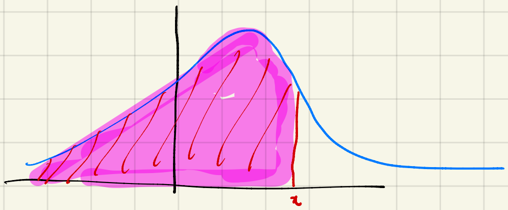
Exam 2 - Review 2.

Review:

Continuous random variables

let X be a r.v $\rightsquigarrow$ we say that X is cnt if F_X is continuous.

$F_X(x) := P(X \leq x)$



Given $F_X(x) \rightsquigarrow \boxed{f_X(x) = \frac{d}{dx}\left(F_X(x)\right)}$

let X r.v with cdf $F_X(x)$

Eg: $F(x) = \begin{cases} 0 & x<0 \\ \frac{x^2}{4} & 0 \leq x < 2 \\ 1 & 2 \leq x \end{cases} \rightsquigarrow f_X(x) = \begin{cases} 0 & x<0 \\ \frac{x}{2} & 0 \leq x \leq 2 \\ 0 & 2 \leq x \end{cases}$

a) $P(X \leq 1) = F(1) = \frac{1}{4}$

b) $P(0.5 \leq X \leq 1) = F(1) - F(0.5)$

$= \frac{1}{4} - \frac{(0.5)^2}{4}$

$=$

e) obtain $f(x)$.

f) $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= \int_0^2 x \cdot \frac{x}{2} dx = \int_0^2 \frac{x^2}{2} dx$

$= \frac{x^3}{3 \times 2}\Big|_0^2 = \frac{2^3}{3 \times 2} = \frac{4}{3}$ //

g) $V(X) = E(X^2) - (E(x))^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \cdot \frac{x}{2} dx = \int_0^2 \frac{x^3}{2} dx$ //

$= \frac{x^4}{4 \times 2}\Big|_0^2 = \frac{2^4}{4 \times 2} = 2$

$\therefore V(X) = E(X^2) - (E(X))^2$

$= 2 - \left(\frac{4}{3}\right)^2$

$= 2 - \frac{16}{9} = \frac{18-16}{9} = \frac{2}{9}$ //

$\sigma_X = \sqrt{V(X)} = \sqrt{2/9}$

$f(x) = \begin{cases} k(x^4 - x^2) & x \in (2,4) \\ 0 & \text{otherwise.} \end{cases}$

What value of k makes f a density function.

$\int_{-\infty}^{\infty} f(x)dx = 1 \iff \int_2^4 k(x^4 - x^2)dx = 1$

$k\left(\frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_2^4 = 1$

$k\left(\left(\frac{4^5}{5} - \frac{4^3}{3}\right) - \left(\frac{2^5}{5} - \frac{2^3}{3}\right)\right) = 1$

Examples of Continuous dist's

Know the pdf

$E(X)$

$V(X)$

σ_X

Know how to calculate probabilities, critical values and percentiles.

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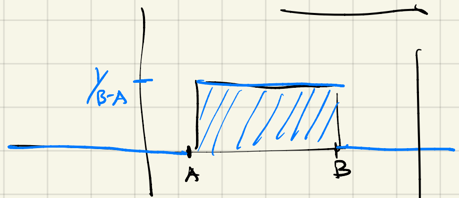
① **Uniform distribution**

$$f(x) = \begin{cases} 1/(B-A) & x \in [A,B] \\ 0 & \text{otherwise.} \end{cases}$$



② **Normal Distribution** $\to \mu$, $\to \sigma^2$

$N(\mu=5, \sigma=1.2)$

$N(\mu=5, \sigma^2=1.2)$

standard normal $\longrightarrow$ $N(\mu, \sigma^2)$

$N(0,1)$

$z = \dfrac{x-\mu}{\sigma} \longleftarrow x$

$z \longmapsto x = \sigma \cdot z + \mu$

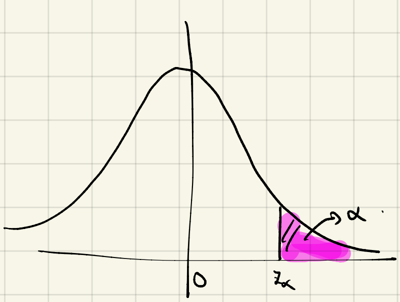
α-critical value for standard normal dist.

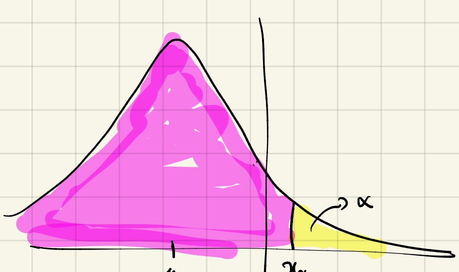
$P(Z > z_\alpha) = \alpha$

$1 - \Phi(z_\alpha) = \alpha \Rightarrow \underline{\Phi(z_\alpha) = 1-\alpha}$

If $X \sim N(\mu, \sigma^2)$ $\quad x_\alpha$.

$\boxed{x_\alpha = \sigma z_\alpha + \mu}$ $\longrightarrow$ same story with percentiles.





Suppose $X \sim N(\mu, \sigma^2)$

$Y = 2X+5$ is also normally dist.

pdf Y is not $2x f_X(x) + 5$ ✗

$$F_Y(y) = P(Y<y) = P(2X+5<y)$$
$$= P(2X < y-5)$$
$$= P\left(X < \frac{y-5}{2}\right)$$
$$= \int_{-\infty}^{\frac{y-5}{2}} f_X(x)\,dx$$

$Y = e^X$, pdf of Y is not $e^{f(x)}$. ✗

$E(Y) = E(2X+5) = E(2X)+5$
$= 2\mu+5$

$V(Y) = V(2X+5) = \underline{4 \cdot \sigma^2}$

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pnorm(x, mean= , sd=) $\rightsquigarrow$ cdf.

qnorm(p, mean= , sd= , lower.tail=True) $\longrightarrow$ p^{th} percentile

lower.tail = False $\longrightarrow$ p^{th} critical value.

$X \sim N(5, \sigma=1.2)$

$P(|X-5| \le c) = 0.98$.

$\Updownarrow$

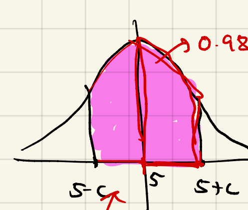
$P(-c < X-5 < c) = 0.98$

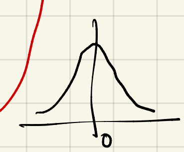
$P(5-c < X < 5+c) = 0.98$

$P(X < 5+c) - P(X < 5-c) = 0.98$

$P\left(\frac{-c}{\sigma} < \frac{X-5}{\sigma} \le \frac{c}{\sigma}\right) = 0.98$

$P\left(\frac{-c}{\sigma} < Z < \frac{c}{\sigma}\right) = 0.98$





$P(X \leq x)$

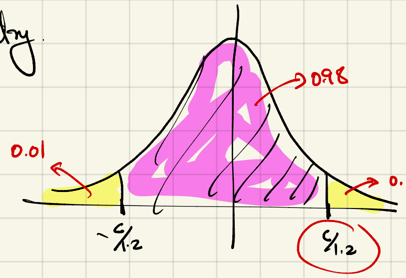
$P(X > x_\alpha) = \alpha$

$P(X < \eta_p) = p$

$\underline{\underline{P(X \geq x)}}$

$$P\left(\frac{-c}{1.2} < Z < \frac{c}{1.2}\right) = 0.98$$

by symmetry



$\frac{c}{1.2} = \left(0.01 \text{ - critical value for } N(0,1)\right)$

$\frac{c}{1.2} = \underline{\underline{Z_{0.01}}} = \text{qnorm}(0.01, \text{lower.tail} = \text{False})$

∴ Suppose

Construct a random sample Gamma(α, β) of size n.

$\{X_1, X_2, \ldots, X_n\}$ $\quad X_i \sim \text{Gamm}(\alpha, \beta)$

$T_0 = X_1 + X_2 + \cdots + X_n$

T_0 will have the normal dist (because of CLT).

T_0 will have gamma dist $(n\alpha, \beta)$

∴ using CLT + fact ⇒ for large α Gamma(α, β) is approx. normal.

Ex: Calculate/graph pdf of $\underline{\underline{\text{Gamma}(100, 2)}}$ in R

bell-shaped graph.

Gamma Dist ⟶ α, β

Exp dist ← Gamma Dist → Chi squared

λ $\quad$ df.

If $Z \sim N(0,1)$ then $Y = Z^2$ has chi-squared dist with $\underline{\underline{df = 1}}$

If $X_1 = \text{Gam}(\alpha_1, \beta)$, $X_2 \sim \text{Gam}(\alpha_2, \beta)$, ..., $X_n \sim \text{Gamm}(\alpha_n, \beta)$

$Y = X_1 + X_2 + \cdots + X_n$ (sum of n-gamma dist).

$Y \sim \text{Gamma}(\alpha_1 + \alpha_2 + \cdots + \alpha_n, \beta)$

If $X \rightsquigarrow f_X(x)$

pdf $X + Y \longrightarrow f_X(x) + f_X(x)$.

$X \sim \text{Ber}(p)$ $\quad Y = \text{Ber}(p)$

$\underline{\underline{X + X}} \overset{?}{=} 2X$ $\quad$ $\underline{\underline{\text{No}}}$

Values of $\underline{\underline{X+Y}} = \{0, 1, 2\}$

Values of $\underline{\underline{2X}} = \{0, 2\}$

y \ x	0	1
0		
1		

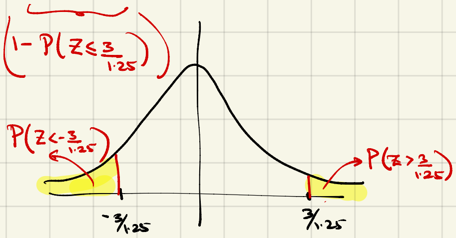
$X \sim N(\mu = 15, \sigma = 1.25)$ $\quad P(|X-15| \geq 3)$

$|X-15| \geq 3 \iff X-15 > 3$ or $X - 15 < -3$

$\iff \frac{X-15}{1.25} > \frac{3}{1.25}$ or $\frac{X-15}{1.25} < \frac{-3}{1.25}$

$\iff Z > \frac{3}{1.25}$ or $Z < \frac{-3}{1.25}$

$P(|X-15| \geq 3) = \underbrace{P\left(Z > \frac{3}{1.25}\right)}_{= \left(1 - P\left(Z \leq \frac{3}{1.25}\right)\right)} + P\left(Z < \frac{-3}{1.25}\right)$



Alternatively

$$P(|X-15| \geq 3) = 1 - P(|X-15| < 3)$$
$$= 1 - P(-3 \leq X - 15 < 3)$$
$$= 1 - P\left(\frac{-3}{1.25} \leq \frac{X-15}{1.25} < \frac{3}{1.25}\right)$$
$$= 1 - P\left(\frac{-3}{1.25} \leq Z \leq \frac{3}{1.25}\right)$$
$$= 1 - \left(P\left(Z \leq \frac{3}{1.25}\right) - P\left(Z \leq \frac{-3}{1.25}\right)\right)$$
$$= \underbrace{1 - P\left(Z \leq \frac{3}{1.25}\right)}_{P\left(Z \geq \frac{3}{1.25}\right)} + \underbrace{P\left(Z \leq \frac{-3}{1.25}\right)}_{P\left(Z \leq \frac{-3}{1.25}\right)}$$

E2 sample

① $P(0 \leq Z \leq c) = 0.258$

$P(Z \leq c) - P(Z \leq 0) = 0.258$

$P(Z \leq c) = 0.258 + \underbrace{P(Z \leq 0)}_{= 0.5}$

$\therefore P(Z \leq c) = 0.258 + 0.5$
$= 0.758$

use qnorm ⟶ qnorm(0.758, lower.tail = True).