PY4612 CLASS SOLUTIONS §1

1) a) Some footballer isn't a biped: $\exists x(Fx \wedge \neg Bx)$

b) No footballer is a biped: $\neg\exists x(Fx \wedge Bx)$

(You could also use: $\forall x(Fx \rightarrow \neg Bx)$, but I think that the $\neg\exists x(Fx \wedge Bx)$ is a closer fit to the English: it is explicitly the negation of $\exists x(Fx \wedge Bx)$, which is "Some footballer is a biped.")

c) Some footballer is taller than Sócrates: $\exists x(Fx \wedge Txa)$

d) No footballer is taller than any biped older than Sócrates.

This one is a little tricky: "any" works interestingly in English. Here are two choices, one translating 'any' as 'some' and the other, as 'all'

$$\neg\exists x(Fx \wedge \exists y((By \wedge Oya) \wedge Txy))$$

this is: x is taller than some footballer older than Sócrates

$$\neg\exists x(Fx \wedge \forall y((By \wedge Oya) \rightarrow Txy))$$

this is: x is taller than every footballer older than Sócrates.

Both are possible translations of the English, but I think that the first one is how I understand the English sentence.

(Compare: No one answered any of the questions:

$$\neg\exists x(P_x \wedge \exists y(Qy \wedge Axy))$$)

2) a) $(Fx)[x/a]$ is Fa

b) $(Fx)[y/a]$ is Fx

c) $(\forall x(Fx \to Lxy))[y/b]$ is $\forall x(Fx \to Lxb)$

d) $(\exists x Fx \land Gx)[x/y]$ is $\exists x Fx \land Gy$

e) $((Fx \to \exists y(Lxy \land (Gz \lor \forall x Hx)))[x/z])[z/a]$ is

$(Fz \to \exists y(Lzy \land (Gz \lor \forall x Hx)))[z/a]$ which is

$Fa \to \exists y(Lay \land (Ga \lor \forall x Hx))$.

What is wrong with $\forall x(Fx \to Lxy)[y/x]$?

The variable x is not free for y in $\forall x(Fx \to Lxy)$, since the highlighted y is inside the scope of the $\forall x$.

3)

$$\dfrac{\dfrac{\dfrac{\forall x(Fx \land \exists y \neg Fy)}{Fa \land \exists y \neg Fy}\,\forall E}{\exists y \neg Fy}\,\land E \qquad \dfrac{[\neg Fa]^1 \qquad \dfrac{\dfrac{\forall x(Fx \land \exists y \neg Fy)}{Fa \land \exists y \neg Fy}\,\forall E}{Fa}\,\land E}{\bot}\,\neg E}{\bot}\,\exists E^1$$

This is a proof. The eigenvariable condition is satisfied in the $\exists E$ step.

(The name a doesn't appear in the highlighted parts.)

$$\dfrac{\dfrac{\forall x(Fx \lor Gx)}{Fa \lor Ga}\,\forall E \qquad \dfrac{\dfrac{[Fa]^1}{\forall x Fx}\,\forall I}{\forall x Fx \lor \forall x Gx}\,\lor I \qquad \dfrac{\dfrac{[Ga]^2}{\forall x Gx}\,\forall I}{\forall x Fx \lor \forall x Gx}\,\lor I}{\forall x Fx \lor \forall x Gx}\,\lor E^{1,2}$$

This is not a proof. In the two $\forall I$ steps, the eigenvariable condition is not satisfied — the a is present in the assumptions of the $\forall I$ inferences.

(We'd hope this isn't a proof! $\forall x(Fx \lor Gx)$ shouldn't give us $\forall x Fx \lor \forall x Gx$!)

$$
\begin{prooftree}
\AxiomC{$\forall x(Fx\vee Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa\vee Ga$}
\AxiomC{$[Fa]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$Fa\vee Ha$}
\AxiomC{$[Ga]^3$}
\AxiomC{$[\neg Ha]^1$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Ga\wedge\neg Ha$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Gx\wedge\neg Hx)$}
\AxiomC{$\neg\exists x(Gx\wedge\neg Hx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Ha$}
\RightLabel{DNE}
\UnaryInfC{Ha}
\RightLabel{$\vee I$}
\UnaryInfC{$Fa\vee Ha$}
\RightLabel{$\vee E^{2,3}$}
\TrinaryInfC{$Fa\vee Ha$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(Fx\vee Hx)$}
\end{prooftree}
$$

This is a proof: the assumptions on which $Fa\vee Ha$ depends are $\forall x(Fx\vee Gx)$ & $\neg\exists x(Gx\wedge\neg Hx)$ — and these don't involve a. The eigenvariable conditions are satisfied.

4) Here are the proofs:

a)

$$
\begin{prooftree}
\AxiomC{$\forall x(Fx\to Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa\to Ga$}
\AxiomC{$[Fa]^1$}
\RightLabel{$\to E$}
\BinaryInfC{Ga}
\AxiomC{$\forall x(Gx\to Hx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Ga\to Ha$}
\RightLabel{$\to E$}
\BinaryInfC{Ha}
\RightLabel{$\to I^1$}
\UnaryInfC{$Fa\to Ha$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(Fx\to Hx)$}
\end{prooftree}
$$

b)

$$
\begin{prooftree}
\AxiomC{$\exists xFx$}
\AxiomC{$[\forall x(Fx\to Gx)]^2$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa\to Ga$}
\AxiomC{$[Fa]^1$}
\RightLabel{$\to E$}
\BinaryInfC{Ga}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists xGx$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\exists xGx$}
\RightLabel{$\to I^2$}
\UnaryInfC{$\forall x(Fx\to Gx)\to\exists xGx$}
\end{prooftree}
$$

c)

$$
\begin{prooftree}
\AxiomC{$\exists xFx$}
\AxiomC{$[Fa]^1$}
\AxiomC{$\forall xGx$}
\RightLabel{$\forall E$}
\UnaryInfC{Ga}
\RightLabel{$\wedge I$}
\BinaryInfC{$Fa\wedge Ga$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx\wedge Gx)$}
\RightLabel{$\exists E$}
\BinaryInfC{$\exists x(Fx\wedge Gx)$}
\end{prooftree}
$$

d)

$$
\begin{prooftree}
\AxiomC{$[\exists x(Fx\wedge Gx)]^2$}
\AxiomC{$\forall x(Fx\to x=a)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fb\to b=a$}
\AxiomC{$[Fb\wedge Gb]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{Fb}
\RightLabel{$\to E$}
\BinaryInfC{$b=a$}
\AxiomC{$[Fb\wedge Gb]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{Gb}
\RightLabel{$=E$}
\BinaryInfC{Ga}
\RightLabel{$\exists E^1$}
\BinaryInfC{Ga}
\RightLabel{$\to I^2$}
\UnaryInfC{$\exists x(Fx\wedge Gx)\to Ga$}
\end{prooftree}
$$

e)

$$
\begin{prooftree}
\AxiomC{$[Rab]^2$}
\AxiomC{$[Rba]^1$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Rab\wedge Rba$}
\AxiomC{$\forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$\forall y\forall z((Ray\wedge Ryz)\to Raz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$\forall z((Rab\wedge Rbz)\to Raz)$}
\RightLabel{$\forall E$}
\UnaryInfC{$(Rab\wedge Rba)\to Raa$}
\RightLabel{$\to E$}
\BinaryInfC{Raa}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists xRxx$}
\AxiomC{$\neg\exists xRxx$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg Rba$}
\RightLabel{$\to I^2$}
\UnaryInfC{$Rab\to\neg Rba$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y(Ray\to\neg Rya)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x\forall y(Rxy\to\neg Ryx)$}
\end{prooftree}
$$

5) a)

$$
\begin{prooftree}
\AxiomC{$\forall x(p \vee Fx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$p \vee Fa$}
\AxiomC{$[\neg(p \vee \forall x Fx)]^3$}
\AxiomC{$[p]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \forall x Fx$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{Fa}
\AxiomC{$[Fa]^2$}
\RightLabel{$\vee E$}
\TrinaryInfC{Fa}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x Fx$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \forall x Fx$}
\AxiomC{$[\neg(p \vee \forall x Fx)]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg\neg(p \vee \forall x Fx)$}
\RightLabel{DNE}
\UnaryInfC{$p \vee \forall x Fx$}
\end{prooftree}
$$

b)

$$
\begin{prooftree}
\AxiomC{$\neg\exists x(Fx \wedge \neg Gx)$}
\AxiomC{$[\neg(\neg Fa \vee Ha)]^4$}
\AxiomC{$[\neg Fa]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fa$}
\RightLabel{DNE}
\UnaryInfC{Fa}
\AxiomC{$[\neg Ga]^2$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Fa \wedge \neg Ga$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \wedge \neg Gx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg Ga$}
\RightLabel{DNE}
\UnaryInfC{Ga}
\AxiomC{$[\neg(\neg Fa \vee Ha)]^4$}
\AxiomC{$[Ha]^3$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg Ha$}
\RightLabel{$\wedge I$}
\BinaryInfC{$Ga \wedge \neg Ha$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Gx \wedge \neg Hx)$}
\AxiomC{$\neg\exists x(Gx \wedge \neg Hx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^4$}
\UnaryInfC{$\neg\neg(\neg Fa \vee Ha)$}
\RightLabel{DNE}
\UnaryInfC{$\neg Fa \vee Ha$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(\neg Fx \vee Hx)$}
\end{prooftree}
$$

c)

$$
\begin{prooftree}
\AxiomC{$[\neg\exists x(Fx \to \forall y Fy)]^6$}
\AxiomC{$[\neg\exists x(Fx \to \forall y Fy)]^6$}
\AxiomC{$[\neg\forall y Fy]^5$}
\AxiomC{$[\neg\exists y \neg Fy]^2$}
\AxiomC{$[\neg Fb]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists y \neg Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fb$}
\RightLabel{DNE}
\UnaryInfC{Fb}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg\exists y \neg Fy$}
\RightLabel{DNE}
\UnaryInfC{$\exists y \neg Fy$}
\AxiomC{$[\neg Fa]^4$}
\AxiomC{$[Fa]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I^3$}
\UnaryInfC{$Fa \to \forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\exists E^4$}
\BinaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^5$}
\UnaryInfC{$\neg\neg\forall y Fy$}
\RightLabel{DNE}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I$ (vacuous)}
\UnaryInfC{$Fa \to \forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^6$}
\UnaryInfC{$\neg\neg\exists x(Fx \to \forall y Fy)$}
\RightLabel{DNE}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\end{prooftree}
$$

This is a really complex proof, and I don't expect you to be able to find a proof this complicated in this module!

PY4612 CLASS SOLUTIONS § 2

1 a) Ma is true in this model, since $I(a) = r$, and $I(M)(r) = 1$.

Mb is false

Bab is false

$Bab \to Bba$ is true (since Bba is true).

b)

$\forall y My$: check the value of My for each possible value of y.

$My\ [y:=r]$ true

$My\ [y:=p]$ false

$My\ [y:=s]$ true

so $\forall y My$ is false; since My is false for some value for y.

$$\exists x Bxa \left\{ \begin{array}{ll} Bxa[x:=r] & \text{true} \\ Bxa[x:=p] & \text{false} \\ Bxa[x:=s] & \text{false} \end{array} \right\}$$

so $\exists x Bxa$ is true, since there is some value of x that makes Bxa true.

$$\exists x (Bxa \wedge Bax) \left\{ \begin{array}{l} Bxa \wedge \underset{\times}{Bax}[x:=r] \text{ — false} \\ \underset{\times}{Bxa} \wedge Bax[x:=p] \text{ — false} \\ \underset{\times}{Bxa} \wedge \underset{\times}{Bax}[x:=s] \text{ — false} \end{array} \right\}$$

so $\exists x (Bxa \wedge Bax)$ is false.

c) • $\forall x \forall y (Bxy \to \neg Byx)$ — is true, since these are all true.

$\forall y (Bxy \to \neg Byx)\ [x := r]$ - true, since these are all true

$\underset{0}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=r][y:=r]$ — true

$\underset{0}{Bxy} \to \underset{0}{\neg} \underset{1}{Byx}\ [x:=r][y:=p]$ — true

$\underset{1}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=r][y:=s]$ — true

$\forall y (Bxy \to \neg Byx)\ [x := p]$ true, since these are all true

$\underset{1}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=p][y:=r]$ - true

$\underset{0}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=p][y:=p]$ — true

$\underset{0}{Bxy} \to \underset{0}{\neg} \underset{1}{Byx}\ [x:=p][y:=s]$ — true

$\forall y (Bxy \to \neg Byx)\ [x := s]$ — true, since these are all true

$\underset{0}{Bxy} \to \underset{0}{\neg} \underset{1}{Byx}\ [x:=s][y:=r]$ - true

$\underset{1}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=s][y:=p]$ - true

$\underset{0}{Bxy} \to \underset{1}{\neg} \underset{0}{Byx}\ [x:=s][y:=s]$ — true

• $\exists x \forall y Bxy$ — for this to be true, we need one of these to be true.

— $\forall y Bxy[x:=r]$ $\forall y Bxy[x:=p]$ $\forall y Bxy[x:=s]$

false since $Bxy[x:=r][y:=r]$ is false.

false since $Bxy[x:=p][y:=p]$ is false.

false since $Bxy[x:=s][y:=s]$ is false.

— So, $\exists x \forall y Bxy$ is false

$\forall x(Mx \vee \exists y(My \wedge Bxy))$ is true iff

$Mx \vee \exists y(My \wedge Bxy)[x:=r]$ is true

↪ which it is, since $Mx[x:=r]$ is true,

$Mx \vee \exists y(My \wedge Bxy)[x:=p]$ is true,

↪ which it is, since $(My \wedge Bxy)[x:=p, y:=r]$ is true,

and $Mx \vee \exists y(My \wedge Bxy)[x:=s]$ is true.

↪ which it is, since $Mx[x:=s]$ is true.

So $\forall x(Mx \vee \exists y(My \wedge Bxy))$ is true!

2

a) $\vdash \forall x \forall y(Rxy \to Ryx)$: Counterexample in M_1, M_3 but not M_2

Since in both M_1 & M_3 we have $Rxy \to Ryx$ false when $[x:k, y:m]$, so $\forall y(Rxy \to Ryx)$ is false when $[x:k]$, & $\forall x \forall y(Rxy \to Ryx)$ is false

b) $\forall x \exists y\, Rxy \vdash \forall x\, Rxx$

The conclusion $\forall x Rxx$ is false in M_1, M_2 and M_3.

The premise $\forall x \exists y Rxy$ is true in M_1 & M_2 but not M_3

So this argument has counterexamples in M_1 & M_2 but not M_3.

c) $\forall x \exists y\, Rxy \vdash \exists x \forall y\, Rxy$

Here, too, the conclusion $\exists x \forall y\, Rxy$ is false in each model & the premise is true in M_1 & M_2 but not M_3. So this argument has counterexamples in M_1 & M_2 but not M_3

3) a) $\exists x Fx, \exists x Gx \vdash \exists x(Fx \land Gx)$ has a counterexample:

$D = \{a, b\}$

	$I(F)$	$I(G)$
a	1	0
b	0	1

$\exists x Fx$ is true, since Fx is true $[x := a]$

$\exists x Gx$ is true, since Gx is true $[x := b]$

$\exists x(Fx \land Gx)$ is false, since $Fx \land Gx$ is false for every value of x.

b) $\exists x Fx, \forall x Gx \vdash \exists x(Fx \land Gx)$ has no counterexample:

If everything in the domain has property G, then whatever has property F, must be a thing which is both F & G.

Here is a proof:

$$\dfrac{\exists x Fx \qquad \dfrac{\dfrac{[Fa]^1 \qquad \dfrac{\forall x Gx}{Ga}\,\forall E}{Fa \land Ga}\,\land I}{\exists x(Fx \land Gx)}\,\exists I}{\exists x(Fx \land Gx)}\,\exists E^1$$

(c) $\vdash \exists x(Fx \to \forall y Fy)$ This has no counterexample: To show that $\exists x(Fx \to \forall y Fy)$ is always true, notice this: either everything in the domain is an F, in which case $\forall y Fy$ is true, & so, for any object at all $Fx \to \forall y Fy$ is true, & hence, so is $\exists x(Fx \to \forall y Fy)$.
Otherwise, not everything is F, so pick something that isn't F. If x is assigned that object as a value, Fx is false, & so $Fx \to \forall y Fy$ is true, & so $\exists x(Fx \to \forall y Fy)$ is true here, too.
So, regardless, in any model, $\exists x(Fx \to \forall y Fy)$ is true, & the argument has no counterexample.

A proof for this formula is hard to construct. Here it is (from last section's solutions).

$$
\begin{prooftree}
\AxiomC{$[\neg\exists x(Fx \to \forall y Fy)]^6$}
\AxiomC{$[\neg\exists x(Fx \to \forall y Fy)]^6$}
\AxiomC{$[\neg\forall y Fy]^5$}
\AxiomC{$[\neg\exists y \neg Fy]^2$}
\AxiomC{$[\neg Fb]^1$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists y \neg Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Fb$}
\RightLabel{DNE}
\UnaryInfC{Fb}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg\exists y \neg Fy$}
\RightLabel{DNE}
\UnaryInfC{$\exists y \neg Fy$}
\AxiomC{$[\neg Fa]^4$}
\AxiomC{$[Fa]^3$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I^3$}
\UnaryInfC{$Fa \to \forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\exists E^4$}
\BinaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^5$}
\UnaryInfC{$\neg\neg\forall y Fy$}
\RightLabel{DNE}
\UnaryInfC{$\forall y Fy$}
\RightLabel{$\to I$ (vacuous)}
\UnaryInfC{$Fa \to \forall y Fy$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^6$}
\UnaryInfC{$\neg\neg\exists x(Fx \to \forall y Fy)$}
\RightLabel{DNE}
\UnaryInfC{$\exists x(Fx \to \forall y Fy)$}
\end{prooftree}
$$

(d) $\forall x \exists y Rxy \vdash \forall y \exists x Rxy$ Has counterexamples. Here is a simple one

$D = \{a, b\}$

$I(R)$	a	b
a	1	0
b	1	0

$\forall x \exists y Rxy$ is true, since $\exists y Rxy$ is true, when $[x := a]$ (since Rxy is true, when $[x := a, y := a]$) and $\exists y Rxy$ is true when $[x := b]$, since Rxy is true when $[x := b, y := a]$

$\forall y \exists x Rxy$ is false, since $\exists x Rxy$ is false when $[x := b]$, since Rxy is false when $[x := b, y := a]$ and when $[x := b, y := b]$ too.

(e) $\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy)), \forall x \exists y Rxy, \forall x \neg Rxx \vdash \perp$

This has counterexamples. Here is one:

$D = \{a, b, c\}$

$I(R)$	a	b	c
a	0	1	1
b	1	0	1
c	1	1	0

In this model $\forall x \exists y Rxy$ & $\forall x \neg Rxx$ are both true. So is $\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy))$.

$\forall x \exists y Rxy$ is true since each column contains a 1.

$\forall x \neg Rxx$ is true since the diagonal is all 0.

$\forall x \forall y (Rxy \to \exists z (Rxz \wedge Rzy))$ is true because in this model R is difference (Rxy iff $x \neq y$) & in this model, whenever x & y are different there is something that differs from both x & y. — since there are 3 things in this domain.

4 Here are the remaining cases of the proof of the soundness theorem.

$\wedge E$

$$\wedge E\,\dfrac{\begin{array}{c}X\\ \Pi\\ A\wedge B\end{array}}{A} \qquad \wedge E\,\dfrac{\begin{array}{c}X\\ \Pi\\ A\wedge B\end{array}}{B}$$

If Π is sound, we have $X \vDash A\wedge B$. So, in any model where X is true, so is $A\wedge B$. It follows that A (and B) are also true in those models, so $X \vDash A$ & $X \vDash B$ too.

$\to E$

$$\to E\,\dfrac{\begin{array}{c}X\\ \Pi_1\\ A\to B\end{array} \qquad \begin{array}{c}Y\\ \Pi_2\\ B\end{array}}{B}$$

If Π_1 & Π_2 are sound, we have $X \vDash A\to B$ & $Y \vDash B$. Take any model where X, Y are all true. Since $X \vDash A\to B$ & $Y \vDash A$, we have $A\to B$ & A true in this model, and so, B is also true. In other words, we have $X, Y \vDash B$, as desired.

$\vee I$

$$\dfrac{\begin{array}{c}X\\ \Pi_1\\ A\end{array}}{A\vee B} \qquad \dfrac{\begin{array}{c}X\\ \Pi_2\\ B\end{array}}{A\vee B}$$

If Π_1 (or Π_2) is sound, we have $X \vDash A$ ($X \vDash B$). So, in any model where X is true, so is A (B). It follows that $A\vee B$ is also true in those models, so $X \vDash A\vee B$, too.

DNE

$$DNE\,\dfrac{\begin{array}{c}X\\ \Pi\\ \neg\neg A\end{array}}{A}$$

If Π is sound, we have $X \vDash \neg\neg A$, so in any model where X is true, so is $\neg\neg A$. It follows that A is true in that model, & so $X \vDash A$ too.

$\neg E$

$$\neg E\,\dfrac{\begin{array}{c}X\\ \Pi_1\\ \neg A\end{array} \qquad \begin{array}{c}Y\\ \Pi_2\\ A\end{array}}{\bot}$$

If Π_1 & Π_2 are both sound, we have $X \vDash \neg A$ & $Y \vDash A$. This means that there is no model where X & Y are all true, since all of the X models have A false, and all of the Y models have A true. So, there is no counterexample to $X, Y \succ \bot$, ie $X, Y \vDash \bot$.

$\bot E$

$$\begin{array}{c} X \\ \Pi \\ \bot \end{array} \quad \bot E \dfrac{}{A}$$

Suppose that Π is sound. That is, $X \vDash \bot$, ie there is no model where each member of X is true. So, there is no counterexample to $X \succ A$ either, we have $X \vDash A$ too.

$\neg I$

$$\begin{array}{c} X, [A] \\ \Pi \\ \bot \end{array} \quad \neg I \dfrac{}{\neg A}$$

Suppose that Π is sound. That is, $X, A \vDash \bot$. So, there is no model where each member of X is true & A is true too. So, in any model where each member of X is true, A must be false (there's no other option) so $\neg A$ is true too. That is, $X \vDash \neg A$, too.

$\forall E$

$$\begin{array}{c} X \\ \Pi \\ \forall x A(x) \end{array} \quad \forall E \dfrac{}{A(t)}$$

Suppose $X \vDash \forall x A(x)$. So, in any model where X is true, so is $\forall x A(x)$, & so, in any such model, so is each standard instance $A(n)$. In any such model, consider $I(t)$. This is an object in the domain, call it d. Since $A(d)$ is true in this model, so is $A(t)$. And hence,

$X \vDash A(t)$, as desired.

$\exists I$

$$\begin{array}{c} X \\ \Pi \\ A(t) \end{array} \quad \exists I \dfrac{}{\exists x A(x)}$$

Suppose $X \vDash A(t)$. So, in any model where X is true, so is $A(t)$, and so there is some object in the domain (namely $I(t)$ in the model under discussion) to choose as an instance of $\exists x A(x)$, & so $\exists x A(x)$ is also true in this model.

$\exists E$

$$\dfrac{\begin{array}{c} X \\ \Pi_1 \\ \exists x A(x) \end{array} \qquad \begin{array}{c} Y, [A(a)] \\ \Pi_2 \\ C \end{array}}{C}$$

Suppose $X \vDash \exists x A(x)$ & $Y, A(a) \vDash C$ (where a doesn't appear in Y, C or $\exists x A(x)$). Consider any model where X, Y are true. Since $X \vDash \exists x A(x)$, we have $\exists x A(x)$ true in that model. So there is a domain element d where $A(d)$ is true in the model. We also have $Y, A(a) \vDash C$. Now, in our model we have Y true, and $A(d)$ true. We are free to assign a any value, without disturbing the truth of $\exists x A(x)$, Y or C, so assign $I(a) = d$, & then we see that $Y, A(a)$ are true, & by $Y, A(a) \vDash C$, we have C in our model, & hence $X, Y \vDash C$ as desired.

PY4612 CLASS SOLUTIONS WEEK 3

1a) To make a model where $\exists x(Fx \to Gx)$, $\forall x Fx$ are both true, & $\forall x Gx$ is false, we need everything in the domain to be an F, not everything to be a G, and at least one object to choose for x to make $Fx \to Gx$ true.

Since everything has to be F, we need at least one thing will have to be G, but to make $\forall x Gx$ false, we need not everything to be G. So, one way to do this is to have two things. One with G, one without.

$D = \{a, b\}$

	$I(F)$	$I(G)$
a	1	1
b	1	0

Here: $\exists x(Fx \to Gx)$: true, since $Fx \to Gx$ is true when $x := a$
$\forall x Fx$: true, since Fx is true when $x := a$, $x := b$
$\forall x Gx$: false, since Gx is false when $x := b$.

b) Here is a counterexample to the argument $\exists x \neg Fx, Ft_1, Ft_2, \ldots \vdash \bot$ where $t_1, t_2, \ldots$ are all the terms in the language.

$D = \{a, b\}$

	$I(F)$
a	1
b	0

We are not told how many names or function symbols there are in this language, so let's specify
$I(c) = a$, for any name c,
$I(f)(d_1, \ldots, d_n) = a$ for all n-place function symbols f.
and $\alpha(x) = a$ for every variable x.

Then, for every term t_i, $I(t_i, \alpha) = a$, and so $I(Ft_i, \alpha) = 1$.

However, $I(\exists x \neg Fx, \alpha) = 1$, since $I(\neg Fx, \alpha[x := b]) = 1$, so we have our counterexample.

c) The argument $\forall x \forall y \forall z((Sxy \wedge Syz) \to Sxz), \forall x \exists y\, Sxy, \forall x \exists y\, Syx, \neg \exists x\, Sxx \succ \perp$ has a counterexample – but it is not straightforward to make.

$\forall x \forall y \forall z((Sxy \wedge Syz) \to Sxz)$: This says that the relation S is transitive, like 'larger than' or 'divides into'.

$\forall x \exists y\, Sxy$: This says that S has no dead-ends – everything is related to something by S.

$\forall x \exists y\, Syx$: And in reverse – everything has something relating to it by S.

$\neg \exists x\, Sxx$: S doesn't relate anything to itself

Here is one model: $D = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$, consisting of all the INTEGERS, the positive + negative (+ zero) whole numbers. And $I(S)$ relates x to y when x is smaller than y.

- If x is smaller than y & y is smaller than z then indeed, x is smaller than z
- Every number x is smaller than some number, $x+1$, for example.
- Every number x has some number smaller, $x-1$, for example.
- Nothing is smaller than itself.

In fact, there is no finite model making these three sentences true. – Since $\forall x \exists y\, Sxy$ is true, any object a must be related to something, b – this b can't be a, since $\neg Saa$, so b is a new object. This b is related to something, c, which can't be b, since $\neg Sbb$, and if it were a, then Sab & Sba, and transitivity, would give Saa, which is not true, & so, c must be yet another object. So, there is a d where Scd, and $d \neq c$, b or a for the same reasons, so there is an e where Sde....

d) With the example from c) in mind, it's not too hard to find a model for d). The trick is to change the domain to be able to model the extra premises.

$\forall x \forall y (Sxy \to \neg Syx)$: antisymmetry — this is satisfied by any ordering like <u>less-than</u> (not less-than-or-equal-to!)

$\forall x \forall y (Sxy \to \exists z (Sxz \wedge Szy))$: this is <u>harder</u>. It says

$\boxed{\text{THIS IS CALLED } \underline{\underline{\text{DENSITY}}}}$

whenever $x \longrightarrow y$ we can find some

'midpoint' z:
$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend left] & z \arrow[r] & y \end{tikzcd}$$

And of course, this happens for <u>all</u> objects related by S, so there is a 'midpoint' between x & z, between z & y, between ...

We have structures like this. Move from the INTEGERS to (for example) the RATIONALS (the fractions $\frac{n}{m}$ where n is an integer & m is an integer >0, where we say $\frac{n}{m} = \frac{n'}{m'}$ iff $n \times m' = m \times n'$).

These are ordered along a line, & satisfy all of our premises, including DENSITY.

2 Here is how I put these statements together:

This is independently proved by way of the step-by-step construction.

We use the fact that $X^* \vdash B \Rightarrow B \in X$, to show that X^* satisfies the nice properties in (ii) (propositional connectives) & (vii) (quantifiers).

$$\cfrac{\cfrac{(i) \qquad \cfrac{\cfrac{(iii)}{(ii)} \quad \cfrac{(iii)}{(vii)}}{}\ (v)}{(vi)}}{(iv)}$$

The witnessed maximal consistent set X^* made in (i) makes the model that is the counterexample to $X \vdash A$.

The conclusion!

Contraposition: $\dfrac{\neg A \to \neg B}{B \to A}$

3. Take the set $X = \{\forall x(Fx \vee Gx), \neg\forall x Fx, \neg\forall x Gx\}$

Expand the set, first using the sequence $Fc_1, \neg Fc_1, \forall x Fx, \exists x Fx, \forall x Gx, \exists x Gx \ldots$

- Fc_1 is consistent with $\forall x(Fx \vee Gx), \neg\forall x Fx, \neg\forall x Gx$ so we add it:

$$\forall x(Fx \vee Gx), \neg\forall x Fx, \neg\forall x Gx, Fc_1$$

- Then $\neg Fc_1$ is inconsistent with the set, so we leave it out.
- Then $\forall x Fx$ is inconsistent with the set, so we leave it out.
- Then $\exists x Fx$ is consistent, & since it is an existentially quantified, we add a witness, too: Fc_2, the next fresh name:

$$\forall x(Fx \vee Gx), \neg\forall x Fx, \neg\forall x Gx, Fc_1, \exists x Fx, Fc_2$$

- Then $\forall x Gx$ is inconsistent with the set, so we leave it out.
- Then $\exists x Gx$ is consistent, & since it is an existentially quantified, we add a witness, too: Gc_3, the next fresh name:

The resulting set is

$$\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, Fc_1, \exists xFx, Fc_2, \exists xGx, Gc_3.$$

Now we expand our starting set $\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx$ using the new sequence

$$\neg Fc_1, Fc_1, \forall xGx, \exists xGx, \forall xFx, \exists xFx \ldots$$

- $\neg Fc_1$ is consistent with $\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx$ so we add it:

$$\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, \neg Fc_1$$

- Then Fc_1 is inconsistent, so we leave it out.
- $\forall xGx$ is inconsistent with our set. We leave it out.
- $\exists xGx$ is consistent with our set, so we add it, with a witness Gc_2.

$$\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, \neg Fc_1, \exists xGx, Gc_2$$

- $\forall xFx$ is inconsistent with our set. We leave it out.
- $\exists xFx$ is consistent with our set, so we add it, with a witness Fc_3

$$\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, \neg Fc_1, \exists xGx, Gc_2, \exists xFx, Fc_3$$

Compare the two sets we constructed.

a) $\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, \neg Fc_1, \exists xGx, Gc_2, \exists xFx, Fc_3$

b) $\forall x(Fx \vee Gx), \neg\forall xFx, \neg\forall xGx, Fc_1, \exists xFx, Fc_2, \exists xGx, Gc_3.$

They are different — & inconsistent with each other ($\neg Fc_1$ versus Fc_1).

④

(a) $\exists x Fx$ — compatible: but neither entails the other — (c)

$\uparrow$ entails

(b) $\exists x \exists y (Fx \wedge Fy \wedge x \neq y)$ - - - - - - incompatible - - - - - - (c) $\exists x \forall y (Fy \rightarrow x = y)$

$\uparrow$ entails

(d) $\exists x \exists y (Fx \wedge Fy \wedge Gx \wedge \neg Gy)$ - - - - incompatible - - - - (c)

Here is why (b) entails (a):

$$
\dfrac{\exists x\exists y(Fx\wedge Fy\wedge x\neq y) \qquad \dfrac{[\exists y(Fa\wedge Fy\wedge a\neq y)]^1 \qquad \dfrac{\dfrac{\dfrac{Fa\wedge Fb\wedge a\neq b}{Fa\wedge Fb}\,\wedge E}{Fa}\,\wedge E}{\exists x Fx}\,\exists I}{\exists x Fx}\,\exists E \qquad}{\exists x Fx}\,\exists E
$$

Here is why (d) entails (b):

$$
\dfrac{\exists x\exists y(Fx\wedge Fy\wedge Gx\wedge Gy) \qquad \dfrac{[\exists y(Fa\wedge Fy\wedge Ga\wedge Gy)]^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[(Fa\wedge Fb)\wedge(Ga\wedge\neg Gb)]^2}{Fa\wedge Fb} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(Fa\wedge Fb)\wedge(Ga\wedge\neg Gb)]^2}{Ga\wedge\neg Gb}\,\wedge E}{Ga}\,\wedge E \qquad [a=b]^1}{Gb}\,{=}E \qquad \dfrac{\dfrac{[(Fa\wedge Fb)\wedge(Ga\wedge\neg Gb)]^2}{Ga\wedge\neg Gb}\,\wedge E}{\neg Gb}}{\bot}}{a\neq b}\,\neg I^1}{Fa\wedge Fb\wedge a\neq b}\,\wedge I}{\exists y(Fa\wedge Fy\wedge a\neq y)}\,\exists I}{\exists x\exists y(Fx\wedge Fy\wedge x\neq y)}\,\exists I}{\exists x\exists y(Fx\wedge Fy\wedge x\neq y)}\,\exists E^2}{\exists x\exists y(Fx\wedge Fy\wedge x\neq y)}\,\exists E^3
$$

Here is why (b) is incompatible with (c):

$$
\begin{prooftree}
\AxiomC{$\exists x\exists y(Fx\wedge Fy\wedge x\neq y)$}
\AxiomC{$[\exists y(Fa\wedge Fy\wedge a\neq y)]^3$}
\AxiomC{$\exists x\forall y(Fy\to x=y)$}
\AxiomC{$[\forall y(Fy\to c=y)]^1$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa\to c=a$}
\AxiomC{$[Fa\wedge Fb\wedge a\neq b]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{Fa}
\RightLabel{$\to E$}
\BinaryInfC{$c=a$}
\AxiomC{$[\forall y(Fy\to c=y)]^1$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fb\to c=b$}
\AxiomC{$[Fa\wedge Fb\wedge a\neq b]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{Fb}
\BinaryInfC{$c=b$}
\RightLabel{$=E$}
\BinaryInfC{$a=b$}
\AxiomC{$Fa\wedge Fb\wedge a\neq b$}
\RightLabel{$\wedge E$}
\UnaryInfC{$a\neq b$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\bot$}
\RightLabel{$\exists E^2$}
\BinaryInfC{$\bot$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

Since (c) is incompatible with (b), & (d) entails (b), (c) is incompatible with (d) too.

(a) is compatible with (c) since both are true in a model in which there is exactly one thing with property F, for example:

$D=\{a,b\}$

	$I(F)$
a	1
b	0

Here, $\exists x Fx$ is true, since Fa is true.

$\exists x\forall y(Fy\to y=x)$

(a) doesn't entail (c), since (a) is also true in a model where there are two different things with property F, while (c) is not true in that model (since in a model like that, (b) is true.

Similarly (c) doesn't entail (a), since in a model with no objects with property F, (c) is true while (a) is not.

$D=\{a\}$

	$I(F)$
a	0

Here $\exists x Fx$ is false, while

$\exists x\forall y(Fy\to y=x)$ is true since

$\forall y(Fy\to y=a)$ is true, since

$Fa\to a=a$ is true.

5 Here are the proofs:

a)

$$\dfrac{a=b \qquad b\neq c}{a\neq c}\,{=}E$$

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{[Fa\wedge Ga]^2}{Ga}\wedge E \qquad \dfrac{\dfrac{\dfrac{\forall x\forall y((Fx\wedge Fy)\to x=y)}{\forall y((Fa\wedge Fy)\to a=y)}\forall E}{(Fa\wedge Fb)\to a=b}\forall E \qquad \dfrac{\dfrac{[Fa\wedge Ga]^2}{Fa}\wedge E \qquad \dfrac{[Fb\wedge\neg Gb]^1}{Fb}\wedge E}{Fa\wedge Fb}\wedge I}{a=b}}{Gb} \qquad \dfrac{[Fb\wedge\neg Gb]^1}{\neg Gb}\wedge E}{\dfrac{\exists x(Fx\wedge Gx) \qquad \dfrac{[\exists x(Fx\wedge\neg Gx)]^3 \qquad \bot}{\bot}\exists I^1}{\dfrac{\bot}{\neg\exists x(Fx\wedge\neg Gx)}\neg I}\exists I^2}$$

c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\exists x Fx \qquad \dfrac{[Fa]^1}{\exists x Fx}\exists I}{\bot}\neg E}{a=b}\bot E}{Fa\to a=b}\to I^1}{\forall y(Fy\to y=b)}\forall I}{\exists x\forall y(Fy\to y=x)}\exists I$$

d) Let's abbreviate

α: $(Fa\wedge\forall y(Fy\to y=a))\wedge Ga$

β: $(Fb\wedge\forall y(Fy\to y=b))\wedge Hb$

γ: $(Fb\wedge\forall y(Fy\to y=b))\wedge(Gb\wedge Hb)$

$$\dfrac{\dfrac{[\beta]^1}{Fb\wedge\forall y(Fy\to y=b)}\wedge E \qquad \dfrac{\dfrac{\dfrac{[\alpha]^2}{Ga}\wedge E \qquad \dfrac{\dfrac{\dfrac{[\beta]^1}{Fb\wedge\forall y(Fy\to y=b)}\wedge E}{Fa\to a=b}\wedge E \qquad \dfrac{\dfrac{[\alpha]^2}{Fa\wedge\forall y(Fy\to y=a)}\wedge E}{Fa}\wedge E}{a=b}\to E}{Gb} \qquad \dfrac{\beta}{Hb}\wedge E}{Gb\wedge Hb}\wedge I}{\gamma}\wedge I$$

$$\dfrac{Ix(Fx,Gx) \qquad \dfrac{Ix(Fx,Hx) \qquad \dfrac{\gamma}{Ix(Fx,Gx\wedge Hx)}\exists I}{Ix(Fx,Gx\wedge Hx)}\exists E^1}{Ix(Fx,Gx\wedge Hx)}\exists E^2$$

PY4612 CLASS SOLUTIONS WEEK 4

1. $f: \omega \to \omega$, which sets $f(x) = x+2$, is injective (since if $x+2 = y+2$, then $x = y$) but not surjective, since there is no natural number x where $x+2 = 1$ (or 0).

$g: \omega \to \omega$, which sets $g(x) = \begin{cases} x/2, & x \text{ even} \\ (x-1)/2, & x \text{ odd} \end{cases}$, is surjective, since for any natural number n, $g(2n) = n$.
But it is not injective, because (for example), $g(0) = g(1) = 0$.

$h: \omega \to \omega$, where $h(x) = x+1$, if x is even, & $x-1$ if x is odd, is bijective since if $h(x) = h(y)$ then if it is <u>even</u> $x = y = h(x)+1$, & if it is <u>odd</u>, $x = y = h(x) - 1$; and for any number n, $n = h(n-1)$ if n is odd, & $h(n+1)$ if n is even.

---

i: is not injective ($i(0) = i(2) = 0$) & not surjective since there is no n where $i(n) = 2$, for example.

2 a) This set is finite. If A has n elements, the set of all sets from elements of A has size 2^n. You can think of it like this. If $A = \{a_1, a_2, a_3, \ldots, a_n\}$, then the set $\{a_2, a_5, \ldots, a_{n-1}\}$ corresponds to one row of the 'truth table' with values 'IN' or 'OUT' depending on whether the element is in or out of the set.

a_1	a_2	a_3	a_4	a_5	$\cdots$	a_{n-1}	a_n
OUT	IN	OUT	OUT	IN	$\cdots$	IN	OUT

Each different choice for $a_1, a_2, \ldots, a_n$ determines a different set, and there are $2 \times 2 \times \cdots \times 2 = 2^n$ different choices for subsets of the set A

b) The set of strings from A is infinite, but countably infinite. It is infinite because the strings

$a \quad aa \quad aaa \quad aaaa \quad \cdots$

are all different, and there are infinitely many of these.

There are many ways to enumerate them. Here is one:

First, list all the strings of size 1. e.g

$a \quad b \quad c$

If A has size n, there are n of these.

Then list all the strings of size 2, e.g.

aa ab ac ba bb bc ca cb cc

List them in "alphabetical order" if you like. There are $n \times n$ of these (here, 9).

Then length 3 (there are $n \times n \times n$ of these), and length 4, etc. This will determine an enumeration of all finite strings from our alphabet.

c) The set of finite trees of objects taken from a countable set C is countably infinite. It's obvious that it's infinite (C is, and any item from C is a baby tree all by itself).

We can't enumerate the trees by first listing the trees of size 1, then those of size 2, etc... since there are infinitely many trees of size 1. We do the same sort of trick we did with pairs of numbers.

First, take our enumeration of the set C, & write it out like this $c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4 \quad \ldots$

Then, enumerate the trees like this:

Trees of size (# of nodes) ≤ 1, from objects < 1 in our list

$$c_0$$

Trees of size ≤ 2, from objects < 2 in our list, not already counted.

$$c_1 \qquad \frac{c_0}{c_0} \qquad \frac{c_1}{c_0} \qquad \frac{c_0}{c_1} \qquad \frac{c_1}{c_1}$$

Trees of size ≤ 3, from objects < 3 in our list, not already counted

$$c_2 \quad \frac{c_0}{c_2} \quad \frac{c_1}{c_2} \quad \frac{c_2}{c_0} \quad \frac{c_2}{c_1} \quad \frac{c_2}{c_2} \quad \frac{\frac{c_0}{c_0}}{c_0} \quad \frac{c_0\ c_0}{c_0} \quad \frac{\frac{c_0}{c_1}}{c_0} \quad \frac{\frac{c_1}{c_1}}{c_0} \quad \cdots$$

$\vdots$

Trees of size $\leq n$, from objects $< n$ in our list, not already counted

$\vdots$

Every tree is on this list somewhere, so this is an enumeration of all the finite trees from C.

d) This set is countably infinite. There are clearly infinitely many formulas in this language. It is countable because any formula can be uniquely associated with a string of symbols from a finite alphabet:

$$\neg \quad \wedge \quad \vee \quad \neg \quad \rightarrow \quad \top \quad (\quad) \quad \forall \quad \exists \quad F \quad G \quad R \quad = \quad a \quad b \quad c$$

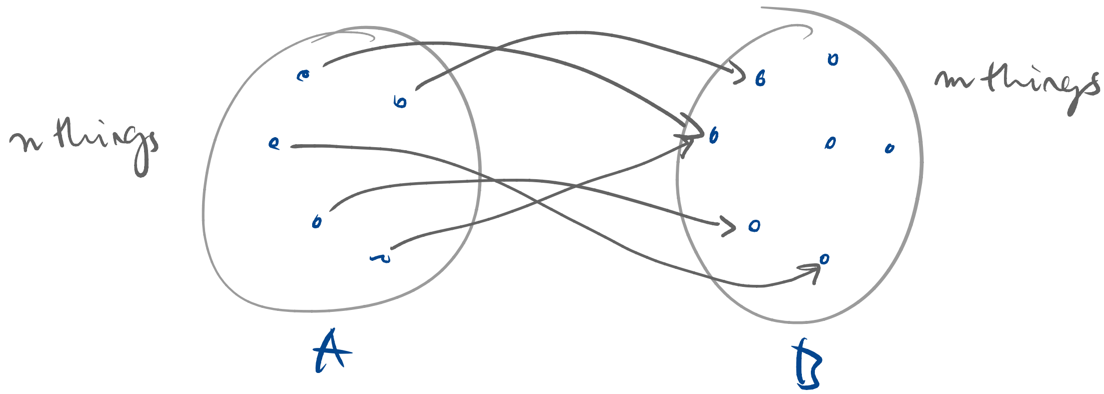
$$x \quad \underbrace{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9}_{\text{for subscripts for the variables}}$$

Not every string of these symbols is a formula (e.g. $\rightarrow F_1 \neg (_2)$) but every formula is a string of symbols, and an enumeration of these strings generates an enumeration of the formulas

e) The proofs in a finite (or countable) language are finite trees of formulas or discharged formulas (bracketed formulas with number flags) they form a countably infinite collection, by **c)**

3. Lets suppose that the sets A & B have size n & m respectively.

a) Each function from A to B corresponds to a choice from B for every element of A.



There are finitely many of these.
For each of the n inputs, you have m choices for the output.
That is, there are $\underbrace{m \times m \times \cdots \times m}_{n \text{ times}} = m^n$

functions.
Eg if A is size 5 & B is size 3, there are $3^5 = 243$.

b) The set of bijections from A to B is also finite, since all bijections are functions. If A & B are not the same size, there are no bijections from A to B. If $n > m$ (if A is bigger than B) then there is no injective function from A to B, and if $m > n$ there is no surjective function from A to B.
If A & B are the same size n, then there are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

bijections from A to B. There are n choices for the first input, $n-1$ choices for the second, $n-2$ for the third,

and so on. If A & B each have 5 elements, there are
$5 \times 4 \times 3 \times 2 = 120$ bijections from A to B.

c) If E is a countable set & A is finite, then there are countably infinitely many functions from A to E.
E.g. If A has only one element, then there are as many functions from A to E as elements of E. A function from A to E is given uniquely by an element of E.

If A has two elements (say a_0, a_1) then a function $f: A \to E$ is given by a pair of elements from E.

$$\left.\begin{aligned} f(a_0) &= e_0 \\ f(a_1) &= e_1 \end{aligned}\right\} \text{a choice for each}$$

But we have seen that pairs of elements from E can be enumerated.

So can triples, so the functions from a 3-element set to E can be enumerated, & so on for functions from a given finite set A to E for any finite size.

d) On the other hand, there are uncountably many functions from E to A, except in very special cases.
If A has size 1, then there is one function from E to A.
(And there are none if A is empty)
If A has size 2 or more, there are uncountably many

functions, because any bitstream determines a different function from E to A. Let two elements of A be a_0 & a_1. Then the bitstream $b_0\ b_1\ b_2\ b_3 \ldots$ gives us the function

$$f(e_0) = a_{b_0} \quad (= a_0 \text{ if } b_0 = 0,\ a_1 \text{ if } b_0 = 1)$$
$$f(e_1) = a_{b_1}$$
$$f(e_2) = a_{b_2}$$
$$\vdots$$

where different bitstreams give us different functions.

So, there must be uncountably many functions from E to A.

e) Similarly, there are uncountably many functions from D to E, when D & E are both countable, since there are uncountably many functions from D to a two-element subset of E.

---

4. To answer these questions concerning counting theories, it helps to apply the completeness theorem.

If $T_1 \neq T_2$ then there either is an A in T_1 & not in T_2, or vice versa, A is in T_2 & not in T_1.

If A is in T_1 & not in T_2, then $T_1 \vdash A$ & $T_2 \nvdash A$, so $T_1 \vDash A$ & $T_2 \nvDash A$. So, there is a model where all of T_2 is true, & A isn't, and hence, not all of T_1 is true.

So, if we have two different theories, there must be a model which is a model of every formula in one of those theories, but not every formula in the other.

In other words, a theory T is determined by its set $Mod(T)$, of models of that theory.

a) So, counting theories in the language given by the atoms p, q, r under $\wedge, \vee, \rightarrow, \neg$.

There are $2 \times 2 \times 2$ different models of this language, corresponding to the 8 different choices of truth values of the atoms, p, q & r. & hence, finitely many

So there are $2^8 = 256$ different theories in this language, because there are 256 different sets of models. Each such set determines a different theory, of all formulas true in all those models.

b) There are uncountably many theories (& in fact, even more than the number of bitstreams!) in the language with atoms

$p_0 \quad p_1 \quad p_2 \quad p_3 \quad \ldots$

Since each model corresponds uniquely to a bitstream, & so, there are uncountably many different sets of models, & uncountably many different theories corresponding to those sets.

c) There are uncountably many theories in the language with countably many variables, quantifiers, connectives & the names a, b, c, the predicates F & G (one-place), R & $=$ (two place).

Here is one way to see how this is true — we will use the predicates F, R, and the name a.

Consider the models with domain $\omega = \{0, 1, 2, 3, \ldots\}$ with

$I(a) = 0$

$I(R)$	0	1	2	3	...
0	0	1	0	0	...
1	0	0	1	0	...
2	0	0	0	1	...
3	⋮	⋮	⋮	⋮	
⋮					

So R relates x to $x+1$ & to no other number.

and where, for a given bitstream $b_0 b_1 b_2 \ldots$

we set $I(F)(n) = b_n$, ie F is true of the numbers corresponding to the 1s in the bitstream. It goes "on, off, on..." along the number line in parallel to the pattern of the bitstream.

Then, in the theory of all formulas true in that model, the sentences

F_0: Fa

F_1: $\exists x_1 (Rax_1 \wedge Fx_1)$

⋮

F_n: $\exists x_1 \exists x_2 \ldots \exists x_n (Rax_1 \wedge Rx_1x_2 \wedge \ldots \wedge Rx_{n-1}x_n \wedge Fx_n)$

are such that F_n is in our theory (is true in the model) iff $b_n = 1$, and furthermore $\neg F_n$ is in the theory iff $b_n = 0$.

So, there are at least as many theories in this language as there are bitstreams. There are uncountably many.

PY4612 CLASS SOLUTIONS WEEK 5

1. * To verify that there is no descending <-chain in $\langle \omega, I \rangle$ we can reason like this. For any number n in ω, there are at most n numbers in ω below n.

$$0 < 1 < 2 < \cdots < n-1 < n$$

So, wherever I start in ω, there any descending sequence of numbers has only a finite length.

* If $O_{<,\omega}$ is the set of all sentences in the language with the nonlogical predicate '<', as well as the identity predicate, variables, quantifiers & connectives, that are true in $\langle \omega, I \rangle$, we can see that the set

$$O_{<,\omega} \cup \{a_1 < a_0, a_2 < a_1, a_3 < a_2, a_4 < a_3, \cdots\}$$

must be consistent because we can show that every finite subset of this set is consistent — and then appeal to the COMPACTNESS THEOREM.

Every finite subset is consistent since if we pick finitely many statements from

$$\{a_1 < a_0, a_2 < a_1, a_3 < a_2, a_4 < a_3, \cdots\}$$

Then pick the largest index n featuring in our choice.

We can see that all of the statements

$$a_n < a_{n-1}, a_{n-1} < a_{n-2}, \cdots, a_1 < a_0$$

Can be interpreted as true in $\langle \omega, I \rangle$ if we interpret the names $a_0, \ldots, a_n$ as follows:

$$I(a_0) = n$$
$$I(a_1) = n-1$$
$$\vdots$$
$$I(a_i) = n-i$$
$$\vdots$$
$$I(a_n) = 0$$

So, all of the statements in our finite choice from that set are true in $\langle \omega, I \rangle$. And so are all of $O_{<,\omega}$.

So, any finite subset of

$$O_{<,\omega} \cup \{a_1 < a_0, a_2 < a_1, a_3 < a_2, a_4 < a_3, \ldots\}$$

is consistent, & by the compactness theorem, so is the whole set.

It follows that the whole set has a model. Call one such model, $\mathfrak{M}$.

In $\mathfrak{M}$ there is an infinite descending $<$-chain.

$$\ldots < a_2 < a_1 < a_0$$

But in $\mathfrak{M}$ all of the facts true in $\langle \omega, I \rangle$ (in the language of predicate logic with identity & $<$) are also true. And in $\langle \omega, I \rangle$ there is no descending $<$-chain.

So, there is no sentence in the language of predicate logic with identity & $<$ that is true in all & only the models with no descending $<$-chain.

d. In the model $\langle R, I \rangle$, where the domain is the set of real numbers, & the language contains the identity predicate, the names 0, 1 & the function symbols + & ×, we can represent other functions, like subtraction.

x minus y equals z

is equivalent to $y+z=x$. So whenever we want to say something about $x-y$, like, for example, that $\phi(x-y)$ for some sentence $\phi(z)$ with a variable z free, we can say $\exists z(y+z=x \wedge \phi(z))$, and this is true if & only if the number $x-y$ has property ϕ.

e. We can do the same sort of thing for division.

x divided by y equals z

is equivalent to $y \times z = x$. So, as before, $\phi(x/y)$ can be expressed by saying $\exists z(y \times z = x \wedge \phi(z))$

Notice, though, that 'x/y' doesn't act quite like a term. Take the case of $1/0$. There is <u>no</u> z such that $0 \times z = 1$. That's another way of saying that there is no number that is $1/0$, or that $1/0$ is "undefined".

Using our definition of $\phi(1/0)$ as $\exists z(0 \times z = 1 \wedge \phi(z))$ we can see that $\phi(1/0)$ is false, since $\exists z(0 \times z = 1 \wedge \phi(z))$ is always false. Let's take a specific example.

If $\phi(z)$ is $z=0$, we see that $1/0=0$ is false,

since it is $\exists z(0 \times z = 1 \wedge z=0)$.

But if $\phi(z)$ is $z \neq 0$, then $1/0 \neq 0$ is false too!

since it is $\exists z(0 \times z = 1 \wedge z \neq 0)$.

But this "$1/0 \neq 0$" isn't the negation of $1/0=0$.
That negation is $\neg \exists z(0 \times z = 1 \wedge z=0)$, and that is true.

* Now for $x<y$, if we had a way of specifying the positive numbers (those >0) we would be in business. $x<y$ iff $\exists z(z \text{ is positive} \wedge x+z=y)$.

How do we say that z is positive? As the hint says, for any real number x, $x \times x$ is not negative. (The product of two negative numbers is positive, the product of two positive numbers is also positive, & $0 \times 0 = 0$.)

Furthermore, if z is positive, then for <u>some</u> number w, $z = w \times w$ (this is the square root of z). So, z is positive if & only if there is some $w \neq 0$ where $z = w \times w$.

So, here is one way to say that $x < y$ in our language:

$$\exists z(\underbrace{\exists w(w \neq 0 \wedge z = w \times w)}_{z \text{ is positive}} \wedge x + z = y)$$

- We have already expressed the idea of a square root

x is "the" square root of y iff $x \times x = y$.

Notice: if $x \times x = y$ then $-x \times -x = y$ too — the notion of square root also fails to define a function, just like division, but this time, by having too many candidates, not just too few.

If you like, you could define

"x is the non-negative square root of y" as

$(x \times x = y) \wedge \exists w(w \times w = x)$.

Yes, I know that this is not a primitive function symbol in our language. We have defined subtraction, & then define $-x$ as $0 - x$ & this works fine.

The countable model theorem tells us that the theory of all sentences in our language that are true in $\langle \mathbb{R}, I \rangle$, also has a model with a countable domain.

What is this domain like? We know that there must be objects in the domain corresponding to all the natural numbers, since we have terms

$$0 \qquad 0+1 \qquad (0+1)+1 \qquad ((0+1)+1)+1 \qquad \cdots\cdots$$

So these terms refer to objects, and we will call these objects

$0, 1, 2, 3, \ldots$ for obvious reasons.

But our domain also has objects corresponding to the negative numbers, for our theory proves $\forall x \forall y \exists z (y + z = x)$, ie $\forall x \forall y \exists z$(z is x minus y), so in particular, we also have

$$-1 \qquad -2 \qquad -3 \qquad -4 \qquad -5 \cdots$$

Furthermore, we have $\forall x \forall y \exists z (y \neq 0 \rightarrow y \times z = x)$, ie $\forall x \forall y \exists z (y \neq 0 \rightarrow z \text{ is } x/y)$, so we have all the fractions n/m whenever $m \neq 0$. — We have all of the rational numbers.

But we have plenty of other numbers, too. We have $\sqrt[n]{x}$ for any numbers for which these roots exist in $\mathbb{R}$, and furthermore, solutions to any polynomial expression which has a solution in $\mathbb{R}$.

In short, our model must contain only numbers which are <u>describable</u> in the vocabulary of our language.

This language contains only countably many expressions, and $\mathbb{R}$ is uncountable, so this model must be missing many numbers. Which ones?

(Not π or e — these are definable in our language)

What $\mathbb{R}$ has that our countable model must miss out on are limits of arbitrary convergent sequences.

If $a_0, a_1, a_2, a_3, \ldots$ are all in $\mathbb{R}$, and

if $\forall \varepsilon > 0 \, \exists n \, \forall m > n \; |a_m - a_n| < \varepsilon$,

{that is $\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n \forall m (m > n \rightarrow (-\varepsilon < a_m - a_n \wedge a_m - a_n < \varepsilon)))$}

then there is a real number a that is the <u>limit</u> of this sequence: ie

$$\forall \varepsilon > 0 \, \exists n \, \forall m > n \; |a_m - a| < \varepsilon$$

There are <u>uncountably many</u> such sequences, and uncountably many such limits. There is no way to describe all of them in our language, & no way that they are all in our countable model.

PY4612 CLASS TASKS 6 SOLUTIONS

1 Here is why $\exists f_1 \forall x\, B f_1(x) x$ & $\exists f_2 \forall x\, B x f_2(x)$ are both true in our Rock-Paper-Scissors model:

$D = \{r, p, s\}$

$\mathcal{I}(B)$	r	p	s
r	0	0	1
p	1	0	0
s	0	1	0

(the other parts of the model are not necessary for the interpretation of our formulas, so we can safely ignore them.)

$\exists f_1 \forall x\, B f_1(x) x$ is true in this model because there is a value we can assign the function variable f_1, which makes $\forall x\, B f_1(x) x$ true:

x	$b(x)$
r	p
p	s
s	r

$\mathcal{I}(\forall x\, B f_1(x) x, \alpha[f_1 : b]) = 1$ because

$\mathcal{I}(B f_1(x) x, \alpha[f_1 : b, x : r]) = 1$, since $\mathcal{I}(f_1(x), \alpha[f_1 : b, x : r]) = p$, and paper beats rock

$\mathcal{I}(B f_1(x) x, \alpha[f_1 : b, x : p]) = 1$, since $\mathcal{I}(f_1(x), \alpha[f_1 : b, x : p]) = s$, and scissors beats paper

$\mathcal{I}(B f_1(x) x, \alpha[f_1 : b, x : s]) = 1$, since $\mathcal{I}(f_1(x), \alpha[f_1 : b, x : s]) = r$, and rock beats scissors.

Similarly, $\exists f_2 \forall x\, B x f_2(x)$ is true, because when we assign f_2 the value l (where $l(r) = s$, $l(p) = r$, $l(s) = p$), then we have

$\mathcal{I}(\forall x\, B x f_2(x), \alpha[f_2 : l]) = 1$ because

$\mathcal{I}(B x f_2(x), \alpha[f_2 : l, x : r]) = 1$, since $l(r) = s$ and rock beats scissors,

$\mathcal{I}(B x f_2(x), \alpha[f_2 : l, x : p]) = 1$, since $l(p) = r$ and paper beats rock,

$\mathcal{I}(B x f_2(x), \alpha[f_2 : l, x : s]) = 1$, since $l(s) = p$ and scissors beats paper.

2. In the model on Domain $D = \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\}$, with predicates R & S interpreted like this:

I(R)	α	β	γ	δ	ϵ	ζ
α	0	0	1	0	0	0
β	0	0	1	0	0	0
γ	0	0	0	1	0	0
δ	0	0	0	0	1	1
ϵ	0	0	0	0	0	0
ζ	0	0	0	0	0	0

I(S)	α	β	γ	δ	ϵ	ζ
α	0	1	1	1	1	1
β	0	0	1	1	1	1
γ	0	0	0	1	1	1
δ	0	0	0	0	1	1
ϵ	0	0	0	0	0	1
ζ	0	0	0	0	0	0

- The sentence Repeat (R, S), ie,

$$\forall x \forall y (Rxy \to Sxy) \wedge \forall x \forall y \forall z ((Rxy \wedge Syz) \to Sxz))$$

is true. We take each conjunct in turn:

$\forall x \forall y (Rxy \to Sxy)$ holds since $I(Rxy \to Sxy, \alpha[x{:}a, y{:}b]) = 1$ for every pair a, b taken from D. Whatever objects we choose as values for x & y, if R holds of that pair, so does S, as we can see by checking the tables for R & S.

The second conjunct $\forall x \forall y \forall z ((Rxy \wedge Syz) \to Sxz)$ is also true: its instances $I((Rxy \wedge Syz) \to Sxz, \alpha[x{:}a, y{:}b, z{:}c]) = 1$ are true for every triple of objects from D, since it is easy to verify that S is a transitive relation, corresponding to the order $\alpha < \beta < \gamma < \delta < \epsilon < \zeta$ (S relates a to b iff a is under b in the order), and so, every instance of $(Sxy \wedge Syz) \to Sxz$ is true. But we have already seen that whenever Rxy holds, so does Sxy, so we also have $(Rxy \wedge Syz) \to Sxz$ for every triple of values for x, y & z.

So, the conjunction is true. We have Repeat (R, S).

- We want to find some R^* where $TC(R, R^*)$ holds.

So, we want a two place relation R^* where $Repeat(R, R^*)$ holds, and also

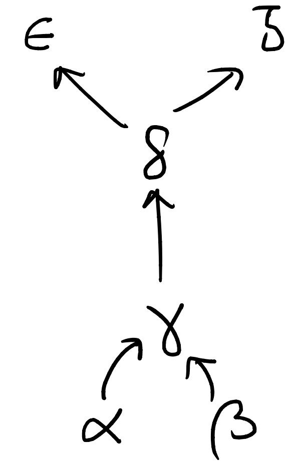
$$\forall Q(Repeat(R,Q) \to \forall x \forall y(R^*xy \to Qxy)).$$

This means we want R^* to be the smallest relation (the weakest, relating the fewest pairs) of all relations Q where $Repeat(R,Q)$. We can do this by relating only the pairs that must be related by any such relation.

The start is the pairs related by R in one step.

$J(R^*)$	α	β	γ	δ	ϵ	ζ
α	0	0	1	0	0	0
β	0	0	1	0	0	0
γ	0	0	0	1	0	0
δ	0	0	0	0	1	1
ϵ	0	0	0	0	0	0
ζ	0	0	0	0	0	0

or in a diagram



But since we have $R\alpha\gamma$ and $R^*\gamma\delta$, the repeat condition says we need to have $R^*\alpha\delta$ too.

And since $R\gamma\delta$ and $R^*\delta\epsilon$, we need $R^*\gamma\epsilon$ too, and hence, $R^*\alpha\epsilon$, Similarly $R^*\alpha\zeta$, $R^*\beta\delta$, $R^*\beta\epsilon$, $R^*\beta\zeta$, $R^*\gamma\epsilon$, $R^*\gamma\zeta$, by chaining these steps together.

These values for R^* are all forced by the Repeat condition.

$I(R^*)$	α	β	γ	δ	ϵ	ζ
α	0	0	1	1	1	1
β	0	0	1	1	1	1
γ	0	0	0	1	1	1
δ	0	0	0	0	1	1
ϵ	0	0	0	0	0	0
ζ	0	0	0	0	0	0

This reasoning shows not just that Repeat(R, R^*) holds, but that each of these relata must hold for any relation Q where Repeat(R, Q) holds. This means that we also have

$$\forall Q(\text{Repeat}(R,Q) \to \forall x \forall y (R^*xy \to Qxy)),$$

and hence, $TC(R, R^*)$.

- $TC(R, S)$ states that S is the TRANSITIVE CLOSURE of R, the 'smallest' transitive binary relation extending R.

$\forall R \exists S\, TC(R,S)$ is true in every second order model, as we can assign a value for S as the transitive closure of the value for R, using the procedure I used in the previous question.

We can make this totally precise, defining the relation R^* (the transitive closure of R) like this.

R_0 is empty — it relates no pairs in the domain

R_1 is just R — R_1xy iff Rxy

Given R_n, define R_{n+1} by setting $R_{n+1}xz$ iff there is some y where $Rxy \wedge R_nyz$

So $R_n xy$ holds iff there is a chain of n steps of the R relation linking x to y.

Then define R^*xy as holding iff $R_n xy$ for some n.

This R^* is the (unique) smallest transitive relation extending R. Since R^* always exists, we have

$\forall R \exists S\, TC(R,S)$, as desired.

Q3. Here is a function-term-free formula equivalent to $(x+y')=(x+y)'$ using the two-place predicate $E_\prime$, for the extension of the successor function, and E_+, a three-place predicate for the extension of the addition function.

I construct it step-by-step.

$(x+y')=(x+y)'$ is equivalent to

$\exists z(E_\prime(y,z) \wedge (x+z)=(x+y)')$, is equiv to

$\exists w(E_+(x,z,w) \wedge \exists z(E_\prime(y,z) \wedge w=(x+y)'))$, which is equiv to

$\exists v(E_+(x,y,v) \wedge \exists w(E_+(x,z,w) \wedge \exists z(E_\prime(y,z) \wedge w=v')))$, which, finally, is equiv. to

$\exists u(E_\prime(v,u) \wedge \exists v(E_+(x,y,v) \wedge \exists w(E_+(x,z,w) \wedge \exists z(E_\prime(y,z) \wedge w=u))))$.