PY2010 PRACTICE FINAL EXAM

INSTRUCTIONS: Complete FOUR of the following six questions, making sure to answer AT LEAST ONE question from EACH of the three sections.

Each question is worth five points.

SECTION 1: PROPOSITIONAL LOGIC

1. Prove each of the following, using natural deduction

   a) $\neg p \lor \neg q \vdash_I \neg(p \land q)$

   b) $p \lor q \vdash_I (p \to q) \to q$

   c) $\neg(p \land \neg q) \vdash_C \neg p \lor q$

2. a) Find each of the detour formulas in this natural deduction proof. Then use the reduction rules to rewrite the proof using no detours.

$$\dfrac{\dfrac{\dfrac{[p \land (p \lor q)]^1}{p}\,{\scriptstyle \land E}}{(p \land (p \lor q)) \to p}\,{\scriptstyle \to I^1} \qquad \dfrac{\dfrac{\dfrac{[p]^2 \qquad \dfrac{[p]^2}{p \lor q}\,{\scriptstyle \lor I}}{p \land (p \lor q)}\,{\scriptstyle \land I}}{p \to (p \land (p \lor q))}\,{\scriptstyle \to I^2} \qquad p}{p \land (p \lor q)}\,{\scriptstyle \to E}}{p}\,{\scriptstyle \to E}$$

   b) Suppose X is a maximal A-avoiding set, in classical propositional logic. Explain why whenever B and C are members of X, then $B \land C$ is a member of X too.

SECTION 2: MODAL LOGIC

3. a) Consider $\Box(p \to q) \succ \Diamond\neg p \to \Diamond q$. Either find an S4 natural deduction proof for this argument, or find an S4 counterexample to it.

   b) Consider $\Box\Diamond p \succ \Diamond\Box p$. Either find an S4 natural deduction proof for this argument, or find an S4 counterexample to it.

   c) Consider $K_a p, K_a K_b q \succ K_a(p \land q)$ in the epistemic logic S5E with two agents, a and b. Either find an S5E-counterexample to this argument, or show that there is no counterexample.

4. Consider actuality models for S5A (models with a distinguished 'actual' world g). Define a new one-place connective $\mathbb{C}$, meaning *in some counterfactual world*, with the following clause:

   – $V(\mathbb{C}A, w) = 1$ if and only if for some $u \in W$ where $u \neq g$, $V(A, u) = 1$.

Verify the following facts about $\mathbb{N}$, in the logic S5AC of actuality and counterfactuality, using this new connective.

a) $\mathbb{C}A \models_{\mathsf{S5AC}} \Diamond A$,

b) $\mathbb{A}A, \neg\mathbb{C}\neg A \models_{\mathsf{S5AC}} \Box A$,

c) $\models_{\mathsf{S5AC}} (\mathbb{C}A \to \mathbb{C}\mathbb{C}A) \wedge (\mathbb{C}\mathbb{C}A \to \mathbb{C}A)$.

SECTION 3: FIRST-ORDER PREDICATE LOGIC

5. For each of the following arguments, either provide a natural deduction proof in classical first-order predicate logic, or a model that serves as a counterexample.

   a) $\exists y \forall x(Rxy \wedge Ryx) \succ \forall x \exists y(Rxy \wedge Ryx)$

   b) $\forall x((Fx \wedge \exists y Oxy) \to \exists y Bxy) \succ \forall x \forall y((Fx \wedge Oxy) \to Bxy)$

   c) $\forall x(Fx \to \exists y(Rxy \vee Sxy)) \succ \forall x(Fx \to (\exists y Rxy \vee \exists y Sxy))$

6. Explain the strategy for proving the *soundness* theorem for first-order predicate logic. As a part of your answer, make sure to explain the strategy as it applies to the natural deduction rules $\forall$I and $\forall$E.

PY2010 PRACTICE EXAM SOLUTIONS

Q1

$$\dfrac{\dfrac{\neg p \lor \neg q \qquad \dfrac{[\neg p]^1 \quad \dfrac{[p\land q]^3}{p}\,\land E}{\bot}\,\neg E \qquad \dfrac{[\neg q]^2 \quad \dfrac{[p\land q]^3}{q}\,\land E}{\bot}\,\neg E}{\bot}\,\lor E^{1,2}}{\neg(p\land q)}\,\neg I^3$$

$$\dfrac{\dfrac{p\lor q \qquad \dfrac{[p\to q]^3 \quad [p]^1}{q}\,\to E \qquad [q]^2}{q}\,\lor E^{1,2}}{(p\to q)\to q}\,\to I^3$$

$$\dfrac{\dfrac{\neg(p\land\neg q) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg(\neg p\lor q)]^3 \quad \dfrac{[\neg p]^1}{\neg p\lor q}\,\lor I}{\bot}\,\neg E}{\neg\neg p}\,\neg I^1}{p}\,\mathrm{DNE} \qquad \dfrac{\dfrac{[\neg(\neg p\lor q)]^3 \quad \dfrac{[q]^2}{\neg p\lor q}\,\lor I}{\bot}\,\neg E}{\neg q}\,\neg I^2}{p\land\neg q}\,\land I}{\dfrac{\bot}{\neg\neg(\neg p\lor q)}\,\neg I^3}\,\neg E}{\neg p\lor q}\,\mathrm{DNE}$$

2.a) I have highlighted the detour formulas in the proof.

$$
\dfrac{
\dfrac{\dfrac{[p\wedge(p\vee q)]^1}{p}\,\wedge E}{(p\wedge(p\vee q))\to p}\,\to I^1
\qquad
\dfrac{\dfrac{\dfrac{[p]^2 \qquad \dfrac{[p]^2}{p\vee q}\,\vee I}{p\wedge(p\vee q)}\,\wedge I}{p\to(p\wedge(p\vee q))}\,\to I \qquad p}{p\wedge(p\vee q)}\,\to E
}{p}\,\to E
$$

Both these instances of $p\to(p\wedge(p\vee q))$ are introduced & immediately eliminated. Using the $\to I/\to E$ transformation, first on the left instance, we get:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \qquad \dfrac{[p]^2}{p\vee q}\,\vee I}{p\wedge(p\vee q)}\,\wedge I}{p\to(p\wedge(p\vee q))}\,\to I \qquad p}{p\wedge(p\vee q)}\,\to E}{p}\,\wedge E
$$

and then, eliminating the second detour we get

$$
\dfrac{\dfrac{p \qquad \dfrac{p}{p\vee q}\,\vee I}{p\wedge(p\vee q)}\,\wedge I}{p}\,\wedge E
$$

in which the $p\wedge(p\vee q)$ is a detour formula, & when we reduce this detour, we get the tiny proof p, which has no detours!

2b) If X is a maximal A-avoiding set (in classical logic), this means that $X \nvdash_c A$, and whenever $B \notin X$, then $X, B \vdash_c A$, for any formula B.

Now suppose for some given formulas B & C, $B, C \in X$. Here is why $B \wedge C \in X$.

If $B \wedge C \notin X$, then $X, B \wedge C \vdash_c A$. But this means we have a proof from X to A, since we can prove $B \wedge C$ from B, C which are themselves members of X.

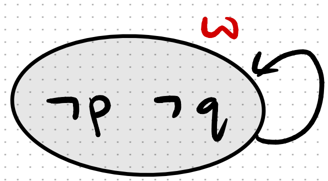
$$\begin{array}{cc} X & \dfrac{B \quad C}{B \wedge C}\,\wedge I \\ & \Pi \\ & A \end{array}$$

But X was A-avoiding, so this can't happen, and hence, $B \wedge C \in X$, as desired.

Q3.

a) The argument from $\Box(p \to q)$ to $\Diamond\neg p \to \Diamond q$ is not valid in S4 (or S5).

Consider the model with one world w where wRw (this relation is reflexive & transitive, so this is an S4 model), and in w, p and q are both false.



So, in this model, $p \to q$ is true at w, and so, $\Box(p \to q)$ holds at w too. On the other hand $\Diamond\neg p$ is true at w while $\Diamond q$ is false at w, so the conclusion $\Diamond\neg p \to \Diamond q$ is false at w, and we have a counterexample.

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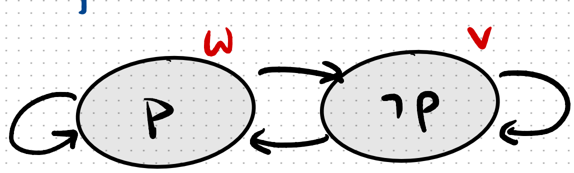
b) The argument from $\Box\Diamond p$ to $\Diamond\Box p$ is invalid in S4 (& also S5).

Consider the following two world S5 models with two worlds, where



p is true in one (w) but not the other (v).

In this model, $\Diamond p$ is true in each world, so $\Box\Diamond p$ holds at w.

In this model $\Box p$ is true in no world, so $\Diamond\Box p$ fails at w and we have a counterexample.

c) The argument from $K_a p$ and $K_a K_b q$ to $K_a(p \wedge q)$ is valid in the epistemic logic S5E.

Suppose we have a model in which there is a world w where $K_a p$ and $K_a K_b q$ are both true. We wish to show that $K_a(p \wedge q)$ is true at w too. To show this, suppose that $w R_a v$. We need to show, then, that $p \wedge q$ is true at v, and then we are done.

p is true at v, since $K_a p$ is true at w and $w R_a v$. Now, since $K_a K_b q$ is also true at w, we have $K_b q$ true at v, and since $v R_b v$ (each relation is reflexive!), we have q true at v too. So, $p \wedge q$ is true at v, giving $K_a(p \wedge q)$ true at w as desired. The argument has no counterexample: it is valid.

Q4 a) Here is why $\mathbb{C}A \vDash_{S5AC} \Diamond A$.

Take any model & any world w in that model, where $\mathbb{C}A$ is true at w. We want to show that $\Diamond A$ is true at w too. Since $\mathbb{C}A$ holds at w, by the definition of $\mathbb{C}$, we have <u>some</u> non-actual ($\neq g$) world v at which A is true. It follows that $\Diamond A$ is true at w, as desired.

b) Here is why $\mathbb{A}A, \neg\mathbb{C}\neg A \vDash_{S5AC} \Box A$

Take any model & any world w in that model, where $\mathbb{A}A$ and $\neg\mathbb{C}\neg A$ are true at w. We want to show that $\Box A$ is true at w too. For $\Box A$ to be true at w, we need to show that A holds at every world in the model.

To show that A holds at g, we use the $\mathbb{A}A$ that holds at w. To show that A holds at any world <u>other</u> than g, we argue like this: Take $v \neq g$. If A doesn't hold at v, then $\neg A$ holds at v, which means $\mathbb{C}\neg A$ holds at w, which contradicts the assumption that $\neg\mathbb{C}\neg A$ also holds at w. So, indeed A holds at v as desired.

c) To explain why $(\mathbb{C}A \to \mathbb{C}\mathbb{C}A) \wedge (\mathbb{C}\mathbb{C}A \to \mathbb{C}A)$ holds in any world in any model, lets check the conditions under which $\mathbb{C}\mathbb{C}A$ is true at a world.

$\mathbb{C}\mathbb{C}A$ is true at w iff $\mathbb{C}A$ is true at some world $v \neq g$.
and $\mathbb{C}A$ is true at v iff A is true at some world $u \neq g$.

So, $\mathbb{C}\mathbb{C}A$ is true at w iff A is true at some world $u \neq g$, and this holds iff $\mathbb{C}A$ is true at w.

So, $\mathbb{C}A$ is true at a world if & only if $\mathbb{C}\mathbb{C}A$ is true at that world. This means we have $\models_{S5AC} (\mathbb{C}A \to \mathbb{C}\mathbb{C}A) \wedge (\mathbb{C}\mathbb{C}A \to \mathbb{C}A)$, as desired.

Q5 a)

$$\dfrac{\exists y\forall x(Rxy\wedge Ryx) \qquad \dfrac{\dfrac{[\forall x(Rxa\wedge Rax)]^1}{Rba\wedge Rab}\,\forall E}{\exists y(Rby\wedge Ryb)}\,\exists I}{\dfrac{\exists y(Rby\wedge Ryb)}{\forall x\exists y(Rxy\wedge Ryx)}\,\forall I}\,\exists E^1$$

b) The argument from

$\forall x((Fx\wedge\exists y\,Oxy)\rightarrow\exists y\,Bxy)$

to $\forall x\forall y((Fx\wedge Oxy)\rightarrow Bxy)$

is invalid. This model serves as a counterexample.

$D=\{a,b\}$

	$I(F)$
a	1
b	0

$I(O)$	a	b
a	1	1
b	0	0

$I(B)$	a	b
a	1	0
b	0	0

In this model $\forall x((Fx\wedge\exists y\,Oxy)\rightarrow\exists y\,Bxy)$ is true, since $Fx\wedge\exists y\,Oxy$ is true when x is assigned the value a, & then, $\exists y\,Bxy$ is true, since Bxy holds when y is assigned the value a too.

However, in this model, $\forall x\forall y((Fx\wedge Oxy)\rightarrow Bxy)$ is false, as $(Fx\wedge Oxy)\rightarrow Bxy$ is false when x is assigned a and y is assigned b, since then Fx and Oxy are true & Bxy is false.

So, this model serves as as a counterexample to the argument.

c)

$$\dfrac{\dfrac{\dfrac{\forall x(Fx\rightarrow\exists y(Rxy\vee Sxy))}{Fa\rightarrow\exists y(Ray\vee Say)}\,\forall E \quad [Fa]^4}{\exists y(Ray\vee Say)}\,\rightarrow E \qquad \dfrac{[Rab\vee Sab]^3 \quad \dfrac{\dfrac{[Rab]^1}{\exists y\,Ray}\,\exists I}{\exists y\,Ray\vee\exists y\,Say}\,\vee I \quad \dfrac{\dfrac{[Sab]^2}{\exists y\,Say}\,\exists I}{\exists y\,Ray\vee\exists y\,Say}\,\vee I}{\exists y\,Ray\vee\exists y\,Say}\,\vee E^{1,2}}{\exists y\,Ray\vee\exists y\,Say}\,\exists E^3$$

$$\dfrac{\dfrac{\exists y\,Ray\vee\exists y\,Say}{Fa\rightarrow(\exists y\,Ray\vee\exists y\,Say)}\,\rightarrow I^4}{\forall x(Fx\rightarrow(\exists y\,Rxy\vee\exists y\,Sxy))}\,\forall I$$

Q6

Here is the high level strategy of the proof of the soundness theorem. We show that if $X \vdash A$ (if there is a proof from X to A) then $X \vDash A$ (there is no counterexample to the argument from X to A, that is, no model, and assignment of values to the variables, in which each member of X is true and A is not true). We show that this is true, by induction on the construction of the proof from X to A. That is, we show that there is no counterexample for an *atomic* proof (consisting solely of the assumption, which is also the conclusion), and then, for each of the different rules in the proof system, we show that assuming that there is no counterexample to the proofs out of which the new proof is built, then there is so counterexample introduced at the last step.

The base case, for atomic proofs, is straightforward. A counterexample to the proof from A to A would be a model and assignment of values according to which A is true and at the very same time, A is not true, which is impossible. So the base case is done.

For the induction steps, we consider the two quantifier rules $\forall$E and $\forall$I. For $\forall$E, this means we have a proof from premises X to a conclusion $\forall xA(x)$, which has no counterexamples, and we end it to prove A(t), for some given term t. We wish to show that there is also no counterexample to the argument from X to A(t). So, take a model M and assignment of values v to the variables in which each member of X is true. We wish to show that A(t) is true in (M,v). Since this model/assignment pair is not a counterexample to the argument from X to $\forall xA(x)$ (which has *no* counterexamples), we know that in this model/assignment pair (M,v) that $\forall xA(x)$ is true. This means that (M,v') makes A true, for any x-variant v' of v. So, choose the x-variant v' that assigns the variable x the value that the term t takes in (M,v). A is true in (M,v'), which means (by the Semantic Substitution Lemma) that A(t) is true in (M,v), as desired.

For $\forall$I, we reason similarly. The argument from X to A(n) has no counterexamples, where n is a name not present in X. We wish to show that the argument from X to $\forall xA(x)$ also has no counterexamples. Suppose, to the contrary, that it does: that means that there's a model/assignment pair (M,v) where each member of X is true and $\forall xA(x)$ isn't true. This means there is some x-variant v' of v, according to which A is not true. Let the value that v' assigns to x be a, and consider the model/assignment (M',v) which is just like M except it assigns the name n the value a. Then by the Semantic Substitution Lemma, (M',v) makes A(n) false since (M,v') makes A false. But since the name n does not appear in X, (M',v) makes each member of X true, since (M,v) makes those formulas true, and (M',v) differs from (M,v) only in the value assigned to n. This would make (M',v) a counterexample to the argument from X to A(n), and our assumption was that this argument has no counterexamples. So, contraposing all this, if $X \vDash A(n)$ then $X \vDash \forall xA(x)$, as desired. This is an explanation of the crucial steps of the proof of the soundness theorem.