Methods of Point Estimation

<u>Goal</u>: Introduce two methods $\rightarrow$ Method of moments
$\rightarrow$ Maximum Likelihood Estimation.

Suppose

Population has distribution X $\rightarrow$ pmf / pdf of X is $f(x)$.

Let $\{X_1, X_2, \ldots, X_n\}$ $\rightarrow$ random sample of size n from population.

Then for $k = 1, 2, 3, \ldots$

① The k^{th} population moment or the k^{th} distribution moment $\rightarrow$ The expected value of the r.v X^k $\Updownarrow$ $E(X^k)$.

② The k^{th} sample moment for the random sample $\{X_1, X_2, \ldots, X_n\}$ $\rightarrow$ $\dfrac{X_1^k + X_2^k + \cdots + X_n^k}{n}$ $\Updownarrow$ $\dfrac{\sum_{i=1}^{n} X_i^k}{n}$

The Method of Moments

If there are m parameters: $\theta_1, \theta_2, \theta_3, \ldots, \theta_m$ that we want to estimate using $\{X_1, X_2, \ldots, X_n\}$

$\Updownarrow$

get m-equations by "equating" the first m population moments to the first m sample moments.

$\Updownarrow$

Pray that you can solve these equations to get estimators $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ for $\theta_1, \theta_2, \ldots, \theta_n$ resp.

<u>Example</u>:

① Sampling from exponential distribution

Suppose $\{X_1, X_2, X_3, \ldots, X_n\}$ random sample from $\exp(\lambda)$.

$\Updownarrow$

only one parameter to estimate, ie λ.

$\Updownarrow$

only need to calculate first moments.

$\boxed{\text{first population moment} \rightarrow E(X) = \frac{1}{\lambda}}$

$\boxed{\text{first sample moment} \rightarrow \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) = \overline{X} \quad \text{sample mean.}}$

$\therefore$ Equating the population moments to sample moments:

$$\frac{1}{\lambda} = \overline{X} \implies \lambda = \frac{1}{\overline{X}}$$

$\boxed{\therefore \text{Method of moments estimator for } \lambda \rightarrow \hat{\lambda} = \frac{1}{\overline{X}}.}$

② Sampling from Normal distribution

ie $\{X_1, X_2, \ldots, X_n\}$ coming from $N(\mu, \sigma^2)$

$\therefore$ want to estimate μ and σ^2 (ie $k=2$).

(i) Calculate k^{th} population moments:

$\boxed{k=1 \rightarrow E(X) = \mu}$

$k=2 \rightarrow E(X^2)$, recall awesome identity

$\Updownarrow$

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2 = E(X^2) - \mu^2$$

$\therefore$ $\boxed{E(X^2) = \sigma^2 + \mu^2.}$

Equating population moments to sample moments:

$$E(X) = \mu = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

$$E(X^2) = \sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$\therefore$ The moment estimators are

$\boxed{\hat{\mu} = \overline{X}}$ and $\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2}$

$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$

③ Sampling from Gamma Distribution

$\{X_1, X_2, \ldots, X_n\}$ from Gamma(α, β)

ie $\alpha \rightarrow$ shape

$\beta \rightarrow$ scale.

Recall: $\boxed{E(X) = \alpha \cdot \beta}$ $\boxed{V(X) = \alpha \cdot \beta^2}$

$\therefore E(X^2) - E(X)^2 = \alpha \cdot \beta^2$

$\implies E(X^2) = \alpha \cdot \beta^2 + (E(X))^2$

$\therefore$ Population moments are

$\boxed{E(X) = \alpha \cdot \beta}$ $\boxed{E(X^2) = \alpha\beta^2 + (\alpha\beta)^2 = \alpha\beta^2(1+\alpha)}$

$\therefore$ Equating sample moments to pop'n moments we get

$\boxed{\overline{X} = \alpha \cdot \beta}$ and $\boxed{\dfrac{\sum_{i=1}^{n} X_i^2}{n} = \alpha\beta^2(1+\alpha)}$

Let $\boxed{A = \overline{X}}$ and $\boxed{B = \frac{1}{n}\sum_{i=1}^{n} X_i^2}$,

need to solve the following for α, β

$\boxed{A = \alpha \cdot \beta \quad \text{and} \quad B = \alpha\beta^2(1+\alpha)}$

$\implies \boxed{\beta = \frac{A}{\alpha}}$, $\therefore B = \alpha \cdot \left(\frac{A}{\alpha}\right)^2 (1+\alpha)$

ie $B = \cancel{\alpha} \cdot \frac{A^2}{\alpha^{\cancel{2}}}(1+\alpha) = \frac{A^2(1+\alpha)}{\alpha}$

$\therefore \alpha \cdot B = A^2 + A^2 \cdot \alpha \implies \alpha(B - A^2) = A^2$

$\implies \boxed{\alpha = \frac{A^2}{(B - A^2)}}$

$\therefore$ The method of moment estimators are:

$\boxed{\hat{\alpha} = \dfrac{\overline{X}^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2}}$, $\boxed{\hat{\beta} = \dfrac{\overline{X}}{\hat{\alpha}}}$

Maximum Likelihood Estimation

Suppose $\{X_1, X_2, \ldots, X_n\}$ $\rightarrow$ random sample from pop'n with pdf/pmf: $f(x)$.

Let $f(x_1, x_2, \ldots, x_n)$ $\rightarrow$ "joint pdf / pmf" of $X_1, X_2, \ldots, X_n$

$\Updownarrow$

X_i's are independent,

$\boxed{f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)}$

Now, given sample data $\rightarrow$ $\{x_1, x_2, \ldots, x_n\}$

$\Updownarrow$

want to estimate parameters $\theta_1, \theta_2, \ldots, \theta_m$

The Likelihood function for sample data $\{x_1, x_2, \ldots, x_n\}$ $\rightarrow$ $f(\underbrace{x_1, x_2, \ldots, x_n}_{\text{fixed sample data}}; \underbrace{\theta_1, \theta_2, \ldots, \theta_m}_{\text{unknown parameters}})$

unknown parameters $\Updownarrow$ vary over all possible choices of parameters.

$\Updownarrow$

"Likelihood function" is function of the parameters $\theta_1, \theta_2, \ldots, \theta_n$ given sample data $\{x_1, x_2, \ldots, x_n\}$

$\boxed{\text{Maximum likelihood Estimation}}$ $\rightarrow$ find choices of params $\hat{\theta}_1, \ldots \hat{\theta}_m$ which maximize the likelihood function for the observed sample data $\{x_1, x_2, x_3, \ldots, x_n\}$

That is,

$\boxed{\text{find } \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \text{ such that } f(x_1, x_2, \ldots, x_n; \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m) \geq f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_m) \text{ for all possible values of } \theta_1, \theta_2, \ldots, \theta_n.}$

$\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ $\rightarrow$ maximum likelihood estimates (mle) for $\theta_1, \theta_2, \ldots, \theta_m$ resp, using $\{x_1, x_2, \ldots, x_n\}$

If we substitute X_i for x_i, get estimators

$\boxed{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \rightarrow \text{maximum likelihood estimators (mle) for } \theta_1, \theta_2, \ldots, \theta_m \text{ resp.}}$

<u>Examples</u>

① Sampling from exponential distribution.

$\{X_1, X_2, \ldots, X_n\}$ rand sample from $\exp(\lambda)$.

$\boxed{\text{pdf of } X_i \rightarrow f(x_i; \lambda) = \lambda e^{-\lambda x_i}}$

$\therefore$ The likelihood function is

$$f(x_1, x_2, \ldots, x_n; \lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdots \lambda e^{-\lambda x_n} = \lambda^n e^{-\lambda(x_1 + x_2 + \cdots + x_n)}$$

$\therefore$ $\boxed{f(x_1, x_2, \ldots, x_n; \lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)}$

$\Updownarrow$

want to maximize this as a function of λ's

If $\boxed{g(\lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)}$,

$$\ln(g(\lambda)) = \ln\left(\lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\right) = \ln(\lambda^n) + \ln\left(\exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\right)$$

$= n\ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$ $\rightarrow$ $\boxed{\text{log-likelihood function.}}$ $\Updownarrow$ easier to maximize.

Let $\boxed{h(\lambda) = \ln(g(\lambda))}$, then

$$h'(\lambda) = \frac{d}{d\lambda}\left(n\ln(\lambda) - \lambda \sum_{i=1}^{n} x_i\right) = n \cdot \frac{1}{\lambda} - \sum_{i=1}^{n} x_i$$

$\therefore h'(\lambda) = 0 \implies n \cdot \frac{1}{\lambda} = \sum_{i=1}^{n} x_i \implies \boxed{\lambda = \dfrac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}}$

Also, $\boxed{h''(\lambda) = -\frac{n}{\lambda^2} < 0}$ $\implies$ $\lambda = \dfrac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}$ is a point of local maxima.

Since $\ln$ is an increasing function,

$\lambda = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1}$ maximizes $h(\lambda)$ $\implies$ $\lambda = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{-1}$ maximizes the likelihood function $g(\lambda)$

$\boxed{\therefore \text{mle for } \lambda \rightarrow \hat{\lambda} = \frac{1}{\overline{X}}}$

Note: this is the same as the moment estimator for λ.

② Sampling from Normal Distribution

$\{X_1, X_2, \ldots, X_n\}$ $\rightarrow$ rand sample from $N(\mu, \sigma^2)$

$\boxed{X_i \text{ has pdf} \rightarrow f(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x_i-\mu)^2/2\sigma^2}}$

The likelihood function is given as

$$\boxed{f(x_1, x_2, \ldots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \cdot e^{-\sum_{i=1}^{n}\left((x_i-\mu)^2/2\sigma^2\right)}}$$

So that log-likelihood function is

$$\ln(f(x_1, x_2, \ldots, x_n; \mu, \sigma^2)) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n + \ln\left(e^{-\sum_{i=1}^{n}\left((x_i-\mu)^2/2\sigma^2\right)}\right)$$
$$= \frac{n}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$\therefore$ $\boxed{h(\mu, \sigma^2) = \frac{n}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2}$

Need to maximize $h(\mu, \sigma^2)$ $\rightarrow$ calculate critical values (ie $\frac{\partial h}{\partial \mu} = 0$ and $\frac{\partial h}{\partial \sigma^2} = 0$)

$\Updownarrow$

use Hessian matrix to show that these are max.

We can show that the mle's are

$\boxed{\hat{\mu} = \overline{X}}$ and $\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2}$

<u>Note</u>: While $\hat{\mu}$ is unbiased, $\hat{\sigma}^2$ is biased!

Properties of MLE's

① Invariance Principle $\rightarrow$ If $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ are mle's for $\theta_1, \theta_2, \ldots, \theta_m$, then $h(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$ is an mle for $h(\theta_1, \theta_2, \ldots, \theta_m)$.

② Large Sample Behavior $\rightarrow$ When sample size is large, the mle $\hat{\theta}$ for θ is atleast approx unbiased (ie $E(\hat{\theta}) \approx \theta$) and has variance that is either small as (or nearly as small as) can be achieved by any estimator.

$\boxed{\therefore \text{The mle } \hat{\theta} \text{ is exactly or atleast approx the MVUE for } \theta.}$

Method of Moments.

Population moments.
$\Updownarrow$
quantities involving the parameters of interest

Sample moment
$\Updownarrow$
quantities involving the random sample.

Want. To repackage the problem of finding estimators.
$\Updownarrow$
problem of finding solutions to a system of equation.

n-linear equations in n-variable $\xrightarrow[\text{some condition}]{\text{under}}$ can guarantee solutions

$\{X_1, X_2, \dots, X_n\}$ random sample popl^n pdf $f(x)$.

If $k = 1, 2, 3, \dots$

k^{th} population moment $= E(X^k)$

For example $k=1$ gives $\boxed{E(X) = \mu}$ = mean of the populat.

$k=2$, Want $E(X^2)$

$$V(X) = E(X^2) - (E(X))^2$$
$$\Rightarrow E(X^2) = V(X) + (E(X))^2$$
$$\boxed{E(X^2) = \sigma^2 + \mu^2}$$

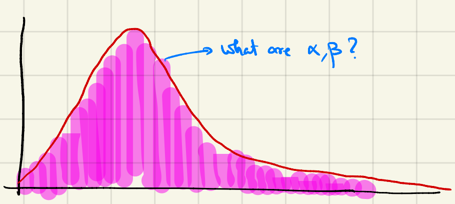
The k^{th} sample moment $=$ Avg of $\{X_i^k \,;\, i=1,2,\dots,n\}$

$$= \frac{X_1^k + X_2^k + \dots + X_n^k}{n}$$

Method of moments: If estimating $\theta_1, \theta_2, \dots, \theta_m$

Calculate the m popl^n moments and the m-sample moments.

And equating these m-moments gives m-equations that involve $\theta_1, \theta_2, \dots, \theta_m$, and $X_1, X_2, \dots, X_n$
$\Updownarrow$
Solve for θ_i's.



data $= \{x_1, x_2, x_3, \dots, x_{150}\}$
$\Updownarrow$
use these 150 numbers to estimate α, β.

Suppose sampling from Gamma(α, β)

$\{X_1, X_2, \dots, X_n\}$ rand sample, $X_i \sim \text{Gamma}(\alpha, \beta)$

Want estimators for α, β.

Need to calculate the first two popl^n and sample moments.

1) $E(X) = \mu_x = \alpha\beta$

2) $E(X^2) = ?$ Note $V(X) = E(X^2) - (E(X))^2$

$$\Rightarrow E(X^2) = V(X) + (E(X))^2 = \alpha\beta^2 + (\alpha\beta)^2$$

$$E(X^2) = \alpha\beta^2(1+\alpha)$$

Equating population moments to sample moments.

$$\alpha \cdot \beta = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X} = A$$

$$\alpha\beta^2(1+\alpha) = \frac{1}{n}\sum_{i=1}^n X_i^2 = B$$

$\alpha, \beta > 0$

$\alpha\beta = A$ and $\alpha\beta^2(1+\alpha) = B$

$$\boxed{\beta = \frac{A}{\alpha}}$$

$$\alpha\left(\frac{A}{\alpha}\right)^2 \cdot (1+\alpha) = B$$

$$\alpha \frac{A^2}{\alpha^2} \cdot (1+\alpha) = B$$

$$A^2(1+\alpha) = \alpha B \Rightarrow A^2 + A^2\alpha = \alpha B$$

$$\Rightarrow \alpha(B - A^2) = A^2$$

$$\Rightarrow \boxed{\alpha = \frac{A^2}{B - A^2}}$$

$\therefore$ Method of moment estimators are

$$\hat{\alpha} = \frac{(\bar{X})^2}{\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2}, \quad \hat{\beta} = \frac{\bar{X}}{\hat{\alpha}}$$

Use R to calculate empirical bias of $\hat{\alpha}$ and $\hat{\beta}$

Given a random sample $\{X_1, X_2, \ldots, X_n\}$ coming from $N(\mu, \sigma^2)$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S_{xx} = \sum_{i=1}^{n}(X_i - \bar{X})^2 \rightsquigarrow \text{squared deviations of } X_i \text{ from the sample mean } \bar{X}.$$

$$S^2 = \frac{1}{(n-1)} S_{xx} \qquad \hat{\sigma}^2 = \frac{1}{n} S_{xx} \quad (\text{if } X \sim N(\mu, \sigma^2))$$

Want to study/understand S_{xx}.

Goal: Calculate $E(S_{xx})$?

$$
\begin{aligned}
S_{xx} &= \sum_{i=1}^{n}(X_i - \bar{X})^2 \\
&= \sum_{i=1}^{n}(X_i - \mu + \mu - \bar{X})^2 \\
&= \sum_{i=1}^{n}\left((X_i - \mu) - (\bar{X} - \mu)\right)^2 \\
&= \sum_{i=1}^{n}\left((X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2\right) \\
&= \sum_{i=1}^{n}(X_i - \mu)^2 - 2(\bar{X} - \mu)\sum_{i=1}^{n}(X_i - \mu) + \sum_{i=1}^{n}(\bar{X} - \mu)^2 \\
&= \sum_{i=1}^{n}(X_i - \mu)^2 - 2(\bar{X} - \mu)(n\bar{X} - n\mu) + n(\bar{X} - \mu)^2 \\
&= \sum_{i=1}^{n}(X_i - \mu)^2 - \cancel{2}n(\bar{X} - \mu)^2 + \cancel{n(\bar{X} - \mu)^2}
\end{aligned}
$$

$$S_{xx} = \sum_{i=1}^{n}(X_i - \mu)^2 - \frac{(\bar{X} - \mu)^2}{1/n}$$

If $Z \sim N(0,1) \Rightarrow Z^2 \sim$ Chi sq dist with $df = 1$.

$$\frac{S_{xx}}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

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Maximum Likelihood Estimator.

Suppose population has distribution whose pmf/pdf : $f(x)$.

Given a random sample:

$\{X_1, X_2, \dots, X_n\} \rightsquigarrow$ independent random variables.

What is the joint pdf of $X_1, X_2, X_3, \dots, X_n$.

$$f(x_1, x_2, \dots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n) = \prod_{i=1}^{n} f(x_i)$$

Likelihood function: suppose you want to estimate $(\theta_1, \theta_2, \dots, \theta_m) \in \Theta_1 \times \Theta_2 \times \dots \times \Theta_m = \Theta$

Given sample data $(x_1, x_2, \dots, x_n)$:

Likelihood function:

$$f(\underbrace{x_1, x_2, \dots, x_n}_{\text{fixed}}; \underbrace{\theta_1, \theta_2, \dots, \theta_m}_{\text{variable}}) = \underbrace{f(\theta_1, \theta_2, \dots, \theta_m \mid_{(x_1, x_2, \dots, x_n)})}_{\text{function of the parameters given sample data } (x_1, x_2, \dots, x_n)}$$

The maximum likelihood Estimates are choices of $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$ which maximize the likelihood function.

$\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$ satisfy

$$\boxed{f(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m) _{(x_1, x_2, \dots, x_n)} \geq f(\theta_1, \theta_2, \dots, \theta_m \mid_{(x_1, x_2, \dots, x_n)}) \quad \forall\ (\theta_1, \theta_2, \dots, \theta_m) \in \Theta}$$

Suppose $\{X_1, X_2, \dots, X_n\}$ random sample from $\exp(\lambda)$.

The X_i's have pdf $f(x_i; \lambda) = \lambda e^{-\lambda x_i}$

Likelihood function:

$$f(x_1, x_2, \dots, x_n; \lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdots \lambda e^{-\lambda x_n}$$

$$= \lambda^n \cdot e^{-\lambda(\sum_{i=1}^{n} x_i)} = \lambda^n \cdot \exp(-\lambda \sum_{i=1}^{n} x_i)$$

$\ln(a \cdot b) = \ln(a) + \ln(b)$, $\ln(a^n) = n \ln(a)$, $\ln(\exp(x)) = x$.

$$\underbrace{g(\lambda) = \lambda^n \cdot \exp(-\lambda \sum_{i=1}^{n} x_i)}_{\text{want to maximize as a function of } \lambda.}$$

Log likelihood function: $h(\lambda) = \ln(g(\lambda))$

$$h(\lambda) = \overset{\ln(\lambda^n)}{n \ln(\lambda)} - \lambda \sum_{i=1}^{n} x_i$$

$$\hat{\lambda} = \frac{1}{\frac{1}{n} \sum_{i=1}^{n} x_i} \quad \text{is a maximizer.}$$

$$\boxed{\hat{\lambda} = \frac{1}{\bar{X}}} \rightsquigarrow \text{MLE for } \lambda.$$

If $\{X_1, \dots, X_n\}$ $N(\mu, \sigma^2)$

$$f(x_1, \dots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x_i-\mu)^2/2\sigma^2} \right)$$

$$= \left(\frac{1}{\sqrt{2\pi} \cdot \sigma} \right)^n e^{-\sum_{i=1}^{n} \left((x_i-\mu)^2/2\sigma^2 \right)}$$

The mle's are:

$$\underbrace{\hat{\mu} = \bar{X}}_{\text{unbiased.}} \qquad \underbrace{\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}_{\text{biased.}}$$

Question: What good is it to accept a biased estimator for σ^2?

Notion of "Goodness of an estimator"

Mean - squared - error.

Suppose $\hat{\theta}$ estimates θ, the mean squared error of $\hat{\theta}$

want to talk about deviations.

$E((\hat{\theta}-\theta)^2)$ = expected value/mean of the squared deviations of $\hat{\theta}$ from θ.

$$E((\hat{\theta}-\theta)^2) = E(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2)$$

$$= E(\hat{\theta}^2) - 2\theta \cdot E(\hat{\theta}) + \underbrace{E(\theta^2)}_{\theta^2 \text{ (constant)}}$$

$V(\hat{\theta}) = E(\hat{\theta}^2) - (E(\hat{\theta}))^2$

$\Rightarrow E(\hat{\theta}^2) = V(\hat{\theta}) + (E(\hat{\theta}))^2$

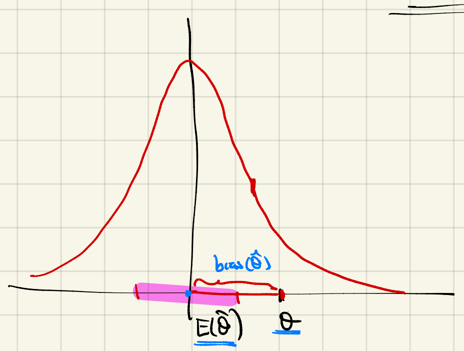
$$= V(\hat{\theta}) + (E(\hat{\theta}))^2 - 2\theta E(\hat{\theta}) + \theta^2$$

$$= V(\hat{\theta}) + (\underbrace{E(\hat{\theta}) - \theta}_{Bias(\hat{\theta})})^2$$

$\therefore$ $$\boxed{MSE(\hat{\theta}) = V(\hat{\theta}) + (Bias(\hat{\theta}))^2}$$

$\therefore$ If $\hat{\theta}$ was unbiased $\Rightarrow MSE(\hat{\theta}) = V(\hat{\theta})$

MVUE's.



Suppose $\{X_1, X_2, \ldots, X_n\}$ is from $N(\mu, \sigma^2)$

$S^2 = \frac{1}{n-1} S_{xx} = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2 \rightarrow$ unbiased estimator for σ^2

$\hat{\sigma}^2 = \frac{1}{n} S_{xx} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \rightarrow$ MLE for σ^2.

Define a new estimator for σ^2

$$\hat{\theta}_k = k \cdot S_{xx} = k \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\hat{\theta}_{1/n} = \hat{\sigma}^2, \quad \hat{\theta}_{1/n-1} = S^2$$

Can you calculate the MSE of $\hat{\theta}_k$?

$MSE(\hat{\theta}_k)$ = function depending k and σ^2.

= quadratic function in k.

Minimizing $MSE(\hat{\theta}_k)$ as a function of k gives the $MSE(\hat{\theta}_k)$ is the smallest when $k = \frac{1}{n+1}$

$\hat{\theta} = \frac{1}{n+1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ is the estimator that minimizes $MSE(\hat{\theta})$.