PROJECT 1: PROOFS AND MODELS FOR THE ‘*NO*’ QUANTIFIER

This project explores how the proofs of the soundness and completeness theorems for first-order predicate logic apply to a language with different fundamental concepts – in this case, a *binary* quantifier expression ‘*NO*’ which combines *two* formulas, instead of the familiar unary quantifiers $\forall$ and $\exists$. ¶ Read through all of these questions before attempting to answer them. Write your answers as clearly and explicitly as you can and explain all your working. ¶ Submit your answers as a PDF file on MMS by the due date: **February 26, 2024**.

The language of predicate logic that we ended up using in the 20th Century (and into the 21st) is a matter of historical contingent choice. We could have done things differently. In this assignment, you will demonstrate your understanding of the soundness and completeness theorems for first-order predicate logic by demonstrating key parts of these theorems with respect to a *different* choice of logical vocabulary.

Instead of the one-place quantifiers $\forall$ and $\exists$, we will consider a *binary* quantifier expression, *NO*. We change the grammar of our language so that if A and B are formulas, and x is a variable, then $\textit{NO}\, x(A, B)$ is a formula, in which the variable x is *bound*. The intended reading of this quantifier should be clear: $\textit{NO}\, x(Fx, Gx)$ is to be read as saying “no F is G”. This assignment concerns the logical language in which we have names, function symbols, variables, predicates, the usual connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$, and $\bot$), and the special binary quantifier $\textit{NO}\, x$ (for each variable x). (So: in this language, we do *not* use the regular quantifiers $\forall x$ and $\exists x$.)

TASK 1: PROOFS (6 POINTS)

This task concerns the proof rules for the *NO* quantifier. Here are the natural deduction introuduction and elimination rules:

$$\cfrac{\begin{matrix} X, [A(n)]^i, [B(n)]^j \\ \vdots \\ \bot \end{matrix}}{\textit{NO}\, x(A(x), B(x))}\ \textit{NOI}^{i,j} \qquad\qquad \cfrac{\textit{NO}\, x(A(x), B(x)) \qquad A(t) \qquad B(t)}{\bot}\ \textit{NOE}$$

In *NO*I, the following condition applies: the name n must appear only in the discharged assumptions $A(n)$ and $B(n)$, and nowhere in any of the remaining undischarged assumptions X in the proof to the contradiction $\bot$.

Here is a simple proof illustrating the use of these rules:

$$\cfrac{\cfrac{\textit{NO}\, x(Fx, Gx) \qquad [Fn]^2 \qquad [Gn]^1}{\bot}\ \textit{NOE}}{\textit{NO}\, x(Gx, Fx)}\ \textit{NOI}^{2,1}$$

The proof shows, as expected, that if no F is G then no G is F. In the first step, we apply the assumption ($\textit{NO}\, x(Fx, Gx)$) to show that a contradiction follows under the supposition of Fn and Gn. Then, since we have proved that Gn and Fn are contradictory, assuming nothing else about n, we can discharge them (in reversed order), to conclude $\textit{NO}\, x(Gx, Fx)$.

Your first task is to write out full natural deduction proofs (using only the standard connective natural deduction rules and these new rules for the *NO* quantifier) for the following arguments:

1. $\textit{NO}\, x(Fx, Fx \land Gx) \succ \textit{NO}\, x(Fx, Gx)$
2. $\textit{NO}\, x(Fx, Gx), \textit{NO}\, x(\neg Gx, Hx) \succ \textit{NO}\, x(Fx, Hx)$
3. $\textit{NO}\, x(Fx, Gx) \lor \textit{NO}\, x(Fx, Hx) \succ \textit{NO}\, x(Fx, Gx \land Hx)$

TASK 2: MODELS (4 POINTS)

Your second task will be to work with the truth conditions for formulas involving the *NO* quantifier in models for first-order predicate logic. These models remain unchanged: they consist of a non-empty domain (a set D) and an interpretation function I assigning appropriate values for each of the names, function symbols and predicates of the language. Given a model, each formula can be assigned a truth value relative to an assignment α of values to the variables. This procedure defines $I(A, \alpha)$ for each formula A in the language. For our new language, we use the following clause to interpret *NO* formulas:

- $I(\textit{NO}\, x(A, B), \alpha) = 1$ iff there is no x-variant α' of α where $I(A, \alpha') = 1$ and $I(B, \alpha') = 1$.

Consider the following model, for the language with the one-place function symbol f, the one-place predicate F and the two-place predicate R.

$D = \{0, 1, 2, 3\}$

	I(f)
0	0
1	3
2	2
3	1

	I(F)
0	1
1	0
2	1
3	0

I(R)	0	1	2	3
0	0	1	1	1
1	1	1	1	0
2	1	1	0	1
3	1	0	1	1

Find the truth values of each of the following formulas in the model, using the clause defined above to interpret the '*NO*' quantifier. Take care to explain each step of your reasoning.

1. $\textit{NO}\, x(Fx, Rxx)$
2. $\textit{NO}\, x\big(Rxy, \textit{NO}\, y(Rxy, \neg Ryx)\big)$
3. $\textit{NO}\, x\big(Fx, \textit{NO}\, y(Ffy, Rxy)\big)$
4. $\textit{NO}\, x\big(\textit{NO}\, y(Fy, Ryx \land Fx), \textit{NO}\, y(Ffy, Ryx \lor Fx)\big)$

TASK 3: SOUNDNESS (4 POINTS)

Your next task is to explain how the *soundness* proof for the language of first-order predicate logic must be modified for our reasoning to apply to this new language. In particular, you must spell out all the relevant steps for the soundness proof concerning the the new natural deduction rules for the *NO* quantifier.

TASK 4: COMPLETENESS (6 POINTS)

Your final task is to explain how the *completeness* proof for first-order predicate logic must be modified for our reasoning to apply to this new language. In particular, in our *original* completeness proof, we relied on the concept of a set of formulas being *witnessed*. This uses the existential quantifier. Our new language does not feature the existential quantifier, so you must (1) find some appropriate substitute for this notion, appropriate to the our new language. Once you have made clear what it is for set of formulas to be *witnessed* in your new sense, you must (2) modify our argument that shows that any A-avoiding set X can be extended into a *witnessed* maximal A-avoiding set so that it works with your new witnessing notion. Then, with this witnessed maximal A-avoiding set in hand, you must (3) show that a set X^* like this, as constructed, gives you the means to define a model $\mathfrak{M}_{X^*}$ which validates exactly the formulas in X^* — that in particular that this fact holds for NO-formulas, with their distinctive definition.

In your reasoning, you can feel free to reproduce any reasoning I gave in the lectures. Your own original work will be the modifications you need to make for the new vocabulary. Be careful to explain all of your reasoning and to spell out each step in your own words.

FINAL INSTRUCTIONS

Project 1 is designed to test your understanding of the soundess and completeness theorems and proofs and models for first-order predicate logic. It will be hard for you to get *everything* right in this assignment, and to explain things clearly. Don't be discouraged if you find it difficult! If you aren't sure about the level of detail you need to show to answer the questions, take my explanations in the course notes as a guide. That's the depth of explanation that I am looking for. (You should also find the solutions to the class tasks useful for examples of how to set out the soundness and completeness arguments.)

In this project (worth 40%) the emphasis is on your depth of understanding of the formal logical tools themselves. In the second project (worth 50%) you will have an opportunity to explore some of the philosophical consequences of these results, and the results and ideas we will be working with through the rest of the course.

If you have any questions, don't hesitate to contact me.

Good luck!

PY4612 Advanced Logic

Project 1: Proofs and models for the ‘NO’ quantifier

26th February 2024

I hereby declare that the attached piece of written work is my own work and that I have not reproduced, without acknowledgement, the work of another.

TASK 1 6/6

1.

$$\dfrac{\dfrac{\textsc{no}\,x(Fx, Fx \wedge Gx) \qquad [Fn]^1 \qquad \dfrac{[Fn]^1 \qquad [Gn]^2}{Fn \wedge Gn}\wedge I}{\bot}\textsc{no}E}{\textsc{no}\,x(Fx, Gx)}\textsc{no}I^{1,2}$$

The eigenvariable condition of the $\textsc{no}I^{1,2}$ step is satisfied because the name n appears only in the assumptions discharged in this step, $\{Fn, Gn\}$, and nowhere in any of the remaining undischarged assumptions of this step, $\{\textsc{no}\,x(Fx, Fx \wedge Gx)\}$.

2.

$$\dfrac{\dfrac{\textsc{no}\,x(\neg Gx, Hx) \qquad \dfrac{\dfrac{\textsc{no}\,x(Fx, Gx) \qquad [Fn]^1 \qquad [Gn]^2}{\bot}\textsc{no}E}{\neg Gn}\neg I^2 \qquad [Hn]^3}{\bot}\textsc{no}E}{\textsc{no}\,x(Fx, Hx)}\textsc{no}I^{1,3}$$

The eigenvariable condition of the $\textsc{no}I^{1,3}$ step is satisfied because the name n appears only in the assumptions discharged in this step and nowhere in any of the remaining undischarged assumptions of this step, $\{\textsc{no}\,x(\neg Gx, Hx), \textsc{no}\,x(Fx, Gx)\}$.

3.

Tiny!

$$\dfrac{\dfrac{\textsc{no}\,x(Fx, Gx) \vee \textsc{no}\,x(Fx, Hx) \qquad \dfrac{[\textsc{no}\,x(Fx, Gx)]^1 \qquad [Fn]^3 \qquad \dfrac{[Gn \wedge Hn]^4}{Gn}\wedge E}{\bot}\textsc{no}E \qquad \dfrac{[\textsc{no}\,x(Fx, Hx)]^2 \qquad [Fn]^3 \qquad \dfrac{[Gn \wedge Hn]^4}{Hn}\wedge E}{\bot}\textsc{no}E}{\bot}\vee E^{1,2}}{\textsc{no}\,x(Fx, Gx \wedge Hx)}\textsc{no}I^{3,4}$$

The eigenvariable condition of the $\textsc{no}I^{3,4}$ step is satisfied because the name n appears only in the assumptions discharged in this step and nowhere in any of the remaining undischarged assumptions of this step, $\{\textsc{no}\,x(Fx, Gx) \vee \textsc{no}\,x(Fx, Hx)\}$.

TASK 2 3½/4

1. Observing line by line, it can be seen that no matter what object is assigned to x, F and R cannot both be true ~~at the same time~~: of the same objects in D.

	$I(F)$
0	1
1	0
2	1
3	0

$I(R)$	0	1	2	3
0	0			
1		1		
2			0	
3				1

✓

Namely, there is no x-variant α' of α where $I(Fx, \alpha') = 1$ and $I(Rxx, \alpha') = 1$. Therefore, $I(\textit{NO}\, x(Fx, Rxx), \alpha) = 1$.

2. $I(\textit{NO}\, x(Rxy, \textit{NO}\, y(Rxy, \neg Ryx)), \alpha) = 1$ iff there is no x-variant α^x of α where
   ✓
   - $I(Rxy, \alpha^x) = 1$, and
   - $I(\textit{NO}\, y(Rxy, \neg Ryx), \alpha^x) = 1$ iff there is no y-variant $\alpha^{x,y}$ of α^x where
     - $I(Rxy, \alpha^{x,y}) = 1$, and
     - $I(\neg Ryx, \alpha^{x,y}) = 1$

   By observing the truth table of $I(R)$, it can be found that R is a symmetric relation. This means that for no pair $x, y \in D$ do we have both Rxy and ¬Ryx. This entails that there is no y-variant $a^{x,y}$ of every x-variant α^x of α such that $I(Rxy, \alpha^{x,y}) = 1$ and $I(\neg Ryx, \alpha^{x,y}) = 1$; thus, $I(\textit{NO}\, y(Rxy, \neg Ryx), \alpha^x) = 1$ for every x-variant α^x of α. On the other hand, there is an x-variant $\alpha^x[x : 1, y : 0]$ of $\alpha[y : 0]$ such that $I(Rxy, \alpha^x) = 1$. Since $I(\textit{NO}\, y(Rxy, \neg Ryx), \alpha^x)$ is true for every x-variant α^x of α, it is true under this assignment. There is an x-variant, e.g., $\alpha^x[x : 1, y : 0]$ of $\boxed{\alpha[y : 0]}$, where $I(Rxy, \alpha^x) = 1$ and $I(\textit{NO}\, y(Rxy, \neg Ryx), \alpha^x) = 1$; therefore,
   ✗ $I(\textit{NO}\, x(Rxy, \textit{NO}\, y(Rxy, \neg Ryx)), \cancel{\alpha}) = 0$. $\alpha[y:0]$ (y is free in the first component). You've proved that this formula is false when y : 0. What about the other values?

3. $I(\textit{NO}\, x(Fx, \textit{NO}\, y(Ffy, Rxy)), \alpha) = 1$ iff there is no x-variant α^x of α where
   - $I(Fx, \alpha^x) = 1$, and
   - $I(\textit{NO}\, y(Ffy, Rxy), \alpha^x) = 1$ iff there is no y-variant $\alpha^{x,y}$ of α^x where
     - $I(Ffy, \alpha^{x,y}) = 1$, and
     - $I(Rxy, \alpha^{x,y}) = 1$

   Suppose $I(\textit{NO}\, x(Fx, \textit{NO}\, y(Ffy, Rxy)), \alpha) = 0$, then we want to find an x-variant α^x of α such that $I(Fx, \alpha^x) = 1$ and $I(\textit{NO}\, y(Ffy, Rxy), \alpha^x) = 1$. There are two x-variant such that the former is true: $\alpha^x[x : 0]$ or $\alpha^x[x : 2]$. For the latter to be true, we must to show that there is no y-variant $a^{x,y}$ of either $\alpha^x[x:0]$ or $\alpha^x[x:2]$ such that $I(Ffy, \alpha^{x,y}) = 1$ and $I(Rxy, \alpha^{x,y}) = 1$. However, for each cases, there is such a y-variant: $\alpha^{x,y}[x:0, y:2]$ and $\alpha^{x,y}[x:2, y:0]$. So, $I(\textit{NO}\, y(Ffy, Rxy), \alpha^x)$ could not be true under $\alpha^x[x:0]$ and $\alpha^x[x:2]$, i.e., it could not be true when $I(Fx, \alpha^x)$ is true. Therefore, $I(\textit{NO}\, x(Fx, \textit{NO}\, y(Ffy, Rxy)), \alpha) = 1$.

4. $I(\textit{NO}\, x(\textit{NO}\, y(Fy, Ryx \wedge Fx), \textit{NO}\, y(Ffy, Ryx \vee Fx)), \alpha) = 1$ iff there is no x-variant α^x of α where
   1. $I(\textit{NO}\, y(Fy, Ryx \wedge Fx), \alpha^x) = 1$ iff there is no y-variant $\alpha^{x,y}$ of α^x where
      1.1. $I(Fy, \alpha^{x,y}) = 1$, and
      1.2. $I(Ryx \wedge Fx, \alpha^{x,y}) = 1$

   and
   2. $I(\textit{NO}\, y(Ffy, Ryx \vee Fx), \alpha^x) = 1$ iff there is no y-variant $\alpha^{x,y}$ of α^x where
      2.1. $I(Ffy, \alpha^{x,y}) = 1$, and
      2.2. $I(Ryx \vee Fx, \alpha^{x,y}) = 1$

   Suppose the opposite, then we must find an x-variant α^x of α such that both 1. and 2. are true.

   $\alpha^x[x:0]$: There is a y-variant $\alpha^{x,y}[x:0, y:2]$ of it such that both 1.1. and 1.2. are true, so 1. is false. Thus, $\alpha^x[x:0]$ is not the x-variant we want.

   $\alpha^x[x:1]$: For 1.2. to be true, its right conjunct $I(Fx, \alpha^{x,y})$ must be true. From the truth table of $I(F)$ we can observe that this requires x to be 0 or 2, but the assignment does not assign x to any of them, so 1.2. is always false under this assignment. It follows that there is no y-variant of $\alpha^x[x:1]$ such that both 1.1. and 1.2. are true because the latter is always false. So, 1. is true. However, there is a y-variant $\alpha^{x,y}[x:1, y:0]$ such that both 2.1 and 2.2 are true:

   $$I(F)(f)(0) = I(F)(0) = 1 \quad \text{and} \quad I(R)(0,1) = 1 \rightarrow I(R)(0,1) \vee I(F)(1) = 1$$

   So, 2. is false in this case. Thus, $\alpha^x[x:1]$ is not the x-variant we want.

   $\alpha^x[x:2]$: There is a y-variant $\alpha^{x,y}[x:2, y:0]$ of it such that both 1.1. and 1.2. are true, so 1. is false. Thus, $\alpha^x[x:2]$ is not the x-variant we want.

   $\alpha^x[x:3]$: The reasoning is similar to the $\alpha^x[x:1]$ case. There is a y-variant $\alpha^{x,y}[x:3, y:0]$ such that both 2.1 and 2.2 are true, so 2. is false. Thus, $\alpha^x[x:2]$ is not the x-variant we want.

   Since there is no x-variant α^x of α such that both 1. and 2. are true, we can conclude that $I(\textit{NO}\, x(\textit{NO}\, y(Fy, Ryx \wedge Fx), \textit{NO}\, y(Ffy, Ryx \vee Fx)), \alpha) = 1$.

Good

TASK 3

2½/4

Nicely Structured, but a key step missing in both parts.

Base case

The assumption rule provides the base case. The atomic proof is a proof for $A \succ A$. Let v be a model such that the conclusion $v(A) = 0$, then the premise is $v(A) = 0$. There is no way to construct a counterexample. Therefore, the assumption rule is truth preserving.

For the inductive hypothesis, assuming that we are given proofs that are truth preserving. We wish to show that whatever proofs we make using an inference step are also truth preserving. Inductive cases of propositional connectives are given in the lecture. Here are the remaining cases of $\textsc{no}$ quantifier.

Inductive case for $\textsc{no}\mathrm{I}^{i,j}$

We assume we have a proof Π for $X, A(n), B(n) \succ \perp$, and we form a new proof using the $\textsc{no}\mathrm{I}^{i,j}$ rule.

$$\dfrac{\begin{array}{c} X, [A(n)]^i, [B(n)]^j \\ \Pi \\ \perp \end{array}}{\textsc{no}\, x(A(x), B(x))}\ \textsc{no}\mathrm{I}^{i,j}$$

We want to show that $X \vDash \textsc{no}\, x(A(x), B(x))$. Suppose that there is a counterexample v, so $v(X) = 1$ and $v(\textsc{no}\, x(A(x), B(x)) = 0$. Since $v(\textsc{no}\, x(A(x), B(x)) = 0$, then if we substitute name n for variable x, $v(A(n))$ and $v(B(n))$ cannot both be true. However, both of them are true because by inductive hypothesis, $v(X, A(n), B(n)) = 1$, i.e., $v(X) = v(A(n)) = v(B(n)) = 1$, which is a contradiction. Therefore, there is no counterexample.

v is a model, with an assignment of values

What does this mean?

Why? Spell this out in words.

No x takes its value by assigning values (in D) to the variable x, not by substituting a name for x.

Inductive case for $\textsc{no}\mathrm{E}$

Assume we have proofs Π_1 for $X \succ \textsc{no}\, x(A(x), B(x))$, Π_2 for $Y \succ A(t)$, and Π_3 for $Z \succ B(t)$. We then form a new proof using the $\textsc{no}\mathrm{E}$ rule.

$$\dfrac{\begin{array}{c} X \\ \Pi_1 \\ \textsc{no}\, x(A(x), B(x)) \end{array} \qquad \begin{array}{c} Y \\ \Pi_2 \\ A(t) \end{array} \qquad \begin{array}{c} Z \\ \Pi_3 \\ B(t) \end{array}}{\perp}\ \textsc{no}\mathrm{E}$$

We want to show $X, Y, Z \vDash \perp$. Suppose there is a counterexample v, so $v(X, Y, Z) = 1$ and $v(\perp) = 0$. Since $v(X, Y, Z) = 1$, then $v(X) = 1$. It follows from the inductive hypothesis that $v(\textsc{no}\, x(A(x), B(x))) = 1$. Similarly, $v(A(t)) = v(B(t)) = 1$. However, $v(\textsc{no}\, x(A(x), B(x)))$ is true only if $v(A(t))$ and $v(B(t))$ are not both true, which yields a contradiction. Therefore, there is no counterexample.

✗ No, that's too quick. Explain why in terms of the Semantics of the No quantifier (we are trying to prove the match between models & proofs — you can't just assume it.)

TASK 4

interesting choice. A set is only witnessed if it's maximal?

Definition 1 (Witnessed set). A set X of formulae in our new language is said to be WITNESSED iff whenever $\text{NO}\,x(A(x), B(x)) \notin X$, there is some name n such that $A(n) \in X$ and $B(n) \in X$.

Could use $\neg \text{NO}\,x(A(x), B(x)) \in X$ instead.

Lemma 1 (Witnessed maximal consistent set expansion). If $X \nvDash A$ and there is an unending supply of names $c_1, c_2, c_3, \ldots$, not present in X or in A, then there is some witnessed maximal consistent set $X^* \supseteq X$ such that $X^* \nvDash A$.

Confusing: 3 different 'A' formulas here.

Proof. We can adopt the construction process used in proving the completeness of propositional logic to construct X^* with an exception case: if $A_n = \neg\, \text{NO}\,x(A(x), B(x))$ for some formula $A(x)$ and some formula $B(x)$, and if $X_{n-1}, \neg\, \text{NO}\,x(A(x), B(x)) \nvDash \bot$, then choose a name c_m (from $c_1, c_2, c_3, \ldots$) not occurring in X_{n-1} or A_n, and set $X_n = X_{n-1} \cup \{\neg\, \text{NO}\,x(A(x), B(x)), A(c_m), B(c_m)\}$.

Why is this A-avoiding? This must be checked.

We can reuse the proofs that such a constructed X^* is consistent and maximal consistent. We only need to provide proof that such a constructed X is witnessed.

If $\text{NO}\,x(A(x), B(x)) \notin X^*$, then by the negation-completeness of X^*, $\neg\, \text{NO}\,x(A(x), B(x)) \in X^*$. Since $\neg\, \text{NO}\,x(A(x), B(x))$ is added to the set along with a witness $\{A(c_m), B(c_m)\}$ as defined in the construction process, it follows that there is some name c such that $A(c) \in X^*$ and $B(c) \in X^*$. ∎

Lemma 2 (Model construction). If X^* is a witnessed maximal consistent set, then there is a model $\mathfrak{M}_{X^*}$, whose domain is the set of all name occurring in X^*, such that for each formula A, $\mathfrak{M}_{X^*} \vDash A$ iff $A \in X^*$.

Proof. Assume that a formula A is true in $\mathfrak{M}_{X^*}$ iff $A \in X^*$. This is the induction hypothesis. The base case and the inductive cases for propositional connectives are given in the propositional completeness proof in the lecture materials, so we will not repeat them. The inductive case for the 'NO' quantifier requires a proof.

That's not the truth condition: that was stated in terms of assignments of values to variables

$A \equiv \text{NO}\,x(A(x), (Bx))$: We prove both direction simultaneously. $\mathfrak{M}_{X^*} \vDash A$ iff $\mathfrak{M}_{X^*} \vDash A(n)$ and $\mathfrak{M}_{X^*} \vDash B(n)$ for no name n (by the truth condition defined in TASK 2). By induction hypothesis, this is the case iff $A(n) \in X^*$ and $B(n) \in X^*$ for no name n. Since X^* is maximal consistent, this holds iff $\text{NO}\,x(A(x), (Bx)) \in X^*$. Therefore, $\mathfrak{M}_{X^*} \vDash \text{NO}\,x(A(x), (Bx))$ iff $\text{NO}\,x(A(x), (Bx)) \in X^*$. ∎

Finally, we complete the proof of the completeness theorem for our new language using these two lemmas. The reasoning is the same as the last part of the proof of the completeness theorem for first-order predicate logic in the lecture notes.

PM2010 PROJECT 1 SOLUTIONS ~ 2024

Task 1 (2 points per proof)

1.

$$\dfrac{\dfrac{No\,x(Fx, Fx\wedge Gx) \quad [Fa]^1 \quad \dfrac{[Fa]^1 \quad [Ga]^2}{Fa\wedge Ga}\,{\scriptstyle \wedge I}}{\bot}\,{\scriptstyle NoE}}{No\,x(Fx, Gx)}\,{\scriptstyle NoI^{1,2}}$$

2.

$$\dfrac{\dfrac{No\,x(\neg Gx, Hx) \quad \dfrac{\dfrac{No\,x(Fx,Gx) \quad [Fa]^2 \quad [Ga]^1}{\bot}\,{\scriptstyle NoE}}{\neg Ga}\,{\scriptstyle \neg I^1} \quad [Ha]^3}{\bot}\,{\scriptstyle NoE}}{No\,x(Fx, Hx)}\,{\scriptstyle NoI^{2,3}}$$

3.

$$\dfrac{No\,x(Fx,Gx)\vee No\,x(Fx,Hx) \quad \dfrac{[No\,x(Fx,Gx)]^1 \quad [Fa]^3 \quad \dfrac{[Ga\wedge Ha]^4}{Ga}\,{\scriptstyle \wedge E}}{\bot}\,{\scriptstyle NoE} \quad \dfrac{[No\,x(Fx,Hx)]^2 \quad [Fa]^3 \quad \dfrac{[Ga\wedge Ha]^4}{Ha}\,{\scriptstyle \wedge E}}{\bot}\,{\scriptstyle NoE}}{No\,x(Fx, Gx\wedge Hx)}\,{\scriptstyle \vee E^{1,2}}$$

Task 2 (1 point per model evaluation — including explanations.)

1. $No\,x(Fx, Rxx)$ is true in the Model iff

$I(Fx, \alpha[x:0]) = 0$ or $I(Rxx, \alpha[x:0]) = 0$ ✓
false true

and $I(Fx, \alpha[x:1]) = 0$ or $I(Rxx, \alpha[x:1]) = 0$ ✓
true!

and $I(Fx, \alpha[x:2]) = 0$ or $I(Rxx, \alpha[x:2]) = 0$ ✓
false true

and $I(Fx, \alpha[x:3]) = 0$ or $I(Rxx, \alpha[x:3]) = 0$
true!

So, the formula is true

2. $No\,x(Rxy, No\,y(Rxy, \neg Ryx))$

Notice that this formula has the first y is free.
This means we might need to know what value y is assigned to evaluate the formula:

$\mathcal{I}(\text{No}\,x\,(Rxy, \text{No}\,y(Rxy, \neg Ryx)), \alpha) = 1$ iff

$\mathcal{I}(Rxy, \alpha[x:0]) = 0$ or $\mathcal{I}(\text{No}\,y(Rxy, \neg Ryx), \alpha[x:0]) = 0$

& $\mathcal{I}(Rxy, \alpha[x:1]) = 0$ or $\mathcal{I}(\text{No}\,y(Rxy, \neg Ryx), \alpha[x:1]) = 0$

& $\mathcal{I}(Rxy, \alpha[x:2]) = 0$ or $\mathcal{I}(\text{No}\,y(Rxy, \neg Ryx), \alpha[x:2]) = 0$

& $\mathcal{I}(Rxy, \alpha[x:3]) = 0$ or $\mathcal{I}(\text{No}\,y(Rxy, \neg Ryx), \alpha[x:3]) = 0$

Notice that $\mathcal{I}(R)$ is symmetric. for <u>any</u> assignment β, $\mathcal{I}(Rxy, \beta) = \mathcal{I}(Ryx, \beta)$, so, for <u>any</u> β, either $\mathcal{I}(Rxy, \beta) = 0$ or $\mathcal{I}(\neg Ryx, \beta) = 0$, so for <u>every</u> α', $\mathcal{I}(\text{No}\,y(Rxy, \neg Ryx), \alpha') = 1$.

So, our original formula is true (under α) iff

$\mathcal{I}(Rxy, \alpha[x:0]) = 0$ ~ true when $\alpha[y] = 0$, only

& $\mathcal{I}(Rxy, \alpha[x:1]) = 0$ ~ true when $\alpha[y] = 3$, only

& $\mathcal{I}(Rxy, \alpha[x:2]) = 0$ ~ true when $\alpha[y] = 2$, only

& $\mathcal{I}(Rxy, \alpha[x:3]) = 0$ — true when $\alpha[y] = 1$, only

— So, for no assignment of a value to y is this true.

It's <u>false</u>, no matter what y is.

3. $\text{No}\, x(Fx, \text{No}\, y(Ffy, Rxy))$ is <u>true</u> iff

$I(Fx, \alpha[x:0]) = 0$ or $I(\text{No}\, y(Ffy, Rxy), \alpha[x:0]) = 0$

NO

& $I(Fx, \alpha[x:1]) = 0$ or $I(\text{No}\, y(Ffy, Rxy), \alpha[x:1]) = 0$

YES this row is checked. ✓

& $I(Fx, \alpha[x:2]) = 0$ or $I(\text{No}\, y(Ffy, Rxy), \alpha[x:2]) = 0$

NO

& $I(Fx, \alpha[x:3]) = 0$ or $I(\text{No}\, y(Ffy, Rxy), \alpha[x:3]) = 0$

YES this row is checked. ✓

So we need to verify that $I(\text{No}\, y(Ffy, Rxy), \alpha[x:0]) = 0$
& $I(\text{No}\, y(Ffy, Rxy), \alpha[x:2]) = 0$

$I(\text{No}\, y(Ffy, Rxy), \alpha[x:0]) = 0$ iff $I(Ffy, \alpha[x:0, y:i]) = 1$
& $I(Rxy, \alpha[x:0, y:i]) = 1$
for some $i \in \{0, 1, 2, 3\}$.

$I(Ffy, \alpha[x:0, y:i]) = 1$ iff $i = 0$ or 2.
$I(Rxy, \alpha[x:0, y:i]) = 1$ iff $i = 1, 2$ or 3, So they both hold when $i = 2$.

$I(\text{No}\, y(Ffy, Rxy), \alpha[x:2]) = 0$ iff $I(Ffy, \alpha[x:2, y:i]) = 1$
& $I(Rxy, \alpha[x:2, y:i]) = 1$
for some $i \in \{0, 1, 2, 3\}$.

$I(Ffy, \alpha[x:2, y:i]) = 1$ iff $i = 0$ or 2.
$I(Rxy, \alpha[x:2, y:i]) = 1$ iff $i = 0, 1$ or 3, So they both hold when $i = 0$.

So, all four options are verified. The original formula is <u>true</u>.

4. $No\ x\ (No\ y\ (Fy, Ryx \wedge Fx), No\ y\ (FFy, Ryx \vee Fx))$ is true iff

$I(No\ y\ (Fy, Ryx \wedge Fx), \alpha[x:0]) = 0$ or $I(No\ y\ (FFy, Ryx \vee Fx), \alpha[x:0]) = 0$

— true since $I(Fy, \alpha[x:0, y:2]) = 1$
$I(Ryx \wedge Fx, \alpha[x:0, y:2]) = 1$ ✓

& $I(No\ y\ (Fy, Ryx \wedge Fx), \alpha[x:1]) = 0$ or $I(No\ y\ (FFy, Ryx \vee Fx), \alpha[x:1]) = 0$

— true since $I(Fy, \alpha[x:1, y:0]) = 1$
$I(Ryx \wedge Fx, \alpha[x:1, y:0]) = 1$ ✓

& $I(No\ y\ (Fy, Ryx \wedge Fx), \alpha[x:2]) = 0$ or $I(No\ y\ (FFy, Ryx \vee Fx), \alpha[x:2]) = 0$

— true since $I(Fy, \alpha[x:2, y:0]) = 1$
$I(Ryx \wedge Fx, \alpha[x:2, y:0]) = 1$ ✓

& $I(No\ y\ (Fy, Ryx \wedge Fx), \alpha[x:3]) = 0$ or $I(No\ y\ (FFy, Ryx \vee Fx), \alpha[x:3]) = 0$

— true since $I(Fy, \alpha[x:3, y:0]) = 1$
$I(Ryx \wedge Fx, \alpha[x:3, y:0]) = 1$ ✓

So, our original formula is true.

Task 3 4 points: 2 for explaining the soundness of NoE
2 for explaining the soundness of NoI.

Consider the rule NoE:

$$\frac{\text{No}\,x\,(A(x), B(x)) \qquad A(t) \qquad B(t)}{\bot}\ \text{NoE}$$

If this rule is applied in a proof as follows:

$$\frac{\begin{matrix} X \\ \Pi_1 \\ \text{No}\,x\,(A(x), B(x)) \end{matrix} \qquad \begin{matrix} Y \\ \Pi_2 \\ A(t) \end{matrix} \qquad \begin{matrix} Z \\ \Pi_3 \\ B(t) \end{matrix}}{\bot}\ \text{NoE}$$

where we may assume (the inductive step of the soundness proof) that $X \models \text{No}\,x\,(A(x), B(x))$, that $Y \models A(t)$, and that $Z \models B(t)$, and we want to verify (using the evaluation rule for No x) that $X, Y, Z \models \bot$.

So, suppose (for the sake of a reductio) that we have some model $\mathcal{M} = \langle D, I \rangle$, and assignment α of variables, such that in $\mathcal{M}, \alpha$, all of the members of X, Y, Z are true. Since we have $Y \models A(t)$ & $Z \models B(t)$, we know that $A(t)$ and $B(t)$ both hold in $\mathcal{M}, \alpha$. By the semantic assignment lemma, $A(x)$ and $B(x)$ both hold in $\mathcal{M}, \alpha[x:d]$, where $I(t, \alpha) = d$.

However, $X \models \text{No}\,x\,(A(x), B(x))$, which contradicts $I(A(x), \alpha[x:d]) = 1$ and $I(B(x), \alpha[x:d]) = 1$, so we have refuted the supposition that X, Y, Z jointly hold in some model, and hence $X, Y, Z \models \bot$, completing the verification of the soundness of NoE.

Consider now the rule NoI.
(in which n satisfies the usual eigenvariable conditions)

$$\dfrac{\begin{matrix} X, [A(n)]^i, [B(n)]^j \\ \vdots \\ \bot \end{matrix}}{No\,x\,(A(x), B(x))}\,NoI$$

We wish to verify that this rule is sound: ie, that if $X, A(n), B(n) \vDash \bot$, then $X \vDash No\,x(A(x), B(x))$.

So, suppose that $X, A(n), B(n) \vDash \bot$, and that $\mathfrak{M}, \alpha$ is a model & assignment in which each member of X holds. We wish to show that $No\,x\,(A(x), B(x))$ holds in $\mathfrak{M}, \alpha$. Consider any x-variant $\alpha[x:d]$ of α (for any $d \in D$ you like). Here is why $I(A(x), \alpha[x:d]) = 0$ or $I(B(x), \alpha[x:d]) = 0$. — If not, then $I(A(x), \alpha[x:d]) = 1$ & $I(B(x), \alpha[x:d]) = 0$, and so, by the semantic assignment lemma, $I'(A(n), \alpha) = 1$ & $I'(B(n), \alpha) = 0$ where we choose I' by setting $I' = I$ except for $I'(n) = d$. (This shifts no components of X, $A(x)$ or $B(x)$, by the eigenvariable condition.) This means in this model $\mathfrak{M}' = \langle D, I' \rangle$, and α, $X, A(n), B(n)$ are all true, which is impossible! — So, as we wished, we must have $I(A(x), \alpha[x:d]) = 0$ or $I(B(x), \alpha[x:d]) = 0$, for each $d \in D$, and hence, $I(No\,x\,(A(x), B(x)), \alpha) = 1$ in $\mathfrak{M}, \alpha$, as we wished to show. NoI is sound.

The rest of the soundness proof proceeds exactly as before: it is an induction on the construction of the proof Π from X to A, where we prove that $X \vDash A$.

The base case is the assumption: A, which is the trivial verification, $A \vDash A$, and the other rules $(\wedge, \vee, \to, \neg, \bot, =)$ are verified in the same way as we did in class.

Task 4

This task breaks down into three components.

1: Modifying the definition of being a witnessed set.
2: Showing that if $X \nvdash A$ then there is a maximal A-avoiding witnessed set, using that new definition
3: Showing that No-formulas are correctly interpreted in the model $\mathcal{M}_{X^*}$ defined from a maximal A-avoiding witnessed set so-constructed.

2 points for each component.

(1) Here is a definition: X is witnessed iff whenever $\neg No\, x(B(x), C(x))$, then for some term t, $B(t) \in X$ and $C(t) \in X$.

We can verify that if X' is maximal, A-avoiding, and witnessed, then $No\, x(B(x), C(x)) \in X'$ iff for every term t, either $B(t) \notin X'$ or $C(t) \notin X'$.

For the L-to-R direction, if $No_x(B(x), C(x)), B(t), C(t) \in X'$, then a swift NoE step shows that $\bot \in X'$ (since maximal sets are closed under provable consequence) & hence, that $A \in X'$, contradicting the assumption that X' is A-avoiding.

For the R-to-L direction, if either $B(t) \notin X'$ or $C(t) \notin X'$ for each term t, then since X' is witnessed, we must have $\neg No\, x(B(x), C(x)) \notin X'$, which, by maximality gives $No\, x(B(x), C(x)) \in X'$ as desired.

So, maximal A-avoiding witnessed sets get 'No' formulas 'right'.

(2) We wish to show that if $X \nvdash A$ then there is a maximal A-avoiding witnessed set $X^* \supseteq X$. We use the same construction we used in class, modifying it only in the step where we add witnesses.

So, we extend our language with fresh names $c_0, c_1, c_2 \ldots$, enumerating the formulas $B_0, B_1, \ldots$ etc and define $X_0 = X$ and X_{n+1} from X_n,

Setting
$$X_{n+1} = \begin{cases} X_n \cup \{B_n\} & \text{if } X_n, B_n \nvdash A \\ X_n & \text{otherwise,} \end{cases}$$

except we supplement the first clause if $B_n = \neg \mathrm{No}\,x(C(x), D(x))$

— then we define $X_{n+1} = X_n \cup \{B_n, C(c_m), D(c_m)\}$

for the next fresh name c_m. (ie c_m is not in X_n, B_n or A).

We must verify $X_n, \neg \mathrm{No}\,x(C(x), D(x)), C(c_m), D(c_m) \nvdash A$,

under the condition that $X_n, \neg \mathrm{No}\,x(C(x), D(x)) \nvdash A$.

Here is why: Suppose $X_n, \neg \mathrm{No}\,x(C(x), D(x)), C(c_m), D(c_m) \vdash A$.

Take that proof (Π) and extend it like this:

$$
\dfrac{\neg \mathrm{No}\,x(C(x),D(x)) \qquad \dfrac{\dfrac{[\neg A]^k \qquad \begin{array}{c} X_n, \neg \mathrm{No}\,x(C(x),D(x)), [C(c_m)]^i, [D(c_m)]^j \\ \Pi \\ A \end{array}}{\bot}\,\neg E}{\mathrm{No}\,x(C(x),D(x))}\,\mathrm{No}I^{j,k}}{\dfrac{\dfrac{\bot}{\neg\neg A}\,\neg I^k}{A}\,DNE}\,\neg E
$$

$\longleftarrow$ Eigenvariable condition satisfied since c_m is fresh in $X, A, \neg \mathrm{No}\,x(C(x),D(x))$

This is a proof from $X_n, \neg \mathrm{No}\,x(C(x), D(x))$ to A, which is not possible. It follows, then, that $X_n, \neg \mathrm{No}\,x(C(x), D(x)), C(c_m), D(c_m) \nvdash A$, as desired.

This completes the verification that if $X \nvdash A$, then there is a maximal, witnessed A-avoiding $X^* \supseteq X$, by the usual construction, taking $X^* = \bigcup_{n=0}^{\infty} X_n$.

③ Finally, we must show that the model $\mathfrak{M}_{X^*}$, defined from a maximal, witnessed A-avoiding set X^*, (with the domain consisting of the term classes from X^*, setting $I(n) = \bar{n}$, for names n, $id(x) = \bar{x}$, for variables x, $I(f)(\bar{t}_1, \dots \bar{t}_n) = \overline{ft_1 \dots t_n}$ for n-ary function symbols f, $I(F)(\bar{t}_1, \dots, \bar{t}_n) = 1$ iff $Ft_1 \dots t_n \in X^*$ for n-ary predicates F) satisfies the condition: $I(B, id) = 1$ iff $B \in X^*$.

This proof is by induction on the construction of B, and all cases are unchanged except for when B is of the form $No\, x(C(x), D(x))$. Here, the proof proceeds as follows:

$I(No\, x(C(x), D(x)), id) = 1$ iff

$I(C(x), id[x:\bar{t}]) = 0$ or $I(D(x), id[x:\bar{t}]) = 0$ for every term class $\bar{t}$, iff

$I(C(x), id[x:\bar{t}]) = 0$ or $I(D(x), id[x:\bar{t}]) = 0$ for every term t, iff

$I(C(t), id) = 0$ or $I(D(t), id) = 0$ for every term t, (since $I(t, id) = \bar{t}$), iff

$C(t) \notin X^*$ or $D(t) \notin X^*$ for every term t (by the induction hypothesis), iff

$No\, x(C(x), D(x)) \in X^*$, since X^* is witnessed (by the fact proved in part ①.)

This is the only part of the induction proof changed from the original version, so we have verified the remaining component of our completeness proof.