Random Variables

Suppose we have an experiment with sample space S.

$\Downarrow$

we might be interested in a particular property of outcomes $\omega \in S$, as opposed to being interested in the outcome ω itself.

Examples:

① Toss a two sided coin 5-times.

$|S| = 2^5 = 32$.

$S = \{HHHHH, HHHHT, \ldots, TTTTH, TTTTT\}$

Then given an $\omega \in S$, we might be interested in:

a) number of heads in the outcome ω.

b) are there at least two heads in ω.

c) is the number of heads equal to number of tails in ω.

d) are there more heads than tails in ω.

② Roll two dice

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), \ldots$$
$$\vdots$$
$$(6,1), (6,2), \ldots, (6,6)\}$$

$|S| = 36$.

For any $\omega \in S$, we might be interested in

a) is the sum greater than 5.

b) is there at least one even number

c) is the sum divisible by 3.

d) is the max greater than 4.

③ Suppose a bag has 5 red balls, 8 blue balls

We draw 3 balls from the bag.

$$S = \{RRR, RRB, RBB, BBB\}$$
$$RBR \quad BRB$$
$$BRR \quad BBR$$

a) count the number of red balls in a outcome

b) are number of red balls more than number of blue balls?

Let S be the sample space of an experiment

Random variable X $\rightsquigarrow$ A function whose domain is S and range a new sample space (typically a subset of $\mathbb{R}$)

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ie $X: S \longrightarrow \mathbb{R}$

$\omega \longmapsto X(\omega)$

The image of X $\rightsquigarrow$ $X(S) :=$ values of X on the outcomes in S

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Specify a new sample space for the original experiment

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in the context of a certain property of interest.

Suppose $c \in X(S)$ $\rightsquigarrow$ value of X,

then

$$\boxed{\{X=c\} = \{\omega \in S : X(\omega) = c\}}$$

= all outcomes $\omega \in S$ which are mapped to c under X.

Important fact

Let $\mathcal{X} \subseteq \mathbb{R}$ be the set of values of X

then

$$\boxed{S = \bigcup_{c \in \mathcal{X}} \{X=c\}}$$

$\rightarrow$ disjoint union.

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The subsets $\{X=c\} \subseteq S$ form a partition of S.

Example:

Suppose we toss a coin 4 times independently

then

$$S = \{HHHH, HHHT, HHTT, TTTH, TTTT\}$$
$$HHTH \quad HTHT \quad TTHT$$
$$HTHH \quad HTTH \quad THTT$$
$$THHH \quad THTH \quad HTTT$$
$$THHT$$
$$TTHH$$

Let $X(\omega) = \#$ heads in the outcome ω.

$\Updownarrow$

$X: S \longrightarrow \mathbb{R}$ is a random variable

$X(HHHT) = 3$, $X(HHHH) = 4$, $X(HTHT) = X(THTH) = 2$

Let $\mathcal{X}$ = values of X

$= \{0, 1, 2, 3, 4\}$

Calculate $\{X=c\}$ for $c \in \mathcal{X}$.

$\{X=0\} = \{TTTT\}$

$\{X=1\} = \{HTTT, THTT, TTHT, TTTH\}$

$\{X=2\} = \{HHTT, HTHT, HTTH, THTH, THHT, TTHH\}$

$\{X=3\} = \{HHHT, HHTH, HTHH, THHH\}$

$\{X=4\} = \{HHHH\}$

Observe that

$\bigcup_{c \in \mathcal{X}} \{X=c\} = S$ in this case.

We can define probabilities of events $\{X=c\}$

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$$\boxed{P(\{X=c\}) = \sum_{\omega \in \{X=c\}} P(\omega)}$$

$\rightarrow$ probability distribution for the sample space S.

$\therefore$ $P(X=1) = P$(getting exactly one head in 4 independent tosses)

$= P(HTTT) + P(THTT) + P(TTHT) + P(TTTH)$

If we assume all outcomes in S are equally likely

we will have

$P(HTTT) = P(THTT) = P(TTHT) = P(TTTH) = \frac{1}{16}$

$\therefore$ $P(X=1) = 4 \cdot \frac{1}{16}$

Similarly can calculate

$P(X=0) = \frac{1}{16}$, $P(X=2) = \frac{6}{16}$, $P(X=3) = \frac{4}{6}$, $P(X=4) = \frac{1}{16}$

We can organize this information into a table

$x \in \mathcal{X}$	0	1	2	3	4
$P(X=x)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

$\Updownarrow$

"Probability distribution table for X"

Note that

$P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$

$= \frac{16}{16} = 1$

$\parallel$

$P(\bigcup_{c \in \mathcal{X}} \{X=c\}) = P(S) = 1$