Analyzing Volatility Risk in Option Contracts

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REU Final Report

Spring 2019

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https://ise.illinois.edu

1. Introduction

The focus of this project is to analyze the volatility risk in option contracts with different approaches. We start with estimating the implied volatility of a stock in the Black-Scholes. Though this method is widely used in the industry, it is based on assumptions that do not entirely hold in practice. Hence, we consider stochastic volatility models such as the GARCH model in discrete time and a Heston-like volatility model in continuous time. In order to estimate the unobserved volatility, we implement a regression-based method introduced in the paper by Carr and Wu (2010). The tools used to clean and analyze the data are Python and R.

2. Black-Scholes Model and Implied volatility

In this part, we aim to reproduce the "implied volatility" and the "time to expiration vs skew" plot from the article (Kamal and Gatheral, 2006) and identify any patterns on the implied volatility surface. We use Apple call and put data from Cboe.com for this task. The selected data includes option with strike prices ranged from 100 to 425 and with expiration dates from Feb 1st, 2019 to June 18th, 2021.

2.1 Calculation of implied volatility

$$\mathrm{c} = S_0 \mathrm{e}^{-qT} \mathrm{N}(\mathrm{d}_1) - \mathrm{K}\mathrm{e}^{-rT} \mathrm{N}(d_2) \text{ ①} \qquad \mathrm{p} = -S_0 \mathrm{e}^{-qT} \mathrm{N}(-\mathrm{d}_1) + \mathrm{K}\mathrm{e}^{-rT} \mathrm{N}(-d_2) \text{ ②}$$

$$\mathrm{d}_1 = \frac{\ln(S_0) + (r - q + 0.5\sigma^2)T}{\sigma\sqrt{T}} \qquad d_2 = \mathrm{d}_1 - \sigma\sqrt{T}$$

where **K** is the strike price, $\mathbf{S_0}$ the current stock price, **q** the dividend, **r** the risk-free interest rate (LIBOR Rate) and time to expiration **T**. The above equations are European Call ("c") and Put ("p") option prices (i.e. the call and put Black-Scholes formulas). From the Apple option data, we obtain all parameters, such as **K**, $\mathbf{S_0}$, **q**, **r** and **T**. We plug the above values into the Black-Scholes equations, equate equation ① and ② with the corresponding market price of each option. Because some option data does not have a last price, we use Ask price for the calculation. The only unknown in the equation is **σ**, which is the volatility. We will solve the equation numerically and get an array of implied volatilities corresponding to different strike prices.

2.2 Volatility plot and skew plot

Due to the restriction of the data obtained, we decide to use the scatter plot instead of a surface plot to better illustrate the patterns.

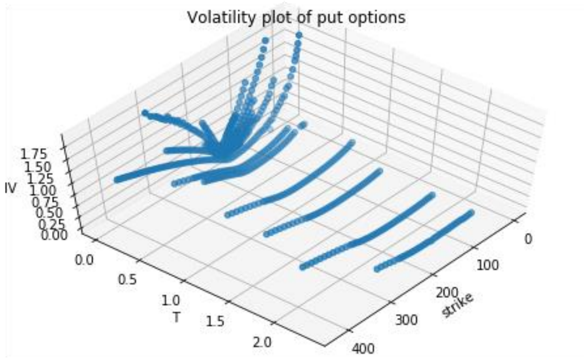

Figure 1. Volatility plot of put options

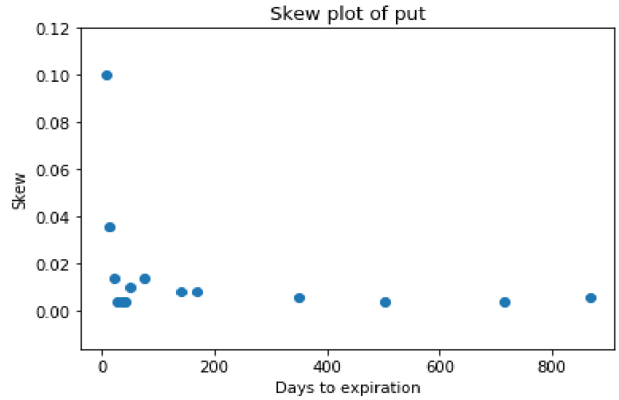

Figure 2. Skew plot of put options

From Figure 1, we can observe that for data points with the same time to expiration, the points of implied volatility show a 'smile' shape. This means the implied volatility increases as the option grows more in-the-money or out-of-the-money and the lowest implied volatility occurs when the option is at-the-money. Also, the 'smile' of short-term volatility has a steeper slope and it grows gentler as time to expiration increases.

The skew is a measure of the volatility surface. It is defined as the difference between the implied volatility at strike price at 105% and 95% of the forward price. Figure 2 shows the inverse relationship between skew and days to expiration described in the reading. The options with shorter days to expiration have higher skew while it decays to a constant level as the days of expiration increases.

2.3 Assumption of Black-Scholes Model

Implied volatility calculated from Black-Scholes Model is widely used in the industry but it has certain assumptions that may not be applicable in practice. One of it is the market efficiency. In other words, the price movement follows a random walk and the current price is independent of the past prices. In the real world, due to human psychological bias in decision making and other factors, the price may have certain relation with the past prices. Hence, other methods to estimate volatility are needed.

3. The GARCH model and alternative approaches

The second part of the research is to implement Generalized ARCH model (aka GARCH) to estimate the volatility of S&P 500 monthly price. The reference for this implementation is the section 3.5 and 3.15 of *Analysis of Financial Time Series* (Tsay, 2010). The data used for this task are S&P 500 monthly prices from 1/1/1950 to 2/1/2019, S&P 500 daily prices from 2014 to 2019 and VIX data till 2/1/2019. The data are from Yahoo Finance.

3.1 Autocorrelation and ARCH effect

We first convert the monthly price to a log return series and generate the autocorrelation plot (ACF and PACF) to analyze current prices correlation with past prices.

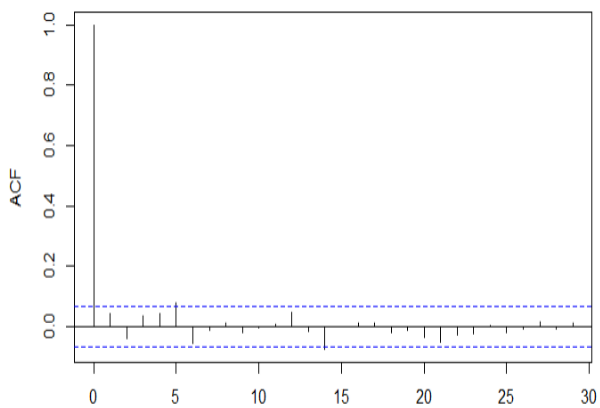

Figure 3. ACF plot

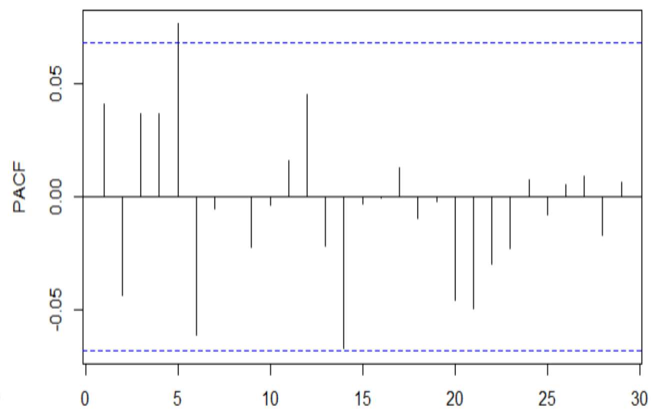

Figure 4. PACF plot

From the ACF and PACF plots above, we see that there is no serial correlation but there are some significant spikes in PACF plot, this suggest that the price changes are not serially independent and exhibit some extent of autocorrelation. Hence, ARCH effect is present and it is valid to use GARCH model.

3.2 GARCH Model and implementation

We implement GARCH model by Bollerslev (1986). Compared to ARCH model, GARCH model requires fewer parameters so it is simpler to determine. For the log return series, the innovation $a_t = r_t - \mu_t$, follows a GARCH (p, q) model:

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

Where ϵ_t is iid random variable with mean 0 and variance 1, $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$ and $\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1$. The coefficients of this model are determined via a maximum likelihood method. Based on the literature, we apply a GARCH (1, 1) model. There are 830 data points. We then split the data into train and test dataset. 810 data points are put to train and 20 are put to test. However, we do not have actual volatility data so we use rolling variance with a window of 15 points, VIX data and volatility calculated from higher frequency observations are used for comparison. After fitting the model, we obtain

$$\sigma_t^2 = 0.0001204 + 0.1190a_{t-1}^2 + 0.8163\sigma_{t-1}^2.$$

The result coefficients satisfy the model assumptions. Then, we predict the next 20 months' volatility variance. The plot is below:

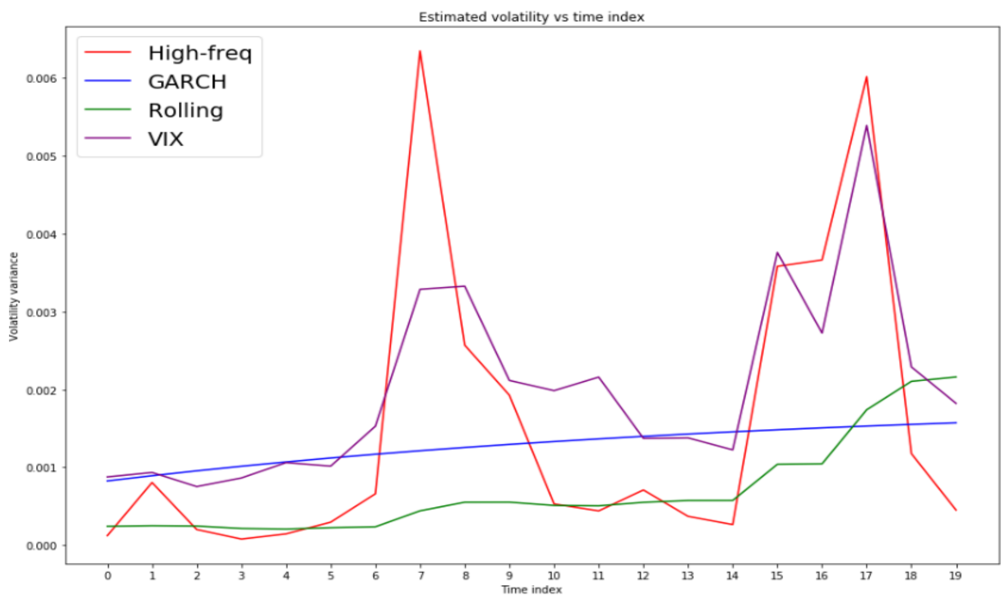

Figure 5. Estimated volatility vs time index

We can see all the lines are very close to each other at the beginning of the prediction. The rolling variance and GARCH prediction are generally close throughout the prediction horizon. The mean square error between them is only $5.33*10^{-7}$. The VIX and prediction by high-frequency data are close to GARCH prediction at the beginning but they deviate more in the later time. There are a few reasons which can explain this deviation. First, the GARCH prediction grows flatter in the long run due to the smoothing effect in the definition of the model which makes it less accurate in longer prediction. Also, calculation of VIX and prediction by high-frequency data is more continuous while GARCH is discrete based on monthly data.

4. Regression-based volatility estimate

The last part of the research is to implement the technique introduced in *Analyzing volatility risk and risk premium in option contracts: A new theory* (Carr and Wu, 2010). We start by considering the following stochastic volatility model:

$$\frac{dS_t}{S_t} = \sqrt{v_t}\, dW_t$$

where $\mathbf{S_t}$ is the stock price at time t. $\mathbf{v_t}$ denotes instantaneous variance rate i.e. the volatility at time t. $\mathbf{W_t}$ denotes a standard Brownian motion. For $\mathbf{v_t}$, it follows a positive, real-valued stochastic process such that there exist a unique solution to the stochastic differential equation.

Starting with the model above and based on properties of the implied volatility along with sophisticated calibration techniques, the authors suggest utilizing options sensitivities (Greeks) to calibrate the implied volatility surface. As a by-product they obtain a the following linear equation satisfied by the option's sensitivities:

$$-\text{Theta} + \text{r} - \text{r} * \text{Delta} = \beta_0 * Vega + \beta_1 * Gamma + \beta_2 * Vanna + \beta_3 * Volga$$

Specifically: Vega is the derivative of the option with respect to the volatility, Delta the derivative of the option with respect to the stock price at 0, Gamma the *second* derivative of the option with respect to the stock price at 0 and Theta (the derivative of the option with respect to time. They also include higher order sensitivities, such as the Vanna and the Volga, defined as follows:

$$\text{Vega} = \frac{\Delta C}{\Delta \sigma} = SN'(d_1)\sqrt{T} \quad where\ N'(d_1) = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}$$

$$\text{Vanna} = \frac{\text{Vega}}{S}\left(1 - \frac{d_1}{\sigma\sqrt{T}}\right) \quad \text{and} \quad \text{Volga} = \text{Vega} * d_1 * \frac{d_2}{\sigma}$$

Where **C** denotes the Black-Scholes option price, **S** denotes the stock price and **T** denotes the time to expiration. $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ are calculated from the Black-Scholes formula and $\boldsymbol{\sigma}$ is the implied volatility. The coefficients of the regression equation are quantities that depend on the model parameters, such as the risk-free rate r, the stock price S, the volatility v and so on.

Our goal is to use market values of the Greeks (Theta, Delta, Vega, Gamma, Vanna, Volga) for S&P 500 to fit the regression line above and estimate the coefficients. Of particular interest is the coefficient of Gamma, which is a function of the volatility, i.e. $\mathrm{vol} = \sqrt{\beta_1 * \frac{2}{\sqrt{S}}}$. By estimating β_1 we can obtain an estimator for the volatility. And by repeating this process with a rolling window, we obtain the daily volatility for a certain time period.

The data used in this part is S&P 500 option daily data from March 2018 to April 2018, along with the corresponding values of the sensitivities. In Figure 6, we plot the estimated volatility for two months by fitting the regression line. The square roots of log returns are also plotted for comparison.

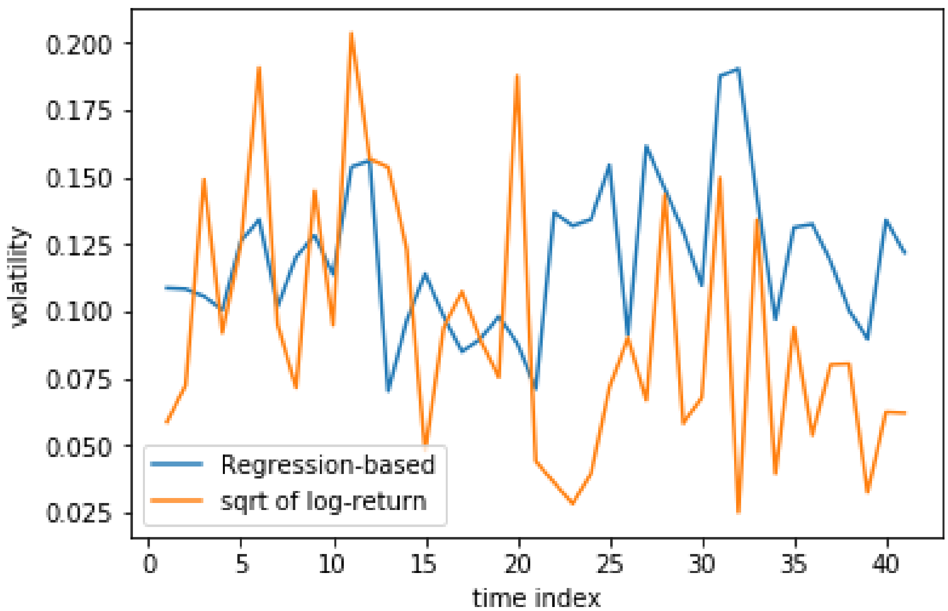

Figure 6. Regression-based volatility plot

From Figure 6, we see that the regression-based estimate is close to the estimate by square roots of log-return. It captures most of the features.

5. Conclusion and future work

In this research, we study the implied volatility surface and its characteristics. The volatility increases as the options go increasingly out-of-money and in-the-money. Also, we learnt that some of the assumptions of Black-Scholes Model may not hold. From the autocorrelation plot, the current price is not completely independent of past prices. Hence, we turn to GARCH model which uses the past volatility to predict future one. GARCH model perform well in short term prediction but due to its smoothing effect in the definition, the prediction will become flatter and inaccurate in the long run. In the end, we implement the new techniques based on regression of Greek letters and produce good estimates of volatility for current dataset. In the future, we would like to test whether the method is robust in longer series with higher frequency.

6. Acknowledgements

I would like to thank Professor Alexandra Chronopoulou, as well as PhD student Qi Zhao, for their kind guidance, help and encouragement. I am grateful to have this opportunity to learn more about volatility in option contracts. I would also like to thank Department of Industrial and Enterprise Systems Engineering for their support on the Research Experience for Undergraduate (REU) program.

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