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**Algorithm 1** Monte Carlo Estimation with Adaptive Polar Grid Search (Table 4)

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**Require:** Number of computational cores (default = 4)

**Ensure:** Estimated bounds for  $\beta$  and corresponding polar coordinates  $\theta$

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1: Initialize: Set random seed and global RNG stream
2: if number of cores not provided then
3:   Set default:  $core \leftarrow 4$ 
4: end if
5: Initialize parallel pool with  $core$  workers
6: Load data:  $\{X_B, y_B, Ey_A, Ey_B\}$  and extract draw  $b = 1$ 
7: Append interaction terms to  $X$  to increase dimension  $D \rightarrow D + 1$ 
8: for each product  $j = 1$  to  $J$  do
9:   for each pair  $t < s$  do
10:    Compute differences  $\Delta y, \Delta Ey_A, \Delta Ey_B$ 
11:    Store reshaped covariates  $X_t, X_s$  and difference  $\Delta X$ 
12:   end for
13: end for
14: for each product  $j$  do
15:   Construct basis using squared terms and 2nd-order interactions
16:   Fit LASSO:  $\hat{E}[\Delta y | X]$  via 10-fold CV
17: end for
18: Store symmetric data: concatenate  $(\Delta X, -\Delta X)$  and  $(\widehat{\Delta y}, -\widehat{\Delta y})$ 
19: Save processed data as dXdE*.mat
20: Set initial polar search region:
21:  $\theta \in [-\pi/2, \pi/2] \times [-\pi, \pi]$ 
22: Initialize tolerance, grid size  $M$ , buffers
23: while Loop 1: Global Search do
24:   Discretize  $\theta$  space into  $M \times M$  grid
25:   Map each  $\theta$  to  $\beta$ 
26:   Parallel: Compute  $Q(\beta)$  for each grid point
27:   Identify 20%-quantile region of lowest  $Q$  values
28:   Update  $\theta$  search bounds accordingly
29:   if  $Q_{\min} = 0$  or tolerance criterion met then
30:     Exit Loop 1
31:   else if range boundary too tight then
32:     Switch range from  $[-\pi, \pi]$  to  $[0, 2\pi]$ 
33:   end if
34: end while
35: while Loop 2: Boundary Expansion do
36:   Expand  $\theta$  bounds with buffer zone
37:   Re-discretize and evaluate  $Q(\beta)$  on finer grid
38:   if No improvement in  $Q_{\min}$  then
39:     Proceed to final refinement loop
40:   end if
41: end while
42: while Loop 3: Final Refinement do
43:   Further narrow  $\theta$  search region
44:   Exclude interior region around current  $Q_{\min}$ 
45:   Compute  $Q(\beta)$  on adaptive fine grid
46:   if Isolated minimum found and grid step  $<$  tolerance then
47:     Stop search
48:   else if Not isolated then
49:     Increase buffer or refine further
50:   end if
51: end while
52: Post-processing:
53: Compute set  $\{\theta : Q(\theta) \leq Q_{\min} + n \log n\}$  and map to  $\beta$ 
54: Calculate min/mean/max of  $\beta$  estimates from this set
55: Save results and log
56: Close parallel pool and cleanup
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