Probability Distributions of Random Variables

Let $S \rightsquigarrow$ sample space of an experiment.

A random variable $X \rightsquigarrow$ function $X : S \longrightarrow \mathbb{R}$

$\Updownarrow$

the values $\mathcal{X}$ of X define a new sample space for the original experiment

$\Updownarrow$

X (or $\mathcal{X}$) extracts specific information about an outcome in S.

Examples:

Experiment	Random variable	Sample values
① Toss a coin five times	$X =$ count the number of heads	$X(HHTTH) = 3 = X(HHHTT)$ $X(HTTTT) = 1$
	$Y =$ proportion of heads in the outcome	$X(HHTTH) = X(HHHTT) = \frac{3}{5}$ $X(HTTTT) = X(THTTT) = \frac{1}{5}$
② Roll a six sided dice 4-times	$X =$ sum of the numbers facing up	$X(1,2,4,4) = 11$ $X(1,1,1,6) = 9$ $X(6,6,6,2) = 20$
	$X =$ average of numbers in the sample	$X(1,2,4,4) = \frac{11}{4}$ $X(1,1,1,6) = 9/4$ $X(6,6,6,2) = 20/4$
③ Choose a box of milk cartons	$X =$ weight of the box	$X(\text{box } 1) = 5.1$ $X(\text{box } 2) = 5.2$ $X(\text{box } 3) = 5.01$

$X =$ # heads in an outcome

$Y =$ proportion of heads in an outcome

$Y = \frac{\text{\# head in an outcome}}{5} = \frac{X}{5}$

$Y(HHHTH) = \frac{X(HHHTH)}{5} = \frac{4}{5}$

$\mathcal{X} = \{4, 5, \dots, 24\}$, $|S| = 6^4 = 1 \times 81$

Vmathcal (X)

1 2 3 4 5 6

$\mathcal{X} = (a, b) \subseteq \mathbb{R}$.

Rolling a dice

$Z =$ # odd numbers that show up.

$Z(1,2,2,2) = 1$

$Z(2,4,6,2) = 0 \rightsquigarrow$ Values of X

$Z(1,3,5,1) = 4$ $\quad \{0,1,2,3,4\}$

Note Given an experiment with sample space S, and an random variable $X : S \longrightarrow \mathbb{R}$

$\Updownarrow$

two sample spaces: S; $\mathcal{X}$ = values of X

$$\lim_{x \to x_0} f(x) = L \qquad L = f(x_0)$$

$$\lim_{x \to x_0} f(x) = f(x_0), \quad \lim_{x \to x_0^+} f(x) = f(x_0) \text{ and } \lim_{x \to x_0^-} f(x) = f(x_0) \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

For simplicity, assume $|S|$ is finite

$\Updownarrow$

$S = \{s_1, s_2, s_3, \dots, s_n\}$

If $X : S \longrightarrow \mathbb{R}$ is a r.v

$\Updownarrow$

Let $\mathcal{X}$ = values of X $= \{x_1, x_2, \dots, x_k\}$

Two sample spaces for original experiment

- S $\Updownarrow$ simple outcomes are $\{s_j\}$ $j = 1, 2, 3, \dots, n$ $\Updownarrow$ event $E \subseteq S$ described in terms of $\{s_j\}$
- $\mathcal{X}$ $\Updownarrow$ simple outcomes are $\{x_i\}$ $i = 1, 2, \dots, k$
  $\{X = x_i\} = \{s \in S : X(s) = x_i\} \subseteq S$
  $\Updownarrow$ events $E \subset \mathcal{X}$ are described in terms of $\{X = x_i\}$

Let P be a probability function for S.

$\Updownarrow$

use P to "induce" a probability function P_X on $\mathcal{X}$

$A \subseteq B$

$\Updownarrow$

Step 1) $\boxed{P_X(\{x_i\}) := P(X = x_i) = \sum_{\omega \in \{X = x_i\}} P(\omega)}$

outcomes in $\mathcal{X}$ — Event in S — outcome in S.

Step 2) Suppose $E \subseteq \mathcal{X}$, ~~$E := \bigcup_{i=1}^{r} \{X = x_i\}$~~ $= \{X = x_1\} \cup \{X = x_2\} \cup \dots \cup \{X = x_r\}$

$$\boxed{P_X(E) := P\left(\bigcup_{i=1}^{r} \{X = x_i\}\right) = \sum_{i=1}^{r} P(X = x_i)}$$

$E = \{x_1, x_2, x_5\} \rightsquigarrow \bigcup_{i=1} \{X = x_i\}$

$P_X(E)$

$E \subseteq \mathcal{X}$; $E \subseteq \{x_1, x_2, x_3, \dots, x_k\}$

$E = \{x_1, x_2, x_5\} \Rightarrow E = \{X = x_1\} \cup \{X = x_2\} \cup \{X = x_5\}$

Note:

① The random variable will be denoted by upper case letters $X, Y, \dots$ etc

② The values of the random variable will be denoted by lower case letters : $x, y, \dots$ etc.

③ when there is no ambiguity, we will use the notation $P(X = x_i)$ instead of $P_X(X = x_i)$.

Distributions of Random Variables

Let $X \rightsquigarrow$ random variable

$\Updownarrow$

P_X : probability function for X.

The cummulative distribution function of X

$\Updownarrow$

$$\boxed{F_X(x) := P(X \le x) \quad \text{for all } x}$$

$F_X : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto F_X(x) = P_X(X \le x)$

When sample space is finite $P(X \le x) = \sum_{\substack{x_i \in \mathcal{X} \\ x_i \le x}} P(X = x_i)$

Example: Suppose we toss a fair coin 4 times.

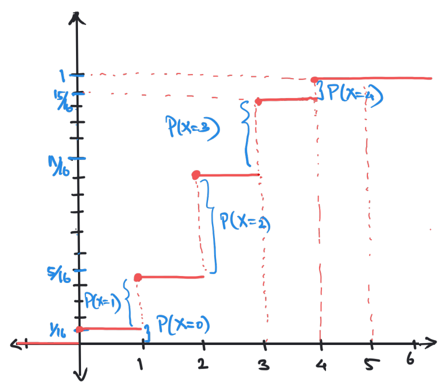
$X =$ # heads in 4 tosses.

We can calculate

$P(X=0) = \frac{1}{16}$, $P(X=1) = \frac{4}{16}$, $P(X=2) = \frac{6}{16}$, $P(X=3) = \frac{4}{16}$, $P(X=4) = \frac{1}{16}$.

We can calculate $F_X(x)$ as

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ 1/16 & \text{if } 0 \le x < 1 \\ 5/16 & \text{if } 1 \le x < 2 \\ 11/16 & \text{if } 2 \le x < 3 \\ 15/16 & \text{if } 3 \le x < 4 \\ 16/16 = 1 & \text{if } 4 \le x < \infty \end{cases}$$



Toss a coin 4 times. $S = \{HHHH, HHHT, \dots, TTTT\}$

$X =$ # heads $\qquad \mathcal{X} = \{0, 1, 2, 3, 4\}$

$E \subseteq \mathcal{X}$, $P_X(E) = ?$

E = atmost two heads, $E = \{0, 1, 2\} \subseteq \mathcal{X}$

We know: $P(HHHH) = \frac{1}{16}$ (assuming all outcomes are equally likely)

$P_X(E) := P$

$\{X = 0\} = \{TTTT\}$ $\quad \{X = 2\} = \{HHTT, \dots, TTHH\}$

$\{X = 1\} = \{HTTT, THTT, TTHT, TTTH\}$

$\{X = 0\} \subseteq S$, $\{X = 1\} \subseteq S$, $\{X = 2\} \subseteq S$.

$P(X=0)$

$P_X(E) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$

Note:

① $F_X(x)$ has jumps at $x = 0, 1, 2, 3, 4$ and the jump equals $P(X = x_i)$.

② $F_X(x)$ is a non-decreasing function

③ $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

④ $F(x) \ge 0 \quad \forall x \in \mathbb{R}$

Note: Every cdf satisfies properties ②, ③ and ④.

Thm: A function $F(x)$ is a cdf if and only if the following three conditions are satisfied

a) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

b) $F(x)$ is a non-decreasing function

c) $F(x)$ is right continuous; ie $\lim_{x \to x_0^+} F(x) = F(x_0)$, $\forall x_0 \in \mathbb{R}$

$X \rightsquigarrow$ a random variable

- X is Discrete if $F_X(x)$ is a step function
  $\Updownarrow$
  The probability mass function pmf is
  $$\boxed{p_X(x) = P(X = x) \quad \forall x}$$
  what if $x \notin \mathcal{X}$ $\quad P(X = x) = 0$
- X is Continuous if $F_X(x)$ is a continuous function.
  $\Updownarrow$
  The probability density function (pdf) $f_X(x)$ satisfies
  $$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \forall x.}$$

Thm: $f_X(x)$ is a pdf (or pmf) of a random variable if and only if

a) $f(x) \ge 0$ for all x

b) $\sum f_X(x) = 1$ or $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

discrete — Continuo

$$P(X = c) = 0$$