Fuzzy Logic: 9/27/22

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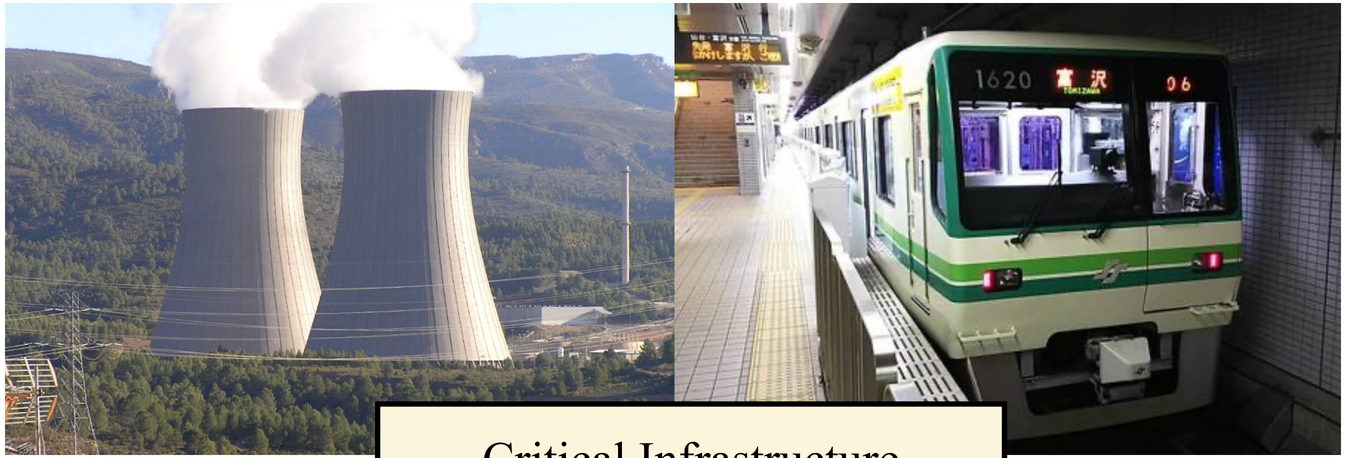
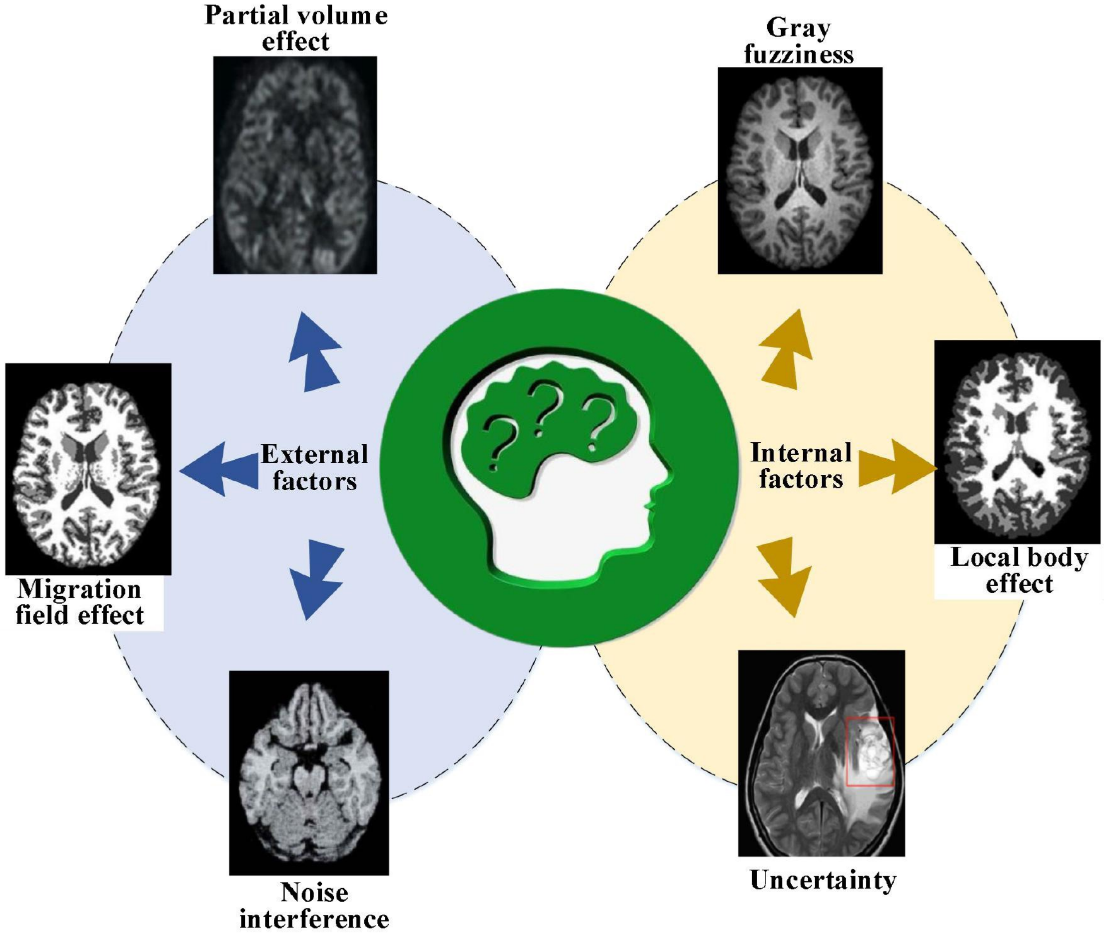
Fuzzy Logic : Why Care?

Fuzzy Logic is present in everyday life!

Commercial Appliances



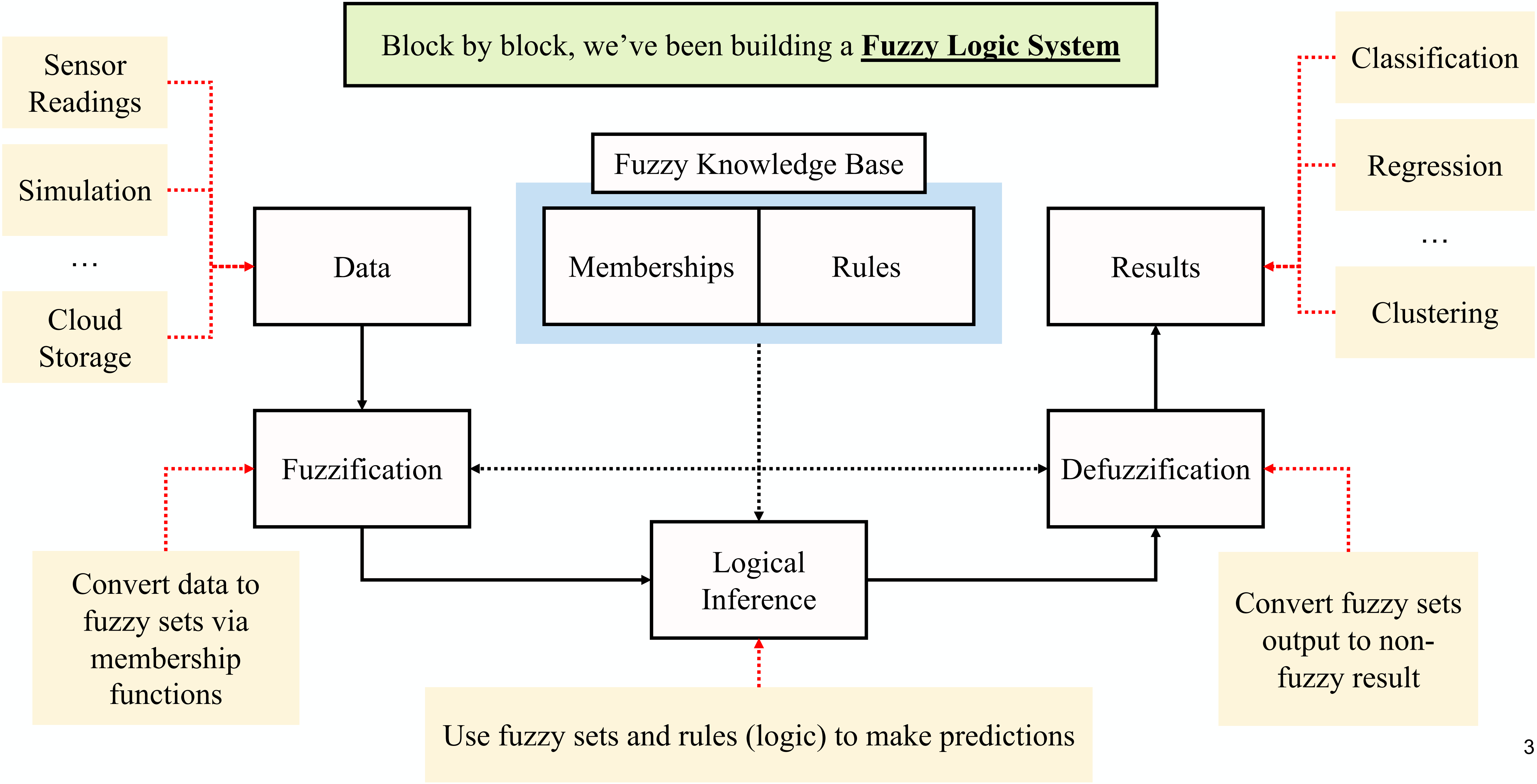
Critical Infrastructure



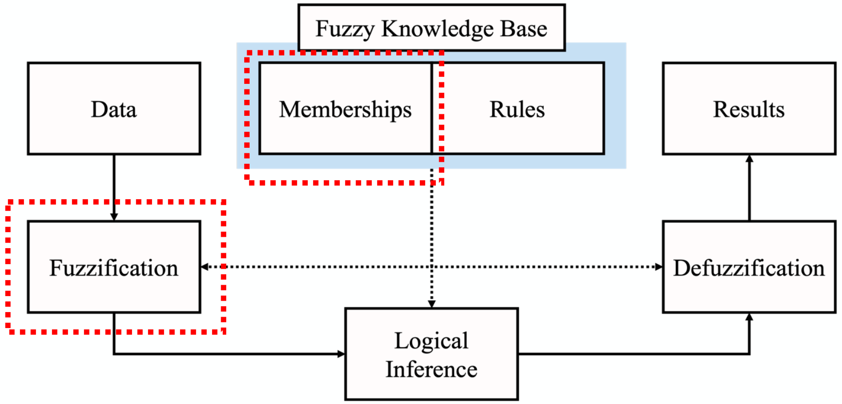
Medical Data Analysis

Fuzzy Logic : The Big Picture

What have we been doing the last few days? ;)



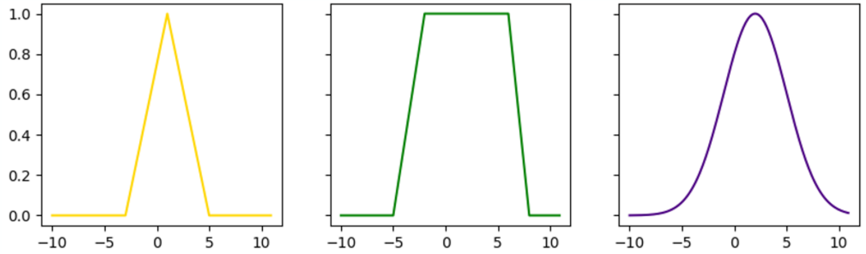
Fuzzy Logic : Fuzzy Sets (Review)



Convert data observations into fuzzy sets via membership functions

Simplifying for now, will go into more detail later this semester ;)

Data → Membership Functions → Fuzzy Sets



Triangle | Trapezoid | Gaussian

$$X = \{x | x \in X\} | X \in \mathbb{R}$$

$$A(X) = \{\mu(x) | x \in X\} | 0 \geq \mu(x) \leq 1$$

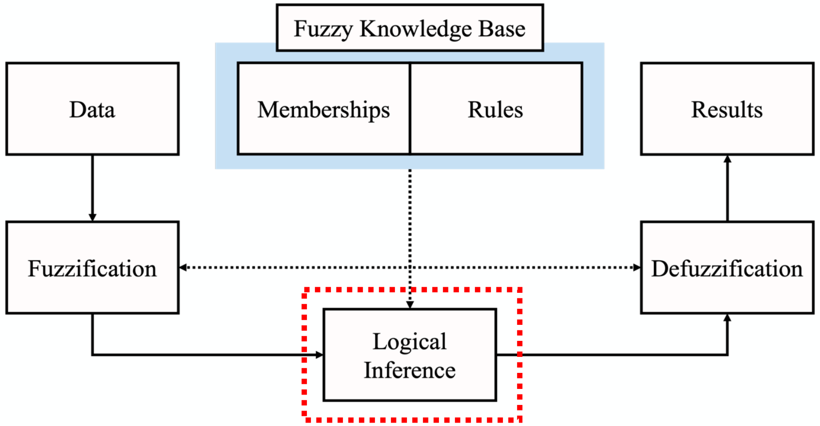
X could be an observation from universe of real numbers

A is a membership function fuzzifies X | A(X) → [0, 1]

Observation	Captured information from sensors, simulation, storage, etc.

Linguistic Variable	Observation expressed by language but defined by a membership function

Fuzzy Logic : Fuzzy Operators (Review)



Assuming two arbitrary linguistic variables (e.g., Sunny and Cloudy)

$$A_1 : X_1 \rightarrow [0, 1] \qquad A_2 : X_2 \rightarrow [0, 1]$$

Intersection (AND)

$$A_1 \wedge A_2$$

Minimum | Product | ….

T-Norms

Union (OR)

$$A_1 \vee A_2$$

Maximum | Sum | ….

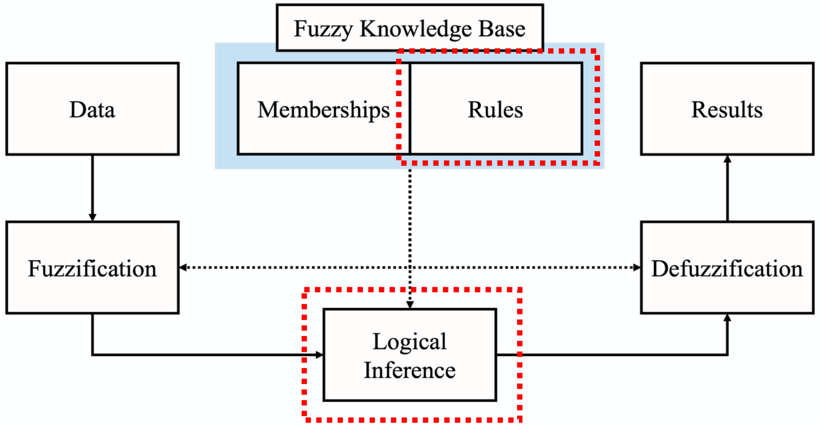
T-Conorms

Compliment (NOT)

$$A^c = 1 - A$$

Many more operators, but these are some essentials for basic logic

Fuzzy Logic : Logical Rules (Review)



Assuming an arbitrary logic rule: **If Sunny and Cloudy then Happy**

$$\text{If } A_1 \text{ and } A_2 \text{ then } B$$

Linguistic Variables (LVs) : **Antecedents** are LVs of "if" and **Consequences** are LVs of "then"

Fuzzy Proposition: Apply fuzzy operators of antecedents (ANDs, ORs, NOTs, etc.)

Fuzzy Implication: Operation defines a **fuzzy relation** mapping propositions to the consequences

$$R(\mu_{A'}, \mu_B) = min(1, 1 - \mu_{A'} + \mu_B)$$

Lukasiewicz

$$R(\mu_{A'}, \mu_B) = min(\mu_{A'}, \mu_B)$$

Correlation Minimum

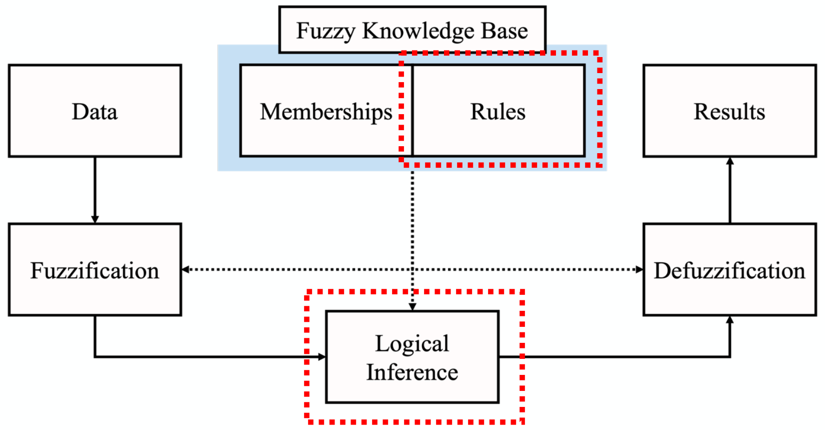
$$R(\mu_{A'}, \mu_B) = \mu_{A'} * \mu_B$$

Correlation Product

$$R : A' \rightarrow B$$

$$A' = \bigwedge_{i=0}^{k} A_i$$

Fuzzy Logic : Logical Rules (Review)



Using Fuzzy Implication (Relation) Operators for Modus Ponens

$$((A \rightarrow B) \wedge A) \rightarrow B$$

Can the following operators follow Modus Ponens?

Lukasiewicz

$$R(\mu_{A'}, \mu_B) = min(1, 1 - \mu_{A'} + \mu_B)$$

A	B	A → B
0	0	1
0	1	1
1	0	0
1	1	1

Correlation Minimum

$$R(\mu_{A'}, \mu_B) = min(\mu_{A'}, \mu_B)$$

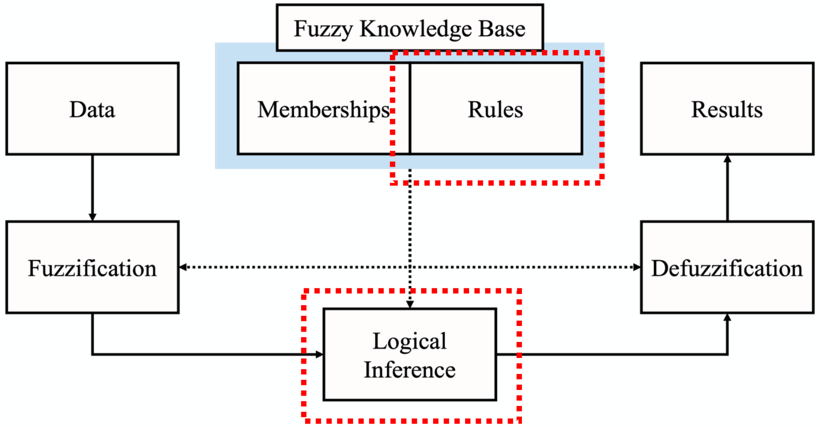
A	B	A → B
0	0	0
0	1	0
1	0	0
1	1	1

Correlation Product

$$R(\mu_{A'}, \mu_B) = \mu_{A'} * \mu_B$$

A	B	A → B
0	0	0
0	1	0
1	0	0
1	1	1

Fuzzy Logic : Rules Of Inference (Ex. 1)



Using **Compositional Rule Of Inference** for generalized of modus ponens

$$B' = A' \circ R = \sup min\{A', R\} = \vee(\wedge(A', R))$$

If Small then Medium → If A then B

$$X = \{1, 2, 3, 4\} \quad Y = \{a, b, c, d\}$$

$$\mathrm{A} = \mathrm{Small} = 1.0/1 + 0.8/2 + 0.0/3 + 0.0/4$$

$$\mathrm{B} = \mathrm{Medium} = 0.0/a + 0.5/b + 1.0/c + 0.5/d + 0.0/e$$

★ $R(\mu_{A'}, \mu_B) = min(1, 1 - \mu_{A'} + \mu_B)$ Lukasiewicz

$$R_z = A \circ B = \begin{bmatrix} \boxed{0.0} & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.2 & \boxed{0.7} & 1.0 & 0.7 & 0.2 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$min(1, 1 - 1.0 + 0) = 0.0$$

$$min(1, 1 - 0.8 + 0.5) = 0.7$$

$$B' = \{\{1.0, 0.8, 0.0, 0.0\} \circ col_i(R) | \forall i\} = \{\boxed{0.2,} 0.7, 1.0, 0.7, 0.2\}$$

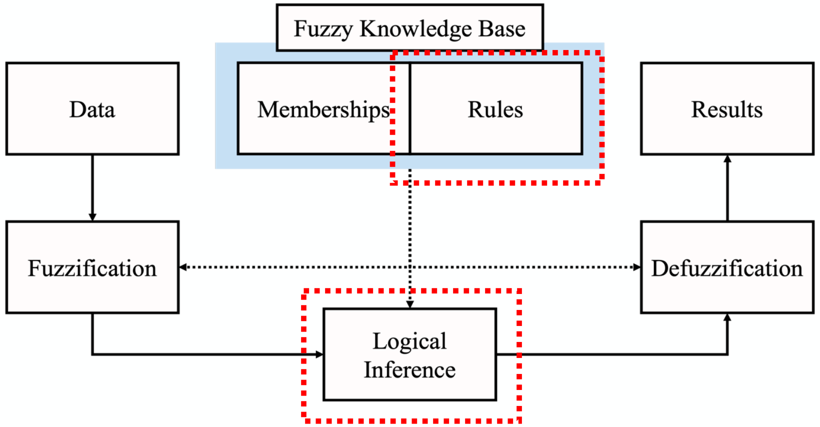
$$\begin{bmatrix} 0.0 & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.2 & 0.7 & 1.0 & 0.7 & 0.2 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$(1.0 \wedge 0.0) \vee (0.8 \wedge 0.2) \vee (1.0 \wedge 0.0) \vee (1.0 \wedge 0.0) = 0.2$$

{0.2, 0.7, 1.0, 0.7, 0.2} != {0.0, 0.5. 1.0, 0.5, 0.0}

B’ != B?? Why? Thoughts?

Fuzzy Logic : Rules Of Inference (Ex. 2)



Using **Compositional Rule Of Inference** for generalized of modus ponens

$$B' = A' \circ R = \sup min\{A', R\} = \vee(\wedge(A', R))$$

If Small then Medium → If A then B

$$X = \{1, 2, 3, 4\} \quad Y = \{a, b, c, d\}$$

$$R_z = A \circ B = \begin{bmatrix} 0.0 & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.0 & 0.5 & 0.8 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$min(1.0, 0.0) = 0.0$$

$$min(0.8, 0.5) = 0.5$$

$$\text{A} = \text{Small} = 1.0/1 + 0.8/2 + 0.0/3 + 0.0/4$$

$$\text{B} = \text{Medium} = 0.0/a + 0.5/b + 1.0/c + 0.5/d + 0.0/e$$

★ $R(\mu_{A'}, \mu_B) = min(\mu_{A'}, \mu_B)$ Correlation-Min

$$B' = \{\{1.0, 0.8, 0.0, 0.0\} \circ col_i(R) | \forall i\} = \{0.0, 0.5, 1.0, 0.5, 0.0\}$$

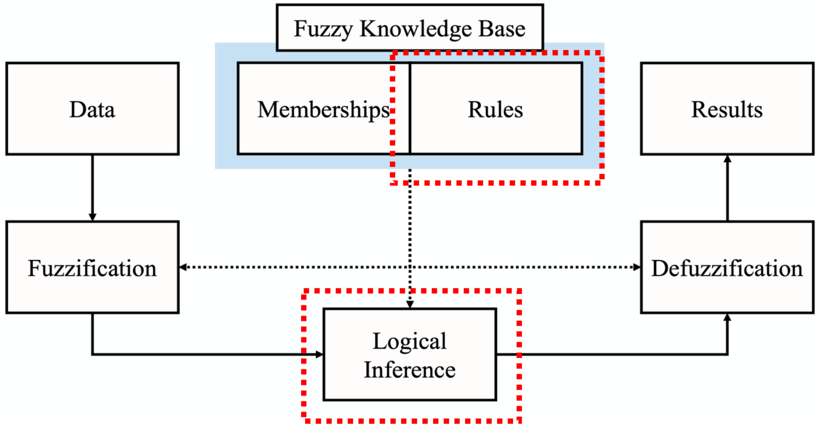
$$\begin{bmatrix} 0.0 & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.0 & 0.5 & 0.8 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$(1.0 \wedge 1.0) \vee (0.8 \wedge 0.8) \vee (0.0 \wedge 0.0) \vee (0.0 \wedge 0.0) = 1.0$$

{0.0, 0.5, 1.0, 0.5, 0.0} == {0.0, 0.5. 1.0, 0.5, 0.0}

B’ == B! Why? Thoughts?

Fuzzy Logic : Rules Of Inference (Ex. 3)



Using **Compositional Rule Of Inference** for generalized of modus ponens

$$B' = A' \circ R = \sup min\{A', R\} = \vee(\wedge(A', R))$$

If Small then Medium → If A then B

$$X = \{1, 2, 3, 4\} \quad Y = \{a, b, c, d\}$$

$$R_z = A \circ B = \begin{bmatrix} \boxed{0.0} & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.0 & \boxed{0.4} & 0.8 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$1.0 * 0.0 = 0.0$$

$$0.8 * 0.5 = 0.4$$

$$\text{A} = \text{Small} = 1.0/1 + 0.8/2 + 0.0/3 + 0.0/4$$

$$\text{B} = \text{Medium} = 0.0/a + 0.5/b + 1.0/c + 0.5/d + 0.0/e$$

★ $R(\mu_{A'}, \mu_B) = \mu_{A'} * \mu_B$ Correlation-Prod

$$B' = \{\{1.0, 0.8, 0.0, 0.0\} \circ col_i(R) | \forall i\} = \{0.0, \boxed{0.5}, 1.0, 0.5, 0.0\}$$

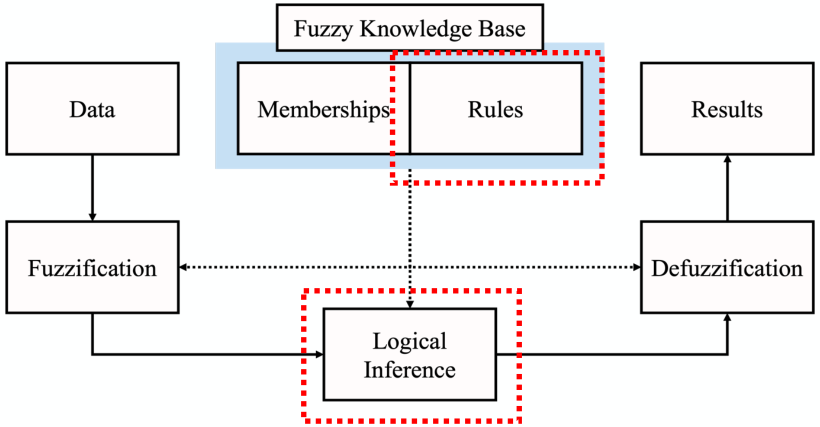
$$\begin{bmatrix} 0.0 & 0.5 & 1.0 & 0.5 & 0.0 \\ 0.0 & 0.4 & 0.8 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$(1.0 \wedge 0.5) \vee (0.8 \wedge 0.4) \vee (0.0 \wedge 0.0) \vee (0.0 \wedge 0.0) = 0.5$$

{0.0, 0.5, 1.0, 0.5, 0.0} == {0.0, 0.5. 1.0, 0.5, 0.0}

B’ == B! Why? Thoughts?

Fuzzy Logic : Rules Of Inference (Ex. 4)



If **NOT** Small then Medium → If NOT A then B

$$X = \{1,2,3,4\} \quad Y = \{a,b,c,d\}$$

$$A = A^c = 0.0/1 + 0.2/2 + 1.0/3 + 1.0/4 \qquad \text{B} = \text{Medium} = 0.0/a + 0.5/b + 1.0/c + 0.5/d + 0.0/e$$

Lukasiewicz

$$R(\mu_{A'}, \mu_B) = min(1, 1 - \mu_{A'} + \mu_B) \qquad B'_z = \{\{0.0, 0.2, 1.0, 1.0\} \circ col_i(R_z) | \forall i\} = \boxed{\{1.0, 1.0, 1.0, 1.0, 1.0\}}$$

Correlation-Min

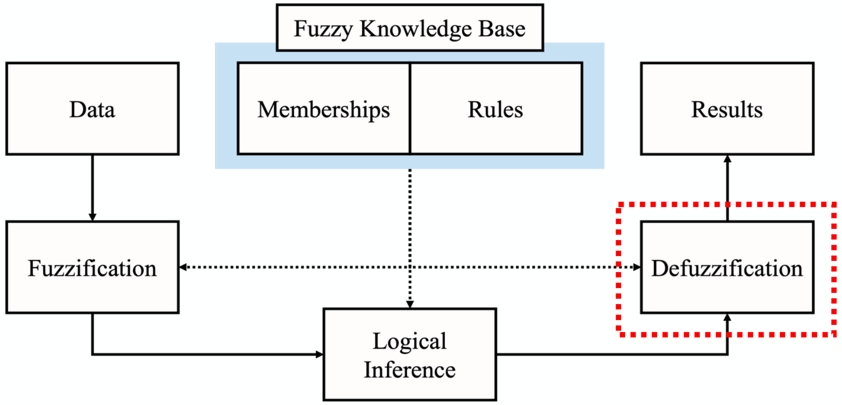
$$R(\mu_{A'}, \mu_B) = min(\mu_{A'}, \mu_B) \qquad B'_{cm} = \{\{0.0, 0.2, 1.0, 1.0\} \circ col_i(R_{cm}) | \forall i\} = \boxed{\{0.0, 0.2, 0.2, 0.2, 0.0\}}$$

Correlation-Prod

$$R(\mu_{A'}, \mu_B) = \mu_{A'} * \mu_B \qquad B'_{cp} = \{\{0.0, 0.2, 1.0, 1.0\} \circ col_i(R_{cp}) | \forall i\} = \boxed{\{0.0, 0.2, 0.2, 0.2, 0.0\}}$$

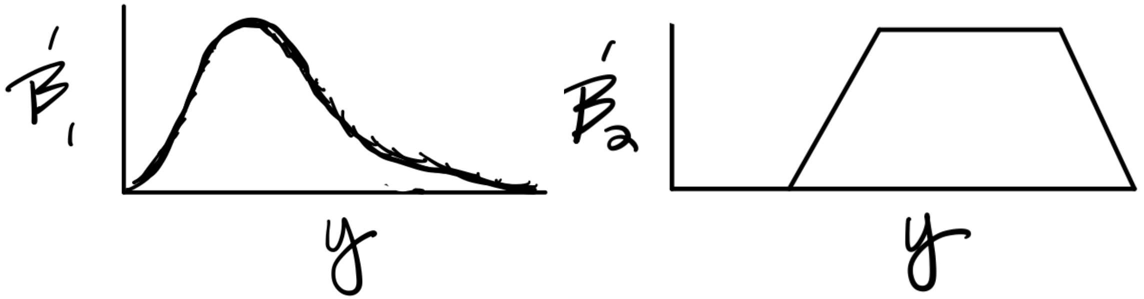
What could these memberships represent? Thoughts?

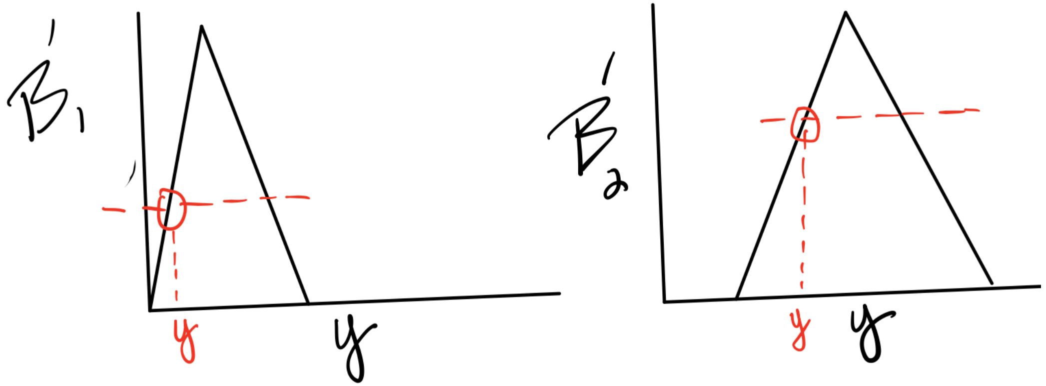
Fuzzy Logic : Defuzzification



Defuzzification starts with an aggregation across all calculated consequences

If A_1 Then B_1, If A_2 Then B_2



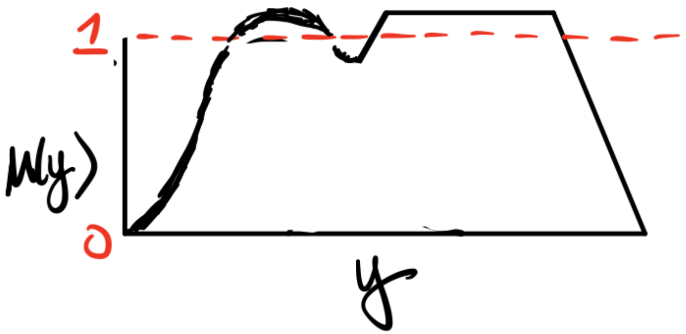


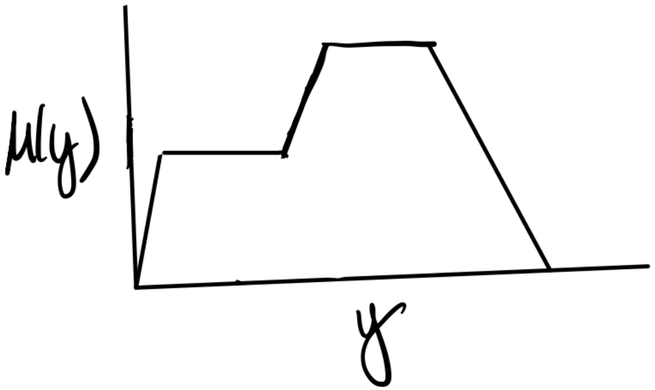
Sum Aggregation

$$B'(y) = \Sigma_{i=0}^{k} B_i'(y)$$

Max Aggregation

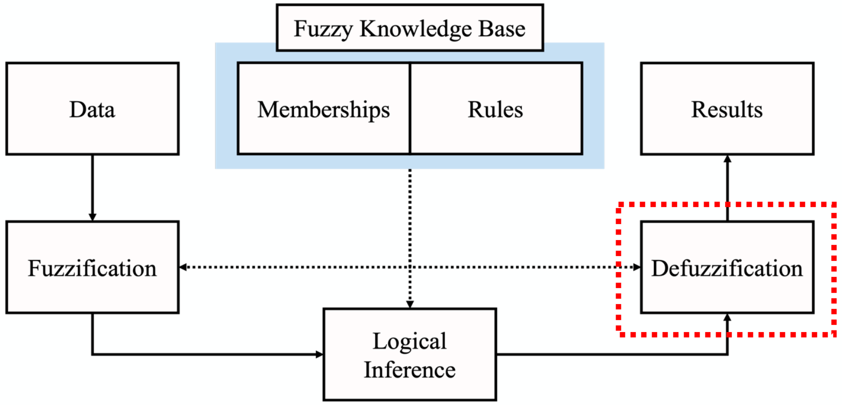
$$B'(y) = \max_{i=0}^{k}\{B_i'(y)\}$$





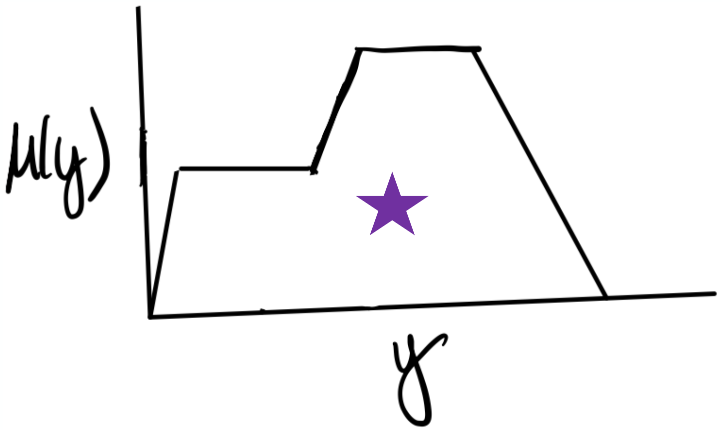
What are **tradeoffs** regarding these aggregation strategies?

Fuzzy Logic : Defuzzification



Defuzzification continues with a conversion from fuzzy sets to crisp outputs

Not many systems can process / use a fuzzy output ☹



$$y_c = \frac{\Sigma_{y \in Y}\, y * B'(y)}{\Sigma_{y \in Y} B'(y)}$$

Centroid Defuzzification	Method that returns the center of the area under the aggregated fuzzy set

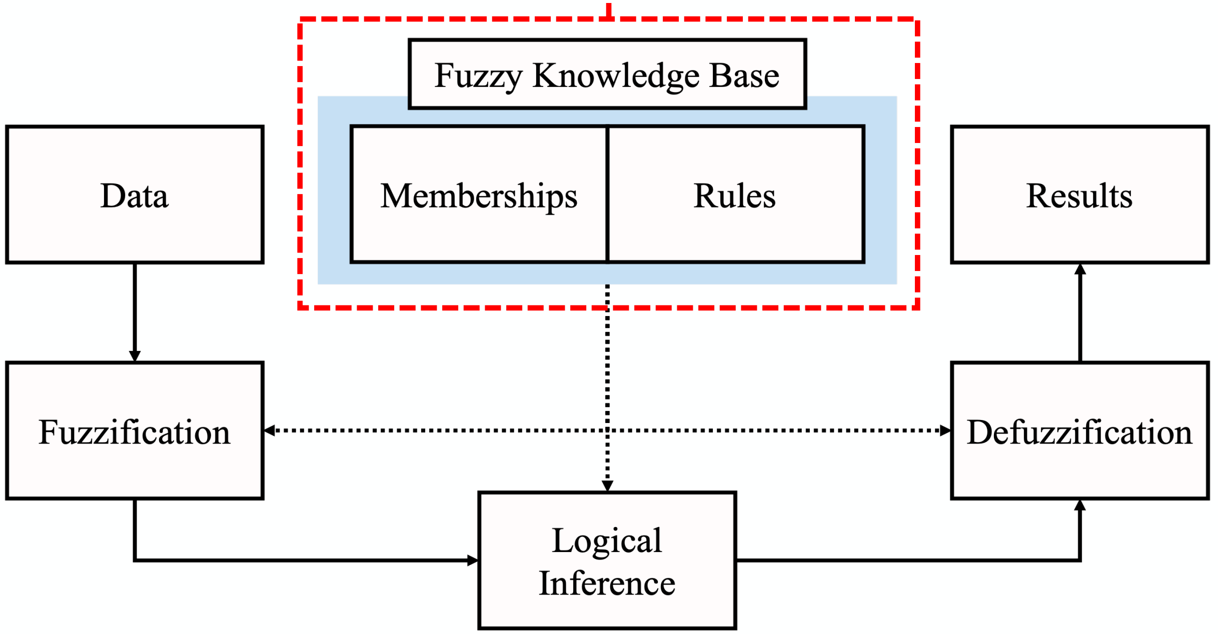
Thoughts on Defuzzification? What are the drawbacks?

Defuzzification Methods

- Bisector of Area
- Center of Area
- Fuzzy Mean
- .....
- First of Maximum

Fuzzy Logic : Example Problem

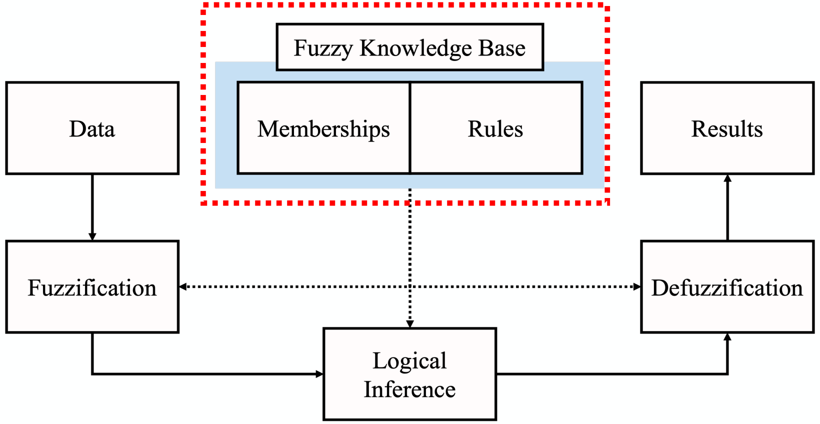
Can we build a simple fuzzy system to predict mood given temperature outside?



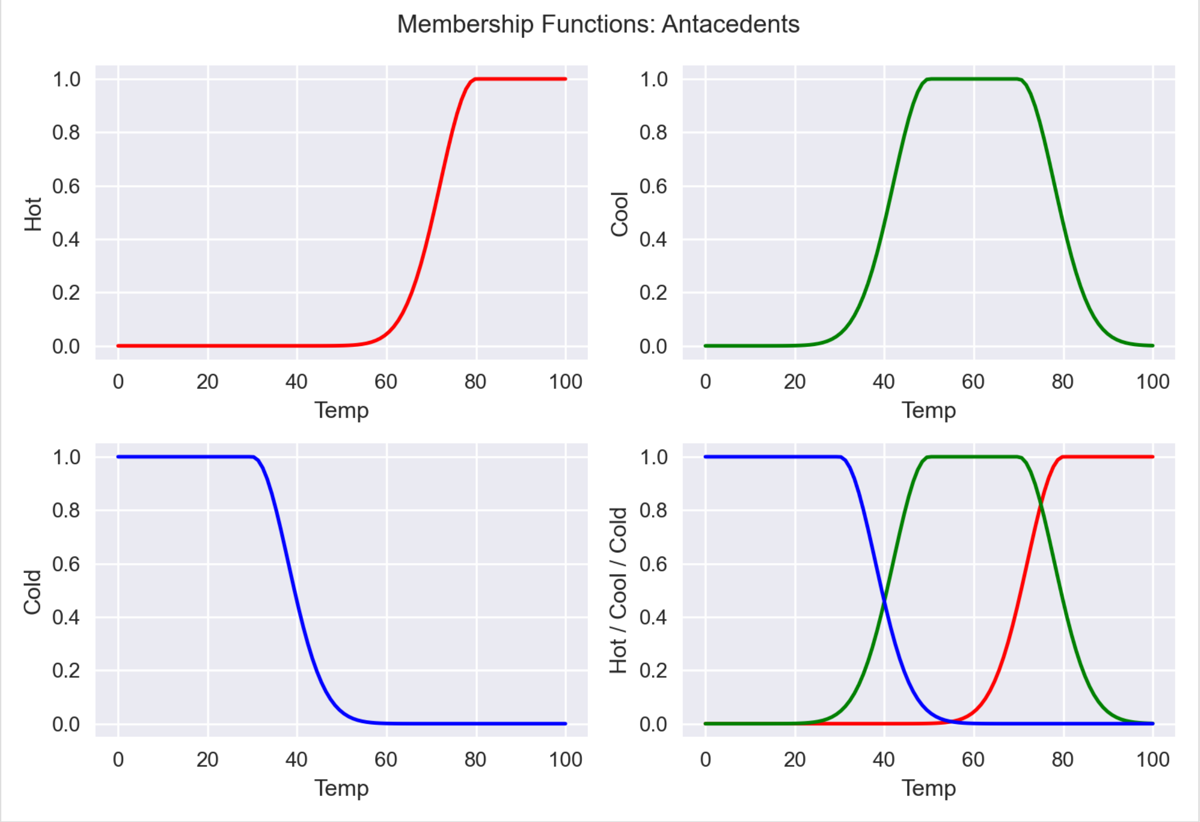
Logical Rules
If **Hot** then **Content**
If **Cool** then **Happy**
If **Cold** then **Sad**

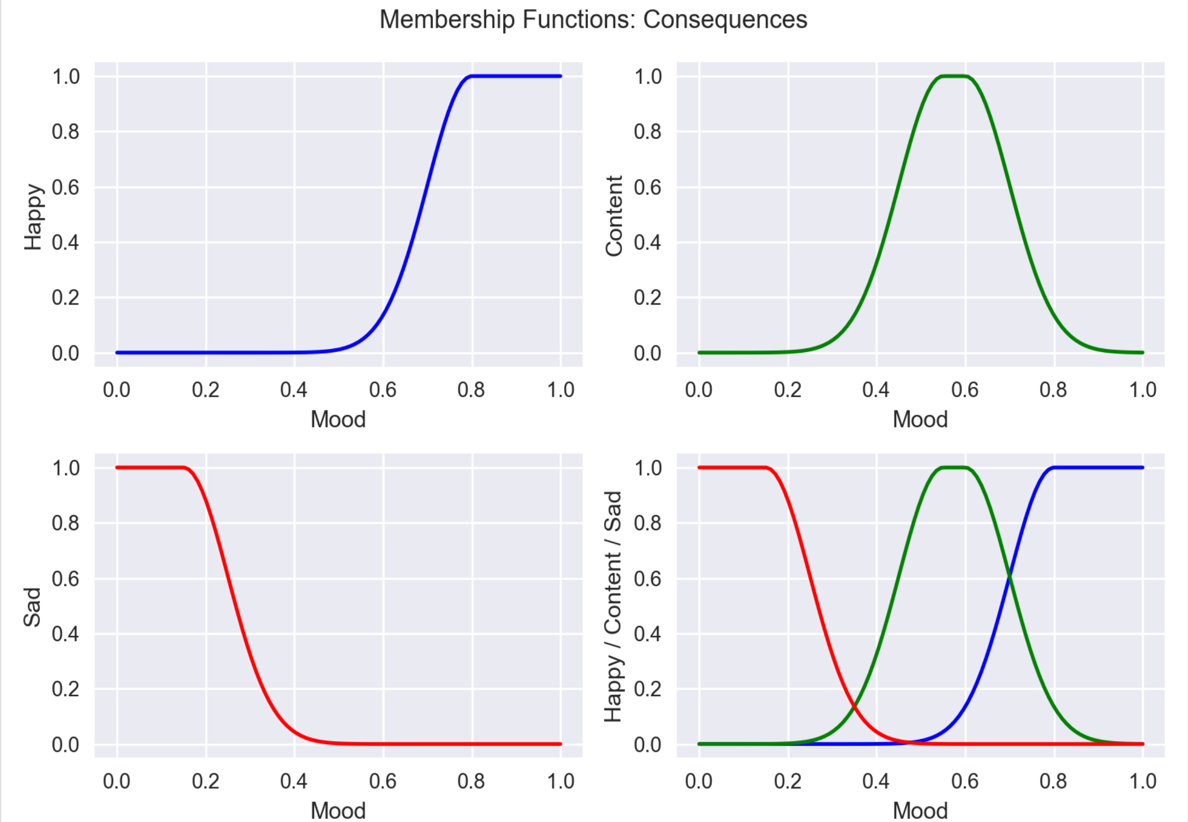
Unlike GA's, Fuzzy systems are typically non-optimizable rule inference systems

Fuzzy Logic : Example Problem



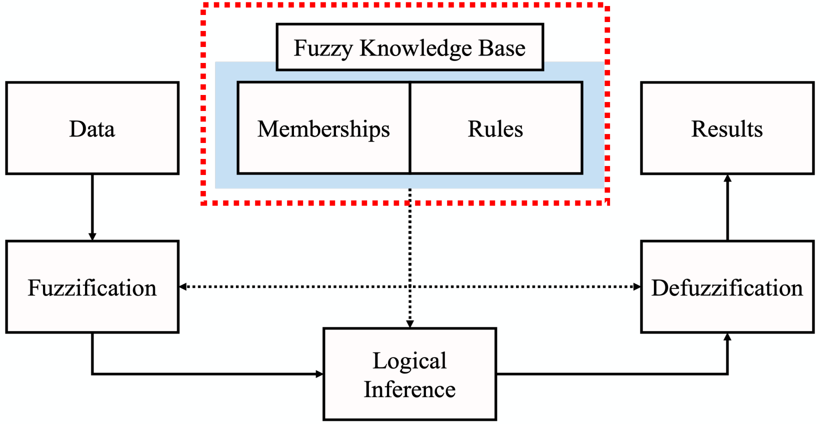
First, we can create the membership functions of each linguistic variable





Logical Rules	If **Hot** then **Content**	If **Cool** then **Happy**	If **Cold** then **Sad**

Fuzzy Logic : Example Problem

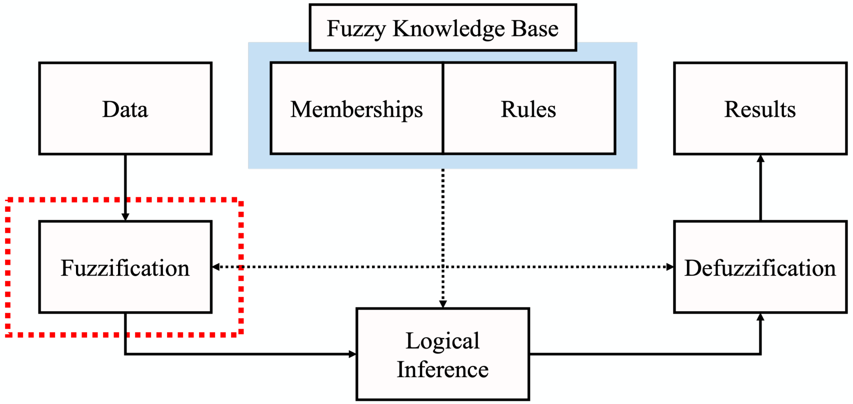


Second, need to specify the fuzzy operators for system functionality: Relation, Aggregation, and Defuzzification

Relation	Aggregation	Defuzzification
$R(\mu_{A'}, \mu_B) = min(\mu_{A'}, \mu_B)$	$\text{B'(y)} = \max_{i=0}^{k}\{B_i'(y)\}$	$y_c = \frac{\Sigma_{y \in Y}\, y * B'(y)}{\Sigma_{y \in Y} B'(y)}$
Correlation Minimum	Max Aggregation	Centroid Defuzzification

Similar to GA's, there's a lot of user specification. However, this allows for some creativity and flexibility for design

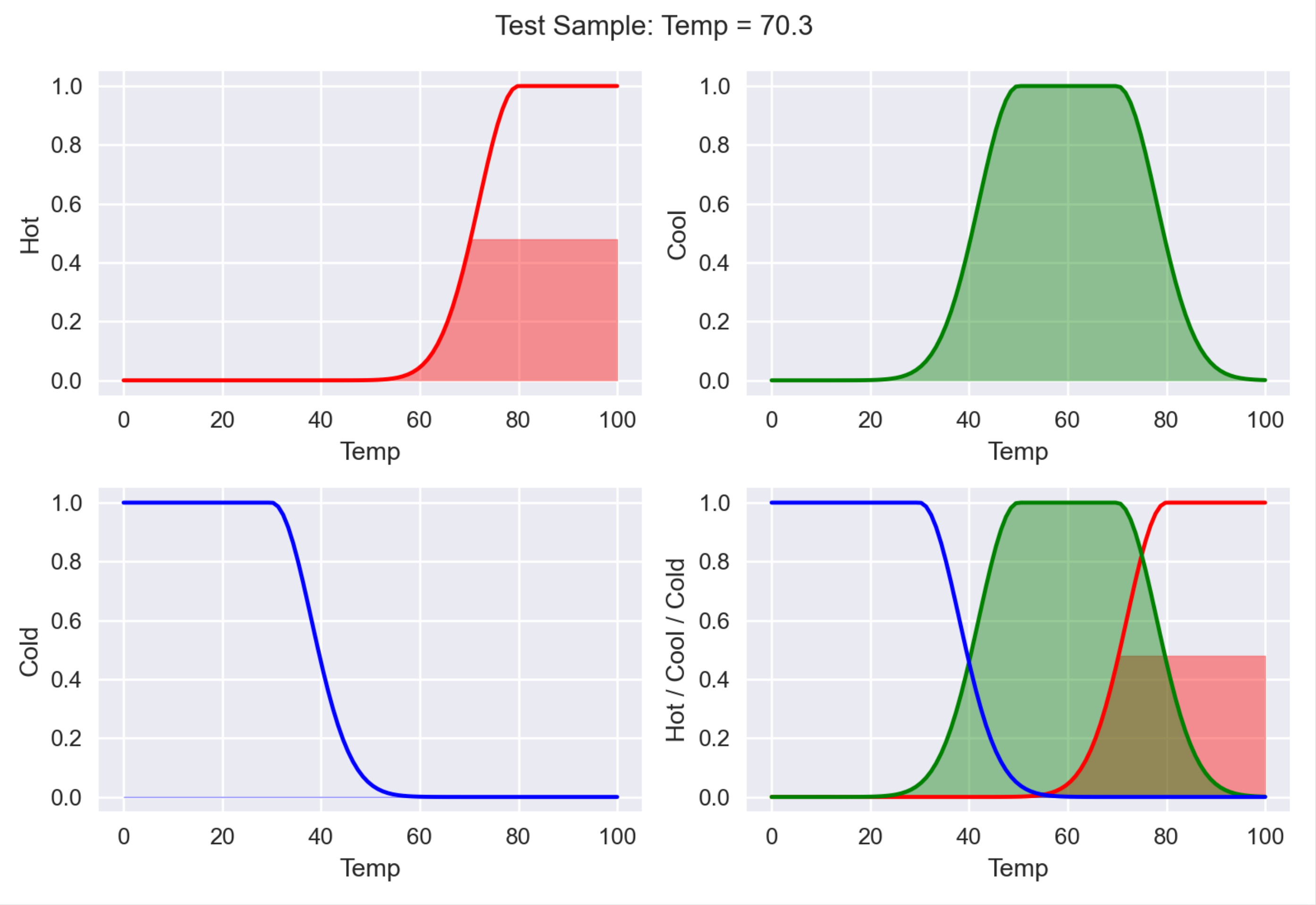
Fuzzy Logic : Example Problem



Third, we can evaluate a test observation through the system. However, since its non-fuzzy, we must invoke Fuzzification.

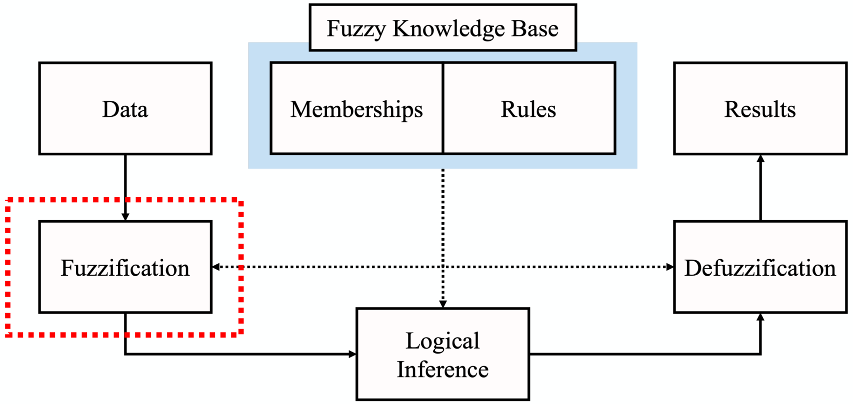
Temp = 70.3

Logical Rules
If **Hot** then **Content**
If **Cool** then **Happy**
If **Cold** then **Sad**

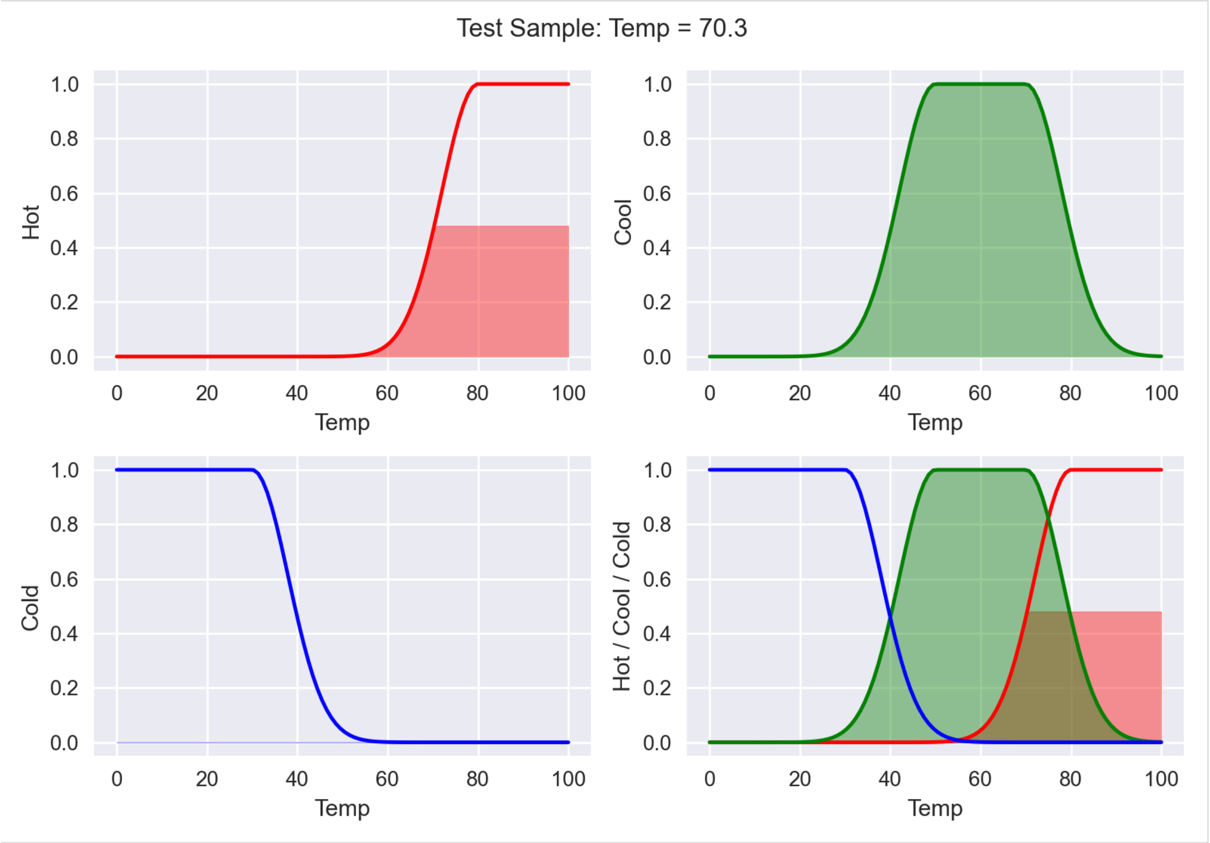


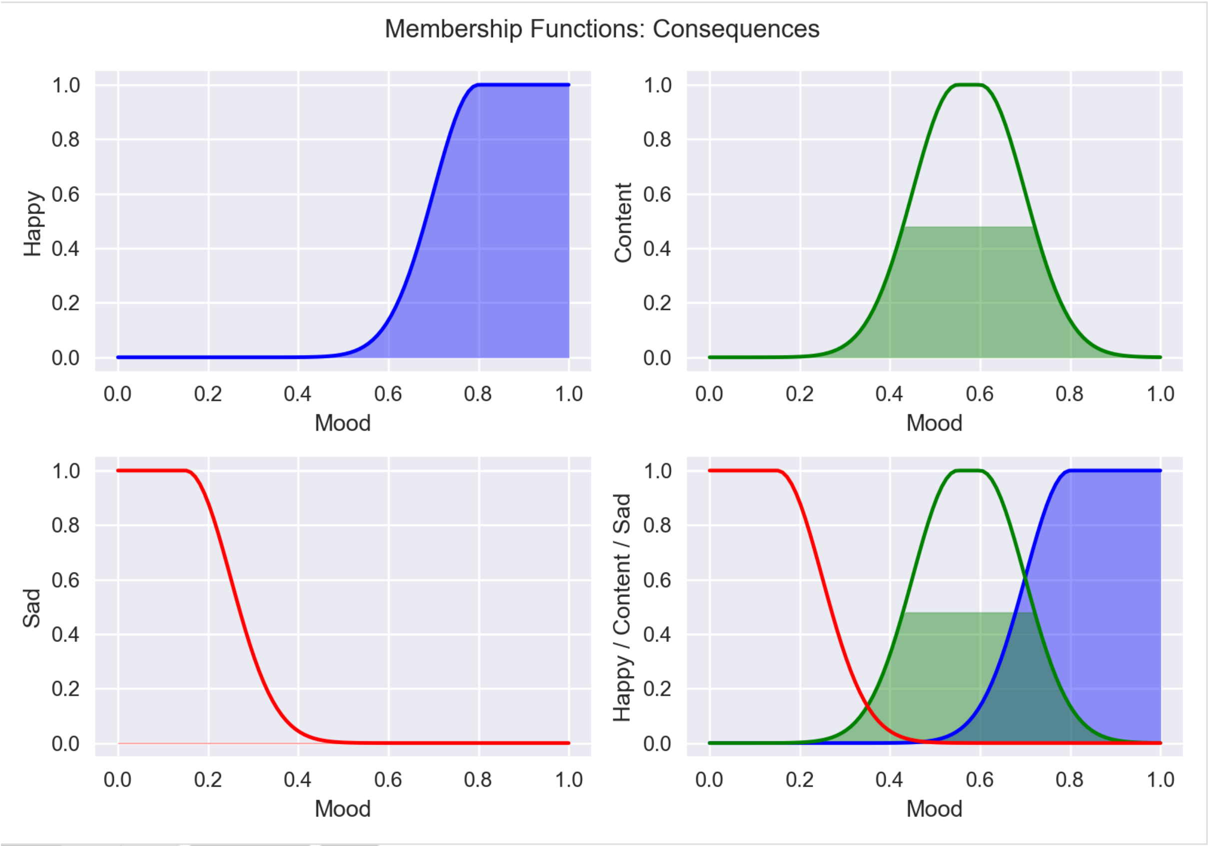
We can explicitly see how this observation interacts with our different antecedent memberships

Fuzzy Logic : Example Problem



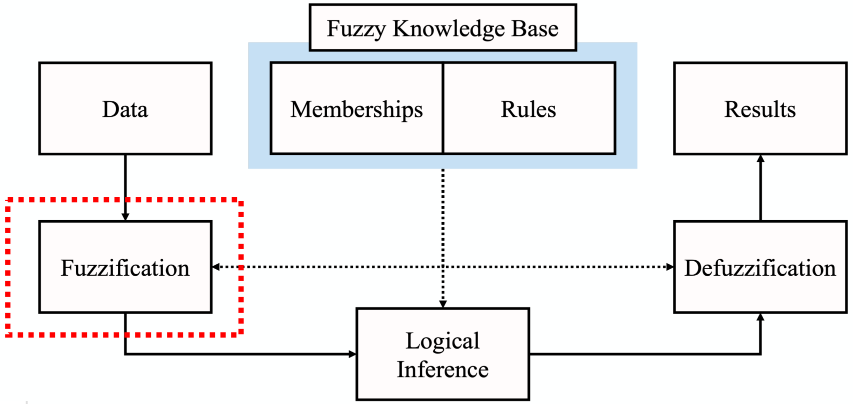
Fourth, we can invoke implication and use the prior calculated relation matrices and estimate a consequence for each rule



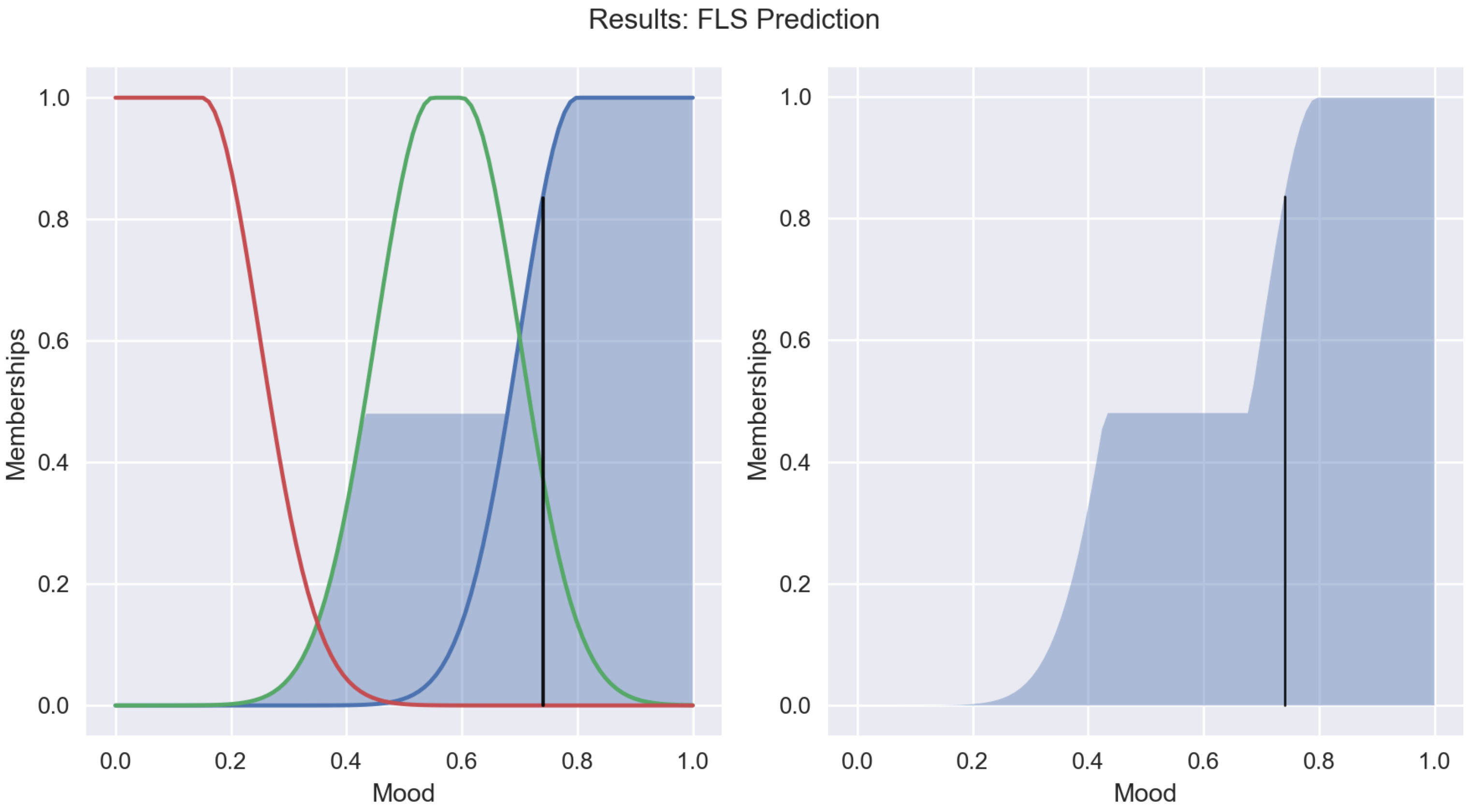


Logical Rules	If **Hot** then **Content**	If **Cool** then **Happy**	If **Cold** then **Sad**

Fuzzy Logic : Example Problem



Lastly, we can aggregate the predicted consequences and apply defuzzification to gather a crisp result



Aggregation gives us a single fuzzy set that expresses information from each rule

Defuzzification gives us a non-fuzzy-set result: scalar value (see the black bar line) that represents our mood prediction

Logical Rules	If **Hot** then **Content**	If **Cool** then **Happy**	If **Cold** then **Sad**