

$$1. E[X] = \int_0^1 \beta(1-x)^{\beta-1} \cdot x \, dx = - \int_0^1 x \, d((1-x)^{\beta})$$

$$= - \int_0^1 (1-x)^{\beta} \, dx + x(1-x)^{\beta} \Big|_0^1 = - \frac{1}{\beta+1} (1-x)^{\beta+1} \Big|_0^1 = \frac{1}{\beta+1}$$

$$E[X^2] = \int_0^1 \beta(1-x)^{\beta-1} \cdot x^2 \, dx = - \int_0^1 x^2 \, d((1-x)^{\beta}) = \int_0^1 (1-x)^{\beta} 2x \, dx$$

$$= \frac{E[X]}{\beta} \cdot 2 = \frac{2}{\beta(\beta+1)} \quad \text{Var}[X] = \frac{2}{\beta(\beta+1)} - \left(\frac{1}{\beta+1}\right)^2 = \frac{\beta+2}{\beta(\beta+1)^2} \hat{\beta}_{ML} = \frac{1}{\bar{X}_n} - 1$$

$$2. \sqrt{n} \left( \bar{X}_n - \frac{1}{\beta+1} \right) \sim N \left( 0, \frac{2}{\beta(\beta+1)} \right) \quad g(t) = \frac{1}{t} - 1 \quad g'(t) = -\frac{1}{t^2}$$

$$\sqrt{n} \left( \frac{1}{\bar{X}_n - 1} - \beta \right) \sim N \left( 0, \frac{\beta+2}{\beta(\beta+1)^2} \cdot \frac{1}{1/(\beta+1)^2} \right)$$

$$\therefore \sqrt{n} \left( \hat{\beta}_{ML} - \beta \right) \sim N \left( 0, \frac{(\beta+2)(\beta+1)^2}{\beta} \right)$$

$$3. \beta_{ML} = \underset{\beta}{\operatorname{argmax}} \sum_i \log \beta + (\beta-1) \log(1-x_i) = \underset{\beta}{\operatorname{argmax}} \ell(x; \beta)$$

$$\frac{\partial \ell(x; \beta)}{\partial \beta} = \frac{n}{\beta} + \sum_i \log(1-x_i) = 0 \Rightarrow \hat{\beta}_{ML} = - \frac{n}{\sum_i \log(1-x_i)}$$

$$4. \quad y_i = -\log(1-x_i) \quad E[y_i] = \int_0^1 -\log(1-x_i) \cdot \beta(1-x_i)^{\beta-1} \, dx_i$$

$$= \int_0^1 \log(1-x_i) \, d((1-x_i)^{\beta})$$

$$= - \int_0^1 (1-x_i)^{\beta} \, d \log(1-x_i)$$

$$= \int_0^1 (1-x_i)^{\beta-1} \, dx_i$$

$$= - \frac{1}{\beta} (1-x_i)^{\beta} \Big|_0^1 = \frac{1}{\beta}$$

$$\begin{aligned}
 E[y_i^2] &= -\int_0^1 [\log(1-x_i)]^2 d(1-x_i)^\beta \\
 &= \int_0^1 (1-x_i)^\beta d[\log(1-x_i)]^2 = \int_0^1 (1-x_i)^\beta \cdot 2 \log(1-x_i) \cdot (-1) dx_i \\
 &= -\int_0^1 2(1-x_i)^{\beta-1} \log(1-x_i) dx_i = \frac{2}{\beta^2}
 \end{aligned}$$

$$\text{Var}[y_i] = \frac{2}{\beta^2} - \left(\frac{1}{\beta}\right)^2 = \frac{1}{\beta^2}$$

$$\sqrt{n}(\bar{y}_n - \frac{1}{\beta}) \sim N(0, \frac{1}{\beta^2}) \quad f(t) = \frac{1}{t} \quad f'(t) = -\frac{1}{t^2}$$

$$\therefore \sqrt{n}(\hat{\beta}_{ML} - \beta) \sim N(0, \frac{1}{\beta^2} \cdot \beta^4) = N(0, \beta^2)$$

$$5. \quad \text{Eff} = \frac{\text{MSE}[\hat{\beta}_M]}{\text{MSE}[\hat{\beta}_{ML}]} = \frac{(\beta+2)(\beta+1)^2}{\beta^3} \xrightarrow{\beta \rightarrow \infty} 1$$

$$6. \quad \eta_{ML} = \frac{1}{\hat{\beta}_{ML}} = -\bar{y}_n$$

$$7. \quad \text{Use LRT} \quad \Lambda = 2(l(x; \beta) - l(x; \beta_0))$$

$$8. \quad \Lambda \sim \chi_1^2 \quad P(\Lambda > \Lambda_0) \leq \alpha$$

$$9. \quad \begin{array}{c} \text{Graph of } \chi_1^2 \text{ distribution with area } \alpha \text{ shaded in the right tail.} \\ \text{Critical value } z_{\frac{1-\alpha}{2}} \text{ is marked on the x-axis.} \end{array} \quad \begin{array}{c} \text{Test statistic:} \\ Z_{\frac{1-\alpha}{2}} \leq \frac{\hat{\beta}_{ML} - \beta}{\beta} \cdot \sqrt{n} \leq Z_{\frac{1+\alpha}{2}} \end{array}$$

$$\frac{\beta_{ML}}{\beta} - 1 \leq \frac{\frac{z_{1+\alpha/2}}{2}}{\sqrt{n}}$$

$$\frac{\beta_{ML}}{\beta} \leq \frac{\frac{z_{1+\alpha/2}}{2}}{\sqrt{n}} + 1$$

$$\beta \geq \frac{\beta_{ML}}{\frac{\frac{z_{1+\alpha/2}}{2}}{\sqrt{n}} + 1}$$

$$\beta \leq \frac{\beta_{ML}}{\frac{\frac{z_{1-\alpha/2}}{2}}{\sqrt{n}} + 1}$$