

4.4 Solve the Boundary Value Problem for 2nd order ODEs

We consider the solution of the following ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x), q(x), r(x)$ are functions of only x .

For example, to solve:

$$xy'' + \sin(x)y' + y = x$$

We convert this equation to:

$$y'' + \frac{\sin(x)}{x}y' + \frac{1}{x}y = 1$$

In this case, $p(x) = \frac{\sin(x)}{x}, q(x) = \frac{1}{x}, r(x) = 1$

Let's say, x is in the range of $[a, b]$

The boundary condition is given by:

$$y(a) = A, y(b) = B$$

First of all, we divide the interval $[a, b]$ into n equal size sub-intervals. We denote the length of each sub-interval as h , then:

$$h = \frac{b - a}{n}$$

The points of $x_0, x_1, x_2, \dots, x_n$

At each points, we what to know the y values of $y_0, y_1, y_2, \dots, y_{n-1}, y_n$

We have:

$$x_1 = x_0 + h$$

Similarly,

$$x_n = x_0 + nh$$

Let's work on this equation:

$$y'' + p(x)y' + q(x)y = r(x)$$

Here, the concept of finite difference comes in:

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

This is called forward difference approximation.

You can also write:

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

This is called backward difference approximation.

Either method is fine, just pick one and be consistent.

$$y''(x_i) = \frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}]$$

Prove the above eq.

$$\begin{aligned} y''(x_i) &= \frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}] = \frac{1}{h^2} [(y_{i+1} - y_i) - (y_i - y_{i-1})] = \frac{1}{h} \left[\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h} \right] \\ &= \frac{1}{h} (y'_i - y'_{i-1}) = \frac{\Delta y'}{\Delta x} = \frac{dy'}{dx} = y'' \end{aligned}$$

Let give it a summary:

$$y'' + p(x)y' + q(x)y = r(x), (1)$$

$$y'(x_i) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{h}, (2)$$

$$y''(x_i) = \frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}], (3)$$

Plug $x = x_i$ into eq. (1), we get:

$$y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = r(x_i), (4)$$

Plug Eq. (2-3) into Eq. (4), we get:

$$\frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}] + p(x_i) \frac{y_{i+1} - y_i}{h} + q(x_i)y_i = r(x_i), (5)$$

Multiply both parts by h^2 :

$$[y_{i+1} - 2y_i + y_{i-1}] + hp(x_i)(y_{i+1} - y_i) + h^2q(x_i)y_i = h^2r(x_i), (6)$$

Eventually, after simplification, you will get:

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i, (7)$$

$$a_i = 1$$

$$b_i = -2 - hp(x_i) + h^2q(x_i)$$

$$c_i = 1 + hp(x_i)$$

$$d_i = h^2r(x_i)$$

Then, we can relate y_i with y_{i+1} and y_{i-1} using Eq. (7) by:

$$y_i = \frac{d_i - a_i y_{i-1} - c_i y_{i+1}}{b_i}$$

$$y_i = \frac{d_i - a_i y_{i-1} - c_i y_{i+1}}{b_i}, \quad (8)$$

You first give an initial guess of y_1, \dots, y_{n-1} , then you can plug this initial guess into Eq. (8), then you get updated values of y_1, \dots, y_{n-1} .