Fourier series

If $f(x)$ is a function defined on $[-\pi,\pi]$, then $f(x)$ can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)], (1)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx, (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx, (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx, (4)$$

Let's prove this.

Integral both parts on $[-\pi,\pi]$ for Eq. (1)

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\, dx$$

$$\int_{-\pi}^{\pi}\cos(nx)\, dx = \frac{1}{n}\sin(nx)\ \Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi}\sin(nx)\, dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

Let's calculate a_1

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)], (1)$$

Multiply both parts of Eq. 1 by $\cos(x)$

$$f(x)\cos(x) = a_0\cos(x) + (\sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)])\cos(x), (1)$$

Integral both parts on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$
$$= \int_{-\pi}^{\pi} a_0\cos(x)\,dx + \int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx + \cdots$$
$$+ \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} a_0\cos(x)\,dx = a_0\int_{-\pi}^{\pi}\cos(x)\,dx = 0$$

$$\int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx = a_1\int_{-\pi}^{\pi}\cos^2(x)\,dx = a_1\int_{-\pi}^{\pi}(1-\sin^2(\mathrm{x}))\,dx$$
$$= a_1\int_{-\pi}^{\pi} 1\,dx - a_1\int_{-\pi}^{\pi}\sin^2(x)\,dx$$

$$\int_{-\pi}^{\pi}\cos^2(x)\,dx = 2\pi - \int_{-\pi}^{\pi}\sin^2(x)\,dx$$

$$\int_{-\pi}^{\pi}\sin^2(x)\,dx + \int_{-\pi}^{\pi}\cos^2(x)\,dx = 2\pi$$

$$\int_{-\pi}^{\pi}[\sin^2(x) + \cos^2(x)]\,dx = 2\pi$$

$$\int_{-\pi}^{\pi}\cos^2(x)\,dx = \int_{-\pi}^{\pi}\cos(\mathrm{x})\,d\sin(x) = \sin(x)*\cos(x)\,\Big| - \int_{-\pi}^{\pi}\sin(\mathrm{x})\,d\cos(x) = \int_{-\pi}^{\pi}\sin(\mathrm{x})^2\,dx$$
$$= \int_{-\pi}^{\pi}(1-\cos^2(x)\,dx$$

$$\int_{-\pi}^{\pi}\cos^2(x)\,dx = \int_{-\pi}^{\pi}(1-\cos^2(x))\,dx$$

$$\int_{-\pi}^{\pi}\cos^2(x)\,dx = \pi$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$

$$= 0 + \int_{-\pi}^{\pi} a_1 \cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} a_2 \cos(2x)\cos(x)\,dx + \cdots$$

$$+ \int_{-\pi}^{\pi} b_1 \sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx + \cdots$$