Algorithm 1 Monte Carlo Estimation with Polar Grid Search (`Table_4_nlogn`)
___
Require: X_B, y_B, EyA_B, Ey_B data tensors, number of cores c, tolerance `nlogn`, random seed
Ensure: Estimated parameters $\hat{\beta}$ and bounds
 1: **Initialize:** Set global RNG stream and open parallel pool with c cores
 2: Load precomputed simulation data for $b = 1$ draw
 3: **Augment Data:** Append interaction term to X to increase dimension $D \rightarrow D + 1$
 4: **for** each $j \in \{1, \ldots, J\}$ **do**
 5:     **for** each (t, s) with $t < s \leq T$ **do**
 6:         Compute differences: $\Delta y = y_{jt} - y_{js}$, and $\Delta X = X_{jt} - X_{js}$
 7:         Store reshaped covariates into flattened matrices
 8:     **end for**
 9: **end for**
10: **Estimate** $E[\Delta y \mid \Delta X]$ using LASSO for each product j:
11: **for** $j = 1$ to J **do**
12:     Construct basis using squared terms and second-order interactions
13:     Fit LASSO regression with 10-fold CV to get predicted $\widehat{\Delta y}$
14: **end for**
15: Save ΔX, $\widehat{\Delta y}$ and their symmetric counterparts $(-\Delta X, -\widehat{\Delta y})$
16: **Initialize** Polar Grid Search: $\Theta = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$, step size τ
17: **while** not converged (Loop 1) **do**
18:     Discretize Θ into $M \times M$ grid
19:     **for all** $\theta = (\theta_1, \theta_2)$ in grid **in parallel do**
20:         Convert to β: $\beta = [\cos\theta_1 \cos\theta_2, \cos\theta_1 \sin\theta_2, \sin\theta_1]$
21:         Compute objective $Q(\beta) = \|\Delta\hat{y} - \Delta X\beta\|^2$
22:     **end for**
23:     Identify subregion with Q below 20%-quantile, refine Θ
24:     **if** minimum $Q = 0$ or iteration limit reached **then**
25:         Break
26:     **end if**
27: **end while**
28: **while** refinement condition (Loop 2) **do**
29:     Expand boundary region with buffer
30:     Evaluate $Q(\beta)$ on finer adaptive grid
31:     **if** minimum Q doesn't decrease **then**
32:         Break and proceed to final refinement
33:     **end if**
34: **end while**
35: **while** tolerance not met (Loop 3) **do**
36:     Narrow search region excluding previous minima
37:     Re-evaluate $Q(\beta)$ on even finer grid
38:     **if** region of θ minimizing Q is isolated and tolerance is small **then**
39:         Conclude search
40:     **else**
41:         Shrink grid or expand buffer
42:     **end if**
43: **end while**
44: **Recover** final parameter bounds:
45: Compute $\hat{\beta} = f(\theta)$ for all θ with $Q \leq Q_{\min} + $ `nlogn`
46: Save: mean and bounds of $\hat{\beta}$ and θ
47: **Output:** Save $\hat{\beta}_{\min}, \hat{\beta}_{\text{mean}}, \hat{\beta}_{\max}$, as well as θ bounds and full θ-Q surface