HW1

Exercise 1.

Question 1. $Q_D(p)$ is p^{th} population quantile such that $P(D \leq Q_D(p)) = p$.

In order word:

$$\int_0^{Q_D(p)} \lambda e^{-\lambda D} dD = 1 - e^{-\lambda Q_D(p)} = p$$

$$Q_D(p) = -\frac{1}{\lambda} \ln(1-p)$$

Question 2 First empirical moment of the exponential distribution:

$$\hat{\mu} = \bar{D_n}$$

Population moment of the exponential distribution:

$$E(D_1) = \frac{1}{\lambda}$$

The MOM estimator of λ is : $\hat{\lambda}^{MOM} = \frac{1}{\bar{D_n}}$

Therefore the method of moments-based estimator of $Q_D(p)$:

$$Q_D(p)^{MOM} = -\frac{1}{\hat{\lambda}^{MOM}} \ln(1-p) = -\bar{D_n} \ln(1-p)$$

Question3 From the CLT

$$\sqrt{n}(\bar{D_n} - \frac{1}{\lambda}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{\lambda^2})$$

Hence, by Delta Method we can get, let $g(t) = t * ln(1-p)$ so $g'(t) = ln(1-p)$

$$\sqrt{n}(\ln(1-p)\bar{D_n} + Q_D(p)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

Since $\hat{\lambda}^{MOM} = \frac{1}{\bar{D_n}}$

$$\sqrt{n}(\ln(1-p)\frac{1}{\lambda} + Q_D(p)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

So, the *approximate* $(1-\alpha)$-confidence interval for $Q_D(p)$ is

$$[-\bar{D_n}\ln(1-p) - \frac{z_{1-\alpha/2} \times \ln(1-p)}{\lambda\sqrt{n}}, -\bar{D_n}\ln(1-p) + \frac{z_{1-\alpha/2} \times \ln(1-p)}{\lambda\sqrt{n}}]$$

Question 4. We know that if $D_1, ..., D_n$ are independent exponential random variables with parameter λ, then

$$\sum_{i=1}^{n} D_i \sim \Gamma(n, \lambda)$$

Therefore

$$\lambda \bar{D_n} = \frac{\lambda}{n} \sum_{i=1}^{n} D_i \sim \Gamma(n, n)$$

So, $\lambda \bar{D_n}$ is independent of the parameter λ, which means it is an exact pivot.

From previous question

$$Q_D(p) = -\frac{1}{\lambda} \ln{(1-p)}$$

$$Q_D(0.5) = \frac{1}{\lambda} \ln{(2)}$$

To construct 95% confidence interval, let a and b be the 0.025 and 0.975 quantile of $\Gamma(n, n)$

Therefore

$$P(a < \lambda \bar{D_n} < b) = 0.95$$

$$P(\frac{\bar{D_n} ln(2)}{b} < \frac{1}{\lambda} \ln{(2)} < \frac{\bar{D_n} ln(2)}{a}) = 0.95$$

The confidence interval is $[\frac{\bar{D_n} ln(2)}{b}, \frac{\bar{D_n} ln(2)}{a}]$