ASSIGNMENT 1: PROPOSITIONAL LOGIC — PROOFS AND THREE VALUES

This is a project exploring *models* and *proofs* for propositional logic, and ways to extend and apply the tools you have learned in the first few weeks of this module. ¶ Read through all of these questions before attempting to answer them. Write your answers as clearly and explicitly as you can and explain all your working. ¶ Submit your answers as a PDF file on MMS by the due date: **October 10, 2023**, and provide a *hardcopy* to Greg either at his office, Edgecliffe 105 (there will be an envelope on the door if Greg is away from the office) or in a lecture class, by **3pm October 11**, to assist with marking. ¶ Each submission must be *anonymous*, and must start with the project coversheet, available on Moodle.

» «

Many people, when they are introduced to the tools and techniques in logic that we have seen so far, have some questions about *negation*. Consider these two arguments:

$$p \land \neg p \succ q \qquad p \succ q \lor \neg q$$

They are both classically valid. There is no two-valued valuation where $p \land \neg p$ is true, and q is false. There is no two-valued valuation where p is true and $q \lor \neg q$ is false. But not everyone agrees that these answers are right, either on grounds of *relevance* (q need not have anything to do with $p \land \neg p$, and p need not have anything to do with $q \lor \neg q$), or on grounds of *paradox* or *vagueness* ($p \land \neg p$ might, in some sense, be *true* in some circumstances; and $q \lor \neg q$ might, in some sense, be *untrue*).[1]

» «

Consider these three-valued truth tables:

$\land$	0	n	1
0	0	0	0
n	0	n	n
1	0	n	1

$\lor$	0	n	1
0	0	n	1
n	n	n	1
1	1	1	1

$\to$	0	n	1
0	1	1	1
n	n	n	1
1	0	n	1

$\neg$	
0	1
n	n
1	0

$\bot$
0

[1] If you would like to read more of an introduction to K3 and LP, read Chapter 7 of Graham Priest's *An Introduction to Non-Classical Logic* (Edition 2; Cambridge University Press, 2008). K3 gets its name from Stephen Cole Kleene, who motivated K3 by reflecting on the behaviour of *partial functions* in mathematics. (These are functions that, for some inputs, fail to return a value. An example is *division*. 5/0 is undefined.) See Section 64 of Kleene's *Introduction to Metamathematics* (Wooters-Noordhoff, 1952) for his introduction to this 3-valued logic. LP is defined by Graham Priest as the *logic of paradox*, in his 1979 paper "Logic of Paradox," *Journal of Philosophical Logic* vol 8, 219–241.

(BEWARE: These are *not* the same as the three-valued tables have already seen, which are sound for intuitionistic logic.)

For this project we are looking at two *different* ways to use these truth tables to define countermodels and validity for propositional logics.

The logic K3 is defined like this: A K3-counterexample to an argument $X \succ A$ is a valuation v using our three-valued tables, where $v(X) = 1$ and $v(A) = 0$ or n. $X \succ A$ is K3-valid (written $X \models_{K3} A$) if it has no K3-counterexample — that is, there is no way to make the premises have value 1 and the conclusion *not* 1 (that is, the conclusion has value 0 or n).

On the other hand, an LP-counterexample to an argument $X \succ A$ is a valuation v where $v(X) = 1$ or n and $v(A) = 0$. So, $X \succ A$ is LP-valid (written $X \models_{LP} A$) if there is no LP-counterexample: there is no way to make the premises *not* have the value 0 (that is, the premises are each either valued 1 or n) and the conclusion has value 0.

QUESTION 1 (1 POINT)

Use a three-valued truth table to show that $p \wedge \neg p \succ q$ K3-valid but not LP-valid. Then, show that $p \succ q \vee \neg q$ is LP-valid, but not K3-valid.

QUESTION 2 (3 POINTS)

Assess the following arguments for K3-validity and LP-validity. For each argument, explain why it is valid, or why it is invalid, in K3 and in LP.

(a) $p, q \succ p \wedge q$

(b) $p \succ q \to p$

(c) $p \succ q \to q$

(d) $p, p \to q \succ q$

(e) $p \succ (p \wedge q) \vee (p \wedge \neg q)$

(f) $(p \vee q) \wedge (p \vee \neg q) \succ p$

QUESTION 3 (8 POINTS)

We can see how our natural deduction proof rules relate to LP- and K3-validity. For example, the rule $\perp E$ is sound for both K3- and LP-validity, because there is no counterexample (either in K3 or LP) to the inference from $\perp$ to A. There is no valuation v where $v(\perp) = 1$ and $v(A) = 0$ or n (no K3-counterexample), and no valuation where $v(\perp) = 1$ or n and $v(A) = 0$ (no LP-counterexample) since there is no valuation where $v(\perp) = 1$ or n — $v(\perp) = 0$ for every valuation v, according to our rules.

Task 1: Consider the proof rules $\wedge I$, $\wedge E$, $\vee I$, $\neg E$ and $\to E$, from Chapters 2 and 3 of *Logical Methods*, and find, from among these rules, one rule which is *sound* for LP-validity. (And explain, in your own words, *why* it is sound, in exactly the same way as we prove soundness for each rule in the proof of Theorem 15 in Chapter 6.)

Task 2: Find, from among these rules, an example of a rule which is *not* sound for LP-validity. (And also, explain in your own words why it is *not* sound, by providing a *counterexample* to the soundness of this rule.)

Once you've done this for LP-validity, we'll do this for K3-validity:

Task 3: Find a rule which is sound for K3-validity, and explain why it is sound.

To find an example of a rule which is not sound for K3-validity, we need to do some more work than in the case of LP. Here is why the rule $\neg$I is not sound for K3-validity. If $\neg$I *were* sound, then this means that whenever $X, A \vDash_{K3} \bot$, then $X \vDash_{K3} \neg A$. But this is not a fact about K3-validity. For example, we have $p \wedge \neg p \vDash_{K3} \bot$ (since there is no valuation that makes $p \wedge \neg p$ take the value 1), but we do *not* have $\vDash_{K3} \neg(p \wedge \neg p)$. (This argument has a counterexample: set $v(p) = n$. Then $v(\neg(p \wedge \neg p)) = n$ too.) So, $\neg$I is not sound for K3-validity.

Task 4: Explain in your own words why $\rightarrow$I is *also* not sound for K3-validity.

QUESTION 4 (4 POINTS)

Write no more than one page, in which you present some reasons one might prefer using either K3- or LP-validity to classical (Boolean, two-valued) validity when assessing arguments in the language of propositional logic, and then critically assess the reasons you have presented.

» «

That is the end of the questions about K3 and LP. To complete this assignment, we have one remaining question about natural deduction proofs.

QUESTION 5 (4 POINTS)

I have complained in class, repeatedly, about the complexity of the disjunction elimination rule. Some of you may know, though, that in *classical* logic, there is really nothing special about disjunction: we can treat disjunction as a *defined* connective, rewriting $A \vee B$ as $\neg A \rightarrow B$.

Task 1: Show that $A \vee B \vdash_I \neg A \rightarrow B$, by constructing a proof from $A \vee B$ to $\neg A \rightarrow B$.

Task 2: Show that $\neg A \rightarrow B \nvDash_I A \vee B$, by finding a counterexample to the in the following three-valued truth table for intuitionistic logic (the three-valued Heyting lattice discussed in Chapter 6). Remember, a counterexample in a Heyting lattice is a valuation in which the premises are assigned 1 and the conclusion is *not* assigned 1.

$\vee$	0	n	1
0	0	n	1
n	n	n	1
1	1	1	1

$\rightarrow$	0	n	1
0	1	1	1
n	0	1	1
1	0	n	1

$\neg$	
0	1
n	0
1	0

So, we cannot define disjunction by way of the other connectives in intuitionistic logic. But we can in *classical* logic. In fact, we can go further, and show that we can rewrite classical *proofs* that use the disjunction rules to use the other connective rules instead.

Task 3: Show that these two rules are *derived rules* of our natural deduction system:

$$\frac{A}{\neg A \rightarrow B}\,{\neg\!\rightarrow\! I_1} \qquad \frac{B}{\neg A \rightarrow B}\,{\neg\!\rightarrow\! I_2}$$

(This means that any proof using the disjunction introduction rules could be rewritten to *not* use those rules, when we replace $A \vee B$ by $\neg A \rightarrow B$.)

Task 4: Show that this rule:

$$\frac{\begin{array}{c}\\ \Pi_1\\ \neg A \rightarrow B\end{array} \quad \begin{array}{c}[A]^i\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]^j\\ \Pi_3\\ C\end{array}}{C}\,{\neg\!\rightarrow\! E^{i,j}}$$

is a derived rule of our proof system for classical logic, by showing that from the smaller proofs Π_1, Π_2 and Π_3 (with the premises and conclusions indicated), we can construct a new proof of C, *without using the disjunction rules*, and in which the assumptions A and B in Π_2 and Π_3 respectively are discharged.

(**HINT**: To do this, you *must* make use of the Double Negation Elimination rule somewhere, because the rule is *not* a derived rule in our proof system for intuitionistic logic. ***Beware***: this task is not straightforward. Attempt it as best you can. You will be awarded part marks for honest efforts which show understanding of what this task is asking of you.)

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I hereby declare that the attached piece of written work is my own work and that I have not reproduced, without acknowledgement, the work of another.

Marking guide:
- '✓' represents 1 point.
- '✗' represents $\frac{1}{2}$ point.

QUESTION 1

p	q	p	$\wedge$	$\neg$	p	$\succ$	q
0	0	0	0	1	0		0
0	n	0	0	1	0		n
0	1	0	0	1	0		1
n	0	n	n	n	n		0
n	n	n	n	n	n		n
n	1	n	n	n	n		1
1	0	1	0	0	1		0
1	n	1	0	0	1		n
1	1	1	0	0	1		1

The highlighted model such that $v(p \wedge \neg p) = n$ and $v(q) = 0$ is a LP-counterexample, so $p \wedge \neg p \nvDash_{LP} q$.

There is no K3-counterexample since there is no model such that the valuation of the only premise equals 1, i.e. $v(p \wedge \neg p) = 1$. ✓✗

p	q	p	$\succ$	q	$\vee$	$\neg$	q
0	0	0		0	1	1	0
0	n	0		n	n	n	n
0	1	0		1	1	0	1
n	0	n		~~1~~ 0	1	~~0~~ 1	~~1~~ 0
n	n	n		~~1~~ n	n	0	~~1~~ n
n	1	n		1	1	0	1
1	0	1		~~1~~ 0	1	~~0~~ 1	~~1~~ 0
1	n	1		~~1~~ n	n	~~0~~ n	~~1~~ n
1	1	1		1	1	0	1

The highlighted model such that $v(p) = 1$ and $v(q \vee \neg q) = n$ is a K3-counterexample, so $p \nvDash_{K3} q \wedge \neg q$.

There is no LP-counterexample since there is no model such that the valuation of the conclusion equals 0, i.e. $v(q \lor \neg q) = 0$.

QUESTION 2

(a) Suppose the argument has a K3-counterexample, setting $v(p) = 1$ and $v(q) = 1$. Then $v(p \land q) = 1$, which contradicts the assumption because $v(p \land q)$ is supposed to be 0 or n. Hence, the argument is K3-valid, i.e. $p, q \models_{K3} p \land q$.

mixed cases are also possible

Suppose it is LP-invalid, so either both $v(p) = 1$ and $v(q) = 1$ or both $v(p) = n$ and $v(q) = n$. Then $v(p \land q) = 1$ or n, which contradicts the assumption because $v(p \land q)$ is supposed to be 0. Hence, the argument is also LP-valid.

(b) Suppose it is K3-invalid, so it has a K3-counterexample such that $v(p) = 1$. Since $v(p) = 1$, $v(q \to p) = 1$, which contradicts the assumption because the valuation of the conclusion $v(q \to p)$ is supposed to be 0 or n. Hence, the argument is K3-valid.

Similarly, it is LP-valid since $v(p) = 1$ or n leads to $v(q \to p) = 1$ or n but not 0.

$\to$	0	n	1
0	1	1	1
n	n	n	1
1	0	n	1

(c) It is K3-invalid because there is a K3-counterexample. Take a valuation v with $v(p) = 1$ and $v(q) = n$, then the only premise $v(p) = 1$ but the conclusion $v(q \to q) = n$.

It is LP-valid. See the truth table below. No matter what value q takes, $v(q \to q)$ is either 1 or n but not 0.

$\to$	0	n	1
0	1	1	1
n	n	n	1
1	0	n	1

(d) Suppose the argument has a K3-counterexample, setting $v(p) = 1$, $v(p \to q) = 1$, and $v(q) = 0$ or n. Since $v(p) = 1$ and $v(p \to q) = 1$, $v(q)$ can only be 1, which contradicts the assumption that $v(q) = 0$ or n. Therefore, there is no K3-counterexample, so the argument is

K3-valid.

It is LP-invalid since there is a LP-counterexample. Take a valuation ν with $\nu(p) = n$ and $\nu(q) = 0$. Then, the premises $\nu(p) = n$ and $\nu(p \rightarrow q) = n$ but the conclusion $\nu(q) = 0$.

(e) It is K3-invalid since there is a K3-counterexample. Take a valuation ν with $\nu(p) = 1$ and $\nu(q) = n$. Then, $\nu(p \wedge q) = n$, $\nu(\neg q) = n$, and $\nu(p \wedge \neg q) = n$. Thus the conclusion $\nu((p \wedge q) \vee (p \wedge \neg q)) = n$ while the premise $\nu(p) = 1$.

Suppose the argument has a LP-counterexample, setting $\nu(p) = 1$ or n, $\nu((p \wedge q) \vee (p \wedge \neg q)) = 0$. Then, $\nu(p \wedge q) = 0$ and $\nu(p \wedge \neg q) = 0$. Assume $\nu(p) = 1$, then from $\nu(p \wedge q) = 0$ we can derive $\nu(q) = 0$. Subsequently, $\nu(p \wedge \neg q)$ will be 1, which contradicts the derivation of the initial assumption, $\nu(p \wedge \neg q) = 0$, so $\nu(p) \neq 1$. Similarly, assume $\nu(p) = n$, then from $\nu(p \wedge q) = 0$ we can derive $\nu(q) = 0$, so $\nu(\neg q) = 1$, but $\nu(p \wedge \neg q)$ will be n, which contradicts the derivation of the initial assumption $\nu(p \wedge \neg q) = 0$, so $\nu(p) \neq n$. Overall, $\nu(p)$ is neither 1 nor n, which contradicts the initial assumption. Therefore, there is no LP-counterexample, so the argument is LP-valid.

(f) Suppose the argument has a K3-counterexample, setting $\nu((p \vee q) \wedge (p \vee \neg q)) = 1$ and $\nu(p) = 0$ or n. Then, $\nu(p \vee q) = 1$ and $\nu(p \vee \neg q) = 1$. Assume $\nu(p) = 0$, then from $\nu(p \vee q) = 1$ we can derive $\nu(q) = 1$, but this will lead to $\nu(p \vee \neg q) = 0$, which contradicts the derivation of the initial assumption, $\nu(p \vee \neg q) = 1$, so $\nu(p) \neq 0$. Similarly, assume $\nu(p) = n$, then from $\nu(p \vee q) = 1$ we can derive $\nu(q) = 1$, but this will lead to $\nu(p \vee \neg q) = n$, which contradicts the derivation of the initial assumption, $\nu(p \vee \neg q) = 1$, so $\nu(p) \neq n$. Overall, $\nu(p)$ is neither 0 or n, which contradicts the initial assumption. Therefore, there is no K3-counterexample, so the argument is K3-valid.

It is LP-invalid since there is a LP-counterexample. Take a valuation ν with $\nu(p) = 0$ and $\nu(q) = n$. Then, $\nu(p \vee q) = n$, $\nu(\neg q) = n$, and $\nu(p \vee \neg q) = n$. Thus, the premise $\nu((p \vee q) \wedge (p \vee \neg q)) = n$ but the conclusion $\nu(p) = 0$.

QUESTION 3

Task 1

$\vee$I is sound for LP-validity means if $X \vDash_{LP} A$, then $X \vDash_{LP} A \vee B$.

Base case

you do not need to consider the base case to answer the questio

The assumption rule provides the base case. The atomic proof, A, is a proof for $A \succ A$. Let v be a LP model such that the conclusion $v(A) = 0$, then the premise $v(A) = 0$. There is no way to construct an LP-counterexample. Therefore, the assumption rule is LP-valid.

Inductive case for $\vee$I

We assume we have a proof Π for $X \succ A$ that is LP-valid, i.e. $X \vDash_{LP} A$. We then form a new proof using the $\vee$I rule.

$$\cfrac{\begin{array}{c} X \\ \Pi \\ A \end{array}}{A \vee B} \vee I$$

We want to show that $X \vDash_{LP} A \vee B$. Assume there is a LP-counterexample, setting $v(X) = 1$ or n and $v(A \vee B) = 0$. By definition, $v(A \vee B) = 0$ iff $v(A) = 0$ and $v(B) = 0$. $v(X) = 1$ or n and $v(A) = 0$ contradicts the inductive hypothesis, $X \vDash_{LP} A$, since this is a LP-counterexample of it. Therefore, there is no LP-counterexample of $X \succ A \vee B$, so $X \vDash_{LP} A \vee B$ if $X \vDash_{LP} A$.

✓✓

The other $\vee$I case, where the conclusion is $B \vee A$, is similar.

Task 2

Q2 (d) is $\to$E case, which has been shown to be LP-invalid for Atoms. This means $X \vDash_{LP} A, Y \vDash_{LP} A \to B \succ X, Y \nvDash_{LP} B$. ✓✓

Task 3

$\vee$I is sound for K3-validity means if $X \vDash_{K3} A, Y \vDash_{K3} B$, then $X, Y \vDash_{K3} A \wedge B$.

Base case

Same as Task 2's base case proof.

there is no base case in task 2; you don't need to show one either if you prove soundness for a rule (as opposed to a proof system).

Inductive case for $\wedge$I

Assume proofs Π_1 for $X \succ A$ and Π_2 for $Y \succ B$ are both K3-valid, i.e. $X \vDash_{K3} A$ and $Y \vDash_{K3} B$. We then form a new proof using the $\wedge$I rule.

$$\dfrac{\begin{matrix} X \\ \Pi_1 \\ A \end{matrix} \qquad \begin{matrix} Y \\ \Pi_2 \\ B \end{matrix}}{A \wedge B} \wedge I$$

We want to show that $X, Y \vDash_{K3} A \wedge B$. Suppose there is a K3-counterexample, setting $v(X, Y) = 1$ or n and $v(A \wedge B) = 0$. If $v(A \wedge B) = 0$ then by definition either $v(A) = 0$ or $v(B) = 0$. $v(A) \neq 0$ since $v(A) = 0$ contradicts the inductive hypothesis that implies $v(X) = 1$ or n then $v(A) = 1$ or n. Similarly, $v(B) \neq 0$. Thus, neither $v(A) = 0$ or $v(B) = 0$, which contradicts the derivation of the assumption, either $v(A) = 0$ or $v(B) = 0$. Therefore, there is no K3-counterexample, so $X, Y \vDash_{K3} A \wedge B$. ✓✓

Task 4

you cannot use this symbol here (use, e.g. ⇒) instead

If $\to$I were sound for K3-validity, then this means that $X, A \vDash_{K3} B \succ X \vDash_{K3} A \to B$. This has a counterexample, $p, q \vDash_{K3} p \wedge q \succ p \nvDash_{K3} q \to (p \wedge q)$. $p, q \vDash_{K3} p \wedge q$ has been proven ~~proofed~~ in Q2 (a). $p \nvDash_{K3} q \to (p \wedge q)$ since it has a K3-counterexample: $v(p) = 1$ and $v(q) = n$, then $v(q \to (p \wedge q)) = n$. ✓✓

QUESTION 4

The motivation for using K3 and LP instead of classical two-value logic primarily arises from two issues: vagueness and semantic paradoxes. Let's discuss the vagueness first.

We have vague concepts such as *red*, *child*, and *heap*. These concepts do not have a sharp boundary. For example,

This is a good assessment of 3-valued representation, but how do LP/K3 validity handle this? Do their counter-examples make sense?

the right of this colour ribbon is clearly red, and the left is clearly not red; however, the adjacent area can not be told apart. There are objects of which it is neither definitely red nor definitely not red. It is nature to come up with a three-fold division to capture this intermediate region between true and false.

Semantic paradoxes, such as the Liar paradox, reveal a problem: assigning true and false values to

What is your assessment of LP/K3 truth-functionality and -preservation?

statements can lead to contradictions from these statements even though these statements are conform to language rules, i.e. semantically correct statements could be inconsistent in classical logic. K3 and LP can accommodate the inconsistency revealed by semantic paradoxes in respect of validity.

They are not the ultimate answer to these two issues. Introducing a third value to represent the vagueness increases the number of boundaries from one to two, namely, the boundary between true and intermediate, and between intermediate and false. The approach of introducing more values can only increase the resolution of vagueness but cannot eliminate it. Moving on to the second issue, although K3 and LP accommodate the inconsistency of semantic paradoxes in respect of validity, they have made some important logical activities such as determining truth and falsity and searching for truth and falsity more challenging.

References: Concepts without boundaries by Mark Sainsbury and Logical Methods by Greg Restall and Shawn Standefer, pp. 108-110. ✓ ✓✗

You make good points about 3-valued representation but engage little with LP/K3-validity.

QUESTION 5

Task 1

We want to show $A \vee B \vdash_I \neg A \to B$.

$$\dfrac{\dfrac{A \vee B \qquad \dfrac{\dfrac{[A]^1 \qquad [\neg A]^3}{\bot}\,\neg E}{B}\,\bot E \qquad [B]^2}{B}\,\vee E^{1,2}}{\neg A \to B}\,\to I^3$$

✓

Task 2

Take a valuation v with $v(A) = n$, $v(B) = n$, so $v(\neg A) = 0$. Then the premise $v(\neg A \to B) = 1$ but the conclusion $v(A \vee B) = n$. ✓

Task 3

$$\begin{array}{c} \dfrac{A}{A \vee B}\,\vee I \\ \Pi \\ \neg A \to B \end{array}$$

You cannot invoke the $\vee I$ rule in a proof aiming to show that $\vee I$ can be replaced without those rules as made clear in the instructions.

where Π was shown in Task 1.

↳ This was technically shown for $\vdash_I$, but you are supposed to show this for $\vdash_C$. You also need to invoke a theorem on the relationship of the two then, such as 'If $A \vdash_I B$, then $A \vdash_C B$', as we discussed in class.

Similarly,

$$\begin{array}{c} \cfrac{B}{A \lor B}\,{\lor}I \\ \Pi \\ \neg A \to B \end{array}$$

cf. above

Task 4

To start with, we show $\neg A \to B \vdash_I \neg(\neg A \land \neg B)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{[\neg A \land \neg B]^1}{\neg A}\,{\land}E \qquad \neg A \to B}{B}\,{\to}E \qquad \cfrac{[\neg A \land \neg B]^1}{\neg B}\,{\land}E}{\bot}\,{\neg}E}{\neg(\neg A \land \neg B)}\,{\neg}I^1$$

Next, we show $\neg(\neg A \land \neg B) \vdash_{CL} \neg\neg A \lor \neg\neg B$ which is actually an instance of the De Morgan's laws:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\neg\neg A]^1}{\neg\neg A \lor \neg\neg B}\,{\lor}I \qquad [\neg(\neg\neg A \lor \neg\neg B)]^3}{\bot}\,{\neg}E}{\neg\neg\neg A}\,{\neg}I^1}{\neg A}\,\mathrm{DNE} \qquad \cfrac{\cfrac{\cfrac{\cfrac{[\neg\neg B]^2}{\neg\neg A \lor \neg\neg B}\,{\lor}I \qquad [\neg(\neg\neg A \lor \neg\neg B)]^3}{\bot}\,{\neg}E}{\neg\neg\neg B}\,{\neg}I^2}{\neg B}\,\mathrm{DNE}}{\neg A \land \neg B}\,{\land}I \qquad \neg(\neg A \land \neg B)}{\bot}\,{\neg}E}{\neg\neg(\neg\neg A \lor \neg\neg B)}\,{\neg}I^3}{\neg\neg A \lor \neg\neg B}\,\mathrm{DNE}$$

Finally, we show $\neg\neg A \lor \neg\neg B \vdash_{CL} A \lor B$:

$$\cfrac{\neg\neg A \lor \neg\neg B \qquad \cfrac{\cfrac{[\neg\neg A]^1}{A}\,\mathrm{DNE}}{A \lor B}\,{\lor}I \qquad \cfrac{\cfrac{[\neg\neg B]^2}{B}\,\mathrm{DNE}}{A \lor B}\,{\lor}I}{A \lor B}\,{\lor}E^{1,2}$$

This is good proof work, but, unfortunately, not a proof of C. Unfortunately, you are also using the disjunction rules.

PY2010 PROJECT 1 SOLUTIONS

Q1 Here is a 3-valued truth table for $p \wedge \neg p \succ q$

There is no row of this table that gives a K3 counterexample.

There is no row where $p \wedge \neg p$ has value 1, so the argument is K3 valid.

$p \wedge \neg p \vDash_{K3} q$

p	$\wedge$	$\neg p$	q
0	0	1	0
0	0	1	n
0	0	1	1
n	n	n	0
n	n	n	n
n	n	n	1
1	0	0	0
1	0	0	n
1	0	0	1

This row gives us a model which is an LP counterexample to the argument:

$v(p \wedge \neg p) = n \quad v(q) = 0$

(when $v(p) = n$)

So, $p \wedge \neg p \nvDash_{LP} q$.

0.5 POINTS

Here is a 3-valued table for $p \succ q \vee \neg q$

The row in red gives a model which is a K3 counterexample to the argument:

$v(p) = 1 \quad v(q \vee \neg q) = n$

(when $v(q) = n$)

So, $p \nvDash_{K3} q \vee \neg q$

p	q	$\vee$	$\neg q$
0	0	1	1
0	n	n	n
0	1	1	0
n	0	1	1
n	n	n	n
n	1	1	0
1	0	1	1
1	n	n	n
1	1	1	0

There is no row of this table that gives an LP counterexample: no row assigns $q \vee \neg q$ the value 0, so the argument is LP valid.

0.5 POINTS

Q2. a) Consider the table for $\wedge$. If $v(p) = v(q) = 1$ then $v(p \wedge q) = 1$ too, so we have $p, q \vDash_{K3} p \wedge q$. $\boxed{0.5}$

Similarly, if $v(p) = 1$ or n & $v(q) = 1$ or n, then $v(p \wedge q) = 1$ or n, too, so we have $p, q \vDash_{LP} p \wedge q$.

b) $p \vDash_{K3} q \to p$ since if $v(p) = 1$ then $v(q \to p) = 1$, no matter what value q is assigned.

Similarly, $p \vDash_{LP} q \to p$ since if $v(q \to p) = 0$ (which we would need for an LP counterexample) then $v(p) = 0$ too, which means v is not a counterexample. $\boxed{0.5}$

c) $p \nvDash_{K3} q \to q$ ($v(p) = 1, v(q) = n$ is a counterexample.)

and $p \vDash_{LP} q \to q$ (since for every choice of $v(q)$, $v(q \to q) = 1$ or n, so there is no LP-counterexample). $\boxed{0.5}$

d) We have $p, p \to q \vDash_{K3} q$, since if we were to have a counterexample, we'd need to have $v(p) = 1$ and $v(p \to q) = 1$, but in that case $v(q) = 1$ too.

However, we have $p, p \to q \nvDash_{LP} q$, since $v(p) = n$, $v(q) = 0$ is a counterexample, since then, $v(p \to q) = n$. $\boxed{0.5}$

e) $p \nvDash_{K3} (p \wedge q) \vee (p \wedge \neg q)$, $(v(p)=1, v(q)=n)$. However, we have $p \vDash_{LP} (p \wedge q) \vee (p \wedge \neg q)$, since if $v(p) = 1$ or n, then $v((p \wedge q) \vee (p \wedge \neg q)) = 1$ or n, since if $v(q) = 1$ or n, then $v(p \wedge q) = 1$ or n, and otherwise, if $v(q) = 0$, then $v(\neg q) = 1$, and so, $v(p \wedge \neg q) = 1$ or n. $\boxed{0.5}$

f) $(p \vee q) \wedge (p \vee \neg q) \vDash_{K3} p$, since if $v((p \vee q) \wedge (p \vee \neg q)) = 1$ then $v(p \vee q) = 1$ & $v(p \vee \neg q) = 1$, which means that $v(p)$ must be 1, since whatever value q has, one of $v(q)$ and $v(\neg q)$ is either 0 or n, and so, is not 1, and a disjunction has value 1 only if one disjunct has value 1.

$(p \vee q) \wedge (p \vee \neg q) \vDash_{LP} p$, $(v(p) = 0, v(q) = n)$. $\boxed{0.5}$

Q3 TASK 1

$\wedge I$ is sound for LP validity

Suppose we have a proof
$$\dfrac{\begin{matrix} X & & Y \\ \Pi & & \Pi \\ A & & B \end{matrix}}{A \wedge B}$$
where $X \vDash_{LP} A$ & $Y \vDash_{LP} B$,

& consider an LP-valuation where $v(X, Y) = 1$ or n and ask whether $v(A \wedge B) = 1$ or n.

Since $v(X, Y) = 1$ or n, & $X \vDash_{LP} A$, we have $v(A) = 1$ or n and since $Y \vDash_{LP} B$, we have $v(B) = 1$ or n, & inspecting the truth table for $A \wedge B$, we see then that $v(A \wedge B) = 1$ or n

So, $\wedge I$ is sound for LP-validity.

The rules $\wedge E$ & $\vee I$ are also sound for LP validity, and the argument to this effect is just like the argument here, except slightly easier, since these are one-premise rules.

2 points

TASK 2

$\neg E$ is not sound for LP validity. Take this simplest example of a proof using $\neg E$ alone:

$$\frac{\neg p \quad p}{\bot}$$

And take the valuation v where $v(p) = n$.

Here $v(p) = v(\neg p) = n$ & $v(\bot) = 0$, and so, the rule $\neg E$ is not sound for LP-validity

In just the same way, $\to E$ is not sound for LP validity. Consider instead the simple proof

$$\frac{p \to q \quad p}{q}\ \to E,$$

and take the valuation $v(p) = n$, $v(q) = 0$. Here, the premises have value n & the conclusion 0, so this rule is also unsound.

$\boxed{\text{2 points}}$

TASK 3

$\wedge E$ is sound for K3 validity. Take a proof

$$\frac{\begin{array}{c} X \\ \Pi \\ A \wedge B \end{array}}{A}\ \wedge E \quad \text{or} \quad \frac{\begin{array}{c} X \\ \Pi \\ A \wedge B \end{array}}{B}\ \wedge E$$

ending in a $\wedge E$ step, and suppose $X \vDash_{K3} A \wedge B$

We want to show that $X \vDash_{K3} A$ & $X \vDash_{K3} B$. So, let's suppose $v(X) = 1$. Since $X \vDash_{K3} A \wedge B$ we have $v(A \wedge B) = 1$. It follows (checking the $\wedge$ table to verify this) that $v(A) = v(B) = 1$, and so, we also have $X \vDash_{K3} A$ & $X \vDash_{K3} B$, as desired.

The rules $\wedge E$ & $\vee I$ are also sound for K3 validity, and the argument to this effect is just like the argument here, except slightly easier, since these are one-premise rules.

$\boxed{\text{2 POINTS}}$

TASK 4

2 points

~~→I is not sound for K3 validity~~

If →I were sound, we would have whenever $X, A \vDash_{K3} B$ then $X \vDash_{K3} A \to B$. But this is not true, in general. For example we do have $p \vDash_{K3} p$, since if $v(p)=1$ then $v(p)=1$ (obviously!), but $\nvDash_{K3} p \to p$, since in the valuation $v(p)=n$, we have $v(p \to p) = n$.

Q4

4 points

There are many different things you could write to answer this question. I won't write a single model answer (because there are so many different options), but I'll sketch some of the things that you could have written, and explain what kinds of points are important to make in any answer.

On one view, the best kinds of valuations—the most realistic accounts of truth valuations—don't assign one and only one of 1 and 0 to each formula. On one view, some things should not be assigned 1 or assigned 0, in such a way that makes these truth tables look OK.

Eg: Vagueness

Paradoxes

Failure of presuppositions.

If you say that, and you think that things are truth functional with tables like these ones, and you think that validity is preservation of truth, then either K3 or LP looks pretty good.

But these are important assumptions, which should then be examined. Is it truth functional? (Probability not like this.) Is p&~p n when p is n? Is that a reasonable assumption for this kind of application of truth values?

Is validity best understood as preservation of 1 or as preservation of 1-and-0, or something else? That's a question that is to be addressed if you want this framework.

Then, too, there are the questions about the behaviour of ->. It's really hard to get that right with a 3 valued framework if you want -> to satisfy modus ponens and if you want identities to be true. It's almost impossible to get right if you want ->I and ->E to be valid, too.

Q5 Here is a proof from $\neg A \vee B$ to $A \to B$

TASK 1

$$
\dfrac{\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{[\neg A]^1 \quad [A]^3}{\bot}\,\neg E}{B}\,\bot E \qquad [B]^2}{B}\,\vee E^{1,2}}{A \to B}\,\to I^3
$$

1 POINT

TASK 2

$$
\dfrac{A \to B \;\not\models_{I}\; \neg A \vee B}{n \;\; 1 \;\; n \qquad 0 \;\; n \;\; n \;\; n}
$$

Here is a counterexample in this 3-valued Heyting lattice. If $v(A) = n = v(B)$, then $v(A \to B) = 1$ and $v(\neg A \vee B) = n$.

1 POINT

TASK 3 Rewrite any proof

$$
\dfrac{\begin{array}{c}\Pi \\ A\end{array}}{\neg A \to B}\,\neg\to I_1
$$

using our other rules, like this:

$$
\dfrac{\dfrac{\dfrac{[\neg A]^1 \quad \begin{array}{c}\Pi \\ A\end{array}}{\bot}\,\neg E}{B}\,\bot E}{\neg A \to B}\,\to I^1
$$

0.5

The $\neg\!\to\! I_2$ rule is easier (but stronger). The inference

$$\dfrac{\begin{matrix}\Pi \\ B\end{matrix}}{\neg A \to B}\ \to I$$

is an instance of $\to I$ in which <u>zero</u> instances of $\neg A$ are discharged, so this rule is not just a <u>derived</u> rule but a specific individual rule of our proof system.

0.5

TASK 4.

If we have an existing proof:

$$\dfrac{\begin{matrix}\Pi_1 \\ \neg A \to B\end{matrix} \qquad \begin{matrix}[A]^i \\ \Pi_2 \\ C\end{matrix} \qquad \begin{matrix}[B]^j \\ \Pi_3 \\ C\end{matrix}}{C}\ \neg\!\to\! E^{i,j}$$

we can rewrite it using our other rules like this:

$$\dfrac{\dfrac{[\neg C]^j \qquad \begin{matrix}\dfrac{\dfrac{\dfrac{[\neg C]^j \qquad \begin{matrix}\dfrac{\begin{matrix}\Pi_1 \\ \neg A \to B\end{matrix} \qquad [\neg A]^i}{B}\ \to E \\ \Pi_3 \\ C\end{matrix}}{\bot}\ \neg E}{\neg\neg A}\ \neg I^i}{A}\ \text{DNE} \\ \Pi_2 \\ C\end{matrix}}{\bot}\ \neg E}{\dfrac{\neg\neg C}{C}\ \text{DNE}}\ \neg I^j$$

This is a complicated proof, and it uses $\to E$, and $\neg I$ & DNE twice!

1 point