Second order ODE

The ODE has the form of:

$$y'' = S(x, y, y'), (1)$$

e.g.,

$$y'' = x, (2)$$

What is $y(x)$?

$$y' = \frac{1}{2}x^2 + C, y = \frac{1}{6}x^3 + Cx + D$$

$$y(0) = 0, y'(0) = 0, C = D = 0, y(x) = \frac{1}{6}x^3$$

$$y'' = x + y + y', y'' = \sin(xy) + y'$$

$$y'' = S(x, y, y')$$

$$y(x_0) = y_0, y'(x_0) = y_0'$$

Initial value problem (IVP), when both conditions are given at the same x-value.

$$y(a) = y_0, y(b) = y_1$$

Boundary value problem (BVP), when both conditions are given at the different x-value.

For IVP, we use the same methods as we used to solve 1st order ODE.

For BVP, we can use finite difference method to solve the ODE.

$$y'' = S(x, y, y')$$

$$y(x_0) = y_0, y'(x_0) = y'_0$$

If we define another variable $g = y'$, or $y' = g$

$$y'' = (y')' = g'$$

$$S(x, y, y') = S(x, y, g)$$

$y'' = S(x, y, y') \rightarrow g' = S(x, y, g)$ and $y' = g$

If we introduce g, then the 2[nd] order ODE can be transformed into 2 1[st] order ODEs.

$$\begin{cases} g' = S(x, y, g) \\ y' = g \end{cases}$$

The initial condition is:

$$\begin{cases} y(x_0) = y_0 \\ g(x_0) = g_0 \end{cases}$$

In numerical methods, the solution at x_{i+1} is obtained at the value at x_i, by:

$$\begin{cases} g_{i+1} = g_i + hS_g \\ y_{i+1} = y_i + hS_y \end{cases}$$

For Euler method:

$$y_{i+1} = y_i + hg_i$$

$$g_{i+1} = g_i + hS(x_i, y_i, g_i)$$

For example, if you only do 1 step, we know x_0, y_0, g_0, then we can calculate x_1, y_1, g_1 using the above eq.

$$y_1 = y_0 + hg_0$$

$$g_1 = g_0 + hS(x_0, y_0, g_0)$$

For Euler method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$y_{i+1} = y_i + K_0$$

$$g_{i+1} = g_i + L_0$$

If we introduce g, then the 2nd order ODE can be transformed into 2 1st order ODEs.

$$\begin{cases} g' = S(x, y, g) \\ y' = g \end{cases}$$

The initial condition is:

$$\begin{cases} y(x_0) = y_0 \\ g(x_0) = g_0 \end{cases}$$

In numerical methods, the solution at x_{i+1} is obtained at the value at x_i, by:

$$\begin{cases} g_{i+1} = g_i + hS_g \\ y_{i+1} = y_i + hS_y \end{cases}$$

For Euler method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$y_{i+1} = y_i + K_0$$

$$g_{i+1} = g_i + L_0$$

For Backward Euler method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$K_1 = h(g_i + L_0)$$

$$L_1 = hS(x_i + h, y_i + K_0, g_i + L_0)$$

$$y_{i+1} = y_i + K_1$$

$$g_{i+1} = g_i + L_1$$

For Heun's method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$K_1 = h(g_i + L_0)$$

$$L_1 = hS(x_i + h, y_i + K_0, g_i + L_0)$$

$$y_{i+1} = y_i + \frac{1}{2}(K_0 + K_1)$$

$$g_{i+1} = g_i + \frac{1}{2}(L_0 + L_1)$$

For Midpoint method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$K_1 = h(g_i + \frac{1}{2}L_0)$$

$$L_1 = hS(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_0, g_i + \frac{1}{2}L_0)$$

$$y_{i+1} = y_i + K_1$$

$$g_{i+1} = g_i + L_1$$

For RK4 method:

$$K_0 = hg_i$$

$$L_0 = hS(x_i, y_i, g_i)$$

$$K_1 = h(g_i + \frac{1}{2}L_0)$$

$$L_1 = hS(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_0, g_i + \frac{1}{2}L_0)$$

$$K_2 = h(g_i + \frac{1}{2}L_1)$$

$$L_2 = hS(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1, g_i + \frac{1}{2}L_1)$$

$$K_3 = h(g_i + L_2)$$

$$L_3 = hS(x_i + h, y_i + K_2, g_i + L_2)$$

$$y_{i+1} = y_i + \frac{1}{6}(K_0 + 2K_1 + 2K_2 + K_3)$$

$$g_{i+1} = g_i + \frac{1}{6}(L_0 + 2L_1 + 2L_2 + L_3)$$