Sets, Experiments, and Probability

toss a coin two times.
$\{HH, HT, TH, TT\}$

Experiment
⇣
sample space S
⇣
outcome.

Experiment ⟶ a) repeatable task
b) well defined outcomes. ⟶ Examples
a) Toss a coin → $\{H, T\}$
b) Draw ball from a bag containing balls.
c) Choose a student from this class. ⟵ measure the hieght.
d) Choose three students from this class. (Sarah, Anne, Dominic) (Dominic, Sarah, Anna)
⇣
need to specify if choosing with or without replacement.

Sample Space ⟶ set of all possible outcomes of an experiment
⇣
usually denoted by S.

Examples: Coin toss ⟶ $S = \{H, T\}$
Ball draw ⟶ S = set of all balls in the bag.
Choose a student ⟶ S = all students in this class
Choose three studens ⟶ S = all groups of three students in this class
⇣
does order matter ?

Outcome ⟶ collection of possible outcomes of an experiment
⇣
a subset of sample space S.

Examples: a) Coin toss: $\{H\} \subset S$ ⟶ coin lands heads.
b) Choose a student: $\{Sarah\}$ ⟶ selected student is Sarah.

$\{T\}$ ⟶ coin lands tails.

suppose toss a coin three times

$S = \{HHH, HHT, HTT, TTT, THH, THT, HTH, TTH\}$

E = exactly one head in the three tosses.
$= \{HTT, THT, TTH\} \subset S$

E = atmost one head.
$= \{HTT, THT, TTH, TTT\}$

Set Theory: Basics

Let A, B be two sets.

1) $A \subset B$ ⟶ A "subset" of B ⟶ $x \in A \Rightarrow x \in \cancel{A}$ B (containment)

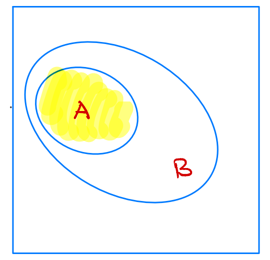
Symbol for subset.

2) $A = B$ ⟶ A "equals" B ⟶ $A \subset B$ and $B \subset A$. (equality)

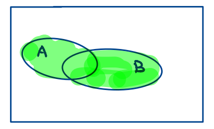
$A = \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, \cdots\}$
$B = \mathbb{Q} = \{\frac{p}{q};\ p, q \in \mathbb{Z}\}$
C = even integers $= \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$

$C \subset A \subset B$

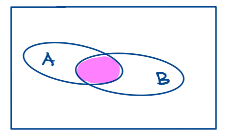
$B \not\subset A$; $\frac{1}{2} \in B$ but $\frac{1}{2} \notin \mathbb{Z}$



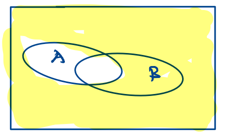
$A \cup B$



$A \cap B$



A



Operations with Sets

a) $A \cup B = \{x : x \in A \boxed{\text{or}}\ x \in B\}$ ⟶ A "union" B.
b) $A \cap B = \{x : x \in A \boxed{\text{and}}\ x \in B\}$ ⟶ A "intersection" B.
c) $A^c = \{x : x \notin A\}$ ⟶ Complement of A
d) ϕ = "empty set" ⟶ set containing no elements.

Let Γ be an indexing set ⟶ finite
⟶ infinite ⟶ countable
⟶ uncountable.

Fix a set S,

$\{A_\alpha : \alpha \in \Gamma\}$ ⟶ collection of subset of S indexed by Γ.

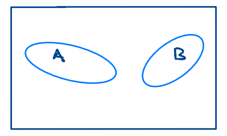
a) $\bigcup_{\alpha \in \Gamma} A_\alpha = \{x \in S : x \in A_\alpha \text{ for some } \alpha \in \Gamma\}$

b) $\bigcap_{\alpha \in \Gamma} A_\alpha = \{x \in S : x \in A_\alpha \text{ for all } \alpha \in \Gamma\}$

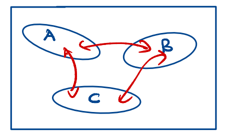
$S = \mathbb{R}$; $A_i = [i, \infty)$ $\bigcup_{i=1}^{\infty} A_i$ $\bigcap_{i=\infty}^{\infty} A_i$

Note:

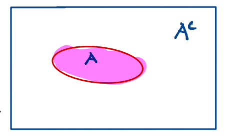
1. A and B are $\boxed{\text{disjoint}}$ if $A \cap B = \phi$
⇣
mutually exclusive.



2. $A_1, A_2, A_3, \ldots$ are $\boxed{\text{pair wise disjoint}}$ if $A_i \cap A_j = \phi$ for all $i \neq j$.



3. $A_1, A_2, A_3, \ldots$ is said to be a $\boxed{\text{partition}}$ of S if
a) $A_1, A_2, A_3, \ldots$ are pairwise disjoint.
b) $\bigcup_{i=1}^{\infty} A_i = S$.



⇣
Note: $A \cup A^c = S$
$A \cap A^c = \phi$.

4. The $\boxed{\text{universal set}}$ ⟶ set of $\boxed{\text{all}}$ objects of interest
⇣
typically the sample space S of an experiment.

Axioms of Probability

Let S be the sample space of an experiment.
⇣
Given $A \subset S$, an event ⟶ want to calculate the chance/probability that A occurs.
⇣
denote this number as $P(A)$.
⇣
the "probability of A"

$P(\{HTT, THT, TTH\})$

Axioms for a Probability function

A probability function P satisfies:

Axiom 1: $P(A) \geq 0$ for any event $A \subset S$.
Axiom 2: $P(S) = 1$, S the sample space.
Axiom 3: For $A_1, A_2, A_3, \ldots$ collection of pair wise disjoint events, we must have

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + \cdots = \sum_{i=1}^{\infty} P(A_i). = P\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$i \in \mathbb{N}$

Properties of a Probability Function P

1. $P(\phi) = 0$ ⟶ Probability of no outcome is zero. $\Rightarrow P(S \cup \phi) = P(S)$
$P(S) + P(\phi) = P(S)$
$P(\phi) = 0$

2. Given an event A ⟶ A^c is set of all outcomes not in A.
⇣
$A^c \cap A = \phi$ and $A^c \cup A = S$
⇣
$P(A) + P(A^c) = P(S) = 1$.
⇣
$$\boxed{P(A^c) = 1 - P(A).}$$

3. Note: Given two events $A, B \subset S$.

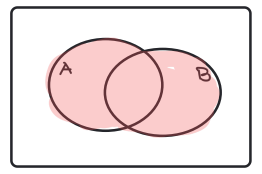
$A \setminus B$ ⟶ A minus B ⟶ $\{x \in A : x \notin B\}$
⇣
$$\boxed{A = (A \setminus B) \cup (A \cap B)}$$
disjoint union.
⇣
$P(A) = P(A \setminus B) + P(A \cap B)$.
⇣
$$\boxed{P(A \setminus B) = P(A) - P(A \cap B)}$$

4. $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
⇣
pair wise disjoint union.
⇣
$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$
$= (P(A) - P(A \cap B)) + P(A \cap B) + (P(B) - P(A \cap B))$

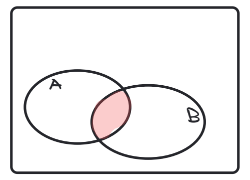
$$\therefore \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Venn Diagrams.

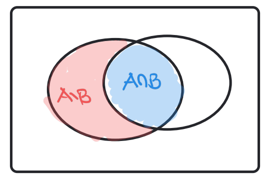
$A \cup B$



$A \cap B$



$A \setminus B$



$B \setminus A$