Conditional Probability and Independence

Conditional Probability $\rightsquigarrow$ decisions under the influence of information.

Given two events $A, B \subseteq S$

Want: "The probability that A occurs given that B has already occured".

denoted by: $P(A|B)$ = Probability of A given B.

Definition: Suppose $A, B \subseteq S$, and $P(B) > 0$ then

$$\boxed{P(A|B) := \frac{P(A \cap B)}{P(B)}}.$$

Example: Assume all outcomes in S are equally likely and that $|S| < \infty$.

Suppose A, B are two non-empty subsets of S.

Want to calculate $P(A|B)$.

Note: $A|B \rightsquigarrow$ assumes B has happened

$\therefore$ sample space reduces to only those outcomes that are in B.

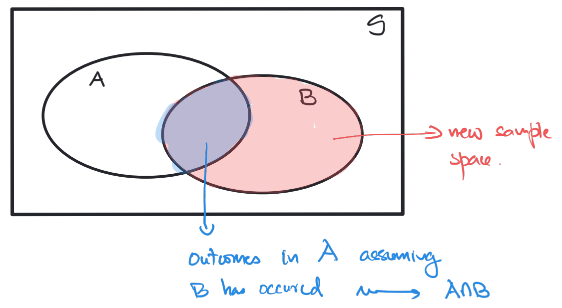
$$\boxed{\therefore P(A|B) = \frac{n(\text{outcomes in } A \text{ given new sample space } B)}{n(\text{outcomes in new sample space, ie } B)}}$$

$$= \frac{n(A \cap B)}{n(B)}$$

$$= \frac{n(A\cap B)/n(S)}{n(B)/n(S)} \quad \text{(divide both num and den by } n(S))$$

$$= \frac{P(A\cap B)}{P(B)} \quad \text{since all outcomes are equally likely.}$$

Visual Heuristic



Multiplication Rule

Calculate $P(A \cap B) = P(\text{events } A \text{ and } B \text{ occur})$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad (\text{provided } P(B) > 0)$$

$$\therefore \boxed{P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)}$$

Alternately:

$$\boxed{P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}}$$

Note:

1. Conditioning is a very important tool.
2. In the formula $P(A|B) = \frac{P(A\cap B)}{P(B)}$, the condition $P(B) > 0$ is important $\rightsquigarrow$ Can lead to paradoxes otherwise!!!

Examples:

1. Consider the following events.

A = Jonathan is carrying an umbrella.

B = it is raining outside

C = it is sunny outside.

Then we can note that

$$P(A) < P(A|B) = P(\text{Jonathan carrying an umbrella} \mid \text{raining outside})$$

and,

$$P(A) > P(A|C) = P(\text{carrying umbrella} \mid \text{sunny outside})$$

$\therefore$ Knowing additional information can affect the probability of an outcome

Now:

Let D = had pasta for breakfast.

We would expect the event D to have no impact on Jonathan carrying an umbrella.

ie w'ed expect $P(A|D) = P(A)$!

Independent Events

$A, B \subseteq S$ are independent iff

$$P(A|B) = P(A) \iff P(B|A) = P(B).$$

Alternately,

$$\boxed{A, B \text{ independent} \iff P(A\cap B) = P(A) \cdot P(B).}$$

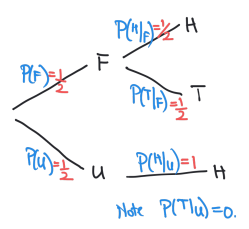
Note: To show that two events are independent, we need to show $P(A\cap B) = P(A)\cdot P(B)$ or be able to calculate $P(A|B)$ or $P(B|A)$ $\rightsquigarrow$ not easy!

Example 2:

Suppose a bag contains two coins
- a fair coin (F)
- a two headed coin. (U)

We randomly choose a coin from the bag and toss it once.

Can draw the following tree diagram:



$S = \{FH, FT, UH\}$

FH: Fair and Heads $= F \cap H$

FT: Fair and tails $= F \cap T$

UH: unfair and Heads $= U \cap T$

Now

$$P(\text{getting } H) = P(F\cap H) + P(U \cap H)$$
$$= P(H|F)\cdot P(F) + P(H|U) \cdot P(U)$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2} = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

$$\therefore \boxed{P(H) = 3/4}$$

Now, $P(H|F) = \frac{1}{2} \neq \frac{3}{4} = P(H) \implies H$ and F are not independent events

Also,

$$\boxed{P(\text{coin is fair} \mid \text{coin landed } H) = P(F|H)}$$

$$= \frac{P(F\cap H)}{P(H)} = \frac{P(H|F)\cdot P(F)}{P(H)}$$

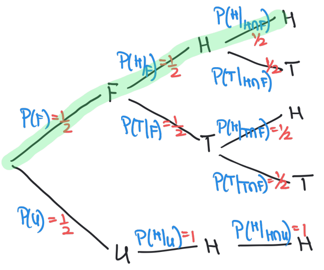
$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{3/4} = \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{3} = \frac{1}{3}$$

$$\therefore P(F) = \tfrac{1}{2} > \tfrac{1}{3} = P(F|H)$$

This makes intuitive sense

"our confidence that the coin is fair should be reduced given the information that it has landed heads".

Suppose now we randomly choose a coin from the bag and toss it two times (independently).

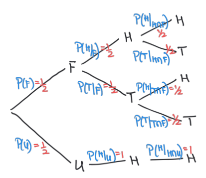


Note: $P(HHF) = P(H \cap (H \cap F))$

$= P(H|H\cap F) \cdot P(H\cap F)$

$= P(H|(H\cap F)) \cdot P(H|F) \cdot P(F)$

= product along path to HHF



$S = \{FHH, FHT, FTH, FTT, UHH\}$

Want to calculate $P(HH) = P(\text{getting two heads})$

$$= P(FHH) + P(UHH)$$
$$= P(H|HF)\cdot P(HF) + P(H|HU)\cdot P(HU)$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot 1 \cdot 1$$
$$= \tfrac{1}{8} + \tfrac{1}{2} = \tfrac{5}{8}$$

Also,

$$\boxed{P(HH|F) = \frac{P(HHF)}{P(F)} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{4}}$$

$$\therefore \boxed{P(HH) = \tfrac{5}{8} > \tfrac{1}{4} = P(HH|F)}$$ $\rightsquigarrow$ We are less confident of getting two consecutive heads if we knew that the tossed coin is fair!

Also,

$$P(\text{the tossed coin is fair} \mid \text{two consecutive heads are observed}) = P(F|HH)$$

$$= \frac{P(FHH)}{P(HH)} = \frac{P(HH|F)\cdot P(F)}{P(HH)}$$

$$= \frac{\frac{1}{4}\cdot\frac{1}{2}}{5/8} = \frac{1}{8}\cdot\frac{8}{5} = \frac{1}{5}$$

$$\boxed{P(F|HH) = \tfrac{1}{5} < \tfrac{1}{3} = P(F|H) < \tfrac{1}{2} = P(F)}$$

$\therefore$ The confidence that the coin is fair decreases as observed consecutive heads increases from none to one and to two!

Relating $P(A|B)$ to $P(B|A)$

Baye's Theorem!