Preservation of Killing Vector Fields under Ricci Flow

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Who am I

- Master Student at National Taiwan University.
- Interesting in Geometric Flows and their topological applications.

Figure: Taiwan

Figure: Taipei 101

Definition of Ricci flow

Definition

Given a smooth manifold and a family of Riemannian metrics $(\mathcal{M}^n, g(t))_{t\in[0,T]}$. $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ is called a solution of Ricci flow if it satisfies

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)), \tag{1}$$

on $\mathcal{M} \times [0, T]$.

R. Hamilton proved the existence and uniqueness of RF for the compact case in 1982. For a complete non-compact manifold, only some partial progress has been explored.

Some known results for existence

- (W.-X. Shi, 1989) Complete and bounded sectional curvature.
- (G. Gieson, P. Topping, 2010) $\mathcal{C}^\infty$-Riemann surface(could be incomplete) exists an instantaneously complete RF.
- (Y. Lai, 2019) When the initial metric is complete and satisfies

$$\begin{cases} \mathrm{Vol}_g(x,1) \geq v_0 > 0 \quad \text{for all } x \in \mathcal{M}^n, \\ \mathrm{Rm}(g) + \mathcal{I} \in \mathcal{C}_{\mathrm{PIC1}} \Rightarrow \mathrm{Ric}(g) \geq -(n-1), \end{cases}$$

$\exists$ a complete RF $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ starting with $(\mathcal{M}^n, g)$ such that

$$\begin{cases} |\mathrm{Rm}|(t) \leq \dfrac{c}{t}, \\ \mathrm{inj}_{g(t)}(x) \geq \sqrt{\dfrac{t}{c}}, \end{cases} \tag{2}$$

on $\mathcal{M}^n \times (0, T]$.

Remark: (2) is invariant under parabolic scaling.

Some known results for uniqueness

Let $(\mathcal{M}^n, g_1(t))_{t\in[0,T]}$ and $(\mathcal{M}^n, g_2(t))_{t\in[0,T]}$ be two complete RFs with same initial metric.

- (B.-L. Chen, X.-P. Zhu, 2006) If $\exists C > 0$ s.t.

$$\sup_{\mathcal{M}^n\times[0,T]} |\mathrm{Rm}(g_1)| + |\mathrm{Rm}(g_2)| \leq C, \tag{3}$$

  then $g_1 \equiv g_2$ on $\mathcal{M} \times [0, T]$.
- (P. Topping, H. Yin, 2023) $n = 2$. ($(\mathcal{M}^2, g(0))$ could be incomplete)
- (M.-C. Lee, J. Ma, 2019 and 2021) Suppose that $\exists c > 0$ s.t.

$$|\mathrm{Rm}(g_1)|(t) + |\mathrm{Rm}(g_2)|(t) \leq \frac{c}{t}. \tag{4}$$

  Then $g_1 \equiv g_2$ holds if either $g_1(t), g_2(t)$ are uniformly equivalent to $g(0)$ or $\mathrm{Rm}(g(0))$ is of polynomial growth.

Killing vector field on Ricci flow

Assume the RF starting from $(\mathcal{M}^n, g(0))$ is unique in a certain class, and denote the unique RF to be $(\mathcal{M}^n, g(t))_{t\in[0,T]}$.

- $F : (\mathcal{M}^n, g(0)) \to (\mathcal{M}^n, g(0))$ is an isometry.
- $(\mathcal{M}^n, F^* g(t))_{t\in[0,T]}$ is a RF with $F^* g(0) = g(0)$.
- Uniqueness $\Rightarrow$ $F^* g(t) = g(t)$ for all $t \in [0, T] \Leftrightarrow$ $F : (\mathcal{M}^n, g(t)) \to (\mathcal{M}^n, g(t))$ is an isometry for all $t \in [0, T]$.
- X is a Killing vector field on $(\mathcal{M}^n, g(0))$.
- Uniqueness $\Rightarrow$ X is a Killing vector field on $(\mathcal{M}^n, g(t))$ for all $t \in [0, T]$.

Question: Can we prove these properties if we don't know the uniqueness?

Example

- We say $(\mathbb{R}^{n+1}, g)$ to be a rotationally symmetric metric if there is a function $f\colon \mathbb{R}_{\geq 0} \to [0, \infty)$ such that $g = ds^2 + f(s)g_{\text{std}}$, where g_{std} is the standard metric on S^n.
- $X := \dfrac{\partial}{\partial \theta^i}$ is a Killing vector field for $i = 1, \cdots, n$ and $|X|_g^2 = f$.
- Uniqueness $\Rightarrow \exists$ two functions $\psi, f\colon \mathbb{R}_{\geq 0} \times [0, T] \to [0, \infty)$ s.t.

$$g(s, \theta^1, \cdots, \theta^n, t) = \psi(s, t)ds^2 + f(s, t)g_{\text{std}}. \tag{5}$$

We called such a flow a **Rotationally symmetric RF**.

Main result

Theorem (M. Hsiao)

Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete RF and X be a Killing vector field w.r.t. $(\mathcal{M}^n, g(0))$. Suppose that $|X|_{g(0)}$ ***is bounded*** *and*

$$|Rm(g(t))| \leq \frac{c}{t},$$

on $\mathcal{M}^n \times (0, T]$. Then X is also a Killing vector field w.r.t. $(\mathcal{M}^n, g(t))$ for all $t \in [0, T]$. Moreover, $|X|_{g(t)}$ is also bounded for all $t \in [0, T]$.

The idea is to generalize P. Lu and G. Tian's argument in the bounded curvature case to the unbounded curvature case.

Main result

Corollary (M. Hsiao)

Let $g_0 = ds^2 + f(s)g_{std}$ be a rotationally symmetric metric on $\mathbb{R}^n$. Suppose that $f(s)$ is bounded, and the RF $(\mathbb{R}^n, g(t))_{t\in[0,T]}$ evolves from g_0 satisfies $|Rm(g(t))| \leq c/t$. Then $(\mathbb{R}^n, g(t))_{t\in[0,T]}$ is a rotationally symmetric RF.

What happens to the unbounded Killing vector field?

Main result

Theorem (M. Hsiao)

Let $(\mathcal{M}, g(t))_{t\in[0,T]}$ be a complete RF with the following conditions:

$$\begin{cases} |Rm(g(t))| \leq \dfrac{c}{t} & \text{on } \mathcal{M}^n \times (0,T], \\ inj_{g(t)}(x) \geq \sqrt{\dfrac{t}{c}} & \text{for all } (x,t) \in \mathcal{M}^n \times (0,T], \\ Ric(x,0) \geq K & \text{on } \mathcal{M}^n. \end{cases}$$

for some constants $c > 0$ and $K \in \mathbb{R}$. Suppose that X is a Killing vector field w.r.t. $g(0)$ and $|X|_{g(0)}(x) = O(\exp(d_0(x,p)^{2-\varepsilon}))$ as $x \to \infty$ for some $\varepsilon \in (0,2)$. Then X is also a Killing vector field w.r.t. $g(t)$ and of the same growth order for all $t \in [0,T]$.

Sketch of proof - Step 1

Proof of the case of a bounded vector field:
Consider the following PDE

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \mathrm{Ric}_{g(t)}\right) X(x,t) = 0, \\ X(x,0) = X(x). \end{cases} \tag{6}$$

Intuition: Note that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |X|^2_{g(t)} &= -2\mathrm{Ric}_{g(t)}(X,X) + 2\mathrm{Ric}_{g(t)}(X,X) - 2g_t(\nabla X, \nabla X) \\ &= -2|\nabla X|^2_{g(t)} \leq 0, \end{aligned}$$

we get $|X|^2_{g(t)} \leq \max_{\mathcal{M}^n} |X|^2_{g(0)} < \infty$.

Sketch of proof - Step 2

Define a time-dependent $(0, 2)$-tensor

$$h_{ij} := \nabla_i X_j + \nabla_j X_i. \tag{7}$$

Remark: $h(\cdot, t) \equiv 0$ if and only if $X(\cdot, t)$ is a Killing v.f. w.r.t. $g(t)$. The equation of $|h|^2$ becomes

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |h_{ij}|^2 = -2|\nabla_k h_{ij}|^2 - 4R_{ik\ell j}h^{k\ell}h^{ij} \leq C(n)|\mathrm{Rm}||h_{ij}|^2. \tag{8}$$

Note that $h(\cdot, 0) \equiv 0$ by the assumption. Suppose the maximal principle holds, we can conclude $h \equiv 0$ on $\mathcal{M}^n \times [0, T]$.

Sketch of proof - Step 3

Then

$$\begin{aligned} 0 =& \nabla_j(\nabla_j X_i + \nabla_i X_j) \\ =& \Delta X_i + \nabla_i \nabla_j X_j - R_{jikj} X_k \\ =& \Delta X_i - \mathrm{Ric}_{ik} X_k \\ =& \frac{\partial}{\partial t} X_i, \end{aligned}$$

this implies X is independent of t and we conclude that X is a Killing vector field $\forall t \in [0, T]$.

Question: When do we have the maximal principle?

Maximal principle

Theorem (M.-C. Lee and L.-F. Tam, 2020)

Let $(\mathcal{M}^n, g(t))_{t\in[0,T]}$ be a complete RF. Let $L, f \in \mathcal{C}^0(\mathcal{M}^n \times [0,T])$ be two continuous functions such that $L(x,t), f(x,t) \le c/t$ for some $c > 0$ and

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f\Big|_{(x_0,t_0)} \le L(x_0,t_0) f(x_0,t_0), \tag{9}$$

whenever $f(x_0,t_0) > 0$ in the sense of barrier. Suppose that $f(x,0) \le 0$ for all $x \in \mathcal{M}^n$, then $f \le 0$ on $\mathcal{M}^n \times [0,T]$.

Take $L := C(n)|\mathrm{Rm}|$ and $f := |h_{ij}|^2$. (Need to prove $|h|^2 \le c/t$)

Appreciation

Thanks for your listening!