PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS
WEEK 1

Q1 These are formulas:

$p \vee q \qquad \neg\neg p \qquad (p \to q) \to ((p \to (q \to r)) \to (p \to r))$

While the others all aren't formulas.

- $p \vee q$ is the disjunction of the atoms p and q.
- $\neg p$ is the negation of p, and $\neg\neg p$ is the negation of $\neg p$.
- $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$ is the hardest one to read, but it can be broken down like this: It is a conditional, with the main connective highlighted here:

$$(p \to q) \to ((p \to (q \to r)) \to (p \to r))$$

So, its antecedent is $p \to q$, a conditional, and its consequent is $(p \to (q \to r)) \to (p \to r)$, also a conditional with antecedent $p \to (q \to r)$ and consequent $p \to r$, and these are also clearly formulas.

The others:

$p \vee q \to r \qquad q \neg p \qquad p \wedge (q \vee r) \to \bot \qquad p \wedge q \wedge r$

are not formulas. $q \neg p$ is just messed up. (Maybe it should say $q \wedge \neg p$?)

The others are ambiguous.

$p \vee q \to r$ between $p \vee (q \to r)$ & $(p \vee q) \to r$

$p \wedge (q \vee r) \to \bot$ between $p \wedge ((q \vee r) \to \bot)$ & $(p \wedge (q \vee r)) \to \bot$

$p \wedge q \wedge r$ between $p \wedge (q \wedge r)$ & $(p \wedge q) \wedge r$

This last case might seem different. After all, $(p \wedge q) \wedge r$ & $p \wedge (q \wedge r)$ seem to "mean the same" in some sense. But for our purposes they still differ in important senses. The main connective of $p \wedge (q \wedge r)$ is the first $\wedge$, the main connective of $(p \wedge q) \wedge r$ is the second.

Q2. In each proof, the assumptions — whether discharged or not, depend only on themselves, and no other formulas. So, the $p \to q$ and the two instances of $p \wedge r$ depend only on themselves.

$$
\dfrac{\dfrac{p \to q \qquad \dfrac{[p \wedge r]^1}{p}\,\wedge E}{q}\,\to E \qquad \dfrac{[p \wedge r]^1}{r}\,\wedge E}{\dfrac{q \wedge r}{(p \wedge r) \to (q \wedge r)}\,\to I^1}\,\wedge I
$$

We can list these dependence relations like this

$p \to q$ depends on $p \to q$
$p \wedge r$ (left) depends on $p \wedge r$ (left)
$p \wedge r$ (right) depends on $p \wedge r$ (right)

Beyond the assumptions, we have (reading down the proof)

p depends on $p \wedge r$ (left)
q depends on $p \to q$ & $p \wedge r$ (left)
r depends on $p \wedge r$ (right)
$q \wedge r$ depends on $p \to q$ & $p \wedge r$ (left & right)

And then we have the discharge of $p \wedge r$ (left & right), so

$(p \wedge r) \to (q \wedge r)$ depends on $p \to q$

Q3 Here are simple proofs that do the job. They aren't the <u>only</u> proofs (if you came up with different proofs you could still be correct), but by and large, these are the most simple, direct proofs for these arguments.

$$\dfrac{\dfrac{\dfrac{p \to q \qquad \dfrac{[r \to p]^2 \quad [r]^1}{p}{\scriptstyle \to E}}{q}{\scriptstyle \to E}}{r \to q}{\scriptstyle \to I^1}}{(r \to p) \to (r \to q)}{\scriptstyle \to I^2}$$

$p \to q \vdash (r \to p) \to (r \to q)$

$$\dfrac{\dfrac{p \quad [q]^1}{p \wedge q}{\scriptstyle \wedge I}}{q \to (p \wedge q)}{\scriptstyle \to I^1}$$

$p \vdash q \to (p \wedge q)$

$$\dfrac{\dfrac{\dfrac{p \wedge (q \to r)}{p}{\scriptstyle \wedge E} \qquad \dfrac{\dfrac{p \wedge (q \to r)}{q \to r}{\scriptstyle \wedge E} \quad [q]^1}{r}{\scriptstyle \to E}}{p \wedge r}{\scriptstyle \wedge I}}{q \to (p \wedge r)}{\scriptstyle \to I^1}$$

$p \wedge (q \to r) \vdash q \to (p \wedge r)$

Q4.

$$\dfrac{\begin{matrix}\Pi_1\\ A\to B\end{matrix} \qquad \begin{matrix}\Pi_2\\ A\to C\end{matrix}}{A\to(B\wedge C)}\,{\scriptstyle \to\wedge} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\to B\end{matrix} \quad [A]^1}{B}\,{\scriptstyle \to E} \qquad \dfrac{\begin{matrix}\Pi_2\\ A\to C\end{matrix} \quad [A]^1}{C}\,{\scriptstyle \to E}}{B\wedge C}\,{\scriptstyle \wedge I}}{A\to(B\wedge C)}\,{\scriptstyle \to I^1}$$

$$\dfrac{\begin{matrix}\Pi\\ A\to(B\to C)\end{matrix}}{(A\wedge B)\to C}\,{\scriptstyle \text{Import}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi\\ A\to(B\to C)\end{matrix} \quad \dfrac{[A\wedge B]^1}{A}\,{\scriptstyle \wedge E}}{B\to C} \qquad \dfrac{[A\wedge B]^1}{B}\,{\scriptstyle \wedge E}}{C}\,{\scriptstyle \to E}}{(A\wedge B)\to C}\,{\scriptstyle \to I^1}$$

$$\dfrac{\begin{matrix}\Pi\\ (A\wedge B)\to C\end{matrix}}{A\to(B\to C)}\,{\scriptstyle \text{Export}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{(A\wedge B)\to C \qquad \dfrac{[A]^2 \quad [B]^1}{A\wedge B}\,{\scriptstyle \wedge I}}{C}\,{\scriptstyle \to E}}{B\to C}\,{\scriptstyle \to I^1}}{A\to(B\to C)}\,{\scriptstyle \to I^2}$$

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WEEK 2

Q1.

$$
\begin{prooftree}
\AxiomC{$[p \vee q]^3$}
\AxiomC{$(p \to \neg r) \wedge (q \to \neg r)$}
\RightLabel{$\wedge E$}
\UnaryInfC{$p \to \neg r$}
\AxiomC{$[p]^1$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg r$}
\AxiomC{$(p \to \neg r) \wedge (q \to \neg r)$}
\RightLabel{$\wedge E$}
\UnaryInfC{$q \to \neg r$}
\AxiomC{$[q]^2$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg r$}
\RightLabel{$\vee E^{1,2}$}
\TrinaryInfC{$\neg r$}
\AxiomC{$[r]^4$}
\RightLabel{$\neg E$}
\BinaryInfC{$\perp$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg(p \vee q)$}
\RightLabel{$\to I^4$}
\UnaryInfC{$r \to \neg(p \vee q)$}
\end{prooftree}
$$

$p \to \neg r$ depends on $(p \to \neg r) \wedge (q \to \neg r)$ (left)

$\neg r$ (left) depends on $(p \to \neg r) \wedge (q \to \neg r)$ (left) & p

$q \to \neg r$ depends on $(p \to \neg r) \wedge (q \to \neg r)$ (right)

$\neg r$ (right) depends on $(p \to \neg r) \wedge (q \to \neg r)$ (right) & q

$\neg r$ (last) depends on $p \vee q$ & $(p \to \neg r) \wedge (q \to \neg r)$ (left & right)

$\perp$ depends on $p \vee q$, $(p \to \neg r) \wedge (q \to \neg r)$ (left & right) & r

$\neg(p \vee q)$ depends on $(p \to \neg r) \wedge (q \to \neg r)$ (left & right) & r

$r \to \neg(p \vee q)$ depends on $(p \to \neg r) \wedge (q \to \neg r)$ (left & right).

Q2

$$
\begin{prooftree}
\AxiomC{$[\neg p]^1$}
\AxiomC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\perp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg p$}
\end{prooftree}
$$

$p \vdash \neg\neg p$

$$
\begin{prooftree}
\AxiomC{$p \to r$}
\AxiomC{$[p \wedge q]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{r}
\AxiomC{$q \to s$}
\AxiomC{$[p \wedge q]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{s}
\RightLabel{$\wedge I$}
\BinaryInfC{$r \wedge s$}
\RightLabel{$\to I^1$}
\UnaryInfC{$(p \wedge q) \to (r \wedge s)$}
\end{prooftree}
$$

$p \to r, q \to s \vdash (p \wedge q) \to (r \wedge s)$

$$
\begin{prooftree}
\AxiomC{$[p \vee q]^3$}
\AxiomC{$p \to r$}
\AxiomC{$[p]^1$}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\AxiomC{$q \to s$}
\AxiomC{$[q]^2$}
\RightLabel{$\to E$}
\BinaryInfC{s}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\RightLabel{$\vee E^{1,2}$}
\TrinaryInfC{$r \vee s$}
\RightLabel{$\to I^3$}
\UnaryInfC{$(p \vee q) \to (r \vee s)$}
\end{prooftree}
$$

$p \to r, q \to s \vdash (p \vee q) \to (r \vee s)$

$$\dfrac{\neg p \vee \neg q \qquad \dfrac{[\neg p]^1 \quad \dfrac{[p\wedge q]^3}{p}{\scriptstyle \wedge E}}{\bot}{\scriptstyle \neg E} \qquad \dfrac{[\neg q]^2 \quad \dfrac{[p\wedge q]^3}{q}{\scriptstyle \wedge E}}{\bot}{\scriptstyle \neg E}}{\dfrac{\bot}{\neg(p\wedge q)}{\scriptstyle \neg I^3}}{\scriptstyle \vee E^{1,2}}$$

$$\neg p \vee \neg q \succ \neg(p \wedge q)$$

$$\dfrac{\neg\neg\neg p \qquad \dfrac{\dfrac{[\neg p]^1 \quad [p]^2}{\bot}{\scriptstyle \neg E}}{\neg\neg p}{\scriptstyle \neg I^1}}{\dfrac{\bot}{\neg p}{\scriptstyle \neg I^3}}$$

$$\neg\neg\neg p \succ \neg p$$

Q3 Here is the proof with a detour formula marked.

$$\dfrac{\dfrac{\dfrac{(p\to r)\wedge(q\to r)}{p\to r}{\scriptstyle \wedge E}}{(p\to r)\vee(q\to r)}{\scriptstyle \vee I} \qquad \dfrac{[p\to r]^1 \quad \dfrac{[p\wedge q]^3}{p}{\scriptstyle \wedge E}}{r}{\scriptstyle \to E} \qquad \dfrac{[q\to r]^2 \quad \dfrac{[p\wedge q]^3}{q}{\scriptstyle \wedge E}}{r}{\scriptstyle \to E}}{\dfrac{r}{(p\wedge q)\to r}{\scriptstyle \to I^3}}{\scriptstyle \vee E^{1,2}}$$

We eliminate the detour like this.

$$\dfrac{\dfrac{\dfrac{(p\to r)\wedge(q\to r)}{p\to r}{\scriptstyle \wedge E} \qquad \dfrac{[p\wedge q]^3}{p}{\scriptstyle \wedge E}}{r}{\scriptstyle \to E}}{(p\wedge q)\to r}{\scriptstyle \to I^3}$$

And this proof doesn't have any detours.

Q4.

(i)

$$\dfrac{\dfrac{p \quad q}{p\wedge q}{\scriptstyle \wedge I} \qquad \neg(p\wedge q)}{\bot}{\scriptstyle \neg E}$$

(ii) $p\vee q \quad \neg p \vee \neg q$ is not inconsistent.

(imagine p is true & $\neg q$ is true)

Q5

If $X \vdash_I \neg A$, then

$$\dfrac{\begin{array}{c} X \\ \Pi \\ \neg A \end{array} \qquad A}{\bot}\,\neg I$$

is a proof for $X, \neg A \succ \bot$.

Conversely, if $X, A \vdash_I \bot$, then

$$\dfrac{\begin{array}{c} X, [A]^1 \\ \Pi \\ \bot \end{array}}{\neg A}\,\neg I^1$$

is a proof for $X \succ \neg A$.

Q6.

$$\dfrac{\dfrac{\neg(p \to q) \qquad \dfrac{\dfrac{\dfrac{(\neg p)^2 \qquad (p)^1}{\bot}\,\neg E}{q}\,\bot E}{p \to q}\,\to I^1}{\bot}\,\neg E^2}{\dfrac{\neg\neg p}{p}\,\text{DNE}}\,\neg I^2$$

Don't be disappointed if you couldn't find this proof!
It is quite tricky to construct.

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WEEK 3

Q2 a)

p	q	$\neg$	$(p$	$\wedge$	$q)$	$\neg$	p	$\wedge$	$\neg$	q
0	0	1	0	0	0	1	0	1	1	0
0	1	1	0	0	1	1	0	0	0	1
1	0	1	1	0	0	0	1	0	1	0
1	1	0	1	1	1	0	1	0	0	1

In these two rows we find valuations that make $\neg(p \wedge q)$ true & $\neg p \wedge \neg q$ false.

$v_1(p)=0$ $v_1(q)=1$

$v_2(p)=1$ $v_2(q)=0$

The argument is invalid.

b)

p	q	r	p	$\to$	q	q	$\to$	r	p	$\to$	r
0	0	0	0	1	0	0	1	0	0	1	0
0	0	1	0	1	0	0	1	1	0	1	1
0	1	0	0	1	1	1	0	0	0	1	0
0	1	1	0	1	1	1	1	1	0	1	1
1	0	0	1	0	0	0	1	0	1	0	0
1	0	1	1	0	0	0	1	1	1	1	1
1	1	0	1	1	1	1	0	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1

Here there is no row where the premises $p \to q$ and $q \to r$ are true & the conclusion $p \to r$ is false. It is valid.

c)

p	q	r	p	$\to$	q	q	$\to$	r	$\neg$	p	$\to$	r
0	0	0	0	1	0	0	1	0	1	0	0	0
0	0	1	0	1	0	0	1	1	1	0	1	1
0	1	0	0	1	1	1	0	0	1	0	0	0
0	1	1	0	1	1	1	1	1	1	0	1	1
1	0	0	1	0	0	0	1	0	0	1	1	0
1	0	1	1	0	0	0	1	1	0	1	1	1
1	1	0	1	1	1	1	0	0	0	1	1	0
1	1	1	1	1	1	1	1	1	0	1	1	1

Here, there is a row where the premises $p \to q$ and $q \to r$ are true and the conclusion $\neg p \to r$ is false.

$v(p)=0$, $v(q)=0$, $v(r)=0$ is a valuation that is a counterexample to the argument.

The argument is INVALID

d)

p	q	$\neg$	$(p$	$\to$	$q)$	p
0	0	0	0	1	0	0
0	1	0	0	1	1	0
1	0	1	1	0	0	1
1	1	0	1	1	1	1

Here, there is only one row where the premise $\neg(p \to q)$ is true, and here, the conclusion p is true too. There is no counterexample, and the argument is VALID.

e)

p	q	$\neg$	$(p$	$\wedge$	$q)$	$\neg$	p	$\vee$	$\neg$	q
0	0	1	0	0	0	1	0	1	1	0
0	1	1	0	0	1	1	0	1	0	1
1	0	1	1	0	0	0	1	1	1	0
1	1	0	1	1	1	0	1	0	0	1

Here, the premise & the conclusion have the same value in every row, so there is no counterexample. So, the argument is valid.

f)

p	q	$((p$	$\to$	$q)$	$\to$	$p)$	$\to$	p
0	0	0	1	0	0	0	1	0
0	1	0	1	1	0	0	1	0
1	0	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1

Here the conclusion is true in every row. There is no counterexample, The argument is valid.

Q2. Suppose A, B, C & D are formulas where $A, B \models_{CL} C \wedge D$, and suppose v is a boolean valuation.

a) Since $A, B \models_{CL} C \wedge D$, if $v(A) = v(B) = 1$ then $v(C \wedge D) = 1$ too, so $v(C \wedge D) = 0$ can't be true. This claim is false.

b) We can have this statement true for some v, but it isn't true in general.

For example: if $A = p$, $B = q$, $C = q$, $D = p$, (clearly $A, B \vDash_{cl} C \wedge D$ here)

if $v(p) = v(q) = 0$ the statement is true.

if $v(p) = 1$ $v(q) = 0$, the statement is false.

So, there is not enough information to say whether its true or false, in general.

c) This statement is true. If $v(C) = 0$ then $v(C \wedge D) = 0$, and since $A, B \vDash_{cl} C \wedge D$, we cannot have $v(A) = v(B) = 1$. So either $v(A) = 0$ or $v(B) = 0$.

d) This statement is also true. If $v(C \wedge D) = 0$, then since $A, B \vDash_{cl} C \wedge D$, we cannot have $v(A) = v(B) = 1$, so if $v(A) = 1$ then $v(B) = 0$.

e) This statement is not, in general, true. Take the example with $A = C = D = p$ & $B = q$. Then $A, B \vDash_{cl} C \wedge D$ & if $v(p) = 1$ & $v(q) = 0$, we have $v(A) = 1$, $v(B) = 0$ & $v(C \wedge D) = 1$, so in this case the statement is false.
On the other hand, in the case where $A = C = p$, $B = D = q$, then if $v(A) = 1$ & $v(B) = 0$ we must have $v(C \wedge D) = 0$, so in that case, the statement is false. In general, there's not enough info.

f) In this case, too, there is not enough information.
If $A = C = p$, $B = D = q$, this statement is true
If $A = p$, $B = C = D = q$, this statement is false

g) This statement is true. If $\nu(A)=\nu(B)=1$ then since $A, B \vDash_{cl} C \wedge D$ we must have $\nu(C\wedge D)=1$, which means that $\nu(\neg(C\wedge D))=0$.

Q3 Here is why $\wedge E$ is sound.

$$\frac{A \wedge B}{A}\wedge E \qquad \frac{A\wedge B}{B}\wedge E$$

If a proof Π for $X \succ A\wedge B$ is truth preserving (if $X \vDash_{cl} A\wedge B$) then so is the proof for $X \succ A$ (ie $X \vDash_{cl} A$) since $X \vDash_{cl} A\wedge B$ tells us that $\nu(X)=1$ then $\nu(A\wedge B)=1$ & since, if $\nu(A\wedge B)=1$ then $\nu(A)=1$, & so if $\nu(X)=1$ then $\nu(A)=1$.

Similarly, if $X \vDash_{cl} A\wedge B$ then $X \vDash_{cl} B$ since if $\nu(A\wedge B)=1$ then $\nu(B)=1$.

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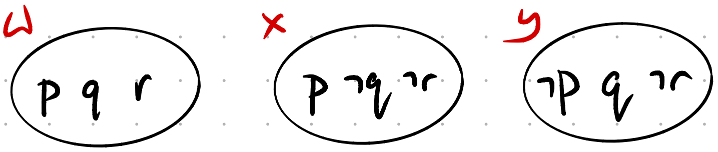
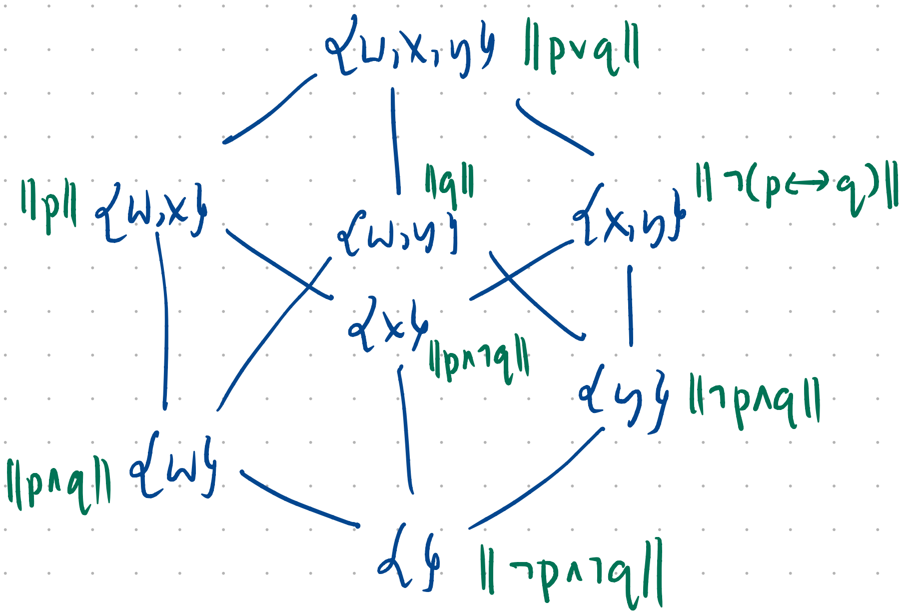
WEEK 4

Q1 Here is a diagram for the model.



w	x	y
$p \vee q$	$p \vee q$	$p \vee q$
$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg(p \leftrightarrow q)$
$\neg(p \wedge \neg q)$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
$\neg(\neg p \wedge q)$	$\neg(\neg p \wedge q)$	$\neg p \wedge q$
$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q)$
$\neg(\neg p \wedge \neg q)$	$\neg(\neg p \wedge \neg q)$	$\neg(\neg p \wedge \neg q)$

Here is one way to arrange the propositions in a diagram



Q2. i) $\Diamond p, \Diamond q \vdash \Diamond(p \wedge q)$ has a counterexample.

w	v
$p \ \neg q$	$\neg p \ q$
$\Diamond p$	$\Diamond p$
$\Diamond q$	$\Diamond q$
$\neg(p \wedge q)$	$\neg(p \wedge q)$
$\neg\Diamond(p \wedge q)$	$\neg\Diamond(p \wedge q)$

$W = \{w, v\}$

$V(p, w) = V(q, v) = 1$

$V(p, v) = V(q, w) = 0$

Here, $\Diamond p, \Diamond q$ are true at w (& v)
& false at w (& v)

ii) $\Diamond p, \Box q \vdash \Diamond(p \wedge q)$ has no counterexample.

If we have any counterexample, we need a world in that model that looks like this.

w
$\Diamond p$
$\Box q$
$\neg\Diamond(p \wedge q)$

That means we need to have q true at world w — and at all worlds, p true at some world, & $p \wedge q$ true at no world — that can't happen — we'd have to have some world v like this:

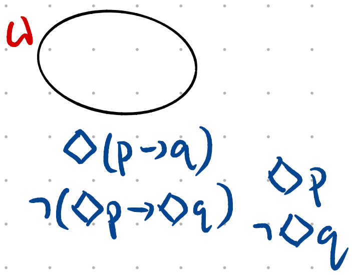
w	v
$\Diamond p$	p
$\Box q$	q
$\neg\Diamond(p \wedge q)$	$\neg(p \wedge q)$

And this violates the rule for interpreting $\wedge$.

So, there is no counterexample.

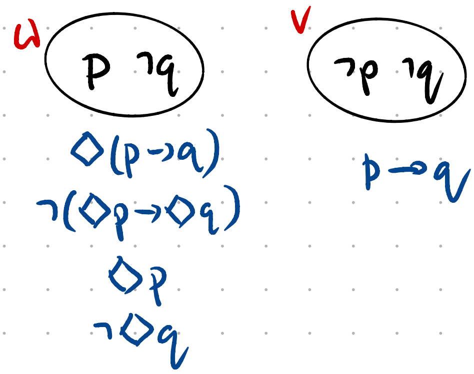
Q2 iii) $\Diamond(p\to q) \vdash \Diamond p \to \Diamond q$ has a counterexample.

To find it, start with a world w where $\Diamond(p\to q)$ is true, and $\Diamond p \to \Diamond q$ is false, like this:



$\neg\Diamond q$ at w means that q is false at w and at every world.

But $\Diamond p$ at w means we need a world where p is true, and $\Diamond(p\to q)$ means we need a world where $p\to q$ is true. We can satisfy this with one extra world:

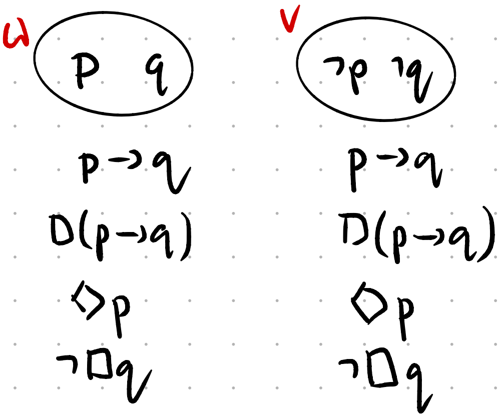


Here, p is true at w, while $p\to q$ is true at v. Our model is

$W=\{w,v\}$ $\quad V(p,w)=1$

$V(p,v)=V(q,w)=V(q,v)=0.$

iv) $\Box(p\to q) \vdash \Diamond p \to \Box q$ has a counterexample. Here's one:



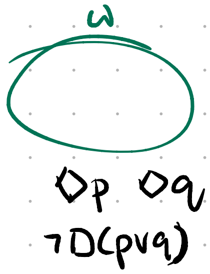
$W=\{w,v\}$

$V(p,w)=V(q,w)=1$

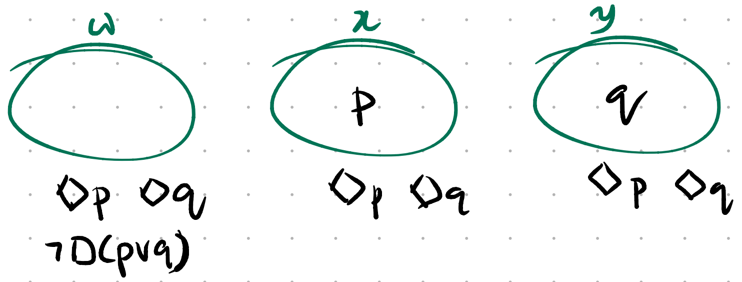
$V(p,v)=V(q,v)=0$

d) $\Box(p \to q) \vdash \Diamond p \to \Diamond q$ has no counterexample. Here's why. If we had a model with a world w where $\Box(p\to q)$ is true & $\Diamond p \to \Diamond q$ is false, then at w, $\Diamond p$ is true & $\Diamond q$ is false. This would mean that $p \to q$ is true at every world, p is true at some world, and q is true at no worlds. That can't happen — if p is true at some world, & q isn't true there, then $p \to q$ would be false there. So, there is no counterexample to this argument.

Q3. i) We start with one world, where $\Diamond p$, $\Diamond q$ are true & $\Box(p \vee q)$ is false.



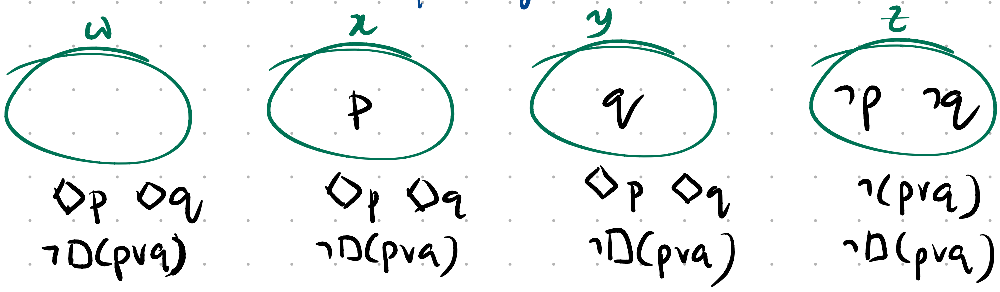
To make $\Diamond p$ & $\Diamond q$ true, we need worlds where p & q are true, so we make one of each



That makes $\Diamond p$ & $\Diamond q$ true in each world.

Now we need to check that $\neg\Box(p \vee q)$ can be true in w.

We need to have some world where $p \vee q$ is not true. Make a new world for that & we get.



And here, $\neg\Box(p \vee q)$ is now true in every world too.

To make this a model, we need to settle the values for p & q at every world. One choice is to make them false everywhere they aren't already true:

w	x	y	z
$\neg p \quad \neg q$	$p \quad \neg q$	$\neg p \quad q$	$\neg p \quad \neg q$
$\Diamond p \quad \Diamond q$	$\Diamond p \quad \Diamond q$	$\Diamond p \quad \Diamond q$	$\neg(p \vee q)$
$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$

This model is a counterexample to the argument.

It could be smaller. Notice that w & z make the same atoms true: we don't need both:

w	x	y
$\neg p \quad \neg q$	$p \quad \neg q$	$\neg p \quad q$
$\Diamond p \quad \Diamond q$	$\Diamond p \quad \Diamond q$	$\Diamond p \quad \Diamond q$
$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$

But we can make it even smaller! The world where p is true & the world where q is true don't need to be different either!

w	x
$\neg p \quad \neg q$	$p \quad q$
$\Diamond p \quad \Diamond q$	$\Diamond p \quad \Diamond q$
$\neg\Box(p \vee q)$	$\neg\Box(p \vee q)$

This is as small as we can get. No one world model will do.

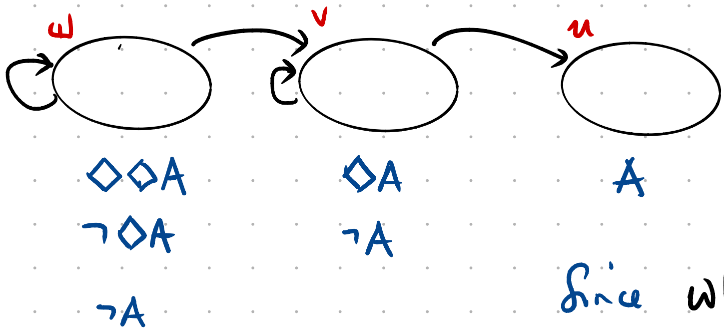
Q3 ii) I will not spell this out as much. We start with a world where $\Box(p \vee q)$ is true & $\Box p \vee \Box q$ is false, so $\Box p, \Box q$ are both false. This means we need a world where p is false, and since $\Box(p \vee q)$ is true at the original world $p \vee q$ is true at this new world, so q must be true. Similarly, we need a world where p is true & q is false. — But now we see these two new worlds are all we need: This model will do.

w	v	u
$\Box(p \vee q)$	$\neg p, q$	$p, \neg q$
$\neg\Box p, \neg\Box q$		

u	v
$\neg p, q$	$p, \neg q$

Q 4 i) $\Diamond\Diamond A \models_{S4} \Diamond A$. Here is why: Any counterexample would be a model with a world where $\Diamond\Diamond A$ is true & $\Diamond A$ is false. If $\Diamond A$ is false there, we have no world accessible from there where A is true. But $\Diamond\Diamond A$ being true there means there must be an accessible world from there where $\Diamond A$ is true, which means there is a world accessible from that world, where A is true. But accessibility is transitive – so this is impossible.

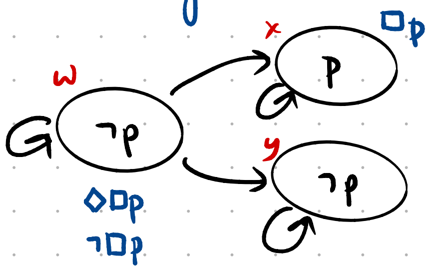
- In a diagram:



Since wRu, we have a clash between $\neg\Diamond A$ at w and A at u.

ii) $\Diamond\Box A \models_{S5} \Box A$. There is no S5 model where $\Diamond\Box A$ is true at a world & $\Box A$ is not true at that world, for if $\Diamond\Box A$ is true at a world, then $\Box A$ is true at some world, & so A is true at all worlds — & hence, $\Box A$ is true at all worlds, — contradicting what we assumed.

iii) $\Diamond\Box A \not\models_{S4} \Box A$ This argument has a counterexample in an S4 model.



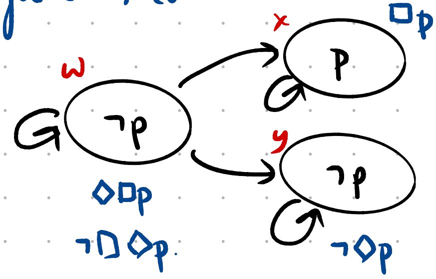
Here, $\Box p$ is true at x, & since wRx, $\Diamond\Box p$ is true at w. But since wRy too, and p is false at y, we don't have $\Box p$ at w.

$W = \{w, x, y\}$ wRw, wRx, wRy, xRx, yRy.
$V(p, x) = 1$, $V(p, w) = V(p, y) = 0$

(It is crucial that we don't have xRy, so we have $\Box p$ at x.)

iv) $\Diamond\Box A \vDash_{S5} \Box\Diamond A$ There is no S5 model where $\Diamond\Box A$ is true at a world & $\Box\Diamond A$ is not true at that world, for if $\Diamond\Box A$ is true at a world, then $\Box A$ is true at some world, & so A is true at all worlds — & hence, $\Diamond A$ is true at all worlds, and so is $\Box\Diamond A$, contradicting what we had assumed.

v) $\Diamond\Box A \nvDash_{S4} \Box\Diamond A$ Our model from d) is a counterexample to this argument, too.

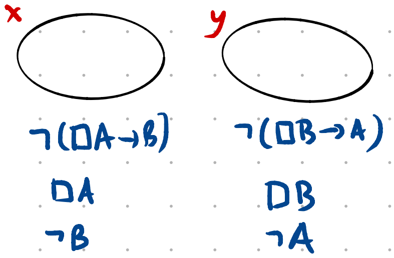


$W = \{w, x, y\}$

$wRw, wRx, wRy, xRx, yRy.$

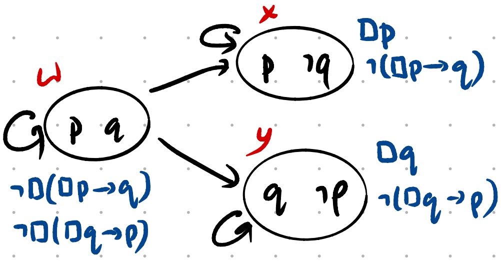
$V(p,x)=1,\ V(p,w)=V(p,y)=0$

vi) $\vDash_{S5} \Box(\Box A \to B) \vee \Box(\Box B \to A)$ There is no counterexample to this formula in an S5 model. To make the formula false at a world we need to falsify $\Box(\Box A \to B)$ & $\Box(\Box B \to A)$, which means that we need one world where $\Box A \to B$ fails & another where $\Box B \to A$ fails. This is impossible —



$\Box A$ at x contradicts $\neg A$ at y, and $\Box B$ at y contradicts $\neg B$ at x.

vii) $\nvDash_{S4} \Box(\Box A \to B) \vee \Box(\Box B \to A)$ Here is a counterexample:



$W = \{w, x, y\}$

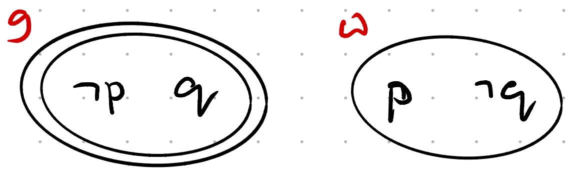
$wRw, wRx, wRy, xRx, yRy.$

$V(p,w) = V(p,x)$
$= V(q,w) = V(q,y) = 1$

$V(q,x) = V(p,y) = 0$

PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS WEEK 5

Q1. (i) $\Diamond(p \wedge \mathbb{A}q) \nvDash_{SSA} \mathbb{A}p \wedge \Diamond q$. Here is a counterexample.



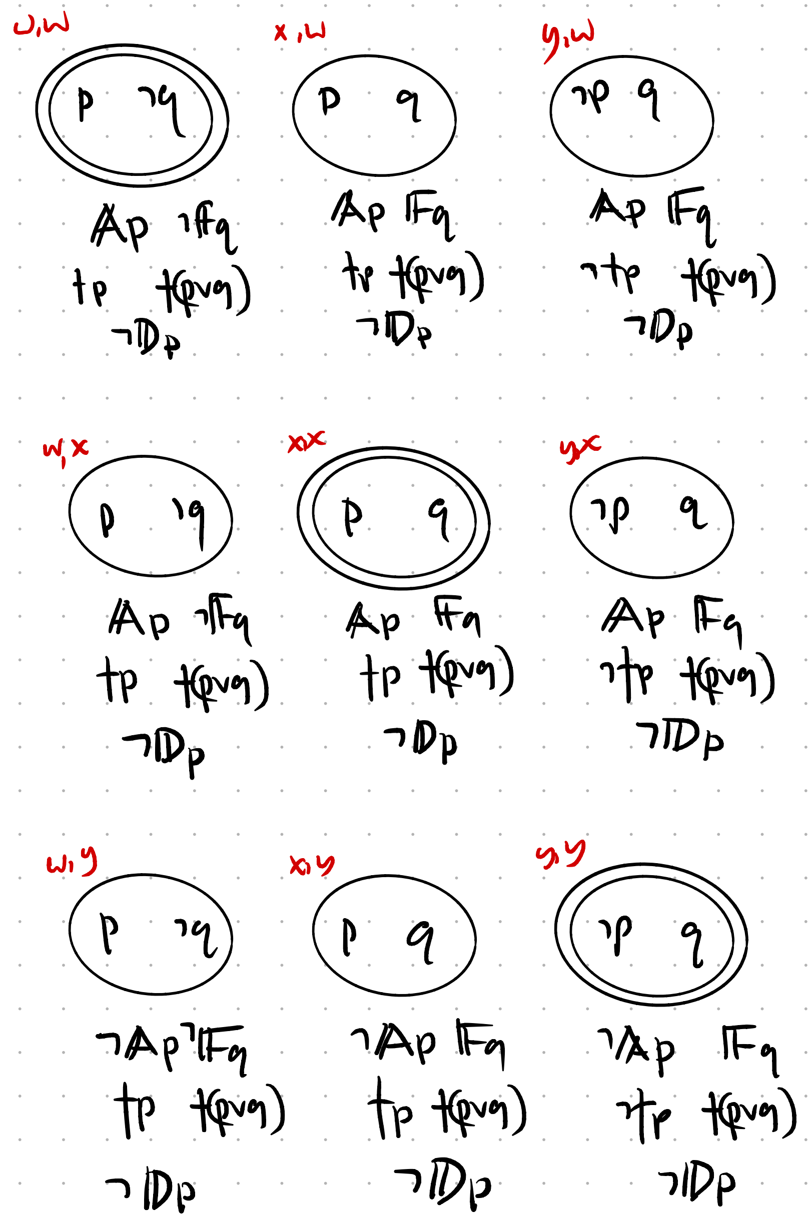
g	ω
$\mathbb{A}q$	$\mathbb{A}q$
$\neg(p \wedge \mathbb{A}q)$	$p \wedge \mathbb{A}q$
$\Diamond(p \wedge \mathbb{A}q)$	$\Diamond(p \wedge \mathbb{A}q)$
$\neg\mathbb{A}p$	$\neg\mathbb{A}p$
$\neg(\mathbb{A}p \wedge \Diamond q)$	$\neg(\mathbb{A}p \wedge \Diamond q)$

At world ω, or at world g, $\Diamond(p \wedge \mathbb{A}q)$ is true, while $\mathbb{A}p \wedge \Diamond q$ is false. The argument is invalid.

(ii) $\Diamond(p \wedge \mathbb{A}q) \vDash_{SSA} \Diamond p \wedge \mathbb{A}q$. Here is why the argument is valid.

Suppose $\Diamond(p \wedge \mathbb{A}q)$ is true at a world ω. This means that there is some world v where $p \wedge \mathbb{A}q$ is true. So at this world, p is true and so is $\mathbb{A}q$. So, at the actual world g, q is true. This means at our original world ω, $\mathbb{A}q$ is true, and since p is true at v, $\Diamond p$ is true at ω too. So, $\Diamond p \wedge \mathbb{A}q$ is true at ω — the argument, therefore is valid, since in any model, in any world, if $\Diamond(p \wedge \mathbb{A}q)$ is true there, so is $\Diamond p \wedge \mathbb{A}q$.

Q2 Here is a diagram for this double indexed model.



Q3(i)

$$\dfrac{\dfrac{\Box p}{p}\,\Box E \qquad \dfrac{\Box q}{q}\,\Box E}{\dfrac{p\wedge q}{\Box(p\wedge q)}}\,\wedge I$$

(ii)

$$\dfrac{\Diamond\Diamond p \qquad \dfrac{[\Diamond p]^1 \quad \neg\Diamond p}{\bot}\,\neg E}{\bot}\,\Diamond E^1$$

(iii)

$$\dfrac{\dfrac{\dfrac{\Diamond\Diamond p \qquad \dfrac{[\Diamond p]^1 \quad [\neg\Diamond p]^2}{\bot}\,\neg E}{\bot}\,\Diamond E^1}{\neg\neg\Diamond p}\,\neg I^2}{\Diamond p}\,DNE$$

PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS

WEEK 7

1) iv) $(\exists x Fx \land Gx)[x/y]$ is $\exists x Fx \land Gy$

v) $((Fx \to \exists y(Lxy \land (Gz \lor \forall x Hx)))[x/z])[z/a]$ is

$(Fz \to \exists y(Lzy \land (Gz \lor \forall x Hx)))[z/a]$ which is

$Fa \to \exists y(Lay \land (Ga \lor \forall x Hx))$.

What is wrong with $\forall x(Fx \to Lxy)[y/x]$?

The variable x is not free for y in $\forall x(Fx \to Lxy)$, since the highlighted y is inside the scope of the $\forall x$.

2)

$$\dfrac{\dfrac{\dfrac{\forall x(Fx \land \exists y\neg Fy)}{Fa \land \exists y \neg Fy}\,\forall E}{\exists y\neg Fy}\,\land E \qquad \dfrac{[\neg Fa]^1 \qquad \dfrac{\dfrac{\forall x(Fx\land\exists y\neg Fy)}{Fa\land\exists y\neg Fy}\,\forall E}{Fa}\,\land E}{\bot}\,\neg E}{\bot}\,\exists E^1$$

This is a proof. The eigenvariable condition is satisfied in the $\exists E$ step.

(The name a doesn't appear in the highlighted parts.)

$$
\dfrac{
  \dfrac{\forall x(Fx \vee Gx)}{Fa \vee Ga}\;\forall E
  \qquad
  \dfrac{\dfrac{[Fa]^1}{\forall x Fx}\;\forall I}{\forall x Fx \vee \forall x Gx}\;\vee I
  \qquad
  \dfrac{\dfrac{[Ga]^2}{\forall x Gx}\;\forall I}{\forall x Fx \vee \forall x Gx}\;\vee I
}{\forall x Fx \vee \forall x Gx}\;\vee E^{1,2}
$$

This is ~~not~~ not a proof. In the two $\forall I$ steps, the eigenvariable condition is not satisfied — the a is present in the assumptions of the $\forall I$ inferences.

(We'd hope this isn't a proof! $\forall x(Fx \vee Gx)$ shouldn't give us $\forall x Fx \vee \forall x Gx$!)

$$
\dfrac{
\dfrac{
  \dfrac{\forall x(Fx \vee Gx)}{Fa \vee Ga}\;\forall E
  \qquad
  \dfrac{[Fa]^2}{Fa \vee Ha}\;\vee I
  \qquad
  \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Ga]^3 \quad [\neg Ha]^1}{Ga \wedge \neg Ha}\;\wedge I}{\exists x(Gx \wedge \neg Hx)}\;\exists I \qquad \neg\exists x(Gx \wedge \neg Hx)}{\bot}\;\neg E}{\neg\neg Ha}\;\neg I^1}{Ha}\;DNE}{Fa \vee Ha}\;\vee I
}{Fa \vee Ha}\;\vee E^{2,3}
}{\forall x(Fx \vee Hx)}\;\forall I
$$

This is a proof: the assumptions on which $Fa \vee Ha$ depends are $\forall x(Fx \vee Gx)$ & $\neg\exists x(Gx \wedge \neg Hx)$ — and these don't involve a. The eigenvariable conditions are satisfied.

iv)

$$
\begin{prooftree}
\AxiomC{$p \wedge \exists x Fx$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\exists x Fx$}
\AxiomC{$p \wedge \exists x Fx$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\AxiomC{$[Fa]^1$}
\RightLabel{$\wedge I$}
\BinaryInfC{$p \wedge Fa$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(p \wedge Fx)$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\exists x(p \wedge Fx)$}
\end{prooftree}
$$

v)

$$
\begin{prooftree}
\AxiomC{$[Rab]^2$}
\AxiomC{$[Rba]^1$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Rab \wedge Rba$}
\AxiomC{$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$\forall y \forall z((Ray \wedge Ryz) \to Raz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$\forall z((Rab \wedge Rbz) \to Raz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$(Rab \wedge Rba) \to Raa$}
\RightLabel{$\to E$}
\BinaryInfC{Raa}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x Rxx$}
\AxiomC{$\neg \exists x Rxx$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg Rba$}
\RightLabel{$\to I^2$}
\UnaryInfC{$Rab \to \neg Rba$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y(Ray \to \neg Rya)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x \forall y(Rxy \to \neg Ryx)$}
\end{prooftree}
$$

4) i)

$$
\begin{prooftree}
\AxiomC{$\forall x(p \vee Fx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$p \vee Fa$}
\AxiomC{$[\neg(p \vee \forall x Fx)]^3$}
\AxiomC{$[p]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \forall x Fx$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{Fa}
\AxiomC{$[Fa]^1$}
\RightLabel{$\vee E$}
\TrinaryInfC{Fa}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x Fx$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \forall x Fx$}
\AxiomC{$[\neg(p \vee \forall x Fx)]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg\neg(p \vee \forall x Fx)$}
\RightLabel{DNE}
\UnaryInfC{$p \vee \forall x Fx$}
\end{prooftree}
$$

iii)

$$
\begin{prooftree}
\AxiomC{$\neg \exists x(Gx \wedge \neg Hx)$}
\AxiomC{$\neg \exists x(Fx \wedge \neg Gx)$}
\AxiomC{$[\neg(\neg Fa \vee Ha)]^4$}
\AxiomC{$[\neg Fa]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fa$}
\RightLabel{DNE}
\UnaryInfC{Fa}
\AxiomC{$[\neg Ga]^2$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Fa \wedge \neg Ga$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \wedge \neg Gx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg Ga$}
\RightLabel{DNE}
\UnaryInfC{Ga}
\AxiomC{$[\neg(\neg Fa \vee Ha)]^4$}
\AxiomC{$[Ha]^3$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg Ha$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Ga \wedge \neg Ha$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Gx \wedge \neg Hx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^4$}
\UnaryInfC{$\neg\neg(\neg Fa \vee Ha)$}
\RightLabel{DNE}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(\neg Fx \vee Hx)$}
\end{prooftree}
$$

5 iii)

$$
\begin{prooftree}
\AxiomC{$[\neg\exists x(Fx\to\forall y Fy)]^6$}
\AxiomC{$[\neg\exists x(Fx\to\forall y Fy)]^6$}
\AxiomC{$[\neg\forall y Fy]^5$}
\AxiomC{$[\neg\exists y\neg Fy]^2$}
\AxiomC{$[\neg Fb]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists y\neg Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fb$}
\RightLabel{DNE}
\UnaryInfC{Fb}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg\exists y\neg Fy$}
\RightLabel{DNE}
\UnaryInfC{$\exists y\neg Fy$}
\AxiomC{$[\neg Fa]^4$}
\AxiomC{$[Fa]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I^3$}
\UnaryInfC{$Fa\to\forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\RightLabel{$\exists E^4$}
\BinaryInfC{$\exists x(Fx\to\forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^5$}
\UnaryInfC{$\neg\neg\forall y Fy$}
\RightLabel{DNE}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I$ (vacuous)}
\UnaryInfC{$Fa\to\forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^6$}
\UnaryInfC{$\neg\neg\exists x(Fx\to\forall y Fy)$}
\RightLabel{DNE}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\end{prooftree}
$$

PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS
WEEK 8

1 i) $\exists x Fx, \exists x Gx \succ \exists x(Fx \wedge Gx)$ has a counterexample.

$D = \{a, b\}$

	$I(F)$	$I(G)$
a	1	0
b	0	1

x:a	x:b	
Fx	$\neg Fx$	$\exists x Fx$
$\neg Gx$	Gx	$\exists x Gx$
$\neg(Fx \wedge Gx)$	$\neg(Fx \wedge Gx)$	$\neg\exists x(Fx \wedge Gx)$

ii) $\exists x Fx, \forall x Gx \succ \exists x(Fx \wedge Gx)$ has no counterexample:
If everything in the domain has property G, then whatever has property F, must be a thing which is both F & G.

Here is a CQ proof:

$$\dfrac{\exists x Fx \qquad \dfrac{\dfrac{[Fa]^1 \qquad \dfrac{\forall x Gx}{Ga}\,\forall E}{Fa \wedge Ga}\,\wedge I}{\exists x(Fx \wedge Gx)}\,\exists I}{\exists x(Fx \wedge Gx)}\,\exists E^1$$

(iii) $\vdash \exists x(Fx \to \forall y Fy)$ This has no counterexample: To show that $\exists x(Fx \to \forall y Fy)$ is always true, notice this: either everything in the domain is an F, in which case $\forall y Fy$ is true, & so, for any object at all $Fx \to \forall y Fy$ is true, & hence, so is $\exists x(Fx \to \forall y Fy)$.

Otherwise, not everything is F, so pick something that isn't F. If x is assigned that object as a value, Fx is false, & so $Fx \to \forall y Fy$ is true, & so $\exists x(Fx \to \forall y Fy)$ is true here, too.

So, regardless, in any model, $\exists x(Fx \to \forall y Fy)$ is true, & the argument has no counterexample.

A CQ-proof for this formula is hard to construct. Here it is (from last week's solutions).

$$
\begin{prooftree}
\AxiomC{$[\neg\exists x(Fx\to\forall y Fy)]^6$}
\AxiomC{$[\neg\forall y Fy]^5$}
\AxiomC{$[\neg\exists y\neg Fy]^2$}
\AxiomC{$[\neg Fb]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists y\neg Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fb$}
\RightLabel{DNE}
\UnaryInfC{Fb}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg\exists y\neg Fy$}
\RightLabel{DNE}
\UnaryInfC{$\exists y\neg Fy$}
\AxiomC{$[\neg Fa]^4$}
\AxiomC{$[Fa]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I^3$}
\UnaryInfC{$Fa\to\forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\RightLabel{$\exists E^4$}
\BinaryInfC{$\exists x(Fx\to\forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^5$}
\UnaryInfC{$\neg\neg\forall y Fy$}
\RightLabel{DNE}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I$ (vacuous)}
\UnaryInfC{$Fa\to\forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\AxiomC{$[\neg\exists x(Fx\to\forall y Fy)]^6$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^6$}
\UnaryInfC{$\neg\neg\exists x(Fx\to\forall y Fy)$}
\RightLabel{DNE}
\UnaryInfC{$\exists x(Fx\to\forall y Fy)$}
\end{prooftree}
$$

(iv) $\forall x \exists y Rxy \vdash \forall y \exists x Rxy$ Has counterexamples. Here is a simple one:

$D = \{a, b\}$

$I(R)$	a	b
a	1	0
b	1	0

x: a y: a Rxy $\exists y Rxy$ $\forall x \exists y Rxy$ $\exists x Rxy$ $\neg\forall y \exists x Rxy$	x: a y: b $\neg Rxy$ $\exists y Rxy$ $\forall x \exists y Rxy$ $\neg\exists x Rxy$ $\neg\forall y \exists x Rxy$
x: b y: a Rxy $\exists y Rxy$ $\forall x \exists y Rxy$ $\exists x Rxy$ $\neg\forall y \exists x Rxy$	x: b y: b $\neg Rxy$ $\exists y Rxy$ $\forall x \exists y Rxy$ $\neg\exists x Rxy$ $\neg\forall y \exists x Rxy$

(v) $\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy)), \forall x \exists y Rxy, \forall x \neg Rxx \vdash \bot$

This has counterexamples. Here is one:

$D = \{a, b, c\}$

$I(R)$	a	b	c
a	0	1	1
b	1	0	1
c	1	1	0

In this model $\forall x \exists y Rxy$ & $\forall x \neg Rxx$ are both true. So is $\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy))$.

$\forall x \exists y Rxy$ is true since each column contains a 1.

$\forall x \neg Rxx$ is true since the diagonal is all 0.

$\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy))$ is true because in this model R is difference (Rxy iff $x \neq y$) & in this model, whenever x & y are different there is something that differs from both x & y — since there are 3 things in this domain.

2) In the model with the domain $D = \{0,1,2,3,\ldots\}$ of all natural numbers, where O is true of the odd numbers, Sxy is true when x is smaller than y, and a_n names the number n.

a) Oa_1 true Sa_1a_1 false
Oa_2 false Sa_1a_2 true
Sa_2a_1 false

b) $\forall y Sa_0y$ is false: Sa_0y is false when $y: 0$.

c) $\exists x \forall y Sxy$ is false: Sxy is false when $y: 0$ no matter what value x is assigned.

d) $\exists x Sxa_1$ is true: Sxa_1 is true when $x: 0$

e) $\exists y Sa_{10\,000}y$ is true: $Sa_{10\,000}y$ is true when $x: 10\,001$, for example.

f) $\forall x \exists y Sxy$ is true: No matter what value x is assigned, $\exists y Sxy$ is true, since Sxy is true when y is assigned a number larger than whatever x is assigned.

g) $\forall x (Ox \to \exists y Syx)$ is true: Assign x any value.
If the value is 0, then Ox is false, so $Ox \to \exists y Syx$ is true.
If the value isn't 0, then $\exists y Syx$ is true, since Syx is true when y is assigned 0 (given ~~that~~ x isn't assigned 0), so in this case $Ox \to \exists y Syx$ is also true. So $\forall x(Ox \to \exists y Syx)$ is true.

h) $\exists y \forall x (Ox \to Syx)$ is true for similar reasons: $\forall x (Ox \to Syx)$ is true when y is assigned 0.

i) $\forall x \forall y (Sxy \to \exists z(Sxz \wedge Szy))$ is false: Assign $x: 0$ & $y: 1$. There is no z where Sxz & Szy are both true.

PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS WEEK 9

Q1. A good one-paragraph explanation for how the *soundness* theorem is proved should contain the following ideas:

a) *A statement of what is to be proved*: that if there is a CQ proof from *X* to *A* then there is no first-order model that serves as a counterexample to the argument from *X* to *A*.

b) The *structure* of how we prove this: by induction on the construction of the proof from *X* to *A*.

c) The base case: There is no counterexample to an argument from *A* to *A* (an assumption).

d) The induction steps: We show that a proof or proofs constructed so far has no counterexample, neither does a proof we can construct from these using one inference rule. (An *excellent* answer will give some sense of the distinctive behaviour of the quantifier rules, and the role of the semantic substitution lemma.)

Q2. A *good* one-paragraph explanation for how the *completeness* theorem is proved should contain the following ideas:

a) *A statement of what is to be proved*: that if there is no CQ proof from *X* to *A* then there is some first-order model that serves as a counterexample to the argument from *X* to *A*.

b) The *structure* of how we prove this: by first constructing a *witnessed maximal A-avoiding set X'* that extends the set X, then showing that we can use *X'* so-defined to construct a model in which each member of *X* is true and *A* is not, the counterexample we're after.

c) An indication of how the *witnesses* are found: we construct *X'* in a language in with a supply of new names, and witnesses are added whenever we add an existentially quantified formula to our set.

d) How the *model* is defined from *X'*. The domain consists of the terms in the new language, and we interpret a predicate *F* as holding of the object *t* iff *Ft* is in *X'*. The witnessing condition is required to show that a quantified formula is in X' iff it holds in the model we have defined.

2. If x is not free in B, $B[x/n] = B$, and $(A \to B)[x/n]$ is $A[x/n] \to B$

So we have this proof

$$
\dfrac{
  [\exists x\, A]^2
  \qquad
  \dfrac{
    \dfrac{\forall x(A \to B)}{A[x/n] \to B}\,\forall E
    \qquad
    [A[x/n]]^1
  }{B}\,\to E
}{
  \dfrac{B}{\exists x A \to B}\,\to I^2
}\,\exists E^1
$$

Therefore: $\forall x(A \to B) \vdash_{fo} \exists x A \to B$.

Conversely, we have this proof,

$$
\dfrac{
  \dfrac{
    \dfrac{
      \exists x A \to B
      \qquad
      \dfrac{[A[x/n]]^1}{\exists x A}\,\exists I
    }{B}\,\to E
  }{A[x/n] \to B}\,\to I
}{\forall x(A \to B)}\,\forall I
$$

so we also have

$\exists x A \to B \vdash_{fo} \forall x(A \to B)$

& so, the two formulas are equivalent.

Similarly, here are proofs for the other equivalences:

$(A \to B)[x/n] = A \to B[x/n]$, as x isn't free in A.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(A \to B)}{A \to B[x/n]}\,\forall E \qquad [A]^1}{B[x/n]}\,\to E}{\forall x B}\,\forall I}{A \to \forall x B}\,\to I^1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to \forall x B \qquad [A]^1}{\forall x B}\,\to E}{B[x/n]}\,\forall E}{A \to B[x/n]}\,\to I}{\forall x(A \to B)}\,\forall I$$

$$\dfrac{\exists x(A \land B) \qquad \dfrac{\dfrac{[A \land B[x/n]]^1}{A}\,\land E \qquad \dfrac{\dfrac{[A \land B[x/n]]^1}{B[x/n]}\,\land E}{\exists x B}\,\exists I}{A \land \exists x B}\,\land I}{A \land \exists x B}\,\exists E^1$$

$$\dfrac{\dfrac{\exists x A \land B}{\exists x A}\,\land E \qquad \dfrac{\dfrac{[A[x/n]]^1 \qquad \dfrac{\exists x A \land B}{B}\,\land E}{A[x/n] \land B}\,\land I}{\exists x(A \land B)}\,\exists I}{\exists x(A \land B)}\,\exists E^1$$

$$\dfrac{\exists x(A\to B) \qquad \dfrac{\dfrac{\dfrac{[A\to B[x/n]]^2 \quad [A]^1}{B[x/n]}\,{\to}E}{\exists x B}\,\exists I}{A\to\exists x B}\,{\to}I}{A\to\exists x B}\,\exists I^2$$

So $\exists x(A\to B)\vdash_{CQ} A\to\exists x B$.

We cannot prove $A\to\exists x B\vdash_{CQ}\exists x(A\to B)$ without DNE (and the proof is complex)

Here is a semantic argument, showing $A\to\exists x B\models_{FO}\exists x(A\to B)$

Suppose (M,v) is a model & variable assignment where $A\to\exists x B$ is true. Suppose $\exists x(A\to B)$ is <u>false</u> in (M,v). Then for every x variant v' of v, $A\to B$ is false in (M,v'). The variable x isn't free in A, so this means that A is true in (M,v), and B is false in (M,v') for each v', ie $\exists x B$ is false in (M,v), which contradicts the truth of $A\to\exists x B$.

$$\dfrac{\forall x A\vee B \qquad \dfrac{\dfrac{\dfrac{[\forall x A]^1}{A[x/n]}\,\forall E}{A[x/n]\vee B}\,\vee I}{\forall x(A\vee B)}\,\forall I \qquad \dfrac{\dfrac{[B]^2}{A[x/n]\vee B}\,\vee I}{\forall x(A\vee B)}\,\forall I}{\forall x(A\vee B)}\,\vee E^{1,2}$$

So $\forall x A\vee B\vdash_{CQ}\forall x(A\vee B)$

The converse is easiest to prove semantically

Suppose $\forall x(A\vee B)$ holds in (M,v), where x is not free in B. So, $A\vee B$ holds in (M,v') for every x-variant v' of v. If B holds in any of these, it holds in all of them (as x isn't free in B), & so, if it holds in <u>none</u> A holds in all, so in (M,v) $\forall x A\vee B$, as desired.

$$
\dfrac{
  \dfrac{\dfrac{\dfrac{\forall x(A\wedge B)}{A[x/n]\wedge B[x/n]}\,\forall E}{A[x/n]}\,\wedge E}{\forall x A}\,\forall I
  \qquad
  \dfrac{\dfrac{\dfrac{\forall x(A\wedge B)}{A[x/n]\wedge B[x/n]}\,\forall E}{B[x/n]}\,\wedge E}{\forall x B}\,\forall I
}{\forall x A\wedge\forall x B}\,\wedge I
\qquad\qquad
\dfrac{\dfrac{
  \dfrac{\dfrac{\forall x A\wedge\forall x B}{\forall x A}\,\wedge E}{A[x/n]}\,\forall E
  \qquad
  \dfrac{\dfrac{\forall x A\wedge\forall x B}{\forall x B}\,\wedge E}{B[x/n]}\,\forall E
}{A[x/n]\wedge B[x/n]}\,\wedge I}{\forall x(A\wedge B)}\,\forall I
$$

$$
\dfrac{
  \exists x A\vee\exists x B
  \qquad
  \dfrac{[\exists x A]^3 \qquad \dfrac{\dfrac{[A[x/n]]^1}{A[x/n]\vee B[x/n]}\,\vee I}{\exists x(A\vee B)}\,\exists I}{\exists x(A\vee B)}\,\exists E^1
  \qquad
  \dfrac{[\exists x B]^4 \qquad \dfrac{\dfrac{[B[x/n]]^2}{A[x/n]\vee B[x/n]}\,\vee I}{\exists x(A\vee B)}\,\exists I}{\exists x(A\vee B)}\,\exists E^2
}{\exists x(A\vee B)}\,\vee E^{3,4}
$$

$$
\dfrac{
  \exists x(A\vee B)
  \qquad
  \dfrac{[A[x/n]\vee B[x/n]]^3 \qquad \dfrac{\dfrac{[A[x/n]]^1}{\exists x A}\,\exists I}{\exists x A\vee\exists x B}\,\vee I \qquad \dfrac{\dfrac{[B[x/n]]^2}{\exists x B}\,\exists I}{\exists x A\vee\exists x B}\,\vee I}{\exists x A\vee\exists x B}\,\vee E^{1,2}
}{\exists x A\vee\exists x B}\,\exists E^3
$$

These equivalences are all proved in Logical Methods (Theorem 35).

Here are the proofs!

$$
\begin{prooftree}
\AxiomC{$[\exists x A(x)]^2$}
\AxiomC{$\forall x \neg A(x)$}
\RightLabel{$\forall E$}
\UnaryInfC{$\neg A(b)$}
\AxiomC{$[A(b)]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg \exists x A(x)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\neg \exists x A(x)$}
\AxiomC{$[A(b)]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x A(x)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg A(b)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x \neg A(x)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\exists x \neg A(x)$}
\AxiomC{$[\neg A(b)]^1$}
\AxiomC{$[\forall x A(x)]^2$}
\RightLabel{$\forall E$}
\UnaryInfC{$A(b)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg \forall x A(x)$}
\end{prooftree}
$$

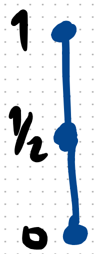
$$
\begin{prooftree}
\AxiomC{$\neg \forall x A(x)$}
\AxiomC{$[\neg \exists x \neg A(x)]^2$}
\AxiomC{$[\neg A(b)]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x \neg A(x)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg A(b)$}
\RightLabel{DNE}
\UnaryInfC{$A(b)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x A(x)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg \exists x \neg A(x)$}
\RightLabel{DNE}
\UnaryInfC{$\exists x \neg A(x)$}
\end{prooftree}
$$

To show that every formula is logically equivalent to a formula in prenex normal form, you use each of the equivalences we have proved, together with the equivalence of formulas under renaming of bound variables. ($\exists x A(x)$ is logically equivalent to $\exists y A(y)$, when x is free for y and y is free for x in A; and similarly for $\forall x A(x)$ and $\forall y A(y)$.)

Given any formula, we rewrite it into an equivalent formula in prenex normal form, by working from the innermost quantifiers, transforming the formula into an equivalent one in which the quantifier does not occur under any other connective. Here, we use the equivalences to push the quantifiers “outward”. A negated existential [universal] becomes a universal [existential] negation. In the case of binary connectives, for example, if I have $\forall x A \rightarrow B$, then I first replace $\forall x A$ by $\forall z A[x/z]$, where z is a variable that is not free in B, and then I can use the equivalence between $\forall z A[x/z] \rightarrow B$ and $\exists z(A[x/z] \rightarrow B)$ to transform this formula into one in which the quantifier is not inside the conditional. I keep going in this way, with all the quantifiers, until no quantifier is under any connective. The result is in prenex normal form.

PY2010 INTERMEDIATE LOGIC CLASS TASK SOLUTIONS
WEEK 10

Q1



$\to$	0	½	1
0	1	1	1
½	0	1	1
1	0	½	1

$\neg$	
0	1
½	0
1	0

i) $\neg\neg p \succ p$

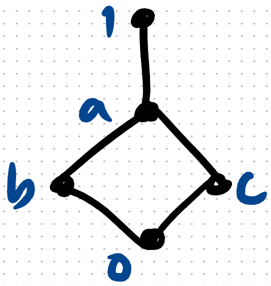
$\neg$	$\neg$	p		p
1	0	½		½

If $v(p) = \frac{1}{2}$, $v(\neg\neg p) = 1$, & so, this is a H_3 counterexample

ii) $\neg(p \wedge q) \succ \neg p \vee \neg q$

$\neg$	p	$\wedge$	q		$\neg$	p	$\vee$	$\neg$	q
1	0	0	0		1	0	1	1	0
1	0	0	½		1	0	1	0	½
1	0	0	1		1	0	1	0	1
1	½	0	0		0	½	1	1	0
0	½	½	½		0	½	0	0	½
0	½	½	1		0	½	0	0	1
1	1	0	0		0	1	1	1	0
0	1	½	½		0	1	0	0	½
0	1	1	1		0	1	0	0	1

This argument has no H_3 counterexample



$\to$	0	c	b	a	1
0	1	1	1	1	1
c	b	1	b	1	1
b	c	c	1	1	1
a	0	c	b	1	1
1	0	c	b	a	1

$\neg$	
0	1
c	b
b	c
a	0
1	0

$\neg(p \wedge q) \vdash \neg p \vee \neg q$

$1 \; b \; 0 \; c \qquad c \; b \; a \; b \; c$

If $v(p) = b$, $v(q) = c$,
$v(\neg(p\wedge q)) = 1$ &
$v(\neg p \vee \neg q) = a$,
So this is a H_5 counterexample.

iii) $\neg p \vee q \vdash p \to q$ has no Heyting counterexample

We have an intuitionistic proof!

$$\dfrac{\dfrac{\neg p \vee q \qquad \dfrac{\dfrac{[\neg p]^1 \quad [p]^3}{\bot}\,\neg E}{q}\,\bot E \qquad [q]^2}{q}\,\vee E}{p \to q}\,\to I^3$$

iv) $p \to q \vdash \neg p \vee q$ is H_3 invalid.

p	$\to$	q		$\neg$	p	$\vee$	q
$\frac{1}{2}$	1	$\frac{1}{2}$		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$v(p)=v(q)=\frac{1}{2}$ is a counterexample.

v) $\vdash \neg p \vee \neg\neg p$ has no H_3 counterexample:

$\neg$	p	$\vee$	$\neg$	$\neg$	p
1	0	1	0	1	0
0	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$
0	1	1	1	0	1

But it does have an H_5 counterexample.

$\neg$	p	$\vee$	$\neg$	$\neg$	p
c	b	a	b	c	b

When $v(p)=b$;

$v(\neg p \vee \neg\neg p) = a \ (\neq 1)$

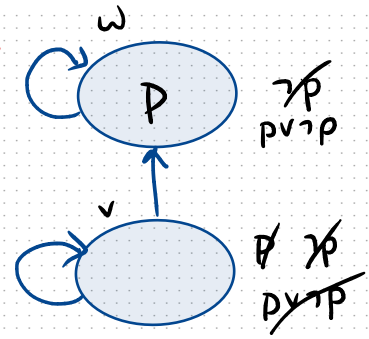
vi) $(p \to q) \to p \vdash p$ is H_3 invalid.

p	$\to$	q	$\to$	p		p
$\frac{1}{2}$	0	0	1	$\frac{1}{2}$		$\frac{1}{2}$

$v(p)=\frac{1}{2}$ $v(q)=0$ is a counterexample.

Q2

i)



Take this Kripke model where vRw, and

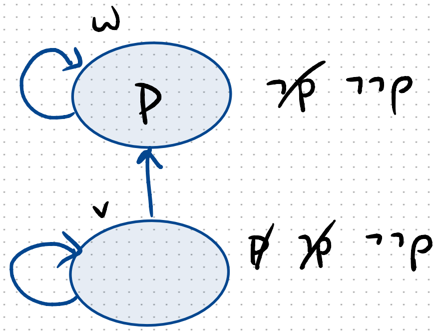
$V(p, v) = 0$, $V(p, w) = 1$

Then $V(\neg p, v) = 0$ (as p is true later, at w)

and so $V(p \vee \neg p, v) = 0$ too.

This model is a counterexample to $\succ p \vee \neg p$, at world v.

ii)

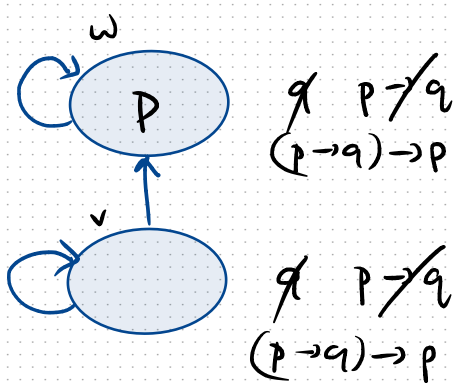


Since, in this model $V(\neg p, w) = V(\neg p, v) = 0$

we have $V(\neg\neg p, v) = 1$, so this model

is a counterexample to $\neg\neg p \succ p$ at world v.

iii)



If we also set $V(q, v) = V(q, w) = 0$,

the model is also a counterexample to

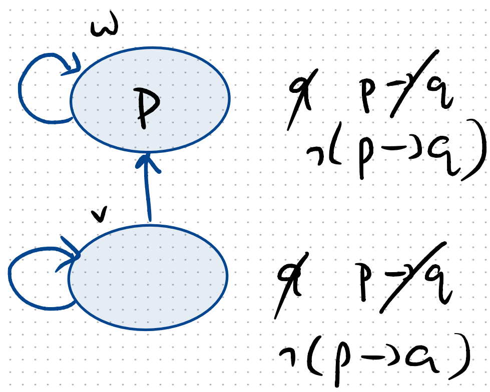
$(p \to q) \to p \succ p$ at v, since

$V(p \to q, v) = V(p \to q, w) = 0$ (as p is true at w

while q isn't) & so $V((p \to q) \to p, v) = 1$

while $V(p, v) = 0$.

iv)



In this same model,

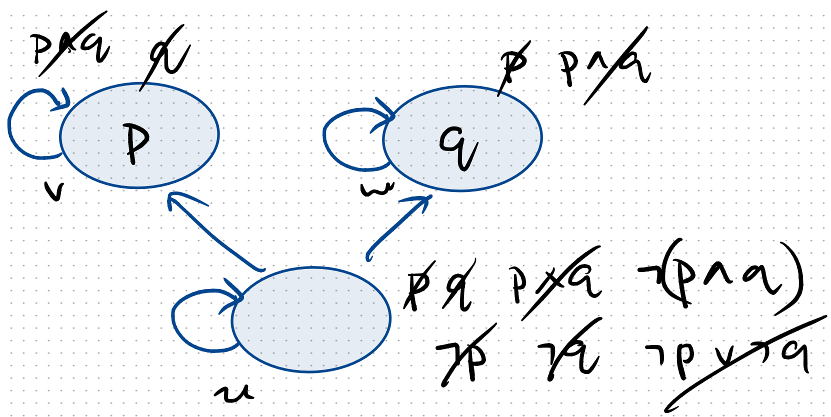
$V(\neg(p \to q), v) = 1$ (since $p \to q$

is false everywhere) &

so, v is a counterexample

to $\neg(p \to q) \succ p$

v)



Take this model, with $W = \{u, v, w\}$, where uRv, uRw but $v \not R w$ $w \not R v$.

Set p true at v only, while q is true at w only. So $p \wedge q$ is true nowhere & $\neg(p \wedge q)$ is true everywhere. Since p is true at v & uRv, $\neg p$ isn't true at u, & similarly, since q is true at w & uRw, $\neg q$ isn't true at u. So $\neg p \vee \neg q$ isn't true at u either.

So in this model, $\neg(p \wedge q)$ holds at u while $\neg p \vee \neg q$ doesn't. It is a counterexample to the argument.