Fourier series

If $f(x)$ is a function defined on $[-\pi, \pi]$, then $f(x)$ can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, (4)$$

Let's prove this.

Integral both parts on $[-\pi, \pi]$ for Eq. (1)

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\, dx$$

$$\int_{-\pi}^{\pi} \cos(nx)\, dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\, dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Let's calculate a_1

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

Multiply both parts of Eq. 1 by cos (x)

$$f(x)\cos(x) = a_0\cos(x) + (\sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)])\cos(x),(1)$$

Integral both parts on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$
$$= \int_{-\pi}^{\pi} a_0\cos(x)\,dx + \int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx + \cdots$$
$$+ \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} a_0\cos(x)\,dx = a_0\int_{-\pi}^{\pi}\cos(x)\,dx = 0$$

Review of trigonometry:

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\int_{-\pi}^{\pi} a_1\cos(x)\cos(x)\,dx = a_1\int_{-\pi}^{\pi}\cos(x)\cos(x)\,dx$$
$$= \frac{1}{2}a_1\int_{-\pi}^{\pi}[\cos(2x) + \cos(0)]\,dx$$
$$= \frac{1}{2}a_1[\int_{-\pi}^{\pi}\cos(2x)\,dx + \int_{-\pi}^{\pi}1dx] = \frac{1}{2}a_1\left[\frac{1}{2}\int_{-\pi}^{\pi}dsin(2x) + 2\pi\right] = a_1\pi$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx$$
$$= 0 + a_1\pi + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx + \cdots + \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx$$
$$+ \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} a_2\cos(2x)\cos(x)\,dx = a_2\int_{-\pi}^{\pi}\cos(2x)\cos(x)\,dx$$
$$= \frac{1}{2}a_2\int_{-\pi}^{\pi}[\cos(3x) + \cos(x)]\,dx$$
$$= \frac{1}{2}a_2[\int_{-\pi}^{\pi}\cos(3x)\,dx + \int_{-\pi}^{\pi}\cos(x)\,dx] = \frac{1}{2}a_2\left[\frac{1}{3}\int_{-\pi}^{\pi}dsin(3x) + \int_{-\pi}^{\pi}dsin(x)\right] = 0$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx = 0 + a_1\pi + 0 + \cdots + \int_{-\pi}^{\pi} b_1\sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2\sin(2x)\cos(x)\,dx + \cdots$$

Review of trigonometry:

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\int_{-\pi}^{\pi} b_1 \sin(x)\cos(x)\,dx = \frac{1}{2}b_1 \int_{-\pi}^{\pi}[\sin(2x) + \sin(0)]dx = \frac{1}{2}b_1[-\frac{1}{2}\int_{-\pi}^{\pi} dcos(2x)$$
$$= \frac{1}{2}b_1\left[-\frac{1}{2}\cos(2x)\ \Big|_{-\pi}^{\pi}\right] = 0\ ]$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx = 0 + a_1\pi + 0 + \cdots + 0 + \int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx = b_2\int_{-\pi}^{\pi}\sin(2x)\cos(x)\,dx = \frac{1}{2}b_2\int_{-\pi}^{\pi}[\sin(3x) + \sin(x)]dx$$
$$= \frac{1}{2}b_2[-\frac{1}{3}\cos(3x)\,\Big|_{-\pi}^{\pi} - \cos(x)\,\Big|_{-\pi}^{\pi}] = 0$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)dx = 0 + a_1\pi + 0 + \cdots + 0 + 0 + \cdots 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(1\cdot x)dx$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2\cdot x)dx$$

…

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(1\cdot x)dx$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2\cdot x)dx$$

…

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

If $f(x)$ is a function defined on $[-\pi, \pi]$, then $f(x)$ can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], (1)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, (4)$$

If $f(x)$ function is not defined on $[-\pi, \pi]$, but is defined on $[-L, L]$, we can still find the Fourier series by making a change in variables.

Let's define:

$$t = \frac{\pi x}{L}, or\ x = \frac{Lt}{\pi}$$

Then, when x=-L, t=-π, when x=L, t=π. In this case, t ranges from $-\pi$ to π

$$f(x) = f\left(\frac{Lt}{\pi}\right) = g(t)$$

Then, $g(t)$ is a function defined on $[-\pi, \pi]$. We can use our previous formula to express $g(t)$ as:

$$g(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], (5)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt, (6)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt)\, dt, (7)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt)\, dt, (8)$$

Then, substitute $g(t)$ with $f(x)$, t with $\frac{\pi x}{L}$, $dt = d\frac{\pi x}{L} = \frac{\pi}{L}dx$, into Eq. (5-8), we get:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (9)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, (10)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, (11)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, (12)$$

In summary, if $f(x)$ is defined on $[-L, L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (9)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, (10)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, (11)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, (12)$$

If $f(x)$ is defined on $[0, 2L]$, what to do?

In practice, we can define:

$$t = \frac{\pi(x - L)}{L}$$

Then, when $x = 0$, $t = -\pi$; when $x = 2L$, $t = \pi$. In this case, t ranges from $-\pi$ to π.

Again,

$$f(x) = f\left(\frac{tL}{\pi} + L\right) = g(t)$$

Then, $g(t)$ is a function defined on $[-\pi, \pi]$. We can use our previous formula to express $g(t)$ as:

$$g(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], (13)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt, (14)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt)\, dt, (15)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt)\, dt, (16)$$

Then, substitute $g(t)$ with $f(x)$, t with $\frac{\pi(x-L)}{L}$, $dt = d\frac{\pi(x-L)}{L} = \frac{\pi}{L} dx$, into Eq. (13-16), we get:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (17)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L} \int_{0}^{2L} f(x)\, dx, (18)$$

$$a_n = \frac{1}{L} \int_{0}^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, (19)$$

$$b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, (20)$$

In summary, if $f(x)$ is defined on $[0, 2L]$, then it can be represented by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)], (21)$$

where, a_0, a_n, b_n are constants, and are given by:

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\, dx, (22)$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, (23)$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, (24)$$

Let's first do some example, what is the period of:

$$\text{A}\sin(nx)$$

Period is $T = \frac{2\pi}{n}$, amplitude is A.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)]$$

What is the period when n=1?

$$T = \frac{2\pi}{\frac{\pi}{L}} = 2L$$

Then, the period is given by:

$$T_n = \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$$

What is the amplitude? We define the amplitude of $f(x)$ as:

$$A_n = \sqrt{a_n^2 + b_n^2}$$

In summary, the amplitude for period of $\boldsymbol{T_n = \frac{2L}{n}}$ is calculated by:

$$\boldsymbol{A_n = \sqrt{a_n^2 + b_n^2}}$$

Discrete Fourier Transformation

Most of the time, $f(x)$ is unknown, but is given by a list of discrete observational data points. If $f(x)$ is represented by a number of data points:

Number of data points = np

x values: $x_1, x_2, \dots, x_{np}$

y values: $y_1, y_2, \dots, y_{np}$

Increment of x: $\Delta x = x_2 - x_1$