Fourier series

If $f(x)$ is a function defined on $[-\pi, \pi]$, then $f(x)$ can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, (4)$$

Let's prove this.

Integral both parts on $[-\pi, \pi]$ for Eq. (1)

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\, dx$$

$$\int_{-\pi}^{\pi} \cos(nx)\, dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\, dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Let's calculate a_1

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

Multiply both parts of Eq. 1 by cos (x)

$$f(x)\cos(x) = a_0\cos(x) + (\sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)])\cos(x)\,, (1)$$

Integral both parts on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$
$$= \int_{-\pi}^{\pi} a_0\cos(x)\,dx + \int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx + \cdots$$
$$+ \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} a_0\cos(x)\,dx = a_0\int_{-\pi}^{\pi}\cos(x)\,dx = 0$$

Review of trigonometry:

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx = a_1\int_{-\pi}^{\pi}\cos(x)\cos(x)\,dx$$
$$= \frac{1}{2}a_1\int_{-\pi}^{\pi}[\cos(2x) + \cos(0)]\,dx$$
$$= \frac{1}{2}a_1[\int_{-\pi}^{\pi}\cos(2x)\,dx + \int_{-\pi}^{\pi}1dx] = \frac{1}{2}a_1\left[\frac{1}{2}\int_{-\pi}^{\pi}d\sin(2x) + 2\pi\right] = a_1\pi$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$
$$= 0 + a_1\pi + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx + \cdots + \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx$$
$$+ \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx = a_2\int_{-\pi}^{\pi}\cos(2x)\cos(x)\,dx$$
$$= \frac{1}{2}a_2\int_{-\pi}^{\pi}[\cos(3x) + \cos(x)]\,dx$$
$$= \frac{1}{2}a_2[\int_{-\pi}^{\pi}\cos(3x)\,dx + \int_{-\pi}^{\pi}\cos(x)\,dx] = \frac{1}{2}a_2\left[\frac{1}{3}\int_{-\pi}^{\pi}d\sin(3x) + \int_{-\pi}^{\pi}d\sin(x)\right] = 0$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx = 0 + a_1\pi + 0 + \cdots + \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

Review of trigonometry:

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\int_{-\pi}^{\pi} b_1 \sin(x)\cos(x)\,dx = \frac{1}{2}b_1 \int_{-\pi}^{\pi}[\sin\,(2x) + \sin\,(0)]dx = \frac{1}{2}b_1[-\frac{1}{2}\int_{-\pi}^{\pi} dcos(2x)$$

$$= \frac{1}{2}b_1\left[-\frac{1}{2}\cos(2x)\ \Big|_{-\pi}^{\pi}\right] = 0\,]$$

$$\int_{-\pi}^{\pi} f(x)\cos\,(x)dx = 0 + a_1\pi + 0 + \cdots + 0 + \int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx = b_2 \int_{-\pi}^{\pi} \sin(2x)\cos(x)\,dx = \frac{1}{2}b_2 \int_{-\pi}^{\pi}[\sin\,(3x) + \sin\,(x)]dx$$

$$= \frac{1}{2}b_2[-\frac{1}{3}\cos(3x)\Big|_{-\pi}^{\pi} - \cos\,(x)\Big|_{-\pi}^{\pi}] = 0$$

$$\int_{-\pi}^{\pi} f(x)\cos\,(x)dx = 0 + a_1\pi + 0 + \cdots + 0 + 0 + \cdots 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\,(1\cdot x)dx$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\,(2\cdot x)dx$$

…

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\,(nx)\,dx$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\,(1\cdot x)dx$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\,(2\cdot x)dx$$

…

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\,(nx)\,dx$$

If $f(x)$ is a function defined on $[-\pi, \pi]$, then $f(x)$ can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, (4)$$

If $f(x)$ function is not defined on $[-\pi, \pi]$, but is defined on $[-L, L]$, we can still find the Fourier series by making a change in variables.

Let's define:

$$t = \frac{\pi x}{L}, or\ x = \frac{Lt}{\pi}$$

Then, when x=-L, t=-π, when x=L, t=π. In this case, t ranges from $-\pi$ to π

$$f(x) = f\left(\frac{Lt}{\pi}\right) = g(t)$$

Then, $g(t)$ is a function defined on $[-\pi, \pi]$. We can use our previous formula to express $g(t)$ as:

$$g(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], (5)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt, (6)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt)\, dt, (7)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt)\, dt, (8)$$

Then, substitute $g(t)$ with $f(x)$, t with $\frac{\pi x}{L}$, $dt = d\frac{\pi x}{L} = \frac{\pi}{L}dx$, into Eq. (5-8), we get:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (9)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, (10)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, (11)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, (12)$$

In summary, if $f(x)$ is defined on $[-L, L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (9)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, (10)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, (11)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, (12)$$

If $f(x)$ is defined on $[0,2L]$, what to do?

In practice, we can define:

$$t = \frac{\pi(x-L)}{L}$$

Then, when $x = 0$, $t = -\pi$; when $x = 2L$, $t = \pi$. In this case, t ranges from $-\pi$ to π.

Again,

$$f(x) = f\left(\frac{tL}{\pi} + L\right) = g(t)$$

Then, $g(t)$ is a function defined on $[-\pi, \pi]$. We can use our previous formula to express $g(t)$ as:

$$g(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], (13)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt, (14)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt)\, dt, (15)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt)\, dt, (16)$$

Then, substitute $g(t)$ with $f(x)$, t with $\frac{\pi(x-L)}{L}$, $dt = d\frac{\pi(x-L)}{L} = \frac{\pi}{L} dx$, into Eq. (13-16), we get:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (17)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L} \int_{0}^{2L} f(x)\, dx, (18)$$

$$a_n = \frac{1}{L} \int_{0}^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, (19)$$

$$b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, (20)$$

In summary, if $f(x)$ is defined on $[0, 2L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (21)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\, dx, (22)$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, (23)$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, (24)$$

Let's first do some example, what is the period of:

$$\mathrm{A}\sin(nx)$$

Period is $T = \frac{2\pi}{n}$, amplitude is A.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)]$$

What is the period when n=1?

$$T = \frac{2\pi}{\frac{\pi}{L}} = 2L$$

Then, the period is given by:

$$T_n = \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$$

What is the amplitude? We define the amplitude of $f(x)$ as:

$$A_n = \sqrt{a_n^2 + b_n^2}$$

In summary, the amplitude for period of $\boldsymbol{T_n = \frac{2L}{n}}$ is calculated by:

$$\boldsymbol{A_n = \sqrt{a_n^2 + b_n^2}}$$

The period of $sin(x) + cos(x)$ is 2π

The amplitude of $sin(x) + cos(x)$ is $\sqrt{2} = 1.41414$

Discrete Fourier Transformation

Most of the time, $f(x)$ is unknown, but is given by a list of discrete observational data points. If $f(x)$ is represented by a number of data points:

Number of data points = np

x values: $x_1, x_2, \dots, x_{np}$

f values: $f_1, f_2, \dots, f_{np}$

Increment of x: $\Delta x = x_2 - x_1$

Then, we can use numerical methods to calculate the integration of a_0, a_n, and b_n.

In summary, if $f(x)$ is defined on $[0, 2L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (25)$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{2L}\sum_{i=1}^{i=np-1} f(x_i)\,\Delta x = \frac{\Delta x}{2L}\sum_{i=1}^{i=np-1} f(x_i) = \frac{\Delta x}{2L}\sum_{i=1}^{i=np-1} f_i, (26)$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\sum_{i=1}^{i=np-1} f(x_i)\cos\left(\frac{n\pi x_i}{L}\right)\Delta x = \frac{\Delta x}{L}\sum_{i=1}^{i=np-1} f_i \cos\left(\frac{n\pi x_i}{L}\right), (27)$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\sum_{i=1}^{i=np-1} f(x_i)\sin\left(\frac{n\pi x_i}{L}\right)\Delta x = \frac{\Delta x}{L}\sum_{i=1}^{i=np-1} f_i \sin\left(\frac{n\pi x_i}{L}\right), (28)$$

$$T_n = \frac{2L}{n}$$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

What is the maximum period of T_n?

It is $2L$

What is the minimum period of T_n?

The Nyquist rule says,

$$T_n = 2\Delta x$$

We need at least 2 elements, or $2\Delta x$ to finish one cycle, the minimum period we can get is $2\Delta x$.

$$\frac{2L}{n} = 2\Delta x$$

$$n = \frac{L}{\Delta x}$$

$$\Delta x = \frac{2L}{np - 1}$$

We can get:

$$n = \frac{np - 1}{2}$$

So, n ranges from 1 to $\frac{np-1}{2}$

And, T_n ranges from $2L$ to $2\Delta x$

If np is an even number, then, in practice, we could remove 1 data point from the data. Usually, we remove the last data point. In this case, when we write our code, we need to first test whether np is odd or even number. If np is an even number, then we do $np = np - 1$.

In summary again, if $f(x)$ is defined on $[0, 2L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\frac{np-1}{2}} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (29)$$

$$a_0 = \frac{\Delta x}{2L} \sum_{i=1}^{i=np-1} f_i, (30)$$

$$a_n = \frac{\Delta x}{L} \sum_{i=1}^{i=np-1} f_i \cos\left(\frac{n\pi x_i}{L}\right), (31)$$

$$b_n = \frac{\Delta x}{L} \sum_{i=1}^{i=np-1} f_i \sin\left(\frac{n\pi x_i}{L}\right), (32)$$

$$T_n = \frac{2L}{n}, (33)$$

$$A_n = \sqrt{a_n^2 + b_n^2}, (34)$$

n ranges from 1 to $(np - 1)/2$

np is the number of points of $f(x)$

Δx is the increment of x

$2L$ is the length of x, or the length of the data