Continuous Random Variables (III)

Goal: Study the Gamma Distribution

Useful when modelling → component lifetimes → modelling waiting times

Gamma Distribution

The gamma function is defined as

$$\boxed{\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}\cdot e^{-t}\,dt \quad , \text{ for } \alpha>0}$$

Properties of Gamma Function

① $\boxed{\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha) \quad \text{for } \alpha>0}$

Can check that $\Gamma(1)=1$ and using induction we have $\boxed{\Gamma(n)=(n-1)!}$ for $n\in\mathbb{N}$.

② $\boxed{\Gamma(\tfrac{1}{2})=\sqrt{\pi}}$.

For $\alpha,\beta>0$, let

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^\alpha\,\Gamma(\alpha)}\cdot x^{\alpha-1}\cdot e^{-x/\beta} & \text{if } x\geq 0\\ 0 & \text{if } x\leq 0\end{cases}$$

Note: ① $f(x;\alpha,\beta)\geq 0 \quad \forall x\in\mathbb{R}$

② $\int_{-\infty}^{\infty} f(x;\alpha,\beta)\,dx=1$ (do a change of variables $y=\frac{x}{\beta}$)

We say X has the Gamma Distribution with shape parameter α, and scale parameter β

if the pdf of X is $f(x;\alpha,\beta)$.

Note:

① When $\beta=1$ → "Standard Gamma distribution" with shape param: α.

pdf: $$f(x;\alpha)=\begin{cases}\dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

② $X\sim \text{Gamma}(\alpha,\beta)$ then

a) $\boxed{E(X)=\mu_X=\alpha\cdot\beta}$

b) $\boxed{V(X)=\sigma_X^2=\alpha\beta^2}$

c) $\boxed{\sigma_X=\sqrt{\alpha}\cdot\beta}$

③ The cdf of $X\sim\text{Gamma}(\alpha,\beta)$

$$F_X(x;\alpha,\beta)=\begin{cases}\int_0^x f(t;\alpha,\beta)\,dt & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$$

$$\boxed{P(X\leq x)=F_X(x;\alpha,\beta)=F\left(\tfrac{x}{\beta};\alpha\right)}$$

cdf of Standard Gamma with param: α.

ie $$F(x;\alpha)=\begin{cases}\int_0^x \dfrac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\,dt & \text{if } x\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

If T has Standard Gamma with Shape param: α

$X=\beta T$ has Gamma distribution with shape: α, scale: β.

Special Cases of Gamma Distribution

① Exponential Distribution → set $\alpha=1$, $\beta=\frac{1}{\lambda}$

Get Exponential Distribution with param $\lambda>0$.

pdf: $$f(x;\lambda)=\begin{cases}\lambda\cdot e^{-\lambda x} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

cdf: $$F_X(x;\lambda)=\begin{cases}1-e^{-\lambda x} & x\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

Note

① If X has exponential Distribution with param $\lambda>0$ →

$\boxed{E(X)=\mu_X=\frac{1}{\lambda}}$

$\boxed{V(X)=\sigma_X^2=\frac{1}{\lambda^2}}$

$\boxed{\sigma_X=\frac{1}{\lambda}}$

② If we have a "Poisson Process" with rate α → The exponential distribution with $\lambda=\alpha$

models the distribution of "elapsed time" between the occurrence of two successive events.

③ Also, if $X\sim\text{Exp}(\lambda)$

$$P(X\geq t+t_0 \mid X\geq t_0)=\frac{P[\{X\geq t+t_0\}\cap\{X\geq t_0\}]}{P(X\geq t_0)}$$

$$=\frac{P(X\geq t+t_0)}{P(X\geq t_0)}=\frac{1-F(t+t_0;\lambda)}{1-F(t_0;\lambda)}=e^{-\lambda t}=P(X\geq t)$$

That is, if X was modelling the lifetime of a component

The distribution of additional lifetime is exactly the same as the original distribution of lifetime.

The exp distribution has "memoryless property".

② Chi-Squared Distribution

X is said to have Chi-squared dist with param: ν (degrees of freedom) → $X\sim\text{Gamma}(\alpha,\beta)$ with $\alpha=\frac{\nu}{2}$, $\beta=2$.

pdf: $$f(x;\nu)=\begin{cases}\dfrac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\,x^{\nu/2-1}\,e^{-x/2} & \text{if } x\geq 0\\ 0 & \text{otherwise.}\end{cases}$$

$\boxed{E(X)=\mu_X=\alpha\beta=\nu}$

$\boxed{V(X)=\sigma_X^2=\alpha\beta^2=2\nu}$

$\boxed{\sigma_X=\sqrt{2\nu}}$

Note: ① Chi-square distribution → plays important role in statistical inference

② If $X\sim N(\mu,\sigma^2)$ then $\left(\frac{X-\mu}{\sigma}\right)^2$ has chi-sq distribution with $\nu=1$.