Continuous Random Variables (II)

Goal: Study the Normal Distribution $\Updownarrow$ useful in modelling measurements

Normal Distribution

$\Updownarrow$

Defined using two parameters

- $\mu \rightsquigarrow$ mean
- $\sigma^2 \rightsquigarrow$ variance

$\Updownarrow$

Denoted as $N(\mu,\sigma^2)$

Note: $\sigma = \sqrt{\sigma^2}$ is the standard deviation.

Suppose X is normally distributed $\rightsquigarrow$ ie $X \sim N(\mu,\sigma^2)$

$\Updownarrow$

the pdf of X is defined as

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad x \in (-\infty,\infty)}$$

To calculate the cdf of X,

$$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2}\,dt}$$

$$\therefore \boxed{P(a \le X \le b) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{a}^{b} e^{-(t-\mu)^2/2\sigma^2}\,dt} \rightsquigarrow \text{Not an easy integral to calculate!}$$

Note:

① If $X \sim N(\mu,\sigma^2)$, then

$$\boxed{E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2}$$

② When $\mu = 0$ and $\sigma^2 = 1$ $\rightsquigarrow$ Standard Normal Dist $\Updownarrow$ $Z \sim N(0,1)$

If $Z \sim N(0,1)$, the standard normal distribution

$\Updownarrow$

pdf: $$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad z \in (-\infty,\infty)}$$

To be specific about parameters

$\Updownarrow$

denote pdf of $X \sim N(\mu,\sigma^2)$ $\rightsquigarrow$ $f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad x \in (-\infty,\infty)$

If $X \sim N(\mu,\sigma^2)$ and $Z \sim N(0,1)$

$\Updownarrow$

$$P(X \le x) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{(x-\mu)}{\sigma}} e^{-s^2/2}\,ds$$

Change of variable with $s = \frac{(t-\mu)}{\sigma}$

$$= P\left(Z \le \frac{(x-\mu)}{\sigma}\right)$$

$\therefore$ **If $X \sim N(\mu,\sigma^2)$ then $\frac{X-\mu}{\sigma}$ has the standard normal distribution.**

If $Z \sim N(0,1)$ $\rightsquigarrow$ cdf of Z is denoted as $\Phi(z)$

ie $$\boxed{P(Z \le z) = \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt}$$

③ If $X \sim N(\mu,\sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$\therefore$

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$\Updownarrow$

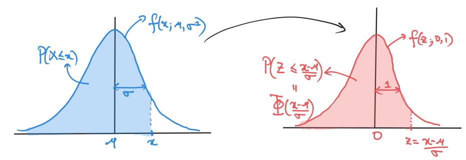
$\therefore$ Can use the standard normal cdf to calculate probabilities for the random variable $X \sim N(\mu,\sigma^2)$

$\Updownarrow$

$X \sim N(\mu,\sigma^2) \longrightarrow Z \sim N(0,1)$

$x \longmapsto z = \frac{x-\mu}{\sigma} \rightsquigarrow$ "z-score" of x

$x = z\sigma + \mu \longleftarrow z$



Percentiles and Critical Values

Suppose $Z \sim N(0,1) \rightsquigarrow$ standard normal distribution

$\Updownarrow$

For $\alpha \in (0,1)$, $z_\alpha \rightsquigarrow \alpha^{th}$ critical value $\Updownarrow$ satisfies

$$\boxed{P(Z \ge z_\alpha) = \alpha} \quad \text{ie} \quad \boxed{\int_{z_\alpha}^{\infty} f(z;0,1)\,dz = \alpha}$$

$\Updownarrow$

ie $1 - P(Z \le z_\alpha) = \alpha \Rightarrow 1 - \Phi(z_\alpha) = \alpha$

Let $p \in (0,1) \rightsquigarrow (100p)^{th}$ percentile, $\eta(p)$ satisfies

$\Updownarrow$

$$\boxed{P(Z \le \eta(p)) = p}$$

$\Updownarrow$

ie $\Phi(\eta(p)) = p$

Suppose $X \sim N(\mu,\sigma^2)$ and $\alpha \in (0,1)$

$\Updownarrow$

$$\boxed{\alpha^{th} \text{ critical value for } X := x_\alpha = \sigma \cdot z_\alpha + \mu}$$

$z_\alpha = \alpha^{th}$ critical value for $Z \sim N(0,1)$.

Approximating Bin(n,p) using Normal Distribution

Let $X \sim Bin(n,p)$ ie $\mu_X = np$ and $\sigma_X^2 = n \cdot p \cdot q$

$\Updownarrow$

If the $Bin(n,p)$ is not too skewed

$$\boxed{P(X \le x) \approx \Phi\left(\frac{x + 0.5 - \mu_X}{\sigma_X}\right)}$$

$\Updownarrow$

The right hand side does not involve terms of the form $\binom{n}{x}$ $\rightsquigarrow$ which can be difficult to compute.

(4) from 4.1 Devore

$$f(x) = \begin{cases} \dfrac{x}{\theta^2} \cdot e^{-x^2/2\theta^2} & x > 0 \\ 0 & \text{otherwise } (x \le 0) \end{cases}$$

Note: $f(x;\theta) = \begin{cases} \dfrac{x}{\theta^2} \cdot e^{-x^2/2\theta^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$E(X)$, $V(X)$

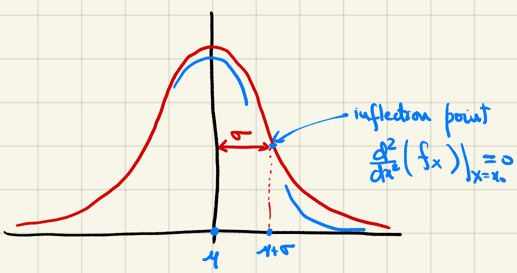
Talk about: Normal Distribution

If X has Normal Distribution with parameters $\mu \rightsquigarrow$ mean
$\sigma^2 \rightsquigarrow$ variance

pdf:

$$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2} \qquad x \in (-\infty, \infty)$$

Graph of $f(x;\mu,\sigma^2)$: $\rightsquigarrow$ Bell shaped curve



The cdf:

$$F_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{x} \underbrace{e^{-(t-\mu)^2/2\sigma^2}}\, dt$$

$\parallel$

$P(X \le x)$

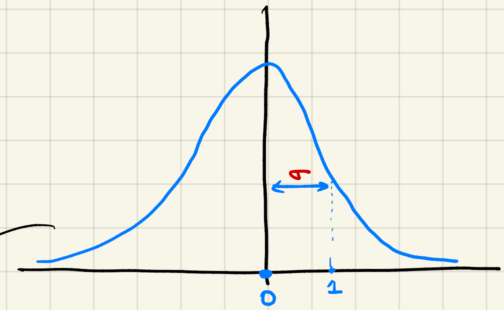
The Standard Normal Distribution

$\mu = 0$, $\sigma^2 = 1$

If $Z \sim N(\mu=0, \sigma^2=1)$

$$\boxed{f(z;0,1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}}$$

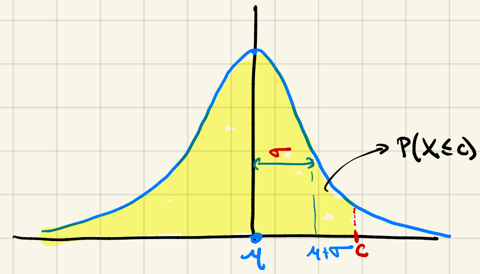
Graph of $f(z;0,1)$



$X \sim N(\mu,\sigma^2) \longrightarrow Z \sim N(0,1)$

$x \longmapsto \frac{x-\mu}{\sigma} \rightsquigarrow$ "z-score" of x.

$x = \mu + \sigma \cdot z \longleftarrow z$



To Calculate Probabilities

$P(X \le c) = P($random variable takes values less than or equal to $c)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{c} e^{-(t-\mu)^2/2\sigma^2}\, dt.$$

Change of variables $s = \frac{t-\mu}{\sigma}$

$t = -\infty$	$s = -\infty$
$t = c$	$s = \frac{c-\mu}{\sigma}$

$ds = \frac{dt}{\sigma}$

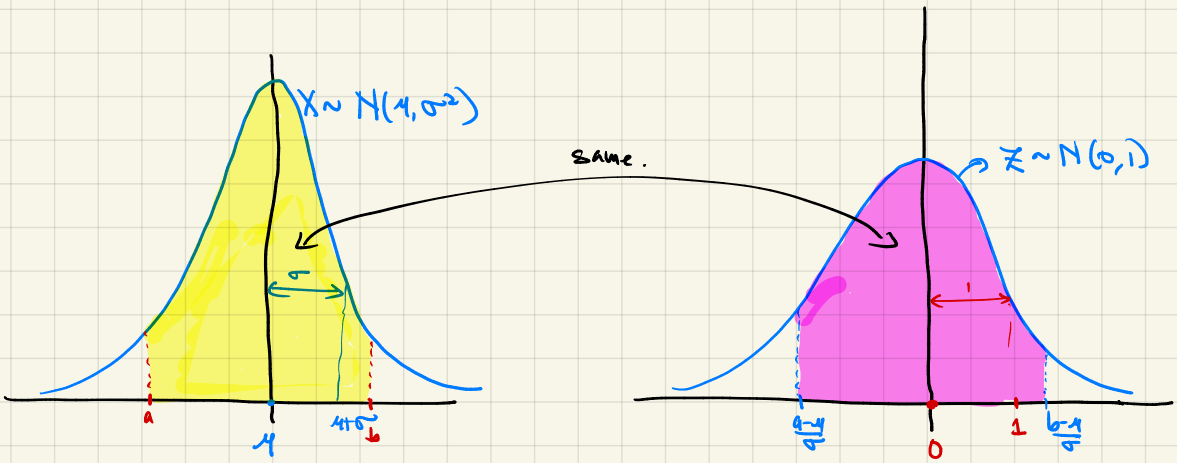
$$P(X \le c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\left(\frac{t-\mu}{\sigma}\right)^2 \cdot \frac{1}{2}} \cdot \frac{dt}{\sigma} \rightarrow ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(c-\mu)/\sigma} e^{-s^2/2}\, ds.$$

$$= \int_{-\infty}^{(c-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\, ds \longrightarrow \text{pdf of the standard normal distribution}$$

$$= P\left(Z \le \left(\frac{c-\mu}{\sigma}\right)\right)$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}\, dx = 1}$$



$$P(a \le X \le b) = P(X \le b) - P(X \le a)$$

$$= F_X(b) - F_X(a)$$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

$p \in (0,1)$ then $(100)\cdot p^{\text{th}}$ percentile for Z

$\Updownarrow$

$\eta(p)$ satisfying

$$P(Z \le \eta(p)) = p.$$

$$\boxed{\Phi(\eta(p)) = p}$$

Critical Values and Percentiles

$Z \sim N(0,1)$

$\alpha \in (0,1) \rightsquigarrow \alpha^{\text{th}}$ critical value for Z

$\Updownarrow$

z_α and satisfies

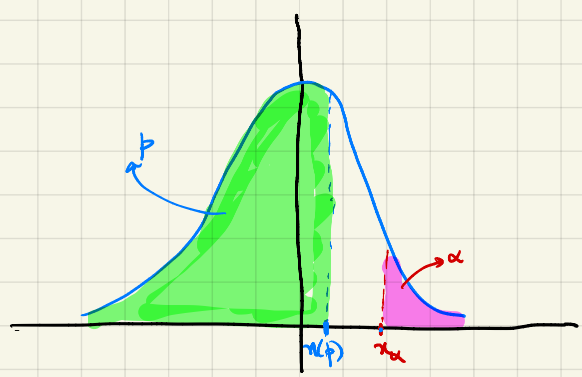
$$P(Z > z_\alpha) = \alpha$$

$$1 - P(Z \le z_\alpha) = \alpha$$

$$1 - F_Z(z_\alpha) = \alpha.$$

$$\boxed{\alpha = 1 - \Phi(z_\alpha)}$$

need to calculate.



$z_\alpha = \eta(1-\alpha)$

If $X \sim N(\mu, \sigma^2)$

α^{th} critical value for X $\rightsquigarrow$ $x_\alpha = \sigma \cdot z_\alpha + \mu$

z_α: α^{th} critical value for $Z \sim N(0,1)$.

$(100 \cdot p)^{\text{th}}$ percentile for X $= \sigma \cdot \eta(p) + \mu$

$\eta(p)$: $(100p)^{\text{th}}$ percentile for $Z \sim N(0,1)$