# STAT5703 HW1 Ex2

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## Exercise 2.

### Question 1.

Based on the equation: $(E(X))^2 + D(X) = E(X^2)$. We can get the Method of Moments estimator for $\lambda$:

$$\hat{\lambda_M}^2 + \hat{\lambda_M} = E(X^2) = \mu_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

After simplification,

$$(\hat{\lambda_M} + \frac{1}{2})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 + \frac{1}{4}$$

$$\hat{\lambda_M} = \frac{-1 + \sqrt{1 + \frac{4}{n}\sum_{i=1}^{n} X_i^2}}{2}$$

### Question 2.

Let $g(t) = \frac{-1+\sqrt{1+4t}}{2}$, so $g(\bar{X_n^2}) = \lambda$ Because $Var(X_i^2) = 4\lambda^3 + 6\lambda^2 + \lambda$, we can get the asymptotic distribution of $\bar{X_n}$:

$$\sqrt{n}(\bar{X_n} - \mu_2) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, 4\lambda^3 + 6\lambda^2 + \lambda)$$

Using Delta Method,

$$\sqrt{n}(\hat{\lambda_M} - \lambda) = \sqrt{n}(g(\bar{X_n}) - g(\mu_2)) \xrightarrow[n\to\infty]{\mathcal{D}} g'(\mu_2)\mathcal{N}(0, 4\lambda^3 + 6\lambda^2 + \lambda)$$

Because $g'(t) = \frac{1}{\sqrt{1+4t}}$, $\mu_2 = \lambda^2 + \lambda$

$$\sqrt{n}(\hat{\lambda_M} - \lambda) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{4\lambda^3 + 6\lambda^2 + 1}{4\lambda^2 + 4\lambda + 1})$$

### Question 3.

The methods of moments estimator for $\hat{\lambda_M}$ using the first moment is:

$$\hat{\lambda_{M1}} = \bar{X_n}$$

The methods of moments estimator for $\hat{\lambda_M}$ using the second moment is:

$$\hat{\lambda_{M2}} = \frac{-1 + \sqrt{1 + \frac{4}{n}\sum_{i=1}^{n} X_i^2}}{2}$$

Because both of the two estimators are unbiased:

$$Eff(\lambda_{M1}, \lambda_{M2}) = \frac{MSE(\hat{\lambda_{M1}})}{MSE(\hat{\lambda_{M2}})} = \frac{Var(\hat{\lambda_{M1}})}{Var(\hat{\lambda_{M2}})} \xrightarrow[n\to\infty]{} \frac{\lambda}{\frac{4\lambda^3+6\lambda^2+1}{4\lambda^2+4\lambda+1}} = \frac{4\lambda^3 + 4\lambda^2 + \lambda}{4\lambda^3 + 6\lambda^2 + 1} < 1$$

So the methods of moments estimator for $\hat{\lambda_M}$ using the first moment is more efficient.

**Question 4.**

The asymptotic distribution of $\bar{X_n}$ is:

$$\sqrt{n}(\bar{X_n} - \mu) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \lambda)$$

Because $\hat{\lambda_{M1}} = \bar{X_n}$, using the Delta Method: The asymptotic distribution of $\hat{\lambda_{M1}}$ using the first moment is:

$$\sqrt{n}(\hat{\lambda_{M1}} - \lambda) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \lambda)$$

Then, for *approximate* $(1-\alpha)$-confidence interval,

$$L(\lambda) = \lambda - \frac{\lambda z_{1-\alpha/2}}{\sqrt{n}}$$

$$R(\lambda) = \lambda + \frac{\lambda z_{1-\alpha/2}}{\sqrt{n}}$$

So, the *approximate* $(1-\alpha)$-confidence interval for $\hat{\lambda_{M1}}$ is $[\lambda - \frac{\lambda z_{1-\alpha/2}}{\sqrt{n}}, \lambda + \frac{\lambda z_{1-\alpha/2}}{\sqrt{n}}]$

According to point 2, the asymptotic distribution of $\hat{\lambda_{M1}}$ using the second moment is:

$$\sqrt{n}(\hat{\lambda_{M2}} - \lambda) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{4\lambda^3 + 6\lambda^2 + 1}{4\lambda^2 + 4\lambda + 1})$$

Then, for *approximate* $(1-\alpha)$-confidence interval,

$$L(\lambda) = \lambda - \frac{(4\lambda^3 + 6\lambda^2 + 1)z_{1-\alpha/2}}{\sqrt{n}(4\lambda^2 + 4\lambda + 1)}$$

$$D(\lambda) = \lambda + \frac{(4\lambda^3 + 6\lambda^2 + 1)z_{1-\alpha/2}}{\sqrt{n}(4\lambda^2 + 4\lambda + 1)}$$

So, the *approximate* $(1-\alpha)$-confidence interval for $\hat{\lambda_{M2}}$ is $[\lambda - \frac{(4\lambda^3+6\lambda^2+1)z_{1-\alpha/2}}{\sqrt{n}(4\lambda^2+4\lambda+1)}, \lambda + \frac{(4\lambda^3+6\lambda^2+1)z_{1-\alpha/2}}{\sqrt{n}(4\lambda^2+4\lambda+1)}]$

While both the estimators have the same midpoint. The Method of Moments estimator for $\lambda$ based on the first moment has a smaller upper bound, bigger lower bound and smaller confidence intervals.