Conic Finance.
A short introduction, 2016

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A One Price Economy

Three central themes in mathematical finance are:
1. No arbitrage
2. Risk management
3. Performance measures

No arbitrage provides a framework for pricing derivative contracts.
Risk management identifies and assesses uncertainty related to financial positions
Performance measures allows for the attractiveness of different investments to be compare

Research problem

It is of interest to establish the general relationship between these three themes, in particular the general relationship between no-arbitrage pricing, risk measures, and acceptability indices considering realistic features such as transaction costs.

First Fundamental Theorem of Asset Pricing (1) (2)

A discrete (or finite state) market, on a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure, $\mathcal{P}$.

(A risk-neutral measure, also called an equilibrium measure, or equivalent martingale measure, is a probability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure.)

Corollary:

In a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure.

(1) Harrison, J. M.; Pliska, S. R. (1981). Stochastic Processes and their Applications 11 (3): 215–260.

(2) Delbaen, F.; Schachermayer, W. (1994). Mathematische Annalen 300 (1): 463–520.

Under No Arbitrage (or Risk neutrality) one can price, among others,
- All sorts of derivatives (forward contracts, options, etc)
- European style Call and Put options, by a closed formula due to Black and Scholes
- American style and other options, by Cox, Ross and Rubinstein (1979) binomial trees model.

Risk-neutral valuation of Options

The pay-off of an European Call option on a stock:
$\max(P_T - K, 0)$
The pay-off of European Put option: $\max(K - P_T, 0)$
where T is maturity date, K strike price, P_t spot price of
underlying asset, $t \in [0, T]$.

In a risk-neutral world:

1. The expected return on all securities is the market risk-free interest rate r.

2. The present value of any cash flow is the discounted expected value at the risk-free interest rate

Hence the (risk-neutral) value at time t of a European
Call option: $C_t = e^{-r(T-t)} \cdot E[\max(P_T - K, 0)]$
Put option: $Pu_t = e^{-r(T-t)} \cdot E[\max(K - P_T, 0)]$
where $T - t$ is the time to maturity.

Black-Scholes option pricing formula

Assuming: 1) Stock price follows a GBM with constant drift μ and variance σ, i.e.,

$$dP_t = \mu P_t dt + \sigma P_t dW$$

2) No Arbitrage or equiv. that valuation of options is done considering a risk-neutral measure.

3) There is unlimited borrowing and lending at a risk-free constant interest rate r, and the market is *frictionless*

4) The underlying stock pays no dividend and the option is European.

Under these hypothesis, Fisher Black and Myron Scholes presented in 1973 a closed formula for computing the price of an European option on a stock with no dividends, for given strike price K, current price P_t, and maturity date T.

By risk-neutrality the price of the stock satisfies

$$dP = rPdt + \sigma PdW$$

(r replaces μ).

Solving the equation (Ito's Lemma) leads to

$$d \ln P_t = \left(\frac{r - \sigma^2}{2} \right) dt + \sigma dW$$

Black-Scholes option pricing formula

The difference in $\ln P$ follows a normal distribution (since noise is Wiener process)

$$\ln P_T - \ln P_t = \Phi\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right]$$

Hence the stock price at T follows a log-normal distribution

$$P_T \sim LogNormal\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \ln P_t, \sigma^2(T-t)\right]$$

and its cumulative distribution funct. is

$$F_{P_t}(x) = \Phi\left[\frac{\ln \frac{x}{P_t} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]$$

Black-Scholes option pricing formula

With the previous assumptions (that price follows a log-normal distrib.) the unique cost of the option that does not result in an arbitrage is when its expectation is an integral with respect to the lognormal density of P_t:

$$C_t = e^{-r(T-t)} E[\max(P_t e^L - K, 0)]$$

where $L \sim \Phi((r - \sigma^2/2)(T - t), \sigma^2(T - t))$ and recall that $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.

Evaluating this integral in terms of Φ, we have the BS fmla. for a Call:

$$C_t = P_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) \tag{1}$$

Similar argument gives fmla. for a put:

$$Pu_t = Ke^{-r(T-t)}\Phi(-d_2) - P_t\Phi(-d_1) \tag{2}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(P_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln(P_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

General Approach to Arbitrage Valuation

Interesting fact due to Stephen Ross, (1978) for a General Approach to Arbitrage Valuation:

Assuming No Arbitrage one can value all marketed assets with a linear operator.

No Arbitrage $\iff$ Risk-Neutrality
$\Rightarrow$ There exists a linear valuation operator
$\Rightarrow$ The Law of One Price (Rubinstein, 1976): *If two portfolios yield the same return streams then they must have the same price*

A Two Price Economy

Citigroup Inc. (C) - NYSE ⭐ Wa

42.47 ↓2.91 (6.41%) Jan

Prev Close:	**45.38**
Open:	**42.51**
Bid:	**42.36 x 300**
Ask:	**42.47 x 800**
1y Target Est:	**63.32**
Beta:	**1.44927**

Citigroup Inc. (C) ⭐ Watchlist

42.47 -2.91 (-6.41%) NYSE - As of 4:00PM EST

January 22, 2016 ⌄   In The Money

Calls

Strike ⌃ Filter	Contract Name	Last	Bid	Ask
40.00	C160122C00040000	2.72	2.63	2.90
40.50	C160122C00040500	2.43	2.22	2.48
41.00	C160122C00041000	2.20	1.86	2.18
41.50	C160122C00041500	1.57	1.60	1.71
42.00	C160122C00042000	1.28	1.27	1.39
42.50	C160122C00042500	1.01	1.01	1.10

Conic Finance

Conic Finance is founded on the existence of **two price markets**:
- a price at which one may sell to the market (bid price)
- a price at which one may buy from the market (ask price).

Thus, pricing depends on the direction of trade. The theory is founded on the basis of the concepts of acceptability of stochastic cashflows and distorted expectations.

Markets as a counterparty are modeled as accepting at zero cost a convex cone of random variables containing the nonnegative cash flows (hence the name "conic" finance)

Conic finance goes beyond the law of one-price and risk neutral measures to a world of economies where there are two prices, one for bid and one for ask, and the possibility of arbitrage due to acceptance of non-null bid-ask spread.

Conic Finance is intrinsically connected with liquidity effects and risk behavior of financial markets.

Consider in a One Price Economy a random var. X promised at time T

The value of X at initial date $t_o = 0$ (by risk-neutrality) is

$$v(X) = e^{-rT} E^Q[X]$$

where r is interest rate for period T and Q is a probability measure (e.g. a risk neutral or martingale)

Acceptable Risks

The market will accept at zero cost the time T cash flow

$$Z = X - e^{rT}w \quad \text{for} \;\; w \leq v(X)$$

because one is selling to the market at a price below the risk-neutral market price.

The market will also accept at zero cost the time T cash flow

$$Z' = e^{rT}w - X \quad \text{for} \;\; w \geq v(X)$$

because one is buying from the market for a price above the risk-neutral market price.

In both cases we have

$$E^Q[Z] \geq 0 \qquad E^Q[Z'] \geq 0$$

Acceptable Risks

The zero cost cash flows acceptable to the market are then defined

$$\hat{\mathcal{A}} = \{Z : E^Q[Z] \geq 0\}$$

It is the largest possible convex cone (for it is a half space) containing the nonnegative random variables, and these are always acceptable as they are arbitrage.

Artznert et al (1999)

Any such cone of acceptable risks $\mathcal{A}$ is equivalently defined by a convex set of probability measures $\mathcal{M}$ where by

$$X \in \mathcal{A} \iff E^Q[X] \geq 0, \quad \forall Q \in \mathcal{M}$$

If the market agrees to BUY X for the price b, or
it agrees to SELL X for the price a, then

$$X - be^{rT} \in \mathcal{A} \quad \text{and} \quad ae^{rT} - X \in \mathcal{A}$$

Equivalently, $\forall Q \in \mathcal{M}$

$$E^Q[X] - be^{rT} \geq 0$$
$$ae^{rT} - E^Q[X] \geq 0$$

Bid and Ask Prices

The best bid and ask prices for X provided by the market are

$$bid(X) = e^{-rT} \inf\{E^Q[X] : Q \in \mathcal{M}\}$$

$$ask(X) = e^{-rT} \sup\{E^Q[X] : Q \in \mathcal{M}\}$$

These are two valuation operator for a two price economy
Prospective portfolios are being sold to market at the bid price
$\Rightarrow$ We wish to design portfolios to maximise the portfolio's bid price
$\Rightarrow$ The objective function for portfolio design can be taken as

$$bid(X) = \inf\{E^Q[X] : Q \in \mathcal{M}\}$$

Bid and Ask Prices as distorted expectations

We ask for the following properties for the bid price:

1. to only depend on the distribution function of the random variable

2. (additivity for co-monotone risks)
   $bid(X + Y) \geq bid(X) + bid(Y)$

(This excess of the bid for the sum over the sum of bids can be seen as *benefit from diversification*)

We have $bid(X + Y) = bid(X) + bid(Y)$ when X, Y are co-monotone

Kusuoka (2001)

Under the hypotheses of co-monotone additivity and dependence of acceptability on distribution function, then the bid price is an expectation under a *concave distortion*

In particular there exists a concave distribution funct.
$\Psi : [0, 1] \to [0, 1]$ such that for any risky asset X with distrib. funct. $F_X(x)$ its bid price can be computed by

$$bid(X) = e^{-rT} \int_0^\infty x d\Psi(F_X(x))$$

The ask price can be computed as

$$ask(X) = -e^{-rT} \int_{-\infty}^{0} x d\Psi(F_{-X}(x))$$

and risk neutral price (rn):

$$rn(X) = e^{-rT} \int_{0}^{\infty} x dF_X(x)$$

Since derivative of concave funct. is non-increasing

$$
\begin{aligned}
e^{-rT} \int_{0}^{\infty} x d\Psi(F_X(x)) &\leq e^{-rT} \int_{0}^{\infty} x dF_X(x) \\
&\leq -e^{-rT} \int_{-\infty}^{0} x d\Psi(F_{-X}(x))
\end{aligned}
$$

Cherny & Madan, 2009: $\Psi^\gamma(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}$, $u \in [0,1]$, $\gamma \geq 0$. This is the MINMAXVAR.

Wang, 2003 $\Psi_\Phi^\gamma(u) = \Phi(\Phi^{-1}(u) + \gamma)$, $\gamma \geq 0$, Φ the normal distrib. funct.

Estimating Distorted Expectations

Once we have the distribution function of X we employ the empirical distribution function on a sample $x_1, \ldots, x_N$.
So, if $x_{(n)}$ are the values x_n sorted in increasing order, compute the bid price as (say using Ψ^γ)

$$
\begin{aligned}
bid(X) &= \int_0^\infty x d\Psi^\gamma(F_X(x)) \\
&= \sum_{n=1}^M x_{(n)} \left(\Psi^\gamma \left(\frac{n}{N} \right) - \Psi^\gamma \left(\frac{n-1}{N} \right) \right)
\end{aligned}
$$

An important task is to calibrate γ (the stress level)
[Madan, 2015] For Gaussian random var. with mean of 6.25% and volatility 15% an acceptable $\gamma = 0.25$. These mean and volatility match the SP500 over 30 years.
For other Gaussian vars. map γ to required Sharpe ratios, in portfolio design

Conic Option Valuation

Recall :The pay-off of an European Call option C_t is
$\max(P_T - K, 0)$
If F_{C_t} is distrib. funct. of C_t, we know that its bid price is

$$
\begin{aligned}
bid(C_t) &= \int_0^\infty x d\Psi(F_{C_t}(x)) = \int_K^\infty (x - K) d\Psi(F_{P_t}(x)) \\
&= \int_K^\infty x d\Psi(F_{P_t}(x)) - \int_K^\infty K d\Psi(F_{P_t}(x)) \quad (*)
\end{aligned}
$$

You can likewise express formulas for the ask of a Call, the bid and
ask of a Put.

Recall that under hypothesis of Black-Scholes model (the price follows log-normal distrib.) we have cumulative distribution funct. of P_T:

$$F_{P_t}(x) = \Phi\left[\frac{\ln\frac{x}{P_t} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]$$

Apply the Wang transform to it, and get

$$\Psi_{\Phi}^{\gamma}(F_{P_t}(x)) = \Phi\left[\frac{\ln\frac{x}{P_t} - \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right]$$

Plug this expression into Eq. $(*)$ for $bid(C_t)$ and calculate ...

You should get

$$bid(C_t) = P_t e^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(P_t/K) + (r + \sigma^2/2)(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Homework: Compute $ask(C_t)$, $bid(Pu_t)$, $ask(Pu_t)$.

- Conic Monte Carlo
- Conic Derivative Pricing
- Conic Implied Volatility
- Conic Implied Liquidity
- Conic measures of Diversity/Dependence
- Conic Portfolio Theory

Reading List

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