STAT5703 HW1 Ex1

Chao Huang (ch3474), Wancheng Chen (wc2687), Chengchao Jin (cj2628)

Exercise 1.

Question 1.

To calculate p^{th}, we need to find $Q_D(p)$ such that $P(D \leq Q_D(p)) = p$. Then, the previous function can be transformed to

$$\int_0^{Q_D(p)} \lambda e^{-\lambda D} dD = 1 - e^{-\lambda Q_D(p)} = p$$

So, from this equation, $Q_D(p)$ can be expressed by

$$Q_D(p) = -\frac{1}{\lambda} \ln(1-p)$$

Question 2.

From question(a), we already obtain the equation for $Q_D(p)$ which is $Q_D(p) = -\frac{1}{\lambda}\ln(1-p)$. Then, to find the MLE of $Q_D(p)$, we can find the MLE of λ first and then replace λ with its Maximum Likelihood Estimator $\hat{\lambda}^{MLE}$. $D_1, ..., D_n$ are i.i.d. Exponential random variables with parameter λ, the log-likelihood function is

$$\ell(\lambda; D_1, ..., D_n) = n \ln \lambda - \sum_{i=1}^{n} \lambda D_i$$

The MLE $\hat{\lambda}^{MLE}$ is

$$\hat{\lambda}^{MLE} = \frac{1}{\bar{D_n}}$$

and

$$Q_D(p)^{MLE} = -\frac{1}{\hat{\lambda}^{MLE}} \ln(1-p) = -\bar{D_n} \ln(1-p)$$

Question 3.

$D_1, ..., D_n \overset{i.i.d.}{\sim} Exp(\lambda)$. Then the CLT tells us that

$$\sqrt{n}(\bar{D_n} - \mu) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

Hence, by Delta Method we can get,

$$\sqrt{n}(Q_D(p) + \frac{\ln(1-p)}{\lambda}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

Then, for *approximate* $(1-\alpha)$-confidence interval,

$$L(D) = -\bar{D_n} \ln(1-p) - \frac{z_{1-\alpha/2} \times \ln(1-p)}{\lambda\sqrt{n}}$$

$$R(D) = -\bar{D_n} \ln{(1-p)} + \frac{z_{1-\alpha/2} \times \ln{(1-p)}}{\lambda\sqrt{n}}$$

So, the *approximate* $(1-\alpha)$-confidence interval for $Q_D(p)$ is $[-\bar{D_n} \ln{(1-p)} - \frac{z_{1-\alpha/2} \times \ln{(1-p)}}{\lambda\sqrt{n}}, -\bar{D_n} \ln{(1-p)} + \frac{z_{1-\alpha/2} \times \ln{(1-p)}}{\lambda\sqrt{n}}]$

Question 4.

We know that if $D_1, ..., D_n$ are independent exponential random variables with parameter λ, then

$$\lambda\bar{D_n} \sim \Gamma(n, n)$$

So, $\lambda\bar{D_n}$ is independent of the parameter λ, which means it is an exact pivot. To construct an exact confidence interval of the median, we can first transform $\lambda\bar{D_n}$ to χ^2 distribution. Then,

$$2n\lambda\bar{D_n} \sim \chi^2_{2n}$$

Hence, for any $\alpha \in (0, 1)$,

$$P(\chi^2_{1-\alpha/2,2n} < 2n\lambda\bar{D_n} < \chi^2_{\alpha/2,2n}) = 1 - \alpha$$

Since $Q_D(0.5) = -\bar{D_n} \ln 0.5$, then

$$P(-\frac{\ln 0.5\chi^2_{\alpha/2,2n}}{2n\lambda} < Q_D(0.5) < -\frac{\ln 0.5\chi^2_{1-\alpha/2,2n}}{2n\lambda}) = 1 - \alpha$$

Hence, the $(1-\alpha)$ exact confidence interval of the median is,

$$Q_D(0.5) \in [-\frac{\ln 0.5\chi^2_{\alpha/2,2n}}{2n\lambda}, -\frac{\ln 0.5\chi^2_{1-\alpha/2,2n}}{2n\lambda}]$$