Continuous Random Variables (IV)

Goal: Study Lognormal and Beta distributions

① Lognormal Distribution

We say X has a Lognormal Distribution $\xrightarrow{\text{if}}$ $\ln(X)$ has normal distribution with parameters μ and σ^2

ie $\boxed{\ln(X) \sim N(\mu, \sigma^2)}$

If X has lognormal distribution

pdf: $$f(x;\mu,\sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma x} \cdot e^{-(\ln(x)-\mu)^2/2\sigma^2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note: ① μ and σ^2 $\longrightarrow$ mean and Variance of $\ln(X)$

ie $$E(\ln(X)) = \mu$$ $$V(\ln(X)) = \sigma^2$$

② If X has lognormal distribution

$$E(X) = \mu_X = e^{\mu + \sigma^2/2}$$

$$V(X) = \sigma_X^2 = e^{(2\mu+\sigma^2)}\left(e^{\sigma^2} - 1\right)$$

③ Can use standard normal table to calculate probabilities:

$$F_X(x;\mu,\sigma) = P(X \leq x) = P(\ln(X) \leq \ln(x))$$

$$= P\left(Z \leq \frac{\ln(x)-\mu}{\sigma}\right) \qquad (\because \ln(X) \sim N(\mu,\sigma^2))$$

$$= \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right) \quad \text{for } x > 0.$$

② Beta Distribution

takes values in a finite interval

good to model proportions $\longrightarrow$ naturally lie in $(0,1)$.

X is said to have Beta Distribution with parameters: $\alpha, \beta > 0$ $\xrightarrow{\text{if}}$ pdf of X is

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \qquad x \in (0,1).$$

Note: ① Called Beta distribution because the Beta function

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha,\beta > 0$$

so that $$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } x \in (0,1).$$

② If $X \sim \text{Beta}(\alpha,\beta)$

$$E(X) = \mu_X = \frac{\alpha}{\alpha+\beta}$$

$$V(X) = \sigma_X^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$\longrightarrow$ Notice that both μ_X and σ_X^2 are in $(0,1)$.

③ Depending on values of α, β $\longrightarrow$ the pdf $f(x;\alpha,\beta)$ has different shapes.

If
- a) $\alpha > 1, \beta = 1$ $\longrightarrow$ strictly increasing
- b) $\alpha = 1, \beta > 1$ $\longrightarrow$ strictly decreasing
- c) $\alpha < 1, \beta < 1$ $\longrightarrow$ U-shaped
- d) $\alpha = \beta$ $\longrightarrow$ symmetric about $\frac{1}{2}$, with $\mu_X = \frac{1}{2}$ and $\sigma_X^2 = \frac{1}{4(2\alpha+1)}$
- e) $\alpha = \beta = 1$ $\longrightarrow$ get uniform distribution on $(0,1)$.

Cauchy Distribution

X is said to have Cauchy Distribution with parameter: θ $\xrightarrow{\text{if}}$ the pdf of X is

$$f(x;\theta) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2} \qquad x \in (-\infty,\infty) \quad \theta \in (-\infty,\infty)$$

Note: ① $E(X)$ and $V(X)$ do not exist if X has Cauchy distribution.

② The graph of $f(x;\theta)$ is bell shaped $\longrightarrow$ like the Normal Density.

But has heavier tails (than Normal density).