1. $E[X] = \int_0^1 \beta(1-x)^{\beta-1} \cdot x \, dx = -\int_0^1 x \, d((1-x)^\beta)$

$= -\int_0^1 (1-x)^\beta dx + x(1-x)^\beta \Big|_0^1 = -\frac{1}{\beta+1}(1-x)^{\beta+1} \Big|_0^1 = \frac{1}{\beta+1}$

$E[X^2] = \int_0^1 \beta(1-x)^{\beta-1} \cdot x^2 dx = -\int_0^1 x^2 d(1-x)^\beta = \int_0^1 (1-x)^\beta 2x \cdot dx$

$= \frac{E[X]}{\beta} \cdot 2 = \frac{2}{\beta(\beta+1)}$. $Var[X] = \frac{2}{\beta(\beta+1)} - \frac{1}{(\beta+1)^2} = \frac{\beta+2}{\beta(\beta+1)^2}$ $\hat{\beta}_M = \frac{1}{\overline{X}_n} - 1$

2. $\sqrt{n}(\overline{X}_n - \frac{1}{\beta+1}) \sim N(0, \frac{2}{\beta(\beta+1)})$ $g(t) = \frac{1}{t} - 1$ $g'(t) = -\frac{1}{t^2}$

$\sqrt{n}(\frac{1}{\overline{X}_n} - 1 - \beta) \sim N(0, \frac{\beta+2}{\beta(\beta+1)^2} \cdot \frac{1}{1/(\beta+1)^2}$

$\therefore \sqrt{n}(\hat{\beta}_M - \beta) \sim N(0, \frac{(\beta+2)(\beta+1)^2}{\beta})$

3. $\beta_{ML} = \underset{\beta}{\operatorname{argmax}} \sum_i \log\beta + (\beta-1)\log(1-x_i) = \underset{\beta}{\operatorname{argmax}}\, l(x;\beta)$

$\frac{\partial l(x;\beta)}{\partial \beta} = \frac{n}{\beta} + \sum_i \log(1-x_i) = 0 \Rightarrow \beta_{ML} = -\frac{n}{\sum_i \log(1-x_i)}$

4. $y_i = -\log(1-x_i)$ $E[y_i] = \int_0^1 -\log(1-x_i) \cdot \beta(1-x_i)^{\beta-1} dx_i$

$= \int_0^1 \log(1-x_i) \, d(1-x_i)^\beta$

$= -\int_0^1 (1-x_i)^\beta \, d\log(1-x_i)$

$= \int_0^1 (1-x_i)^{\beta-1} \, dx_i$

$= -\frac{1}{\beta}(1-x_i)^\beta \Big|_0^1 = \frac{1}{\beta}$

$$E[y_i^2] = -\int_0^1 [\log(1-x_i)]^2 \, d(1-x_i)^\beta$$

$$= \int_0^1 (1-x_i)^\beta \, d[\log(1-x_i)]^2 = \int_0^1 (1-x_i)^\beta \cdot 2\log(1-x_i) \cdot$$

$$= -\int 2(1-x_i)^{\beta-1} \cdot \log(1-x_i) \, dx_i = \frac{2}{\beta^2}$$

$$Var[y_i] = \frac{2}{\beta^2} - \frac{1}{\beta^2} = \frac{1}{\beta^2}$$

$$\sqrt{n}\left(\bar{y}_n - \frac{1}{\beta}\right) \sim N\left(0, \frac{1}{\beta^2}\right) \qquad f(t) = \frac{1}{t} \quad f'(t) = -\frac{1}{t^2}$$

$$\therefore \sqrt{n}(\beta_{ML} - \beta) \sim N\left(0, \frac{1}{\beta^2} \cdot \beta^4\right) = N(0, \beta^2)$$

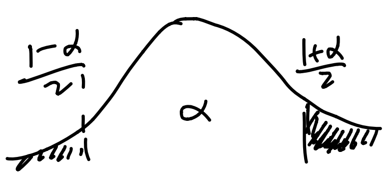
5. $$Eff = \frac{MSE[\hat{\beta}_M]}{MSE[\hat{\beta}_{ML}]} = \frac{(\beta+2)(\beta+1)^2}{\beta^3} \rightarrow \; > 1$$

$$\xrightarrow[\beta \to \infty]{} 1$$

6. $$\eta_{ML} = \frac{1}{\beta_{ML}} = -\bar{y}_n$$

7. Use LRT. $\Lambda = 2(l(x;\beta) - l(x;\beta_0))$

8. $\Lambda \sim \chi_1^2 \qquad P(\Lambda > \Lambda_0) \leq \alpha$

9. 

$$\left[z_{\frac{1-\alpha}{2}}, z_{\frac{1+\alpha}{2}}\right]$$

$$z_{\frac{1-\alpha}{2}} \leq \frac{\beta_{ML} - \beta}{\beta} \cdot \sqrt{n} \leq z_{\frac{1+\alpha}{2}}$$

$$\frac{\beta_{ML}}{\beta} - 1 \leq \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}}$$

$$\frac{\beta_{ML}}{\beta} \leq \frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} + 1$$

$$\beta \geq \frac{\beta_{ML}}{\frac{z_{\frac{1+\alpha}{2}}}{\sqrt{n}} + 1}$$

$$\beta \leq \frac{\beta_{ML}}{\frac{z_{\frac{1-\alpha}{2}}}{\sqrt{n}} + 1}$$