4.4 Solve the Boundary Value Problem for 2nd order ODEs

We consider the solution of the following ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x), q(x), r(x)$ are functions of only x.

For example, to solve:

$$xy'' + \sin(x)\, y' + y = x$$

We convert this equation to:

$$y'' + \frac{\sin(x)}{x} y' + \frac{1}{x} y = 1$$

In this case, $p(x) = \frac{\sin(x)}{x}, q(x) = \frac{1}{x}, r(x) = 1$

Let's say, x is in the range of [a, b]

The boundary condition is given by:

$$y(a) = A, y(b) = B$$

First of all, we divide the interval [a, b] into n equal size sub-intervals. We denote the length of each sub-interval as h, then:

$$h = \frac{b-a}{n}$$

The points of $x_0, x_1, x_2, \dots, x_n$

At each points, we what to know the y values of $y_0, y_1, y_2, \dots, y_{n-1}, y_n$

We have:

$$x_1 = x_0 + h$$

Similarly,

$$x_n = x_0 + nh$$

Let's work on this equation:

$$y'' + p(x)y' + q(x)y = r(x)$$

Here, the concept of finite difference comes in:

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

This is called forward difference approximation.

You can also write:

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

This is called backward difference approximation.

Either method is fine, just pick one and be consistent.

$$y''(x_i) = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$$

Prove the above eq.

$$y''(x_i) = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = \frac{1}{h^2}[(y_{i+1} - y_i) - (y_i - y_{i-1})] = \frac{1}{h}\left[\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h}\right]$$
$$= \frac{1}{h}(y_i' - y_{i-1}') = \frac{\Delta y'}{\Delta x} = \frac{dy'}{dx} = y''$$

Let give it a summary:

$$y'' + p(x)y' + q(x)y = r(x), (1)$$

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{h}, (2)$$

$$y''(x_i) = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}], (3)$$

Plug $x = x_i$ into eq. (1), we get:

$$y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = r(x_i), (4)$$

Plug Eq. (2-3) into Eq. (4), we get:

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + p(x_i)\frac{y_{i+1} - y_i}{h} + q(x_i)y_i = r(x_i), (5)$$

Multiply both parts by h^2:

$$[y_{i+1} - 2y_i + y_{i-1}] + hp(x_i)(y_{i+1} - y_i) + h^2q(x_i)y_i = h^2r(x_i), (6)$$

Eventually, after simplification, you will get:

$$a_iy_{i-1} + b_iy_i + c_iy_{i+1} = d_i, (7)$$

$$a_i = 1$$

$$b_i = -2 - hp(x_i) + h^2q(x_i)$$

$$c_i = 1 + hp(x_i)$$

$$d_i = h^2r(x_i)$$

Then, we can relate y_i with y_{i+1} and y_{i-1} using Eq. (7) by:

$$y_i = \frac{d_i - a_iy_{i-1} - c_iy_{i+1}}{b_i}$$

$$y_i = \frac{d_i - a_i y_{i-1} - c_i y_{i+1}}{b_i}, \qquad (8)$$

$$a_i = 1$$

$$b_i = -2 - hp(x_i) + h^2 q(x_i)$$

$$c_i = 1 + hp(x_i)$$

$$d_i = h^2 r(x_i)$$

You first give an initial guess of $y_1, \dots, y_{n-1}$, then you can plug this initial guess into Eq. (8), then you get updated values of $y_1, \dots, y_{n-1}$.

$$x - y = 1$$

$$x + y = 2$$

Reorganize:

$$x = y + 1$$

$$y = 2 - x$$

(0,0)->(1,2)->(3,1)->....(1.5,0.5)->(1.5,0.5)

$$y_1 = \frac{d_i - a_i y_0 - c_i y_2}{b_i}, \qquad (8)$$

$$y_2 = \frac{d_i - a_i y_1 - c_i y_3}{b_i}, \qquad (8)$$

…

$$y_{n-1} = \frac{d_i - a_i y_{n-2} - c_i y_n}{b_i}, \qquad (8)$$

$$y_i^{(k+1)} = \frac{d_i - a_i y_{i-1}^{(k)} - c_i y_{i+1}^{(k)}}{b_i}, \qquad (8)$$

This is the Jacobi method.

$$y_i^{(k+1)} = \frac{d_i - a_i y_{i-1}^{(k+1)} - c_i y_{i+1}^{(k)}}{b_i}, \qquad (8)$$

This is the Gauss-Seidel method.

$$a_i = 1$$

$$b_i = -2 - hp(x_i) + h^2 q(x_i)$$

$$c_i = 1 + hp(x_i)$$

$$d_i = h^2 r(x_i)$$

i is the index for the nodal points, i ranges from 0 to n. k is the index for the iteration, and k ranges from 0 to any value that you want until you get the precise solution.

Let say, we want to find a solution of

$$y'' = x$$

$$y(0) = 0, y(1) = 1$$

Analytical solution:

$$y = \frac{1}{6}x^3 + \frac{5}{6}x$$

y(0.9)=0.8715

For x ranging from 0 to 1.

$$y'' + p(x)y' + q(x)y = r(x)$$