COLUMBIA UNIVERSITY
Statistical Inference and Modeling
Quiz 3

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Name:

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You have 20 minutes to answer the following questions. Good luck!

Question 1 (4 points)

$W = (X, Y, Z)$

$W_A = (X, Y)$

$W_B = Z$

Let X, Y and Z be jointly normally distributed as $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.25 \\ 0 & 0.25 & 1 \end{bmatrix}.$$

$= \Sigma_{AA}$ $\quad = \Sigma_{AB}$ $\quad = \Sigma_{BB}$

Give the law of $(X, Y)|Z$ and compute the partial correlation of X and Y.

1 pt $\quad \mathbb{E}\left[\begin{pmatrix} X \\ Y \end{pmatrix} \Big| Z\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} \cdot 1 \, (z - 0) = \begin{pmatrix} 0 \\ \frac{1}{4} z \end{pmatrix}$

1,5 $\quad \operatorname{Var}\left[\begin{pmatrix} X \\ Y \end{pmatrix} \Big| Z\right] = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} \cdot 1^{-1} \cdot \begin{pmatrix} 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1/16 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1/2 \\ 1/2 & 15/16 \end{pmatrix}$$

0,5 $\quad$ Hence $\quad \begin{pmatrix} X \\ Y \end{pmatrix} \Big| Z \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 1/4 \, z \end{pmatrix}, \begin{pmatrix} 1 & 1/2 \\ 1/2 & 15/16 \end{pmatrix} \right)$

1 pt $\quad \rho_{partial}(X, Y) = \dfrac{1/2}{1 \cdot \sqrt{15/16}} = \dfrac{2}{\sqrt{15}}$

Question 2 (6 points)

The tables below report historical records of rainfall in Tel Aviv during the months of December, January and February. They report the frequencies of transitions from the states $\{0, 1\} = \{\text{“dry”}, \text{“wet”}\}$.

Table 1: Daily records of rainfall in Tel Aviv during the months of December, January and February

		Actual day		
		Dry	Wet	Total
Previous	Dry	1049	350	1399
day	Wet	351	687	1038

We use the data in Table 1 to fit a first order Markov model.

1. Give an estimate of $\mathbb{P}(X_1 = X_2 = X_3 = X_4 = X_5 = 1, X_6 = 0 | X_1 = 1)$.
2. Give a confidence interval for p_{01}.
3. Imagine a friend tells you that it is equally likely to have a nice day or a rainy day after a day of rain. What is the underlying null hypothesis of your friends assertion? how would you test it?
4. Give the likelihood ratio test for the testing whether a zeroth order model fits sufficiently well the data. What will be its asymptotic distribution?

1pt 1. $(\hat{p}_{11})^4 \hat{p}_{10} = \left(\frac{687}{1038}\right)^4 \cdot \frac{351}{1038}$

1pt 2. $\sqrt{n_{0\cdot}}\left(\hat{p}_{01} - p_{01}\right) \xrightarrow{D} N(0, p_{01}(1-p_{01}))$

So $CI_{1-\alpha}(p_{01}) = \left[\hat{p}_{01} \pm z_{1-\alpha/2} \cdot \sqrt{\frac{\hat{p}_{01}(1-\hat{p}_{01})}{n_{0\cdot}}}\right]$

$= \left[\frac{350}{1399} \pm 1{,}96 \sqrt{\frac{350 \cdot 1049}{1399}}\right]$

2pt. 3. $H_0 : p_{10} = 1/2$, can use test statistic based on asymptotic distribution i.e.

2pt $\sqrt{n_{1\cdot}} \frac{\hat{p}_{10} - 1/2}{\sqrt{\hat{p}_{10}(1-\hat{p}_{10})}} \overset{H_0}{\sim} N(0,1)$

4. $\Lambda_n = 2\left\{ n_{00} \log\left(\frac{\hat{p}_{00}}{\hat{p}_0}\right) + n_{01} \log \frac{\hat{p}_{01}}{\hat{p}_1} + n_{10} \log \frac{\hat{p}_{10}}{\hat{p}_0} + n_{11} \log \frac{\hat{p}_{11}}{\hat{p}_1} \right\}$

$\xrightarrow{D} \chi^2_1$