Continuous Random Variables (I)

Let X be a continuous random variable
$\Updownarrow$
ie the cdf F_X is a continuous function.

Let $f_X(x) \rightsquigarrow$ probability density function (pdf) of X
$\Updownarrow$
satisfies: $\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt}$

Recall:

① $f_X(x)$ must satisfy:

(i) $f_X(x) \geq 0$ for all $x \in (-\infty, \infty)$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

② The expected value of X is

$$\boxed{E(X) = \mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

③ The variance of X is

$$\boxed{V(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx}$$

④ The standard deviation of X is

$$\boxed{\sigma_X = \sqrt{\sigma_X^2}}$$

Important Identities:

① $E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$
$\Downarrow$
$\boxed{E(aX+b) = aE(X)+b}$, $\boxed{V(aX+b) = a^2 V(X)}$

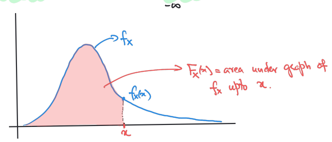
②

$$\boxed{V(X) = E(X^2) - (E(X))^2}$$

Calculating Probabilities

The cdf is defined as

$$F_X(x) := P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$



Note:

① $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$ $(a \leq b)$

$= \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$

$= F_X(b) - F_X(a)$

② $P(X = c) = P(c \leq X \leq c) = F_X(c) - F_X(c)$
$= 0$
$\Updownarrow$
Probability that a continuous rv take a particular value is zero !!!

③ $P(X > c) = 1 - P(X \leq c) = 1 - F_X(c)$

We can use the cdf F_X to calculate $f_X(x)$
$\Updownarrow$
If the derivative $F_X'(x)$ exist at x

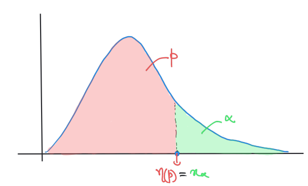
$$\boxed{F_X'(x) = f_X(x)}$$

Let $p \in (0,1) \rightsquigarrow$ The "$(100\cdot p)^{th}$" Percentile for the distribution of X
$\Updownarrow$
$\eta(p)$ and satisfies
$\Updownarrow$

$$p = F_X(\eta(p)) = \int_{-\infty}^{\eta(p)} f_X(x)\,dx$$

Let $\alpha \in (0,1) \rightsquigarrow$ the α^{th} critical value for the distribution of X is
$\Updownarrow$
x_α and satisfies
$\Updownarrow$

$$\alpha = P(X > x_\alpha) = 1 - F_X(x_\alpha)$$



Note: $(100p)^{th}$ percentile $\xrightarrow{\text{same as}}$ $(1-p)^{th}$ critical value.

Example: Uniform Continuous Distribution

Let $[A,B] \subseteq \mathbb{R}$ be a closed interval $(A < B)$
$\Updownarrow$
Define:

$$\boxed{f(x) = \begin{cases} \frac{1}{B-A} & x \in [A,B] \\ 0 & \text{otherwise} \end{cases}} \rightsquigarrow \text{pdf of the uniform cont distribution}$$

then

$$\boxed{F(x) = \begin{cases} 0 & \text{if } x < A \\ \frac{x-A}{B-A} & \text{if } x \in [A,B] \\ 1 & \text{if } x > B. \end{cases}} \rightsquigarrow \text{cdf of the uniform cont distribution.}$$

Let X have the uniform distribution on $[A,B]$
$\Updownarrow$
then

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_A^B \frac{x}{(B-A)}\,dx = \frac{(B+A)}{2}$$

$$V(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 f(x)\,dx = \int_A^B \frac{(x - \frac{B+A}{2})^2}{B-A}\,dx = \frac{(B-A)^2}{12}$$

If $a, b \in [A,B]$ and $a < b$

$$\boxed{P(a \leq X \leq b) = \int_a^b \frac{dx}{B-A} = \frac{(b-a)}{(B-A)}}.$$