# Probability and Counting

Suppose

a) $S \rightsquigarrow$ sample space of an exp with finitely many outcomes.

ie $S$ is a finite set.

b) Every outcome in $S$ is equally likely.

ie if $S = \{s_1, s_2, s_3, \dots, s_n\}$

$P(\{s_i\}) = \frac{1}{n}$ for all $i = 1, 2, 3, \dots, n$.

**Examples:** ① Sample one element from $\{A, B, C\}$

$S = \{A, B, C\}$

$P(\{A\}) = P(\{B\}) = P(\{C\}) = \frac{1}{3}$.

$\therefore$ Every outcome is equally likely.

② Sample one element from $\{A, A, B, C\}$

$S = \{A, B, C\}$

$P(\{A\}) = \frac{2}{4}$, $P(\{B\}) = \frac{1}{4} = P(\{C\})$.

Every outcome of this exp is not equally likely.

Suppose $S \rightsquigarrow$ finite sample space.

$A \subset S \rightsquigarrow$ an event.

If every outcome in $S$ is equally likely.

$$\boxed{P(A) = \text{“Probability that } A \text{ occurs.”} = \frac{n(A)}{n(S)}.}$$

where, $n(A) = \#$ of objects in $A$.

**Note:** In this scenario $\rightsquigarrow$ calculating probability

same as counting outcomes in an event.

Elementary, but not easy!

# Counting

“Fundamental Theorem of Counting”

Suppose there are $k$ tasks : $T_1, T_2, \dots, T_k$

can be performed in $n_1, n_2, \dots, n_k$

Let $T$ be the task of performing $T_1, T_2, \dots, T_k$ sequentially.

Then the total number of ways to perform $T$ is

$$\boxed{\#\text{ways to perform } T = n_1 \times n_2 \times n_3 \times \cdots \times n_k = \begin{pmatrix}\#\text{ways to}\\ \text{do } T_1\end{pmatrix} \times \begin{pmatrix}\#\text{ways to}\\ \text{do } T_2\end{pmatrix} \times \cdots \times \begin{pmatrix}\#\text{ways to}\\ \text{do } T_k\end{pmatrix}}$$

Typically we want

“To count the number of ways of selecting $k$ objects from a set of $n$ objects”

**Example:** Select 3 numbers from $\{0,1,2,3,4,5,6,7,8,9\}$

Four possibilities.

| | Without Replacement | With Replacement |
|---|---|---|
| ordered | (1,1,2) not allowed. (1,2,3) different from (2,1,3). | (1,1,2) allowed (1,1,2) different from (1,2,1). |
| unordered. | (1,1,2) not allowed. (1,2,3) same as (2,1,3) same as (3,1,2). | (1,1,2) allowed. (1,1,2) same as (1,2,1) same as (2,1,1) |

Number of possible arrangements of size $k$ from $n$ objects.

| | Without replacement | With replacement |
|---|---|---|
| Ordered | ${}^nP_k = \frac{n!}{(n-k)!}$ “n permute k” | $n^k$ |
| Unordered. | ${}^nC_k = \frac{n!}{(n-k)!\,k!}$ “n choose k” | ${}^{n+k-1}C_k$ |

For Proofs $\rightsquigarrow$ Use Fundamental Theorem.

(1) Ordered without Replacement

$\underline{n} \times \underline{(n-1)} \times \underline{(n-2)} \cdots \times \underline{(n-k+1)} \rightsquigarrow$ total ways

$n \times (n-1) \times (n-2) \times \cdots (n-k+1) = \frac{n!}{(n-k)!} := {}^nP_k$.

② Unordered Without replacement

**Step 1:** ${}^nP_k = \frac{n!}{(n-k)!}$ ordered arrangements.

**Step 2:** Each ordered arrangement can be rearranged $k!$ times.

get rid of repeats by dividing by $k!$

$\therefore$ total ways $= \frac{{}^nP_k}{k!} = \frac{n!}{(n-k)!\,k!} := {}^nC_k = \binom{n}{k}$.

③ Ordered with replacement

total # of ways $= \underline{n} \times \underline{n} \times \underline{n} \times \underline{n} \cdots \times \underline{n}$

$= n^k$

④ Unordered with replacement

**Want:** number of unordered arrangements of size $k$ from $n$ objects with replacement

reformulate

# ways to choose $k$ “walls” from $(n+k-1)$ choices.

ie ${}^{n+k-1}C_k = \binom{n+k-1}{k}$ ways.

**Example:** $n = 10$, $k = 3$

$S = \{0,1,2,3,4,5,6,7,8,9\}$

$n+k-1 = 12 \rightsquigarrow$ _ _ _ _ _ _ _ _ _ _ _ _

Then,

$(1,1,2) \rightsquigarrow$ 1 X X 2 X 3 4 5 6 7 8 9

$(0,0,7) \rightsquigarrow$ X X 1 2 3 4 5 6 7 X 8 9

$(5,9,7) \rightsquigarrow$ 1 2 3 4 5 X 6 7 X 8 9 X