WEEK 1 EXERCISES 1

1. [Chapter 1, Basic Question 3.]

   Which of these are formulas in our formal language Form, and which are not?

   $$p \lor q \qquad p \lor q \to r \qquad \neg\neg p \qquad q\neg p \qquad p \land (q \lor r) \to \bot$$

   $$(p \to q) \to ((p \to (q \to r)) \to (p \to r)) \qquad p \land q \land r$$

   For those that aren't formulas, are they ambiguous? (Could they be made into correct formulas in different ways by adding parentheses?) If they are, disambiguate them, by listing all of the different ways they can be made formulas, and consider for yourself the different things they could *mean*.

2. [Chapter 2, Basic Question 1 (i).]

   Read this proof from top to bottom, and at every inference step, list which assumptions each formula depends on.

   $$\dfrac{\dfrac{\dfrac{p \to q \qquad \dfrac{[p \land r]^1}{p}\land E}{q}\to E \qquad \dfrac{[p \land r]^1}{r}\land E}{q \land r}\land I}{(p \land r) \to (q \land r)}\to I^1$$

3. [Chapter 2, Basic Question 2, (iii,iv,v).]

   Construct proofs for the following arguments:

   - From the assumption $p \to q$ to $(r \to p) \to (r \to q)$.
   - From p to $q \to (p \land q)$.
   - From $p \land (q \to r)$ to $q \to (p \land r)$.

4. [Chapter 2, Challenge Question 2, (three parts).]

   Show that the following rules are derived rules of the proof system defined in Chapter 2.

   (a) $\dfrac{A \to B \qquad A \to C}{A \to (B \land C)}\ (\to\land)$

   (b) $\dfrac{A \to (B \to C)}{(A \land B) \to C}\ (\mathit{Import})$

   (c) $\dfrac{(A \land B) \to C}{A \to (B \to C)}\ (\mathit{Export})$

KEY CONCEPTS AND SKILLS

- ☐ You can identify premises and conclusions in a course of reasoning presented in a natural language argument.
- ☐ You understand the definitions of the concepts *partial order* and *tree*. You know how to check if a partial order is also a tree, and you can construct examples of partial orders that aren't trees. You can represent finite trees in tree diagrams.
- ☐ You can represent the structure of reasoning of simple arguments in the form of a tree, distinguishing premises and conclusions, individual inference steps and recognising the ultimate conclusion of a proof.
- ☐ You can construct formulas in the formal propositional language Form. You know how to read formulas, recognising conjunction ($\wedge$), disjunction ($\vee$), the conditional ($\rightarrow$) and negation ($\neg$), and you are able to detect whether something is actually a formula or if it is not formed using the formation rules of the formal language Form.
- ☐ You can identify the main connective of a complex formula, and the subformulas of a formula.
- ☐ You should be able to *read* tree proofs using the rules $\wedge$E, $\wedge$I, $\rightarrow$E and $\rightarrow$I. You should be able to check that a proof follows the rules, and you should be able to keep track of which assumptions are active at each stage of the proof.
- ☐ You should be able to *construct* simple tree proofs using the rules $\wedge$E, $\wedge$I, $\rightarrow$E and $\rightarrow$I.
- ☐ You can perform *reductions* on tree proofs which involve detours, using the reduction steps.

WEEK 2 EXERCISES

2

1. [Chapter 3, Basic Question 1 (second proof).]

   Read this proof, from top to bottom, and at every step, list which assumptions each formula depends on.

$$
\begin{prooftree}
\AxiomC{$[p \lor q]^3$}
\AxiomC{$(p \to \neg r) \land (q \to \neg r)$}
\RightLabel{$\land E$}
\UnaryInfC{$p \to \neg r$}
\AxiomC{$[p]^1$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg r$}
\AxiomC{$(p \to \neg r) \land (q \to \neg r)$}
\RightLabel{$\land E$}
\UnaryInfC{$q \to \neg r$}
\AxiomC{$[q]^2$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg r$}
\RightLabel{$\lor E^{1,2}$}
\TrinaryInfC{$\neg r$}
\AxiomC{$[r]^4$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg(p \lor q)$}
\RightLabel{$\to I^4$}
\UnaryInfC{$r \to \neg(p \lor q)$}
\end{prooftree}
$$

2. [Chapter 3, Basic Question 3 (choose one).]

   Construct proofs for *one* of the following arguments:

   - $p \succ \neg\neg p$
   - $p \to r, q \to s \succ (p \land q) \to (r \land s)$
   - $p \to r, q \to s \succ (p \lor q) \to (r \lor s)$
   - $\neg p \lor \neg q \succ \neg(p \land q)$
   - $\neg\neg\neg p \succ \neg p$

3. [Chapter 3, Basic Question 4.]

   Here is a proof for the argument $(p \to r) \land (q \to r) \succ (p \land q) \to r$. It contains a detour formula, marked in blue. Use the reductions to eliminate the detour.

$$
\begin{prooftree}
\AxiomC{$(p \to r) \land (q \to r)$}
\RightLabel{$\land E$}
\UnaryInfC{$p \to r$}
\RightLabel{$\lor I$}
\UnaryInfC{$\color{blue}{(p \to r) \lor (q \to r)}$}
\AxiomC{$[p \to r]^1$}
\AxiomC{$[p \land q]^3$}
\RightLabel{$\land E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{r}
\AxiomC{$[q \to r]^2$}
\AxiomC{$[p \land q]^3$}
\RightLabel{$\land E$}
\UnaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\lor E^{1,2}$}
\TrinaryInfC{r}
\RightLabel{$\to I^3$}
\UnaryInfC{$(p \land q) \to r$}
\end{prooftree}
$$

4. [Chapter 4, Basic Question 1 (i,ii).]

   Recall that X is said to be inconsistent if and only if there is a proof of $\bot$ from X. Which of the following sets are inconsistent? For those that are consistent, prove $\bot$ from those premises. For those sets that aren't, try to explain why they aren't inconsistent.

   (i) $p, q, \neg(p \land q)$

(ii) $p \vee q, \neg p \vee \neg q$

5. [Chapter 4, Basic Question 3 (second part).]

   Complete the proof of Theorem 3 in Chapter 4, by showing that

   - $X \vdash_I \neg A$ if and only if $X, A \vdash_I \bot$.

6. [Chapter 4, Basic Question 4 (i).]

   Use DNE to find arguments to show that the following *classical validity* holds.

   (i) $\neg(p \to q) \vdash_C p$.

KEY CONCEPTS AND SKILLS

☐ You should be able to *read* tree proofs using any or all of the rules ($\wedge$E, $\wedge$I, $\to$E, $\to$I, $\neg$E, $\neg$I, $\bot$E, $\vee$E, $\vee$I). You should be able to check that a proof follows these rules, and you should be able to keep track of which assumptions are active at each stage of the proof.

☐ You should be able to *construct* simple tree proofs using all the rules.

☐ You should be able to perform *reductions* on tree proofs which involve detours, using the reduction steps.

☐ You should be able to reason about and verify simple general facts about provability ($\vdash_I$), such as the facts expressed in Theorems 1, 2 and 3 in Chapter 4.

☐ You should be able to recognise detour formulas and detour sequences in non-normal proofs, make a one step reduction of the detour, including detours involving a formula being introduced in a $\bot$E rule and eliminated in an elimination rule.

☐ You should be able to explain the significance of the normalisation theorem, and the subformula property.

☐ You can do simple proofs involving DNE.

WEEK 3 EXERCISES

3

1. [Chapter 5, Basic Question 2 (choose three).]

   Choose three of these arguments. For the three you choose, which are valid? If an argument is valid, explain why (using valuations), and if it isn't, provide a counterexample.

   (a) $\neg(p \wedge q) \succ \neg p \wedge \neg q$

   (b) $p \to q, q \to r \succ p \to r$

   (c) $p \to q, q \to r \succ \neg p \to \neg r$

   (d) $\neg(p \to q) \succ p$

   (e) $\neg(p \wedge q) \succ \neg p \vee \neg q$

   (f) $\succ ((p \to q) \to p) \to p$

2. [Chapter 5, Basic Question 4 (choose three).]

   Suppose that $A, B \models_{CL} C \wedge D$. Let v be a Boolean valuation. Choose three of the following statements, and explain whether the statements are true, false, or you have not enough information to decide.

   (a) If $v(A) = 1$ and $v(B) = 1$, then $v(C \wedge D)$ can be 0.

   (b) If $v(C) = 0$, then $v(A) = 0$.

   (c) If $v(C) = 0$, then $v(A) = 0$ or $v(B) = 0$.

   (d) If $v(C \wedge D) = 0$ and $v(A) = 1$, then $v(B) = 0$.

   (e) If $v(A) = 1$ and $v(B) = 0$, then $v(C \wedge D) = 0$.

   (f) If $v(C \wedge D) = 1$, then $v(A) = 1$.

   (g) If $v(A) = 1$ and $v(B) = 1$, then $v(\neg(C \wedge D)) = 0$.

3. [Chapter 6, Basic Question 1, $\wedge E$ case.]

   Complete the proof of the Soundness Theorem by completing the case for $\wedge E$. (You need to show that if Π, a proof for $X \succ A_1 \wedge A_2$ is truth preserving, then the proof of A_i (whether $i = 1$ or 2), given by extending Π with an $\wedge E$ step, is also truth preserving.)

KEY CONCEPTS AND SKILLS

- ☐ You need to be familiar with the definition of Boolean valuations, given a valuation, you can calculate the value of a complex formula, in terms of the values of its atoms.
- ☐ You should be able to complete truth tables for formulas.

☐ You need to know what it means for a formula to be a tautology, a contradiction or a contingency, and you can use truth tables to test for whether a formula is a tautology or a contradiction or contingent.

☐ You can test arguments using Boolean valuations.

☐ You should be able to reason about and verify simple general facts about validity ($\vdash_I$).

☐ You need to understand—and to clearly state for yourself—the definitions of soundness and completeness and the difference between them.

☐ You should understand the proof of soundness of intuitionistic and classical proofs for Boolean validity. In particular, you should understand the shape of the argument (an induction on the construction of the proof in question), and you should be able to prove particular instances for yourself.

☐ You understand what arguments count as counterexamples to the completeness of intuitionistic provability for Boolean validity, and how adding DNE strengthens the system to give completeness.

☐ You can evaluate formulas and arguments in the simple three valued Heyting algebra, when given the truth tables to work from.

☐ You understand the connections between proofs and models, and inferentialism and representationalism.

WEEK 4 EXERCISES

4

1. [Chapter 7, Basic Question 2.]

   Take the possible worlds model with worlds $W = \{w, x, y\}$ and where $V(p, w) = V(p, x) = 1$, $V(p, y) = 0$, and $V(q, w) = V(q, y) = 1$ and $V(q, x) = 0$ and $V(r, w) = 1$ and $V(r, x) = V(r, y) = 0$. Draw this model in a diagram. Then list out every *set* of worlds (there should be eight of them), and see if you can find a formula A, whose *proposition* in the model is that set. So, for example, for the set $\{w, x\}$ we can choose, p, since $||p|| = \{w, x\}$. But we can choose $p \vee r$ too, since $||p \vee r|| = \{w, x\}$ too. Find formulas for the seven other sets of worlds.

2. [Chapter 7, Basic Question 3 (choose two).]

   Choose two arguments from the following list. Construct possible worlds model counterexamples to the arguments you choose, if there are any. If there isn't a counterexample try to explain why as clearly as possible why the argument is valid.

   (i) $\Diamond p, \Diamond q \succ \Diamond(p \wedge q)$
   (ii) $\Diamond p, \Box q \succ \Diamond(p \wedge q)$
   (iii) $\Diamond(p \to q) \succ \Diamond p \to \Diamond q$
   (iv) $\Box(p \to q) \succ \Diamond p \to \Box q$
   (v) $\Box(p \to q) \succ \Diamond p \to \Diamond q$

3. [Chapter 7, Basic Question 5 (choose one).]

   Use the countermodel generation procedure from section §7.2 to find counterexamples to one of these these two arguments.

   (i) $\Diamond p, \Diamond q \succ \Box(p \vee q)$
   (ii) $\Box(p \vee q) \succ \Box p \vee \Box q$

   Verify that the model you obtain is a counterexample. Can you find a counterexample with fewer worlds?

4. [Chapter 7, Basic Question 11 (choose two).]

   Verify two of the following following facts about S5- and S4-validity. (Choose one S5 fact and one S4 fact.)

   (i) $\Diamond\Diamond A \models_{S4} \Diamond A$
   (ii) $\Diamond\Box A \models_{S5} \Box A$
   (iii) $\Diamond\Box A \not\models_{S4} \Box A$
   (iv) $\Diamond\Box A \models_{S5} \Box\Diamond A$

(v) $\Diamond\Box A \not\models_{S4} \Box\Diamond A$

(vi) $\models_{S5} \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$

(vii) $\not\models_{S4} \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$

KEY CONCEPTS AND SKILLS

- ☐ You should be able to determine the truth or falsity of modal formulas in worlds in a given possible worlds model.
- ☐ You should be able to generate simple possible worlds models as counterexamples to arguments in S5.
- ☐ You should be able to distinguish the different ways to understand the scope of modal formulas, especially in the interaction between necessity and conditionals.
- ☐ You should be able to model propositions as sets of worlds, and determine the propositions corresponding to formulas in a given possible worlds model.
- ☐ You should be able to determine the truth or falsity of modal formulas in worlds in a given model for S4, or for S5E.
- ☐ You should be able to use models for S5E to interpret the information available to agents in simple situations.

WEEK 5 EXERCISES

5

1. [Chapter 8, Basic Question 2 (i,ii).]

   Which (if any) of these two arguments are S5A valid? For those that aren't valid, provide an actuality model as a counterexample. For those that are valid, explain why it is valid.

   (i) $\Diamond(p \land \mathbb{A}q) \succ \mathbb{A}p \land \Diamond q$

   (ii) $\Diamond(p \land \mathbb{A}q) \succ \Diamond p \land \mathbb{A}q$

2. [Chapter 8, Basic Question 3.]

   Consider the following double indexed model, with worlds $W = \{w, x, y\}$, and $V(p, w) = V(p, x) = 1, V(p, y) = 0$ and $V(q, w) = 0$, $V(q, x) = V(q, y) = 1$. Draw a diagram for this model, and evaluate the formulas $\mathbb{A}p$, $\mathbb{F}q$, $\dagger p$ and $\dagger(p \lor q)$ at each world in the model.

3. [Chapter 9, Basic Question 2 (i,ii,iii).]

   Find S4 proofs for the following arguments.

   (i) $\Box p, \Box q \succ \Box(p \land q)$

   (ii) $\Diamond\Diamond p, \neg\Diamond p \succ \bot$

   (iii) $\Diamond\Diamond p \succ \Diamond p$

KEY CONCEPTS AND SKILLS

- ☐ You should be able to evaluate formulas in S5A models, and construct simple counterexamples to invalid formulas.
- ☐ You should understand the difference between fixity and necessity, and fixed actuality (†).
- ☐ You should be able to evaluate formulas in double indexed models, and use these models to construct counterexamples to general validity and real world validity.
- ☐ You should be able to use double indexed models to clarify or model the difference between necessity, a priori knowability and analyticity.
- ☐ You should understand the idea of a rule with *side conditions*, the motivation for the S4 and S5 conditions in the modal rules, and you should be able to read proofs and check that instances of the modal rules satsify or violate the side conditions.

- ☐ You should be able to construct simple proofs in the proof systems for S4 and S5.
- ☐ You need to understand what it means for an operator to be *congruential* (and why the modal operators $\Box$ and $\Diamond$) *are* congruential in S4 and S5.
- ☐ You need to understand the concept of a *formula context* and the difference between *intensional* and *extensional* contexts.

WEEK 7 EXERCISES

7

1. [Chapter 10, Basic Question 2 (iv,v).] Check that you understand how substitution works by performing these substitutions:

   (iv) $(\exists x Fx \land Gx)[x/y]$
   (v) $((Fx \to \exists y(Lxy \land (Gz \lor \forall x Hx)))[x/z])[z/a]$

   What problem is there with performing the substitution $(\forall x(Fx \to Lxy))[y/x]$?

2. [Chapter 10, Basic Question 3]

   Are these trees *proofs*? Check whether the rules are correctly applied, and particularly, whether the eigenvariable conditions are satisfied by each $\forall$I and $\exists$E step.

$$
\begin{prooftree}
\AxiomC{$\forall x(Fx \land \exists y \neg Fy)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa \land \exists y \neg Fy$}
\RightLabel{$\land E$}
\UnaryInfC{$\exists y \neg Fy$}
\AxiomC{$[\neg Fa]^1$}
\AxiomC{$\forall x(Fx \land \exists y \neg Fy)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa \land \exists y \neg Fy$}
\RightLabel{$\land E$}
\UnaryInfC{Fa}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\forall x(Fx \lor Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa \lor Ga$}
\AxiomC{$[Fa]^1$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x Fx$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall x Fx \lor \forall x Gx$}
\AxiomC{$[Ga]^2$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x Gx$}
\RightLabel{$\lor I$}
\UnaryInfC{$\forall x Fx \lor \forall x Gx$}
\RightLabel{$\lor E^{1,2}$}
\TrinaryInfC{$\forall x Fx \lor \forall x Gx$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\forall x(Fx \lor Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Fa \lor Ga$}
\AxiomC{$[Fa]^2$}
\RightLabel{$\lor I$}
\UnaryInfC{$Fa \lor Ha$}
\AxiomC{$[Ga]^3$}
\AxiomC{$[\neg Ha]^1$}
\RightLabel{$\land I$}
\BinaryInfC{$Ga \land \neg Ha$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(Gx \land \neg Hx)$}
\AxiomC{$\neg \exists x(Gx \land \neg Hx)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg\neg Ha$}
\RightLabel{DNE}
\UnaryInfC{Ha}
\RightLabel{$\lor I$}
\UnaryInfC{$Fa \lor Ha$}
\RightLabel{$\lor E^{2,3}$}
\TrinaryInfC{$Fa \lor Ha$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(Fx \lor Hx)$}
\end{prooftree}
$$

3. [Chapter 10, Basic Question 4 (iv,v).]

   Construct proofs for the following arguments:

   (iv) $p \land \exists x Fx \succ \exists x(p \land Fx)$

(v) $\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz), \neg \exists x Rxx \succ \forall x \forall y(Rxy \to \neg Ryx)$

Warning: the proof for the second one will be significantly longer than for the first. Be patient!

4. [Chapter 10, Basic Question 5 (choose one).]

   Try to construct a proof for *one* of the following arguments. (Beware, for these you may need to employ DNE, and they might be complicated. Do the best you can.)

   (i) $\forall x(p \lor Fx) \succ p \lor \forall x Gx$
   (ii) $\neg \exists x(Fx \land \neg Gx), \neg \exists x(Gx \land \neg Hx) \succ \forall x(\neg Fx \lor Hx)$
   (iii) $\succ \exists x(Fx \to \forall y Fy)$

   Note that a proof for the last may be much more difficult to find than proofs for the first two.

KEY CONCEPTS AND SKILLS

☐ You need to understand the difference between predicates, names, variables, and quantifiers.

☐ You should be able to read and understand formulas in the language of predicate logic.

☐ You should understand the definition of *substitution*, when a term is free for a variable in a formula, and why the definition of substitution is restricted to the cases where the terms substituted in the formula be free for the variable substituted *for*.

☐ You should be able to *read* tree proofs using any or all of the quantifier rules, and be able to check that the eigenvariable condition is satisfied when the $\forall$I and $\exists$E rules are used.

☐ You should be able to *construct* simple tree proofs using all the inference rules.

WEEK 8 EXERCISES

8

1. [Chapter 11, Basic Question 3 (choose two).]

   Find one argument on this list that has a *counterexample*, and present that counterexample to the argument, explaining how, according to that model, the premises are true and the conclusion is false. Find another argument on this list that is *valid*, and explain why it is valid, either by giving an argument that there cannot be a counterexample, or by providing a CQ *proof* for the argument.

   i. $\exists xFx, \exists xGx \succ \exists x(Fx \land Gx)$
   ii. $\exists xFx, \forall xGx \succ \exists x(Fx \land Gx)$
   iii. $\succ \exists x(Fx \to \forall yFy)$
   iv. $\forall x\exists yRxy \succ \forall y\exists xRxy$
   v. $\forall x\forall y(Rxy \to \exists z(Rxz \land Rzy)), \forall x\exists yRxy, \forall x\neg Rxx \succ \bot$

2. [Chapter 11, Basic Question 4.]

   Consider a model with the domain $D = \{0, 1, 2, 3, \ldots\}$ of all of the natural numbers. Consider the language with the one-place predicate O (for *is odd*) true of all of the odd numbers, the two-place predicate S (for *smaller than*) such that Sxy is true of x and y just when the number x is smaller than the number y. And let's add a lot of names a_0, a_1, a_2, etc., naming 0, 1, 2, etc. Evaluate the following formulas in this model:

   i. $Oa_1, Oa_2, Sa_1a_1, Sa_1a_2, Sa_2a_1$.
   ii. $\forall ySa_0y$
   iii. $\exists x\forall ySxy$
   iv. $\exists xSxa_1$
   v. $\exists ySa_{10\,000}y$
   vi. $\forall x\exists ySxy$
   vii. $\forall x(Ox \to \exists ySyx)$
   viii. $\exists y\forall x(Ox \to Syx)$
   ix. $\forall x\forall y(Sxy \to \exists z(Sxz \land Szy))$

KEY CONCEPTS AND SKILLS

☐ You should understand the definition of a *model* for a first-order language (with a domain and an interpretation of each predicate and each name in the language), and of an *assignment* of values to the variables. You should also understand why we need both a model and an assignment to assign truth values to every formula.

- ☐ You should be able to check the truth or falsity of a formula in a model.
- ☐ You should understand the concept of validity in first-order predicate logic, and be able to construct counterexamples to simple invalid arguments, and for simple valid arguments, be able to demonstrate that they are indeed valid.
- ☐ You understand the soundness and completeness theorems, and have a grasp of how the soundness theorem is proved.

WEEK 9 EXERCISES

9

1. Write one paragraph, in your own words, explaining how we prove the *soundness* theorem for first-order predicate logic.

2. Write one paragraph, in your own words, explaining how we prove the *completeness* theorem for first-order predicate logic.

3. [Chapter 11, Challenge Question 5 (two clauses).]

   A formula is said to be in PRENEX FORM if and only if it is of the form $Q_1 v_1 \cdots Q_n v_n A$, where A contains no quantifiers, and each Q_i is either $\forall$ or $\exists$, and each v_i is a variable. In this question, you will show that every formula is logically equivalent to a formula in prenex form.

   Say that A and B are LOGICALLY EQUIVALENT in FO iff $A \models_{FO} B$ and $B \models_{FO} A$. **First**, choose *two* of the following pairs of formulas, and prove, for each pair, that the first item is logically equivalent to the second, in FO.

   - $\forall x(A \rightarrow B)$ and $\exists xA \rightarrow B$, where x is not free in B.
   - $\forall x(A \rightarrow B)$ and $A \rightarrow \forall xB$, where x is not free in A.
   - $\forall x(A \lor B)$ and $\forall xA \lor B$, where x is not free in B.
   - $\exists x(A \land B)$ and $\exists xA \land B$, where x is not free in B.
   - $\exists x(A \rightarrow B)$ and $A \rightarrow \exists xB$, where x is not free in A.
   - $\forall x(A \land B)$ and $\forall xA \land \forall xB$.
   - $\exists x(A \lor B)$ and $\exists xA \lor \exists xB$.
   - $\forall x \neg A$ and $\neg \exists xA$.
   - $\exists x \neg A$ and $\neg \forall xA$.

   Next, taking *all* of the logical equivalences as given try explaining why it is that every formula in the language of first-order predicate logic is logically equivalent to some formula in prenex form.

KEY CONCEPTS AND SKILLS

☐ You understand the soundness theorem, and have a grasp of how this theorem is proved, and be able to reconstruct some of the key cases in the proof.

☐ You understand the completness theorem, and have a grasp of how this theorem is proved, and be able to reconstruct some of the key cases in the proof.

WEEK 10 EXERCISES

10

1. [Chapter 6, Basic Question 4 (two cases).]

   Consider the following arguments. Find one argument that has a counterexample in the Heyting Algebra H_3, giving the counterexample. Then, find another argument in the list that has *no* counterexample in H_3, but does have a counterexample in H_5.

   i. $\neg\neg p \succ p$
   ii. $\neg(p \wedge q) \succ \neg p \vee \neg q$
   iii. $\neg p \vee q \succ p \rightarrow q$
   iv. $p \rightarrow q \succ \neg p \vee q$
   v. $\succ \neg p \vee \neg\neg p$
   vi. $(p \rightarrow q) \rightarrow p \succ p$

2. [Chapter 7, Challenge Question 3.]

   To get a feel for intuitionistic Kripke models, it is useful to come up with counterexamples to a few arguments that are classically valid but not provable intuitionistically. Construct intuitionistic Kripke model counterexamples to the following arguments.

   i. $\succ p \vee \neg p$
   ii. $\neg\neg p \succ p$
   iii. $(p \rightarrow q) \rightarrow p \succ p$
   iv. $\neg(p \rightarrow q) \succ p$
   v. $\neg(p \wedge q) \succ \neg p \vee \neg q$

KEY CONCEPTS AND SKILLS

- ☐ You can evaluate formulas and arguments in the example Heyting algebras, when given the truth tables to work from.
- ☐ You should understand intuitionistic Kripke models, and be able to use them to provide counterexamples to intuitionistically invalid arguments.