# Limit Theorem of Hadamard Product Random Matrices

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# A standard result of GUE

First, we review some basics of random matrix theory.

- $\{X_{i,j} : 1 \leq i \leq j\}$ is a family of i.i.d. standard normal random variables, i.e. $X_{i,j} \sim \mathcal{N}(0,1)$.
- $W_N$ is called a **Gaussian Unitary Ensemble** (or shortly, **GUE**) if it is a random matrix with the form

$$W_N(i,j) := N^{-\frac{1}{2}} X_{i\wedge j, i\vee j} \text{ for all } 1 \leq i,j \leq N.$$

# A standard result of GUE

- ESD$[W_N]$ is the **empirical spectral distribution** of $W_N$, which is defined by

$$\mathrm{ESD}[W_N] := \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N}),$$

where $\lambda_i$ are the eigenvalues of $W_N$.

- **Theorem** [Wigner, 1955]

$$\mathrm{ESD}[W_N] \rightharpoonup w \quad \text{as } N \to \infty,$$

where $w$ with density $\frac{1}{2\pi}\sqrt{4 - x^2}$ is called the standard semi-circular distribution.

# Hadamard product and GUE

**Question:** How to generalize this result?

More precisely, what kind of sequence $(a_{i,j})$ would make $\mathrm{ESD}[(a_{i,j} \cdot W_N(i,j))]$ weakly converge to some probability measure?

Chakrabarty gives an idea as follows:

# Hadamard product and GUE

Let $f : [0,1]^2 \to \mathbb{R}$ be a symmetric Riemann integrable function, i.e. $f(x,y) = f(y,x)$. Then define a random matrix

$$Z_{f,N}(i,j) := f\left(\frac{i}{N+1}, \frac{j}{N+1}\right) W_N(i,j),$$

for $1 \le i,j \le N$. In other words, $Z_{f,N} = A_{f,N} \circ W_N$ is a Hadamard product of matrices $A_{f,N}(i,j) = f(i/(N+1), j/(N+1))$ and $W_N$.

# Hadamard product and GUE

- **Theorem** [Chakrabarty, 2017] There exists a symmetric probability measure $\mu_f \in \mathbf{P}_{sym}(\mathbb{R})$ such that

$$\mathrm{ESD}[Z_{f,N}] \rightharpoonup \mu_f \quad \text{as } N \to \infty.$$

- For $f(x,y) = r(x)r(y)$, Chakrabarty shows that $\mu_f^2 = \mu^{\boxtimes 2} \boxtimes \sigma$, where $\mu$ is the law of $r^2(U)$, $U$ is the standard uniform random variable, $\boxtimes$ is the free multiplicative convolution, and $\sigma$ is the Free Poisson distribution with rate $1$.

  **Remark:** The density of $\sigma$ is $\frac{\sqrt{4x-x^2}}{2\pi x} \mathbb{1}_{[0,4]}(x)$.

Chakrabarty A. *Stat Probab Lett* 2017; **127** : 150-157.

# Introduction to Free Probability

- $(\mathcal{A}, \phi)$: a $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ and a tracial linear functional $\phi$. We call such a pair a **Non-Commutative Probability Space** (or shortly, NCPS). An element $a \in \mathcal{A}$ is called a **random variable**.
- For a random variable $a \in \mathcal{A}$, we called $\mu \in \mathbf{P}(\mathbb{R})$ is **the distribution of** $a$ if

$$\phi(a^n) = \int_{\mathbb{R}} t^n d\mu(t) \quad \forall n \geq 0.$$

- $\mathcal{B}$ and $\mathcal{C}$ are two subalgebras of $\mathcal{A}$. We say they are **freely independent** if for any $n \geq 1$, $b_i \in \mathcal{B}$, and $c_i \in \mathcal{C}$ satisfy $\phi(b_i) = \phi(c_i) = 0$ for all $1 \leq i \leq n$, we have

$$\phi(b_1 c_1 b_2 c_2 \cdots b_n c_n) = 0.$$

# Introduction to Free Probability

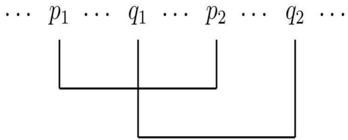
- For $a, b \in \mathcal{A}$, we say $a$ and $b$ are **free** if $span\{1_{\mathcal{A}}, a\}$ and $span\{1_{\mathcal{A}}, b\}$ are free.
- For $a, b \in \mathcal{A}$ and $a, b$ have the distributions $\mu, \nu$, respectively.
  - If the distribution of $a + b$ exists, then we denote it by $\mu \boxplus \nu$, the **free additive convolution of** $a$ **and** $b$.
  - If the distribution of $ab$ exists, then we denote it by $\mu \boxtimes \nu$, the **free multiplicative convolution of** $a$ **and** $b$.
- If $\mathcal{A}$ is a $C^*$-algebra and $\phi$ is **positive state**, then for any self-adjoint element $a$, by **Gelfand transform**, the distribution exists. Moreover, if $a$ is positive, then the distribution is defined on $\mathbb{R}^+$.

**Theorem 2.1 (Existence of Free Product)**

Given $\{(\mathcal{A}_i, \phi_i)\}_{i\in I}$ a family of NCPS. Then there exists a NCPS $(\mathcal{A}, \phi)$ and monomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that $\phi_i = \phi \circ \psi_i$ for all $i \in I$, which satisfy $\{\psi_i(\mathcal{A}_i)\}_{i\in I}$ are free in $(\mathcal{A}, \phi)$. We denote $(\mathcal{A}, \phi)$ by $(*_{i\in I}\mathcal{A}_i, \phi)$ and call it the **free product of** $\{(\mathcal{A}_i, \phi_i)\}_{i\in I}$.

# Introduction to Free Probability

- A partition $\pi$ of the set $[n]$ is called **crossing** if there exist $p_1 < q_1 < p_2 < q_2$ in $S$ such that $p_1 \sim_\pi p_2 \nsim_\pi q_2 \sim_\pi q_1$:




  If $\pi$ is not crossing, then it is called **non-crossing**.

- The set of all non-crossing partitions of $[n]$ is denoted by $NC(n)$.

# Main Theorem I

Now we focus on some specific $f$.

Let $f : [0,1]^2 \to \mathbb{R}$ be a symmetric Riemann integrable function such that

$$f(x,y) = \sqrt{\sum_{k=1}^{N} r_k(x)\tilde{r}_k(y)}, \tag{1}$$

with $r_k, \tilde{r}_k : [0,1] \to \mathbb{R}$ integrable for all $k = 1, \cdots, N$.

It is important and sufficient to study $f$ of this type because by the **Stone-Weierstrass theorem**, these functions are dense in the set of symmetric continuous functions on $[0,1]^2$, and we can approach any symmetric integrable function $f$ through the continuous one.

# Main Theorem I

Let $(\mathcal{A}, \phi_{\mathcal{A}}) = (\mathcal{B}, \phi_{\mathcal{B}}) = (\mathcal{R}([0,1]), dx)$ be two commutative probability spaces. For $f(x,y) = \sqrt{\sum r_k(x)\tilde{r}_k(y)}$, we define random variables $a_k = r_k \in \mathcal{A}$, $b_k = \tilde{r}_k \in \mathcal{B}$, and

$$\eta_f = \sum_{k=1}^{N} a_k b_k \in \mathcal{A} * \mathcal{B}.$$

# Main Theorem I

Given a non-crossing partition $\pi \in NC(n)$, we denote

$$m_\pi(\eta_f) := \phi(\eta_f^{|V_1|})\phi(\eta_f^{|V_2|})\cdots\phi(\eta_f^{|V_r|}),$$

where $\pi = \{V_1, V_2, \cdots, V_r\}$, is a multiplicative function on $NC(n)$.

For example, for $\pi = \{(1,4,8),(2,3),(5),(6,7)\} \in NC(8)$, the moment $m_\pi$ is defined by

$$m_\pi(\eta_f) = \phi(\eta_f^3)\phi(\eta_f^2)^2\phi(\eta_f).$$

# Main Theorem I

**Theorem 3.1 (Hsiao, 2023)**

Let $f : [0,1]^2 \to \mathbb{R}$ and $\eta_f$ be defined as above. Then

$$m_{2n}(\mu_f) = \sum_{\pi \in NC(n)} m_\pi(\eta_f),$$

for all $n \geq 0$. In particular, if $\eta_f$ has the distribution on $\mathbb{R}$, then we have $\mu_f^2 = \eta_f \boxtimes \sigma$, where $\sigma$ is the Free Poisson distribution with rate $1$.

# Main Theorem I

**Remark:**

- This generalizes Chakrabarty's results. ($N = 1$)
- In general, $\eta_f$ may not have the distribution. ($\eta_f$ is not a self-adjoint element, not even a normal element.)

# Main Theorem II

Now, we define a family of probability measure

$$\mathfrak{H} := \left\{ \mu_f \in \mathbf{P}_{sym}(\mathbb{R}) : \begin{array}{l} f : [0,1]^2 \to \mathbb{R} \text{ is symmetric} \\ \text{and Riemann integrable.} \end{array} \right\}.$$

**Theorem 3.2 (Hsiao, 2023)**

1. $\mathfrak{H}$ is a convex set.
2. $\left\{ \sqrt{\nu \boxtimes \sigma} : \nu \in \mathbf{P}(\mathbb{R}^+) \right\} \subsetneq \mathfrak{H} \subsetneq \mathbf{P}_{sym}(\mathbb{R})$.

Remark: This also generalize Chakrabarty's result. Since

$$\left\{ \sqrt{\nu^{\boxtimes 2} \boxtimes \sigma} : \nu \in \mathbf{P}(\mathbb{R}^+) \right\} \subsetneq \left\{ \sqrt{\nu \boxtimes \sigma} : \nu \in \mathbf{P}(\mathbb{R}^+) \right\}.$$

# Application I

Benaych-Georges shows that for any two probability measures $\mu, \nu \in \mathbf{P}(\mathbb{R}^+)$, we have

$$\sqrt{\mu \boxtimes \sigma} \boxplus \sqrt{\nu \boxtimes \sigma} = \sqrt{(\mu \boxplus \nu) \boxtimes \sigma}.$$

Benaych-Georges F. *Ann Inst H Poincaré Probab Statist* 2010; **46**: 644—652.

Here, we give a Hadamard product random matrix model of this equality.

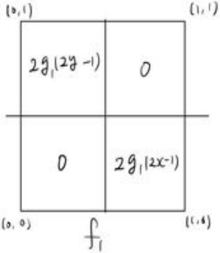
# Application I

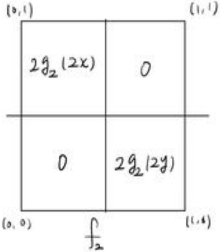
Let $g_1, g_2 : [0,1] \to \mathbb{R}_{\geq 0}$ be two Riemann integrable functions.
Define $f_1, f_2 : [0,1] \to \mathbb{R}$ as follow:

$$f_1(x,y) := \sqrt{2g_1(2x-1)\mathbb{1}_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}(x,y) + 2g_1(2y-1)\mathbb{1}_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}(x,y)},$$

and

$$f_2(x,y) := \sqrt{2g_2(2y)\mathbb{1}_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}(x,y) + 2g_2(2x)\mathbb{1}_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}(x,y)}.$$



# Application I

## Theorem 4.1 (Hisao, 2023)

Let $g_1, g_2, f_1, f_2$ defined as above. Then we have

$$\mu_{f_1} \boxplus \mu_{f_2} = \mu_{\sqrt{f_1^2+f_2^2}}.$$

Furthermore, the above identity can be written as

$$\sqrt{\mu \boxtimes \sigma} \boxplus \sqrt{\nu \boxtimes \sigma} = \sqrt{(\mu \boxplus \nu) \boxtimes \sigma},$$

where $\mu, \nu \in \mathbf{P}(\mathbb{R}^+)$ are two probability measures such that

$$\mu = g_1(U) \text{ and } \nu = g_2(U),$$

with $U$ the standard uniform random variable.

# Application I

**Remark:** By a well-known property of normal distribution, we have

$$Z_{f_1,N} + \tilde{Z}_{f_2,N} \stackrel{d}{=} Z_{\sqrt{f_1^2+f_2^2},N}.$$

But in general, we don't have $\mu_{f_1} \boxplus \mu_{f_2} = \mu_{\sqrt{f_1^2+f_2^2}}$. In other words, $\{Z_{f_1,N}\}$ and $\{\tilde{Z}_{f_2,N}\}$ are not **asymptotically free** for general symmetric Riemann integrable functions $f_1, f_2$.

# Application II

**Definition:** For a symmetric Riemann integrable function $f : [0, 1]^2 \to \mathbb{R}_{\geq 0}$, $f$ is *periodic of type I* if there exists a constant $M_f > 0$ such that for all $(x, y) \in [0, \frac{1}{2}]^2$, we have

$$f(x, y) = f\left(x + \frac{1}{2}, y + \frac{1}{2}\right),$$

and

$$f(x, y)^2 + f\left(x + \frac{1}{2}, y\right)^2 = f(x, y)^2 + f\left(x, y + \frac{1}{2}\right)^2 = 2M_f^2.$$

# Application II

**Definition:** For a symmetric Riemann integrable function $f : [0,1]^2 \to \mathbb{R}_{\geq 0}$, $f$ is *periodic of type II* if there exists a constant $M_f > 0$ such that for all $(x, y) \in [0, \frac{1}{2}]^2$, we have

$$f(x, y) = f(1 - x, 1 - y),$$

and

$$f(x, y)^2 + f(1 - x, y)^2 = f(x, y)^2 + f(x, 1 - y)^2 = 2M_f^2.$$

# Application II

**Theorem 4.2 (Hsiao, 2023)**

Let $f$ be a periodic of type I or type II function. Then we have $\mu_f = w_{M_f}$, where $w_{M_f}$ is the semi-circular distribution with variance $M_f$.

# Future works

- Property of $\mathfrak{H}$: e.g. Is $\mathfrak{H}$ weakly closed?
- Relation between $\mu_{\sqrt{f_1^2+f_2^2}}$ and $\mu_{f_1}, \mu_{f_2}$.
- $\boxplus$-infinitely divisible and Hadamard random matrix.
- When do we have asymptotically freeness?
- More applications in Random matrix theory.

# Reference

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# Appreciation

**Thanks for your listening!**