{cases} \mathcal{L}\{\phi\} = -\lambda_\phi p(x)\phi \\ \mathcal{L}\{\psi\} = -\lambda_\psi p(x)\psi \end{cases}$$

$$\Rightarrow \psi\mathcal{L}\{\phi\} - \phi\mathcal{L}\{\psi\} = (\lambda_\psi - \lambda_\phi)\, p(x)\phi\psi$$

Integrate both side

$\Rightarrow \quad 0 = (\lambda_\psi - \lambda_\phi)\int_a^b p(x)\phi(x)\psi(x)\,dx$ from Ex. 1

$\because \lambda_\psi \neq \lambda_\phi \quad \therefore \int_a^b p(x)\phi(x)\psi(x)\,dx = 0$

Exercise 4.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

for $\lambda > 0$, $\lambda = k^2$

$$y = C_1\cos(kx) + C_2\sin(kx)$$

$$0 = C_1 + 0, \quad y = C_2\sin(kx)$$

$$y'(x) = kC_2\cos(kx)$$

$$y'(\pi) = kC_2\cos(k\pi) = 0$$

For obtaining a non-trivial sol.

$$\Rightarrow \cos(k\pi) = 0$$

$$\Rightarrow k = \frac{2n+1}{2}$$

Eigenvalues: $\lambda = k^2 = \frac{(2n+1)^2}{4}$ ※

Eigenfunctions: $\sin\left(\frac{2n+1}{2}x\right)$ ※

Exercise 5.

From Ex. 4 we know $\varphi_n(x) = \sin\left(\frac{2n+1}{2}\right)x$ , $n \geq 0$

$$C_n = \frac{\int_0^{\pi} f(x)\varphi_n(x)\,dx}{\int_0^{\pi} \varphi_n^2(x)\,dx} = \frac{\int_0^{\frac{\pi}{2}} x\varphi_n(x)\,dx + \int_{\frac{\pi}{2}}^{\pi}\left(\frac{\pi}{2}-x\right)\varphi_n(x)\,dx}{\int_0^{\pi}\varphi_n^2(x)\,dx}$$

a). $\int_0^{\frac{\pi}{2}} x\sin\left(\frac{2n+1}{2}\right)x\,dx$, let $\left(\frac{2n+1}{2}\right) = t$

$$\Rightarrow \int_0^{\frac{\pi}{2}} x\sin tx\,dx = -\frac{x\cos tx}{t}\Bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{\cos tx}{t}\,dx$$

$$= \left(-\frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)}{t}\right) - \frac{\sin(tx)}{t^2}\Bigg|_0^{\frac{\pi}{2}}$$

$$= -\frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)}{t} - \frac{\sin\left(\frac{\pi}{2}t\right)}{t^2} = \frac{\sin\frac{(2n+1)}{4}\pi}{\left(\frac{2n+1}{2}\right)^2} - \frac{\frac{\pi}{2}\cos\frac{2n+1}{4}t}{\left(\frac{2n+1}{2}\right)}$$

$$= \frac{4\sin\left(\frac{2n+1}{4}\right)\pi}{(2n+1)^2} - \frac{\pi\cos\left(\frac{2n+1}{4}\right)\pi}{(2n+1)}$$

b). $$\int_{\frac{\pi}{2}}^{\pi} \left(\frac{\pi}{2}-x\right) \sin\left(\frac{2n+1}{2}\right) x \, dx = \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2} \sin\left(\frac{2n+1}{2}\right) x \, dx - \int_{\frac{\pi}{2}}^{\pi} x \sin\left(\frac{2n+1}{2}\right) x \, dx$$

$$= \frac{-\pi}{2} \frac{2}{2n+1} \cos\left(\frac{2n+1}{2}\right) x \Big|_{\frac{\pi}{2}}^{\pi} + \frac{x \cos tx}{t} \Big|_{\frac{\pi}{2}}^{\pi} + \frac{\sin tx}{t^2} \Big|_{\frac{\pi}{2}}^{\pi}, \quad t = \frac{2n+1}{2}$$

$$= \frac{-\pi}{2n+1}\left(\cos \frac{(2n+1)\pi}{2} - \cos \frac{(2n+1)\pi}{4}\right) + \left(\frac{\pi \cos \pi t}{t} - \frac{\pi \cos \frac{\pi}{2} t}{2t}\right) + \left(\frac{\sin \pi t}{t^2} - \frac{\sin \frac{\pi}{2} t}{t^2}\right)$$

c). $$\int_0^{\pi} \sin^2\left(\frac{2n+1}{2}\right) x \, dx = \frac{x}{2} - \frac{\sin(2tx)}{4t} \Big|_0^{\pi} = \frac{\pi}{2}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} C_n \varphi_n(x) = \sum_{n=0}^{\infty} \frac{2}{\pi}\left(\left(\frac{\sin\left(\frac{t}{2}\pi\right)}{\cancel{t^2}} - \frac{\frac{\pi}{2}\cos\left(\frac{t}{2}\pi\right)}{t}\right) - \frac{\pi}{2n+1}\left(\cos t\pi - \cos \frac{t}{2}\pi\right) + \frac{\pi \cos t\pi}{t} - \frac{\pi \cos \frac{t}{2}\pi}{2t} + \frac{\sin t\pi}{t^2} - \cancel{\frac{\sin \frac{t}{2}\pi}{t^2}}\right) \sin t x$$

$$f(x) = \frac{2}{\pi} \sum_{n=0}^{\infty} \left[ -\frac{\pi}{2n+1}\left(\cos \frac{2n+1}{2}\pi - \cos \frac{2n+1}{4}\pi\right) + \frac{2\pi \cos \frac{2n+1}{2}\pi}{2n+1} - \frac{2\pi \cos \frac{2n+1}{4}\pi}{2n+1} + \frac{4 \sin \frac{2n+1}{2}\pi}{(2n+1)^2} \right] \sin \frac{2n+1}{2} x$$

Exercise 6.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 0 \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} x \, dx$$

$$= \frac{1}{\pi} \cdot \frac{1}{2}x^2 \Big|_{\pi}^{2\pi} = \frac{3}{2}\pi$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 0 \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} x\cos nx \, dx$$

$$= \frac{\cancel{2\pi n\sin(2\pi n)} + \cos(2\pi n) - \cancel{\pi n\sin(\pi n)} - \cos(\pi n)}{\pi n^2}$$

$$= \frac{1-(-1)^n}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 0 \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} x\sin x \, dx$$

$$= \frac{\cancel{\sin(2\pi n)} - 2\pi n\cos(2\pi n) - \cancel{\sin(\pi n)} + \pi n\cos(\pi n)}{\pi n^2}$$

$$= -\frac{2}{n} + \frac{(-1)^n}{n}$$

The Fourier Series:

$$\frac{3}{2}\pi + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-2+(-1)^n}{n}\sin nx \quad \#$$

Exercise 7.

$$(f(x))^2 = f(x)\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)\right]$$

$$\frac{1}{\pi}\int_0^{2\pi}(f(x))^2 = \frac{1}{2}a_0\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\left(\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx\right) + b_n\left(\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx\right)\right] \quad \text{—①}$$

by orthogonality of Fourier Series:

$$\begin{cases} \int_0^{2\pi} f(\theta)\cos(k\theta)\,d\theta = \pi a_k \\ \int_0^{2\pi} f(\theta)\sin(k\theta)\,d\theta = \pi b_k \\ \int_0^{2\pi} f(\theta)\,d\theta = \pi a_0 \end{cases}$$

rewrite ①

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi}(f(x))^2 = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \quad \text{※}$$

Exercise 8.

A Discrete Fourier Transform $F(x_n) = X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$,

with $e^{-\frac{i2\pi}{N}kn} = \cos\left(\frac{2\pi}{N}kn\right) - i\sin\left(\frac{2\pi}{N}kn\right)$

(1) For an even function:

$$F(x_{n_e}) = \sum_{n_e=0}^{N-1} x_{n_e}\left[\cos\left(\frac{2\pi}{N}kn_e\right) - i\sin\left(\frac{2\pi}{N}kn_e\right)\right]$$

$$= \sum_{n_e=0}^{N-1} x_{n_e}\cos\left(\frac{2\pi}{N}kn_e\right), \text{ which is real.}$$

(2) For an odd function:

$$F(x_{n_o}) = \sum_{n_o=0}^{N-1} x_{n_o}\left[\cos\left(\frac{2\pi}{N}kn_o\right) - i\sin\left(\frac{2\pi}{N}kn_o\right)\right]$$

$$= -i\sum_{n_o=0}^{N-1} x_{n_o}\sin\left(\frac{2\pi}{N}kn_o\right), \text{ which is purely imaginary.}$$