Hadamard product of random matrices and their limiting spectral distributions

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2024/01/23

Table of Contents

A standard result of GUE

First, we review some basics of random matrix theory.

- $\{X_{i,j} : 1 \leq i \leq j\}$ is a family of i.i.d. standard normal random variables, i.e. $X_{i,j} \sim \mathcal{N}(0,1)$.
- W_N is called a Gaussian Unitary Ensemble (or shortly, **GUE**) if it is a random matrix with the form

$$W_N(i,j) := N^{-\frac{1}{2}} X_{i\wedge j, i\vee j} \text{ for all } 1 \leq i,j \leq N.$$

A standard result of GUE

- $\mathrm{AED}[W_N] \in \mathbf{P}(\mathbb{R})$ is the average eigenvalue distribution of W_N, which is defined by

$$\mathrm{AED}[W_N] := \frac{1}{N}\mathbb{E}\left[\delta_{\lambda_1(\omega)} + \cdots + \delta_{\lambda_N(\omega)}\right],$$

where λ_i are the eigenvalues of W_N.

- **Theorem** [Wigner, 1955]

$$\mathrm{AED}[W_N] \rightharpoonup w \quad \text{as } N \to \infty,$$

where w with density $\frac{1}{2\pi}\sqrt{4-x^2}$ is called the standard semi-circular distribution.

Hadamard product and GUE

Question: How to generalize this result?

Chakrabarty gives an idea as follows:

Let $f : [0,1]^2 \to \mathbb{R}$ be a symmetric Riemann integrable function (or shortly, **SRIF**), i.e. $f(x,y) = f(y,x)$. Then define a random matrix

$$Z_{f,N}(i,j) := f\left(\frac{i}{N+1}, \frac{j}{N+1}\right) W_N(i,j),$$

for $1 \leq i,j \leq N$. In other words, $Z_{f,N} = A_{f,N} \circ W_N$ is a Hadamard product of matrices $A_{f,N}(i,j) = f(i/(N+1), j/(N+1))$ and W_N.

Hadamard product and GUE

- **Theorem** [A. Chakrabarty, 2017] There exists a symmetric probability measure $\mu_f \in \mathbf{P}_{\text{sym}}(\mathbb{R})$ such that

$$\text{AED}[Z_{f,N}] \rightharpoonup \mu_f \quad \text{as } N \to \infty.$$

- For $f(x,y) = r(x)r(y)$, Chakrabarty shows that $\mu_f^2 = \mu^{\boxtimes 2} \boxtimes \sigma$, where μ is the law of $r^2(U)$, U is the standard uniform random variable, $\boxtimes$ is the free multiplicative convolution, and σ is the Free Poisson distribution with rate 1.

  Remark: The density of σ is $\frac{\sqrt{4x-x^2}}{2\pi x}\mathbb{1}_{(0,4]}(x)$.

Chakrabarty A. *Stat Probab Lett* 2017; **127** : 150-157.

Introduction to Free Probability

- $(\mathcal{A}, \phi)$: a $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ and a tracial linear functional ϕ. We call such a pair a **Non-Commutative Probability Space** (or shortly, NCPS). An element $a \in \mathcal{A}$ is called a **random variable**.
- For a random variable $a \in \mathcal{A}$, we called $\mu \in \mathbf{P}(\mathbb{R})$ is **the distribution of** a if

$$\phi(a^n) = \int_{\mathbb{R}} t^n d\mu(t) \quad \forall n \geq 0.$$

- $\mathcal{B}$ and $\mathcal{C}$ are two subalgebras of $\mathcal{A}$. We say they are **freely independent** if for any $n \geq 1$, $b_i \in \mathcal{B}$, and $c_i \in \mathcal{C}$ satisfy $\phi(b_i) = \phi(c_i) = 0$ for all $1 \leq i \leq n$, we have

$$\phi(b_1 c_1 b_2 c_2 \cdots b_n c_n) = 0.$$

Introduction to Free Probability

- For $a, b \in \mathcal{A}$, we say a and b are **free** if $span\{1_{\mathcal{A}}, a, a^*\}$ and $span\{1_{\mathcal{A}}, b, b^*\}$ are free.
- For $a, b \in \mathcal{A}$ are free and a, b have the distributions μ, ν, respectively.
  - If the distribution of $a + b$ exists, then we denote it by $\mu \boxplus \nu$, the **free additive convolution of a and b.**
  - If the distribution of ab exists, then we denote it by $\mu \boxtimes \nu$, the **free multiplicative convolution of a and b.**
- If $\mathcal{A}$ is a C^*-algebra and ϕ is **positive state**, then for any self-adjoint element a, by **Gelfand transform**, the distribution exists. Moreover, if a is positive, then the distribution is defined on $\mathbb{R}^+$.

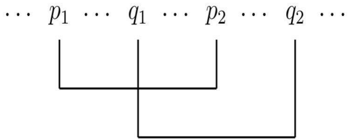
Introduction to Free Probability

Theorem 2.1 (Existence of Free Product)

Given $\{(\mathcal{A}_i, \phi_i)\}_{i \in I}$ a family of NCPS. Then there exists a NCPS $(\mathcal{A}, \phi)$ and monomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that $\phi_i = \phi \circ \psi_i$ for all $i \in I$, which satisfy $\{\psi_i(\mathcal{A}_i)\}_{i \in I}$ are free in $(\mathcal{A}, \phi)$. We denote $(\mathcal{A}, \phi)$ by $(*_{i \in I} \mathcal{A}_i, \phi)$ and call it the **free product of** $\{(\mathcal{A}_i, \phi_i)\}_{i \in I}$.

Introduction to Free Probability

- A partition π of the set $[n]$ is called **crossing** if there exist $p_1 < q_1 < p_2 < q_2$ in S such that $p_1 \sim_\pi p_2 \nsim_\pi q_2 \sim_\pi q_1$:



  If π is not crossing, then it is called **non-crossing**.

- The set of all non-crossing partitions of $[n]$ is denoted by $NC(n)$.

Main Theorem I

Now, we introduce an approximation lemma.

Lemma 1 (M. Hsiao, H-W. Huang, 2023)

Consider a class

$$\mathcal{F} := \left\{ \sum_{k=1}^{N} a_k(x) b_k(y) : \begin{array}{l} \textit{for all } N \geq 1 \textit{ and } a_k, b_k : [0,1] \to \mathbb{R} \\ \textit{is continuous } \forall k = 1, \cdots, N \end{array} \right\}.$$

For any SRIF $f : [0,1]^2 \to \mathbb{R}_{\geq 0}$*, there exists a sequence of SRIF's* $\{f_k\}_{k\geq 1} \subset \mathcal{F}$ *such that* f_k *converges to* f *a.s. as* $k \to \infty$ *and* $f_k \geq 0$ *for all* k*. Moreover, we have*

$$\mu_{f_k} \rightharpoonup \mu_f \textit{ as } k \to \infty.$$

Main Theorem I

Let $(\mathcal{A}, \phi_{\mathcal{A}}) = (\mathcal{B}, \phi_{\mathcal{B}}) = (\mathcal{R}([0,1]), dx)$ be two commutative probability spaces. For $f(x,y) = \sqrt{\sum r_k(x)\tilde{r}_k(y)}$, i.e. $f^2 \in \mathcal{F}$, we define random variables $a_k = r_k \in \mathcal{A}$, $b_k = \tilde{r}_k \in \mathcal{B}$, and

$$\eta_f = \sum_{k=1}^{N} a_k b_k \in \mathcal{A} * \mathcal{B}.$$

Main Theorem I

Given a non-crossing partition $\pi \in NC(n)$, we denote

$$m_\pi(\eta_f) := \phi(\eta_f^{|V_1|})\phi(\eta_f^{|V_2|}) \cdots \phi(\eta_f^{|V_r|}),$$

where $\pi = \{V_1, V_2, \cdots, V_r\}$, is a multiplicative function on $NC(n)$.

For example, for $\pi = \{(1,4,8),(2,3),(5),(6,7)\} \in NC(8)$, the moment m_π is defined by

$$m_\pi(\eta_f) = \phi(\eta_f^3)\phi(\eta_f^2)^2\phi(\eta_f).$$

Main Theorem I

Theorem 3.1 (M. Hsiao, H-W. Huang, 2023)

Let $f^2 \in \mathcal{F}$ be a nonnegative SRIF and η_f be defined as above. Then

$$m_{2n}(\mu_f) = \sum_{\pi \in NC(n)} m_\pi(\eta_f),$$

for all $n \geq 0$. In particular, if η_f has the distribution on $\mathbb{R}$, then we have $\mu_f^2 = \eta_f \boxtimes \sigma$, where σ is the Free Poisson distribution with rate 1.

Main Theorem II

Theorem 3.2 (M. Hsiao, H-W. Huang, 2023)

Let $g \in \mathcal{F}$ be a nonnegative Riemann integrable function. Define a SRIF

$$f(x,y) = \sqrt{2g(2x-1,2y)\mathbf{1}_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}(x,y) + 2g(2y-1,2x)\mathbf{1}_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}(x,y)},$$

then for any $n \geq 1$, we have

$$m_{2n}(\mu_f) = \sum_{\pi \in NC(n)} m_\pi(\eta_{\sqrt{g}}),$$

Moreover, if $\eta_{\sqrt{g}}$ has a distribution μ on $\mathbb{R}$, then $\mu_f = \sqrt{\mu \boxtimes \sigma}$.

Remark: Choose $g(x,y) = r(x)$, then we obtain that $\mu_f = \sqrt{r(U) \boxtimes \sigma}$.

Remark

- This generalizes Chakrabarty's results. ($N = 1$)
- In general, η_f may not have the distribution. (η_f is not a self-adjoint element, not even a normal element.)
- Let $\mathcal{H}$ be the set of all μ_f with SRIF f. Then we show that $\mathcal{H}$ is a convex set. Moreover, we show that

$$\left\{\sqrt{\mu \boxtimes \sigma} : \mu \in \mathbf{P}(\mathbb{R}_{\geq 0})\right\} \subsetneq \mathcal{H} \subsetneq \mathbf{P}_{\mathrm{sym}}(\mathbb{R}).$$

Corollary I

Given two SRIF's f_1, f_2, the property of normal distribution leads to

$$Z_{f_1,N} + \tilde{Z}_{f_2,N} \stackrel{\mathrm{d}}{=} \hat{Z}_{\sqrt{f_1^2+f_2^2},N},$$

for all $N \geq 1$. However, in general, as $N \to \infty$, Main Theorem I shows that we might not have $\mu_{f_1} \boxplus \mu_{f_2} = \mu_{\sqrt{f_1^2+f_2^2}}$. But, since $\mathbb{1}$ is freely independent of any random variable., if we choose $f_1 \equiv c$ to be a constant, we get that

$$\mu_{\sqrt{f^2+c^2}} = \mu_f \boxplus w_c,$$

for all SRIF f.

Corollary I

Corollary 3.3 (M. Hsiao, H-W. Huang, 2023)

Let $g_1, g_2 : [0,1] \to [0,\infty)$ be two Riemann integrable functions, and define

$$f_1(x,y) = \sqrt{2g_1(2y)\mathbf{1}_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}(x,y) + 2g_1(2x)\mathbf{1}_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}(x,y)},$$

and

$$f_2(x,y) = \sqrt{2g_2(2x-1)\mathbf{1}_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}(x,y) + 2g_2(2y-1)\mathbf{1}_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}(x,y)}.$$

Then we have $\mu_{f_1} \boxplus \mu_{f_2} = \mu_{\sqrt{f_1^2+f_2^2}}$ and

$$\sqrt{\lambda_1 \boxtimes \sigma} \boxplus \sqrt{\lambda_2 \boxtimes \sigma} = \sqrt{(\lambda_1 \boxplus \lambda_2) \boxtimes \sigma},$$

where λ_i is the distribution of the random variable $g_i(U)$ for $i = 1, 2$.

Corollary II

Therefore, if $\operatorname{supp}\left(\mu^{\boxtimes 2}\right) \subset [\alpha, M]$ holds for some $\alpha, M > 0$, consider $\lambda_1 := \delta_\alpha$ and $\lambda_2 := \mu^{\boxtimes 2} \boxplus \delta_{-\alpha}$, we get that

$$\mu \boxtimes w = \sqrt{\mu^{\boxtimes 2} \boxtimes \sigma} = \sqrt{\lambda_1 \boxtimes \sigma} \boxplus \sqrt{\lambda_2 \boxtimes \sigma} = w_\alpha \boxplus \sqrt{\lambda_2 \boxtimes \sigma}.$$

Moreover, since λ_2 is a distribution on $\mathbb{R}_{\geq 0}$, we obtain $\nu := \sqrt{\lambda_2 \boxtimes \sigma} \in I_{r,+}$, that is, for any positive integer $m \geq 1$, there exists $\nu_m \in \mathbf{P}(\mathbb{R}^+)$ such that $\nu = \nu_m^{\boxplus m}$.

Corollary II

Corollary 3.4 (M. Hsiao, H-W. Huang, 2023)

Let $\mu \in \mathbf{P}(\mathbb{R}_{\geq 0})$ be compactly supported and $\alpha > 0$ be a constant. TFAE.

1. supp $\left(\mu^{\boxtimes 2}\right) \subset [\alpha, \infty)$.
2. There exists a distribution $\nu \in \mathbf{P}_{\text{sym}}(\mathbb{R})$ with compact support such that $\nu \boxplus w_\alpha = \mu \boxtimes w$ and $\nu^2 \in I_{r,+}$.

Remark: Using some Complex Analysis tools, we can remove the bounded support assumption.

Corollary 3.5 (M. Hsiao, H-W. Huang, 2023)

Let $f : [0,1]^2 \to \mathbb{R}_{\geq 0}$ be a SRIF. Suppose that there is a constant $M_f > 0$ such that

$$\int_0^1 f^2(x,y)dy = M_f^2,$$

for all $x \in [0,1]$. Then $\mu_f = w_{M_f}$, the semi-circular distribution with mean zero and variance M_f.

Proof of Corollary III

Sketch the proof of Corollary III.

By approximation lemma, we may assume that $f^2 \in \mathcal{F}$. If the function f is of the form $f^2(x,y) = M^2 + \sum_{k=1}^{N} a_k(x)b_k(y)$ with $\int_0^1 a_k(x)dx = \int_0^1 b_k(y)dy = 0$ for all $1 \le k \le N$, then by Main Theorem 1., we can show that $\mu_f = w_M$. Therefore, for $f^2 \in \mathcal{F}$ satisfies the assumption, we have the identity

$$\begin{aligned} f^2(x,y) =& M_f^2 + f^2(x,y) - \int_0^1 f^2(x,y)dx - \int_0^1 f^2(x,y)dy + \int_{[0,1]^2} f^2(x,y) \\ =& M_f^2 + \sum_{k=1}^{N} \left(a_k(x) - \int_0^1 a_k(x)dx \right) \left(b_k(y) - \int_0^1 b_k(y)dy \right). \end{aligned}$$

Hence, we conclude that $\mu_f = w_{M_f}$, this completes the proof. $\square$

Future works

- If f is only Riemann integrable on $(0,1)^2$?
- Relation between $\mu_{\sqrt{f_1^2+f_2^2}}$ and μ_{f_1}, μ_{f_2}.
- $\boxplus$-infinitely divisible and Hadamard random matrix.
- Asymptotically freeness?
- Is $\mathcal{H}$ weakly closed?
- More applications.

Reference

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Appreciation

Thanks for your listening!