Continuous Random Variables (IV)

Goal: Study Lognormal and Beta distributions

① Lognormal Distribution

We say X has a Lognormal Distribution $\xrightarrow{\text{if}}$ $\ln(X)$ has normal distribution with parameters μ and σ^2

ie $\boxed{\ln(X) \sim N(\mu, \sigma^2)}$

If X has lognormal distribution

pdf:
$$f(x;\mu,\sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma x} e^{-(\ln(x)-\mu)^2/2\sigma^2} & \text{if } x>0 \\ 0 & \text{otherwise.} \end{cases}$$

Note: ① μ and σ^2 $\longrightarrow$ mean and Variance of $\ln(X)$

ie
$$E(\ln(X)) = \mu$$
$$V(\ln(X)) = \sigma^2$$

② If X has lognormal distribution

$$E(X) = \mu_X = e^{\mu+\sigma^2/2}$$

$$V(X) = \sigma_X^2 = e^{(2\mu+\sigma^2)}\left(e^{\sigma^2}-1\right)$$

③ Can use standard normal table to calculate probabilities:

$$F_X(x;\mu,\sigma) = P(X\le x) = P(\ln(X)\le \ln(x))$$
$$= P\left(Z \le \frac{\ln(x)-\mu}{\sigma}\right) \quad \left(\because \ln(X)\sim N(\mu,\sigma^2)\right)$$
$$= \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right) \quad \text{for } x>0.$$

② Beta Distribution

takes values in a finite interval

good to model proportions $\longrightarrow$ naturally lie in $(0,1)$.

X is said to have Beta Distribution with parameters: $\alpha,\beta>0$ $\xrightarrow{\text{if}}$ pdf of X is

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1}$$
$$x\in(0,1).$$

Note: ① Called Beta distribution because the Beta function

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha,\beta>0$$

so that
$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } x\in(0,1).$$

② If $X \sim \text{Beta}(\alpha,\beta)$

$$E(X) = \mu_X = \frac{\alpha}{\alpha+\beta}$$

$$V(X) = \sigma_X^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$\longrightarrow$ Notice that both μ_X and σ_X^2 are in $(0,1)$.

③ Depending on values of α,β $\longrightarrow$ the pdf $f(x;\alpha,\beta)$ has different shapes.

If
- a) $\alpha>1, \beta=1$ $\longrightarrow$ strictly increasing
- b) $\alpha=1, \beta>1$ $\longrightarrow$ strictly decreasing $(\alpha=1,\ \beta=2)$
- c) $\alpha<1, \beta<1$ $\longrightarrow$ U-shaped
- d) $\alpha=\beta$ $\longrightarrow$ symmetric about $\frac{1}{2}$, with $\mu_X = \frac{1}{2}$ and $\sigma_X^2 = \frac{1}{4(2\alpha+1)}$
- e) $\alpha=\beta=1$ $\longrightarrow$ get uniform distribution on $(0,1)$.

Cauchy Distribution

X is said to have Cauchy Distribution with parameter: θ $\xrightarrow{\text{if}}$ the pdf of X is

$$f(x;\theta) = \frac{1}{\pi}\cdot\frac{1}{1+(x-\theta)^2} \quad x\in(-\infty,\infty),\ \theta\in(-\infty,\infty)$$

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)$$
$$\tan^{-1}(x)\Big|_{-\infty}^{\infty} = \pi$$

Note: ① $E(X)$ and $V(X)$ do not exist if X has Cauchy distribution. //

② The graph of $f(x;\theta)$ is bell shaped $\longrightarrow$ like the Normal Density.

But has heavier tails (than Normal density)

$$f(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2} \qquad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\pi}\left(\frac{x}{1+x^2}\right)dx$$

odd function

$= 0$ ✗

Lognormal Distribution

Y has lognormal distribution: μ, σ^2 $\rightsquigarrow$ if $\ln(Y)$ has the normal dist with parameters μ and σ^2

ie $\ln(Y) \sim N(\mu, \sigma^2)$

What is the pdf of Y ?

$$F_Y(y) = P(Y \leq y) = P(\ln(Y) < \ln(y))$$

$$= \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2} \, dx.$$

to get pdf of Y

$$\frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(\int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-(x-\mu)^2/2\sigma^2} \, dx.\right.$$

$$=$$

Plot various density functions of $Y \sim \text{logNormal}(\mu, \sigma^2)$.