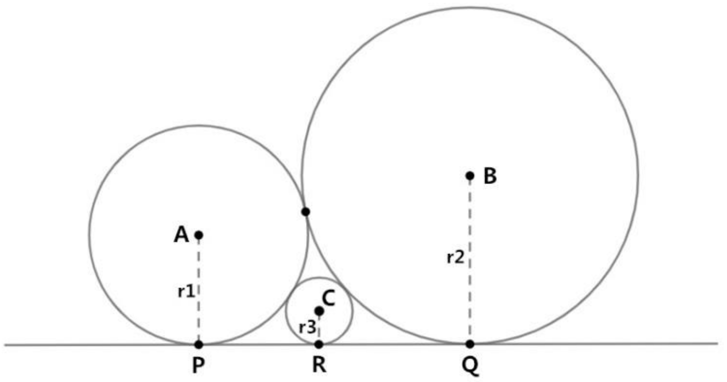
Sangaku Circle Geometry

Background:

- 1630-1850, Japan was isolated. Culture flourished, including Mathematics.
- Geometrical problems were painted on tablets, which were hung in temples
- For all social classes, not limited to mathematicians

Preliminaries:

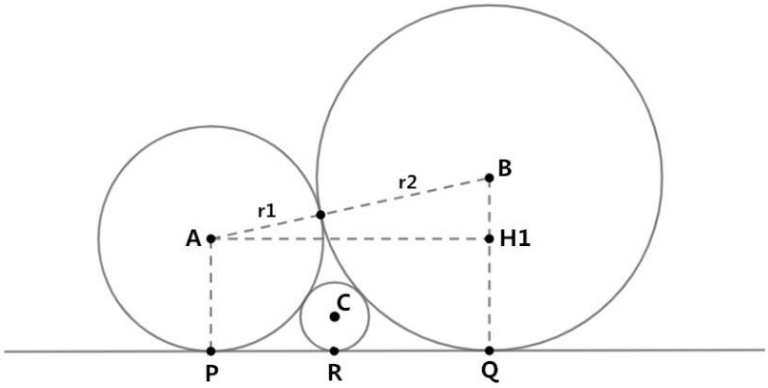
(1) The radius from the centre of the circle to the point of tangency is perpendicular to the tangent.
(2) The line connecting the centres of two tangent circles passes through the point of tangency.
(3) The Pythagorean Theorem



Result 1:

⊙A, ⊙B, and ⊙C have radii r_1, r_2, and r_3 respectively. They are tangent to each other and are all tangent to line PQ. Then r_1, r_2, and r_3 have the following relationship:

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$



Proof:

Construct AH_1 such that $AH_1 \perp BQ$ at H1.
Since AB connects the centres of two tangent circles, AB passes through the point of tangency.
$\mathrm{AB} = r_1 + r_2$

Since ⊙A and ⊙B are tangent to PQ at P and Q respectively, then $AP \perp PQ$, and $BQ \perp PQ$.
From $AH1 \perp BQ$, $AP \perp PQ$, and $BQ \perp PQ$, we have that $PQ = AH_1$, and $QH_1 = AP$.
Thus, in $\triangle ABH_1$,

$$\mathrm{PQ} = \mathrm{AH1} = \sqrt{\mathrm{AB}^2 - {\mathrm{BH}_1}^2} = \sqrt{\mathrm{AB}^2 - (\mathrm{BQ} - \mathrm{QH}_1)^2} = \sqrt{\mathrm{AB}^2 - (\mathrm{BQ} - \mathrm{AP})^2}$$

$$= \sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2}$$

$$= \sqrt{r_1 r_2}$$

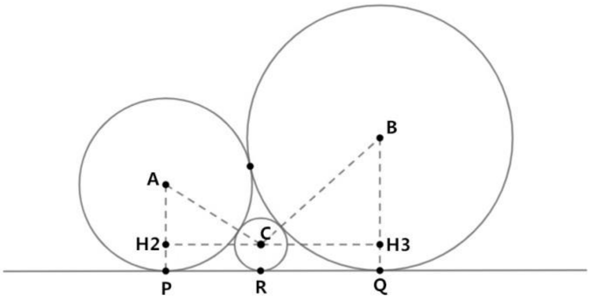
Similarly, we construct right triangles $\triangle ACH_2$ and $\triangle BCH_3$. We get

$PR = \sqrt{r_1 r_3}$, and $QR = \sqrt{r_2 r_3}$

Since $PQ = QR + PQ$,

$$\sqrt{r_1 r_2} = \sqrt{r_2 r_3} + \sqrt{r_1 r_3}$$

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$ as required.

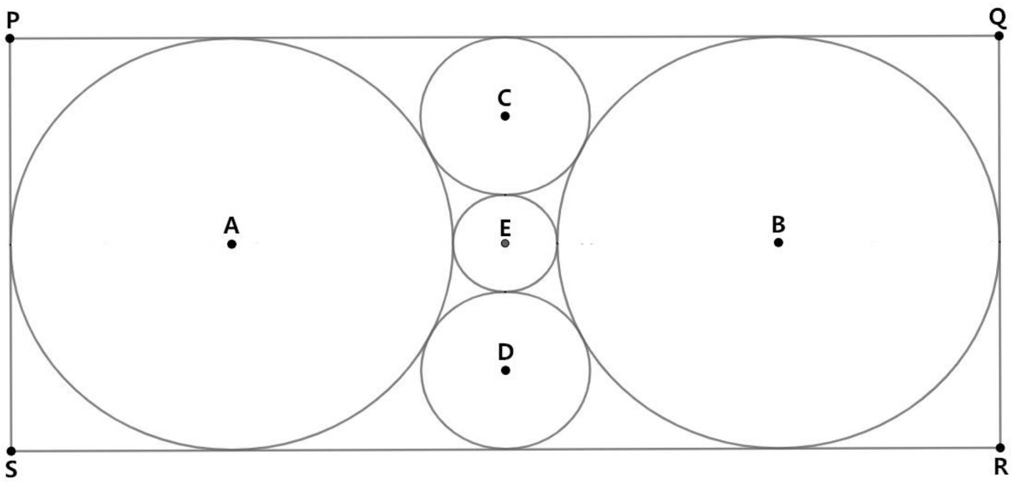


□

Result 2:

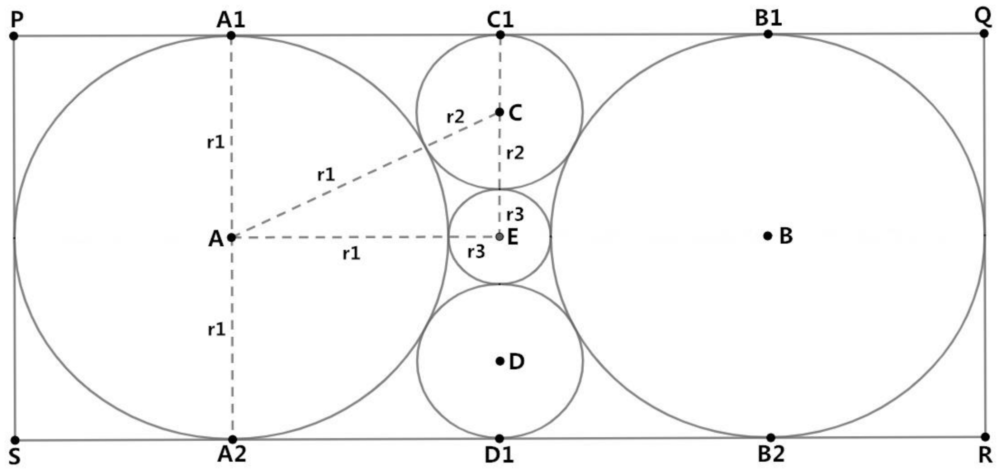
In the diagram, ⊙A, ⊙B, ⊙C, and ⊙D are all tangent to the sides of the rectangle PQRS. ⊙A, ⊙B, ⊙C, and ⊙D are tangent to each other, and are all tangent to ⊙E. If the length of the rectangle PQRS is a and the width is b, then:

$$a = \sqrt{5}b$$



Proof:

Let r_1, r_2, and r_3 be the radii of ⊙A, ⊙C, and ⊙E respectively.
⊙A, ⊙C, and ⊙B is tangent to PQ at A_1, C_1, and B_1 respectively.
⊙A, ⊙D, and ⊙B is tangent to RS at A_2, D_1, and B_2 respectively.



Then, $AA_1 \perp PQ$, and $AA_2 \perp RS$.

Since $\angle P = \angle S = \angle PA_1A = \angle SA_2A_1 = 90°$,

points A_1, A, and A_2 are collinear,

PA_1A_2S is a rectangle, so $A_1A_2 = PS = a = 2r_1$. ⊙A has radius $r_1 = \frac{1}{2}a$.

Similarly, $B_1B_2 = QR = a = r_1$. ⊙B also has radius $r_1 = \frac{1}{2}a$.

Consider ⊙A, ⊙C and their common tangent PQ.

By the work in Result 1, $A_1C_1 = 2\sqrt{r_1r_2}$.

Smilarly,

$B_1C_1 = 2\sqrt{r_1r_2}$

$A_2D_1 = 2\sqrt{r_1 \cdot DD_1}$

$B_2D_1 = 2\sqrt{r_1 \cdot DD_1}$

From $PQ = RS$, we have that

$PA_1 + A_1C_1 + B_1C_1 + QB_1 = SA_2 + A_2D_1 + B_2D_1 + RB_2$

$r_1 + 2\sqrt{r_1r_2} + 2\sqrt{r_1r_2} + r_1 = r_1 + 2\sqrt{r_1 \cdot DD_1} + 2\sqrt{r_1 \cdot DD_1} + r_1$

$DD_1 = r_2$

Hence, ⊙C and ⊙D have the same radius r_2.

Since ⊙A and ⊙B are both tangent to ⊙C and ⊙D,

$AC = BC = AD = BD = r_1 + r_2$

Since ⊙E is tangent to ⊙A, ⊙B, ⊙C, and ⊙D,

$CE = DE = r_2 + r_3$

$AE = BE = r_1 + r_3$

Thus, $\triangle AEC \cong \triangle BEC \cong \triangle AED \cong \triangle BED$

$\angle AEC = \angle BEC = \angle AED = \angle BED$

Since $\angle AEC + \angle BEC + \angle AED + \angle BED = 360°$, we have that $\angle AEC = 90°$

We know that AA_1C_1E is also a rectangle, since the four interior angles are all right angles.

Thus, $A_1C_1 = AE$

$$2\sqrt{r_1r_2} = r_1 + r_3$$

$$r_3 = 2\sqrt{r_1r_2} - r_1$$

Also, $AA_1 = C_1E$

$$r_2 = 2r_1 + r_3$$

$$r_3 = r_2 - 2r_1$$

We now have two ways to write r_3 :

$$\begin{cases} r_3 = 2\sqrt{r_1r_2} - r_1 \\ r_3 = r_1 - 2r_2 \end{cases}$$

Solving the equations, we get that $r_2 = \frac{3-\sqrt{5}}{2}r_1$, $(r_2 < r_1)$.

Therefore, $\frac{a}{b} = \frac{PQ}{PS} = \frac{r_1+2\sqrt{r_1r_2}+2\sqrt{r_1r_2}+r_1}{2r_1} = \frac{2\sqrt{r_1r_2}+r_1}{r_1} = \frac{2\cdot\sqrt{\frac{3-\sqrt{5}}{2}r_1{}^2}+r_1}{r_1} = \frac{(\sqrt{5}-1)r_1+r_1}{r_1} = \sqrt{5}$

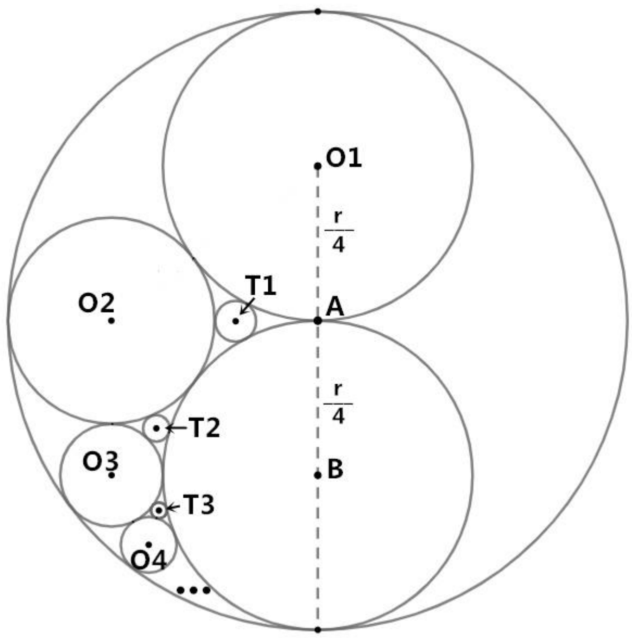
a $= \sqrt{5}b$ as required.

□

Result 3:

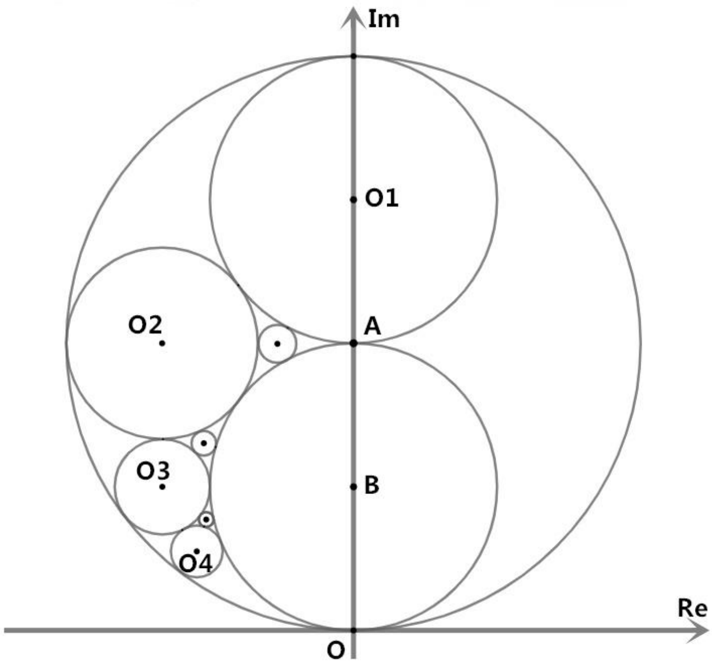
In the diagram, the biggest circle ⊙A has radius r. ⊙A is internally tangent to an infinite number of circles ⊙B, $⊙O_1$, $⊙O_2$, $⊙O_3$, $⊙O_4$ and so on, which are also tangent to each other as shown in the diagram. The tiny in-laid circles $⊙T_1$, $⊙T_2$, $⊙T_3$, and so on, have radii t_1, t_2, t_3, and so on respectively. Each of them is tangent to exactly three circles. The following equation holds:

$$n = \frac{1}{2}\left(\sqrt{\frac{r}{t_n} - 14} + 1\right)$$

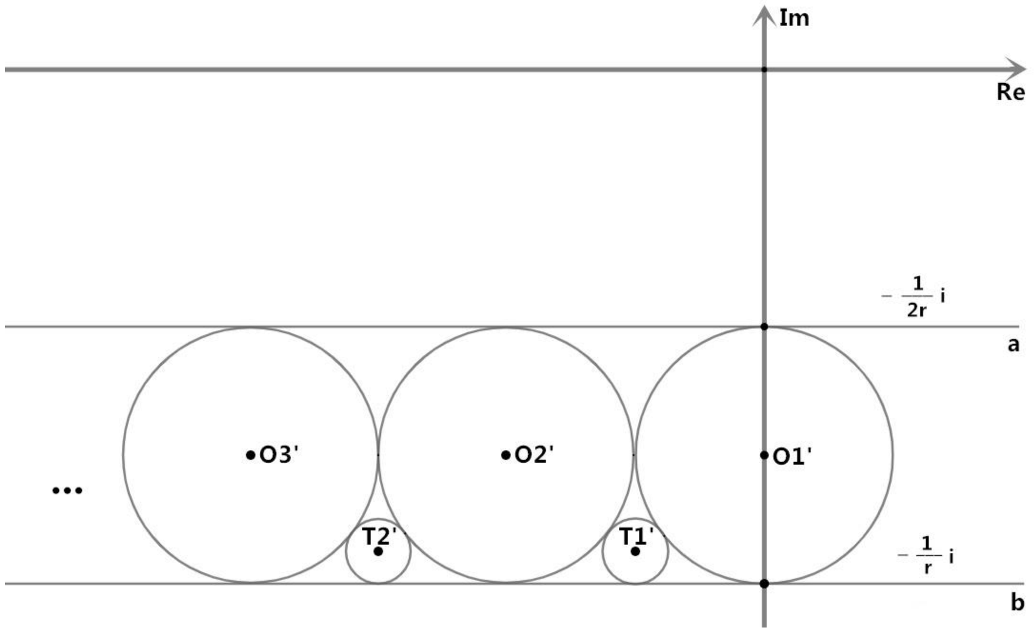


Proof:

Set up the complex plane with the real-axis tangent to $\odot$A, and the imaginary-axis passing through points B, A and O_1, as shown in the diagram.



Map all the circles by the inversion function $f(z) = \frac{1}{z}$.



$\odot$A and $\odot$B are mapped to two parallel lines, a: $w = -\frac{1}{2r}i$, and b: $w = -\frac{1}{r}i$ respectively.

The circles $\odot O_1$, $\odot O_2$, $\odot O_3$, $\odot O_4$, and so on, are mapped to $\odot O_1$', $\odot O_2$', $\odot O_3$', $\odot O_4$', and so on, with radii r_1', r_2', r_3', r_4', and so on respectively. Notice that $\odot O_n$' is tangent to $\odot O_{n-1}$' and $\odot O_{n+1}$' for all $i \geq 2$.

The circles $\odot T_1$, $\odot T_2$, $\odot T_3$, $\odot T_4$, and so on, are mapped to $\odot T_1'$, $\odot T_2'$, $\odot T_3'$, $\odot T_4'$, and so on, with radii t_1', t_2', t_3', t_4', and so respectively. Notice that $\odot T_n'$ is tangent to $\odot O_n'$, $\odot O_{n+1}'$, and b, for all $i \geq 1$.

$\odot O_n'$ has radius $r_n = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{2r}\right) = \frac{1}{4r}$, $n \geq 1$.

By Result 1, $\frac{1}{\sqrt{t_n'}} = \frac{1}{\sqrt{r_n}} + \frac{1}{\sqrt{r_{n+1}}}$

$$\frac{1}{\sqrt{t_n'}} = \frac{1}{\sqrt{\frac{1}{4r}}} + \frac{1}{\sqrt{\frac{1}{4r}}}$$

$\odot T_n'$ has radius $t_n' = \frac{1}{16r}$, $n \geq 1$.

The centre of $\odot T_n'$ is at $c_n' = \frac{1}{4r}(1-2n) - \frac{15}{16r}i$, $n \geq 1$.

Hence, the defining equation of $\odot T_n'$ is $|w - c_n'|^2 = {t_n'}^2$

$$(w - c_n')\overline{w - c_n'} = {t_n'}^2$$

$$w\bar{w} - c_n'\bar{w} - \overline{c_n'}w + \left(|c_n'|^2 - {t_n'}^2\right) = 0$$

Let $= \frac{1}{z}$, so $\bar{w} = \frac{1}{\bar{z}}$.

$$\frac{1}{z\bar{z}} - \frac{c_n'}{\bar{z}} - \frac{\overline{c_n'}}{z} + \left(|c_n'|^2 - {t_n'}^2\right) = 0$$

Multiplying by both sides by $z\bar{z}$,

$$1 - c_n'z - \overline{c_n'z} + \left(|c_n'|^2 - {t_n'}^2\right)z\bar{z} = 0$$

Since $|c_n'|^2 = \left(\frac{1}{4r}\right)^2(1-2n)^2 + \left(\frac{15}{16r}\right)^2 > \left(\frac{1}{16r}\right)^2 = {t_n'}^2$, then $|c_n'|^2 - {t_n'}^2 > 0$.

Dividing both sides by $|c_n'|^2 - {t_n'}^2$,

$$z\bar{z} - \frac{\overline{c_n'}}{|c_n'|^2 - {t_n'}^2}\bar{z} - \frac{c_n'}{|c_n'|^2 - {t_n'}^2}z + \frac{1}{|c_n'|^2 - {t_n'}^2} = 0$$

Rewriting $\frac{1}{|c_n'|^2 - {t_n'}^2}$ as $\frac{|c_n'|^2}{\left(|c_n'|^2 - {t_n'}^2\right)^2} - \frac{{t_n'}^2}{\left(|c_n'|^2 - {t_n'}^2\right)^2}$,

$$z\bar{z} - \frac{\overline{c_n'}}{|c_n'|^2 - {t_n'}^2}\bar{z} - \frac{c_n'}{|c_n'|^2 - {t_n'}^2}z + \frac{|c_n'|^2}{\left(|c_n'|^2 - {t_n'}^2\right)^2} - \frac{{t_n'}^2}{\left(|c_n'|^2 - {t_n'}^2\right)^2} = 0$$

$$\left|z - \frac{t_n'}{|c_n'|^2 - {t_n'}^2}\right|^2 = \left(\frac{t_n'}{|c_n'|^2 - {t_n'}^2}\right)^2$$

Therefore, the radius of $\odot T_n$ is $\frac{t_n'}{|c_n'|^2 - t_n'^2} = \frac{\frac{1}{16r}}{\left(\frac{1}{4r}\right)^2 (1-2n)^2 + \left(\frac{15}{16r}\right)^2 - \left(\frac{1}{16r}\right)^2}$, $n \geq 1$.

Multiplying top and bottom by $16r^2$,

$$t_n = \frac{r}{(1-2n)^2 + \frac{225}{16} - \frac{1}{16}}$$

$$t_n = \frac{r}{(1-2n)^2 + 14}$$

$$(1-2n)^2 + 14 = \frac{r}{t_n}$$

$$(1-2n)^2 = \frac{r}{t_n} - 14$$

$$1 - 2n = -\sqrt{\frac{r}{t_n} - 14} \text{ (since } n \in \mathbb{N},\ 1 - 2n < 0)$$

$$n = \frac{1}{2}\left(\sqrt{\frac{r}{t_n} - 14} + 1\right) \text{ as required.}$$

□

Reference:

https://www.youtube.com/watch?v=XncBGCTgeTk