Continuous Random Variables (I)

Let X be a continuous random variable

$\Updownarrow$

ie the cdf F_X is a continuous function.

Let $f_X(x) \rightsquigarrow$ probability density function (pdf) of X

$\Updownarrow$

satisfies: $\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt}$

Recall:

① $f_X(x)$ must satisfy:

(i) $f_X(x) \geq 0$ for all $x \in (-\infty,\infty)$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

② The expected value of X is

$$\boxed{E(X) = \mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

③ The variance of X is

$$\boxed{V(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx}$$

④ The standard deviation of X is

$$\boxed{\sigma_X = \sqrt{\sigma_X^2}}$$

Important Identities:

① $E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$

$\Updownarrow$

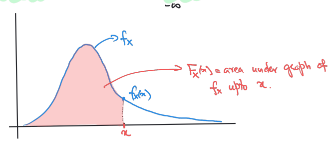
$\boxed{E(aX+b) = aE(X)+b}$, $\boxed{V(aX+b) = a^2 V(X)}$

② $\boxed{V(X) = E(X^2) - (E(X))^2}$

Calculating Probabilities

The cdf is defined as

$$F_X(x) := P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$



Note:

① $P(a \le X \le b) = \int_a^b f_X(x)\,dx \quad (a \le b)$

$= \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$

$= F_X(b) - F_X(a)$

② $P(X=c) = P(c \le X \le c) = \underbrace{F_X(c) - F_X(c)}_{=0}$

$\Updownarrow$

Probability that a continuous rv take a particular value is zero!!!

③ $P(X>c) = 1 - P(X \le c) = 1 - F_X(c)$

$P(X \le c) = P(X<c) + \cancel{P(X=c)}$

$= P(X<c)$

We can use the cdf F_X to calculate $f_X(x)$

$\Updownarrow$

If the derivative $F_X'(x)$ exist at x

$$F_X'(x) = f_X(x)$$

Let $p \in (0,1) \rightsquigarrow$ The $(100 \cdot p)^{th}$ Percentile for the distribution of X

$\Updownarrow$

$\eta(p)$ and satisfies

$\Updownarrow$

$$p = F_X(\eta(p)) = \int_{-\infty}^{\eta(p)} f_X(x)\,dx$$

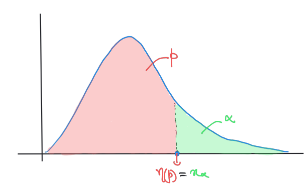
Let $\alpha \in (0,1) \rightsquigarrow$ the α^{th} critical value for the distribution of X is

$\Updownarrow$

x_α and satisfies

$\Updownarrow$

$$\alpha = P(X > x_\alpha) = 1 - F_X(x_\alpha)$$



Note: $(100\,p)^{th}$ percentile $\xrightarrow{\text{same as}}$ $(1-p)^{th}$ critical value.

Example: Uniform Continuous Distribution

Let $[A,B] \subseteq \mathbb{R}$ be a closed interval $(A<B)$

$\Updownarrow$

Define: $\boxed{f(x) = \begin{cases} \frac{1}{B-A} & x \in [A,B] \\ 0 & \text{otherwise} \end{cases}}$ $\rightsquigarrow$ pdf of the uniform cont distribution

then $\boxed{F(x) = \begin{cases} 0 & \text{if } x < A \\ \frac{x-A}{B-A} & \text{if } x \in [A,B] \\ 1 & \text{if } x > B \end{cases}}$ $\rightsquigarrow$ cdf of the uniform cont distribution.

Let X have the uniform distribution on $[A,B]$

$\Updownarrow$

then $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_A^B \frac{x}{(B-A)}\,dx$

$= \frac{(B+A)}{2}$

$V(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 f(x)\,dx = \int_A^B \frac{(x-\frac{B+A}{2})^2}{B-A}\,dx$

$= \frac{(B-A)^2}{12}$

If $a,b \in [A,B]$ and $a<b$

$$\boxed{P(a \le X \le b) = \int_a^b \frac{dx}{B-A} = \frac{(b-a)}{(B-A)}}.$$

Suppose X is a r.v

$F_X \rightsquigarrow$ cdf of X

$F_X(x) := P(X \le x)$

If $F_X(x)$ is a continuous function
$\Updownarrow$
X is a continuous r.v.

The probability density function, $f_X(x)$ satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

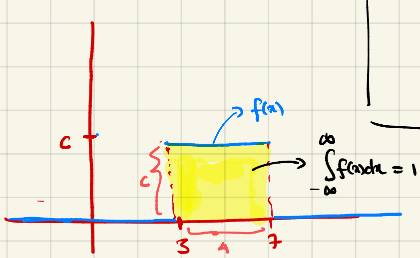
$f_X(x)$ satisfies (i) $f_X(x) \ge 0 \ \forall x \in (-\infty, \infty)$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Example: ① Uniform continuous distribution

Suppose $[3, 7] \subseteq \mathbb{R}$

$$f(x) = \begin{cases} c & x \in [3,7] \\ 0 & \text{otherwise} \end{cases}$$



If f has to be a pdf:

(i) $f(x) \ge 0$ for all $x \in (-\infty, \infty)$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Area $= 4 \cdot c = \int_{-\infty}^{\infty} f(x)dx$

$\therefore \int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow 4 \cdot c = 1$

$\Rightarrow c = \frac{1}{4}$

$\therefore$
$$f(x) = \begin{cases} \frac{1}{4} & x \in [3,7] \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = \cancel{\int_{-\infty}^{3} 0.dx} + \int_{3}^{7} c.dx + \cancel{\int_{7}^{\infty} 0.dx}$$

$$= \int_{3}^{7} c.dx = c(7-3) = \underline{\underline{4 \cdot c}}$$

② from 4.1 Devore

$$f(x) = \begin{cases} \frac{x}{\theta^2} \cdot e^{-x^2/2\theta^2} & x > 0 \\ 0 & \text{otherwise} \cdot (x \le 0) \end{cases}$$

Use R to plot this curve

```
rayliegh <- function(x, theta){
  return(x/θ^2 .    )
}
x <- seq(1, 100, length.out=1000)
y = rayleugh (x, θ=4.2)
```

(i) Show that $f(x)$ is a pdf.

(a) $f(x) \ge 0 \quad \forall x \in (-\infty, \infty)$

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{0} 0.dx} + \underbrace{\int_{0}^{\infty} \frac{x}{\theta^2} \cdot e^{-x^2/2\theta^2}\,dx}_{\text{improper integral of type II.}}$$

$$= \lim_{c \to \infty} \int_{0}^{c} \left(\frac{x}{\theta^2} e^{-x^2/2\theta^2}\right) dx$$

$$\int_{0}^{c} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx$$

$u = x^2/2\theta^2 \qquad x = 0 \quad u = 0$

$du = \frac{2x\,dx}{2\theta^2} \qquad x = c \quad u = \frac{c^2}{2\theta^2}$

$\parallel$

$$\int_{0}^{c^2/2\theta^2} e^{-u}\,du = \frac{e^{-u}}{-1}\Bigg|_{0}^{c^2/2\theta^2} \qquad du = \frac{x\,dx}{\theta^2}$$

$$= \underline{\underline{\left(1 - e^{-c^2/2\theta^2}\right)}}$$

$\therefore$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \lim_{c \to \infty} \left(1 - e^{-c^2/2\sigma^2}\right)$$

$$= 1$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx \rightsquigarrow \text{expected value of } X$$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu_x)^2 \cdot f(x)\,dx \rightsquigarrow \text{Variance of } X.$$

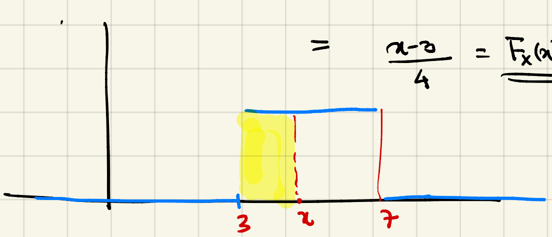
$$f(x) \rightsquigarrow \text{pdf of } X.$$

Examples:

$$f(x) = \begin{cases} \frac{1}{4} & x \in [3\ 7] \\ 0 & \text{otherwise.} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{3} 0\,dt + \int_{3}^{x} \frac{1}{4}\,dt$$

$$= \frac{x-3}{4} = \underline{\underline{F_X(x)}}$$



Example

$$E(X) = \int_{3}^{7} \frac{x}{4}\,dx$$

$$= \frac{1}{4} \cdot \frac{x^2}{2}\Big|_3^7 = \frac{1}{4} \cdot \frac{(7^2-3^2)}{2}$$

$$= \frac{(7-3)(7+3)}{4 \cdot 2} = \frac{\cancel{4} \cdot (7+3)}{\cancel{4} \cdot 2} = \frac{7+3}{2} = \underline{\underline{5}}$$

Useful formula:

$$\boxed{V(X) = E(X^2) - (E(X))^2}$$