Probability Distributions of Random Variables

Let $S \rightsquigarrow$ sample space of an experiment.

A random variable $X \rightsquigarrow$ function $X : S \longrightarrow \mathbb{R}$

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the values $\mathcal{X}$ of X define a new sample space for the original experiment

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X (or $\mathcal{X}$) extracts specific information about an outcome in S.

Examples:

Experiment	Random variable	Sample values
① Toss a coin five times	count the number of heads	$X(HHTTH) = 3 = X(HHHTT)$ $X(HTTTT) = 1$
	proportion of heads in the outcome	$X(HHTTH) = X(HHHTT) = \frac{3}{5}$ $X(HTTTT) = X(THTTT) = \frac{1}{5}$
② Roll a six sided dice 4-times	X = sum of the numbers facing up	$X(1,2,4,4) = 11$ $X(1,1,1,6) = 9$ $X(6,6,6,2) = 20$
	X = average of numbers in the sample	$X(1,2,4,4) = \frac{11}{4}$ $X(1,1,1,6) = 9/4$ $X(6,6,6,2) = 20/4$
③ Choose a box of milk cartons	X = weight of the box	$X(\text{box } 1) = 5.1$ $X(\text{box } 2) = 5.2$ $X(\text{box } 3) = 5.0$

Note: Given an experiment with sample space S, and an random variable $X : S \longrightarrow \mathbb{R}$

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two sample spaces: S and $\mathcal{X}$ = values of X

For simplicity, assume $|S|$ is finite

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$S = \{s_1, s_2, s_3, \ldots, s_n\}$

If $X : S \longrightarrow \mathbb{R}$ is a r.v

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Let $\mathcal{X}$ = values of X $= \{x_1, x_2, \ldots, x_k\}$

Two sample spaces for original experiment

S

⇕

Simple outcomes are $\{s_j\}$ $j = 1, 2, 3, \ldots, n$

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event $E \subseteq S$ described in terms of $\{s_j\}$

$\mathcal{X}$

⇕

Simple out comes are $\{x_i\}$ $i = 1, 2, \ldots, k$

$\{X = x_i\} = \{s \in S : X(s) = x_i\} \subseteq S$

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events $E \subset \mathcal{X}$ are described in terms of $\{X = x_i\}$

Let P be a probability function for S.

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use P to "induce" a probability function P_X on $\mathcal{X}$

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Step 1) $\boxed{P_X(\{x_i\}) := P(X = x_i) = \sum_{\omega \in \{X = x_i\}} P(\omega)}$

outcomes in $\mathcal{X}$ — Event in S — outcome in S

Step 2) Suppose $E \subseteq \mathcal{X}$,

$$\boxed{P_X(E) := P\Big(\bigcup_{x \in E} \{x_i\}\Big) = \sum_{x \in E} P_X(\{x_i\})}$$

Note:

① The random variable will be denoted by upper case letters $X, Y, \ldots$ etc

② The values of the random variable will be denoted by lower case letters: $x, y, \ldots$ etc.

③ When there is no ambiguity, we will use the notation $P(X = x_i)$ instead of $P_X(X = x_i)$.

Distributions of Random Variables

Let $X \rightsquigarrow$ random variable

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P_X: probability function for X.

The cumulative distribution function of X

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$$\boxed{F_X(x) := P(X \leq x) \quad \text{for all } x}$$

$F_X : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto F_X(x) = P_X(X \leq x)$

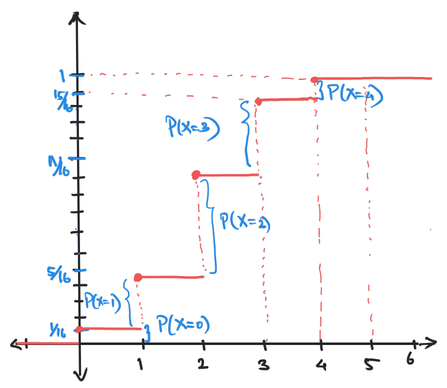
Example: Suppose we toss a fair coin 4 times. X = # heads in 4 tosses.

We can calculate

$P(X=0) = \frac{1}{16}$, $P(X=1) = \frac{4}{16}$, $P(X=2) = \frac{6}{16}$, $P(X=3) = \frac{4}{16}$, $P(X=4) = \frac{1}{16}$

We can calculate $F_X(x)$ as

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ 1/16 & \text{if } 0 \leq x < 1 \\ 5/16 & \text{if } 1 \leq x < 2 \\ 11/16 & \text{if } 2 \leq x < 3 \\ 15/16 & \text{if } 3 \leq x < 4 \\ 16/16 = 1 & \text{if } 4 \leq x < \infty \end{cases}$$



Note:

① $F_X(x)$ has jumps at $x = 0, 1, 2, 3, 4$ and the jump equals $P(X = x_i)$.

② $F_X(x)$ is a non-decreasing function

③ $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

④ $F(x) \geq 0 \quad \forall x \in \mathbb{R}$

Note: Every cdf satisfies properties ②, ③ and ④.

Thm: A function $F(x)$ is a cdf if and only if the following three conditions are satisfied

a) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

b) $F(x)$ is a non-decreasing function

c) $F(x)$ is right continuous, ie $\lim_{x \to x_0^+} F(x) = F(x_0)$, $\forall x_0 \in \mathbb{R}$

$X \rightsquigarrow$ a random variable

X is Discrete

if $F_X(x)$ is a step function

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The probability mass function pmf is

$$\boxed{f_X(x) = P(X = x) \quad \forall x}$$

X is Continuous

if $F_X(x)$ is a continuous function

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The probability density function (pdf) $f_X(x)$ satisfies

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \forall x.}$$

Thm: $f_X(x)$ is a pdf (or pmf) of a random variable if and only if

a) $f(x) \geq 0$ for all x

b) $\sum_x f_X(x) = 1$ or $\int_{-\infty}^{\infty} f(x)\,dx = 1$