STAT5703 HW1 Ex3

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Exercise 3.

Question 1.

As $R_1 - \mu$ has a zero mean distribution, all moments with odd orders are zero. Therefore, we have,

$$
\begin{aligned}
\gamma = \mathbf{E}[R_1^3] &= E[(R_1 - \mu + \mu)^3] \\
&= \mathbf{E}[(R_1 - \mu)^3 + 3(R_1 - \mu)^2\mu + 3(R_1 - \mu)\mu^2 + \mu^3] \\
&= 3\mu\mathbf{E}[(R_1 - \mu)^2] + \mu^3 \\
&= 3\mu Var[R_1 - \mu] + \mu^3 \\
&= \mu^3 + 3\mu\sigma^2
\end{aligned}
$$

Question 2.

(a) Since $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ has the distribution of $\mathbf{N}(\mu, \sigma^2/n)$, similarly to Q1, we can derive $\mathbf{E}[\bar{R}^3] = \mu^3 + 3\mu\frac{\sigma^2}{n}$. So the bias is $\mathbf{E}[\hat{\gamma} - \gamma] = -\frac{n-1}{n}\mu\sigma^3$.

(b) $\hat{\gamma}$ is not consistent. Since $\bar{R} \sim N(\mu, \frac{\sigma^2}{n})$, we have,

$$
\begin{aligned}
\Pr[|\bar{R}^3 - (\mu^3 + 3\mu\frac{\sigma^2}{n})| \geq \epsilon] &= 1 - \Pr[|\bar{R}^3 - (\mu^3 + 3\mu\frac{\sigma^2}{n})| \leq \epsilon] \\
&= 1 - \Phi(\sqrt{n}\frac{(\mu^3 + 3\mu\frac{\sigma^2}{n} + \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\
&\quad + \Phi(\sqrt{n}\frac{(\mu^3 + 3\mu\frac{\sigma^2}{n} - \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\
&\to 1 - \Phi(\sqrt{n}\frac{(\mu^3 + \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) + \Phi(\sqrt{n}\frac{(\mu^3 - \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\
&\to 1 - \Phi(\infty) + \Phi(-\infty) \\
&= 1 - 1 + 0 = 0, \textit{as } n \to \infty \textit{ with fixed } \epsilon
\end{aligned}
$$

So $\hat{\gamma}$ converges to $\mu^3 + 3\mu\frac{\sigma^2}{n} \to \mu^3$, so it is not consistent to the estimated parameter $\gamma = \mu^3 + 3\mu\sigma^3$.

Question 3.

Since we have $\mathbf{E}[R_1R_2R_3] = \mu^3$ and $\mathbf{E}[\hat{\gamma}] = \mu^3 + \frac{3\mu\sigma^2}{n}$, we have $3\mu\sigma^2/n = \mathbf{E}[\hat{\gamma}] - \mathbf{E}[R_1R_2R_3]$.Therefore, we can choose $n\hat{\gamma} - (n-1)R_1R_2R_3$ as the unbias estimator, whose mean is exactly μ^3.

Question 4.

(a) Since $\mathbf{E}[\tilde{\gamma} - \gamma] = n * \frac{1}{n}\mathbf{E}[R_1^3] - \gamma = 0$, the bias is 0.

(b) $\tilde{\gamma}$ is consistent. Using LLT, $\tilde{\gamma} \overset{p}{\sim} \mathbf{E}[R_1^3] = \gamma$. So it's consistent.

Question 5.

Since the minimal sufficient statistics for normal distributions are $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ and $\overline{R^2} = \frac{1}{n}\sum_{i=1}^{n} R_i^2$. And they are also complete statistics. According to the Rao-Blackwell, we only need to find the conditional expection of an unbiased estimator by setting the two statistics as the condition. Therefore $\gamma_{UVME} = \mathbf{E}[\tilde{\gamma}|\bar{R}, \overline{R^2}]$. In the following, we use T to denote the condition. We have,

$$\begin{aligned}
\mathbf{E}[\tilde{\gamma}|T] &= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n} R_i^3|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}(R_i - \bar{R} + \bar{R})^3|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[(R_i - \bar{R})^3 + 3(R_i - \bar{R})^2\bar{R} \\
&\quad + 3(R_i - \bar{R})\bar{R}^2 + \bar{R}^3]|T]
\end{aligned}$$

By using symmmetry of the conditional distribution, one can prove that all (conditional) moments of $R_i - R$ which have odd orders are zero. Therefore, we have,

$$\begin{aligned}
\mathbf{E}[\tilde{\gamma}|T] &= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[3(R_i - \bar{R})^2\bar{R} + \bar{R}^3]|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[3R_i^2\bar{R} - 6R_i\bar{R}^2 + 3\bar{R}^3 + \bar{R}^3]|T] \\
&= \mathbf{E}[\frac{3}{n}\bar{R}\sum_{i=1}^{n} R_i^2 - 2\bar{R}^3|T] \\
&= 3\bar{R}\overline{R^2} - 2(\bar{R})^3
\end{aligned}$$