Sets, Experiments, and Probability

$\boxed{\text{Experiment}}$ ⇝ a) repeatable task
b) well defined outcomes.
⇝ Examples
a) Toss a coin
b) Draw ball from a bag containing balls.
c) Choose a student from this class.
d) Choose three students from this class.
need to specify if choosing with or without replacement.

$\boxed{\text{Sample Space}}$ ⇝ set of all possible outcomes of an experiment
usually denoted by S.

Examples: Coin toss ⇝ $S = \{H, T\}$
Ball draw ⇝ S = set of all balls in the bag.
Choose a student ⇝ S = all students in this class.
Choose three studens ⇝ S = all groups of three students in this class
does order matter ?

$\boxed{\text{Outcome}}$ ⇝ collection of possible outcomes of an experiment
a subset of sample space S.

Examples: a) Coin toss : $\{H\} \subset S$ ⇝ coin lands heads.
b) Choose a student : $\{\text{Sarah}\}$ ⇝ selected student is sarah.

Set Theory : Basics

Let A, B be two sets.

1) $A \subset B$ ⇝ A "subset" of B ⇝ $x \in A \Rightarrow x \in A$ (containment)

2) $A = B$ ⇝ A "equals" B ⇝ $A \subset B$ and $B \subset A$. (equality)

Operations with Sets

a) $A \cup B = \{x : x \in A \ \boxed{\text{or}}\ x \in B\}$ ⇝ A $\boxed{\text{"union"}}$ B.
b) $A \cap B = \{x : x \in A \ \boxed{\text{and}}\ x \in B\}$ ⇝ A $\boxed{\text{"intersection"}}$ B.

c) $A^c = \{x : x \notin A\}$ ⇝ $\boxed{\text{Complement}}$ of A

d) ϕ = "empty set" ⇝ set containing no elements.

Let Γ be an indexing set → finite
→ infinite → countable
→ uncountable.

Fix a set S,
$\{A_\alpha : \alpha \in \Gamma\}$ ⇝ collection of subset of S indexed by Γ.

a) $\bigcup_{\alpha \in \Gamma} A_\alpha = \{x \in S : x \in A_\alpha \text{ for some } \alpha \in \Gamma\}$

b) $\bigcap_{\alpha \in \Gamma} A_\alpha = \{x \in S : x \in A_\alpha \text{ for all } \alpha \in \Gamma\}$

Note:

① A and B are $\boxed{\text{disjoint}}$ if $A \cap B = \phi$.
mutually exclusive.

② $A_1, A_2, A_3, \ldots$ are $\boxed{\text{pair wise disjoint}}$ if $A_i \cap A_j = \phi$ for all $i \neq j$.

③ $A_1, A_2, A_3, \ldots$ is said to be a $\boxed{\text{partition}}$ of S if
a) $A_1, A_2, A_3, \ldots$ are pairwise disjoint.
b) $\bigcup_{i=1}^{\infty} A_i = S$.

④ The $\boxed{\text{Universal set}}$ ⇝ set of $\boxed{\text{all}}$ objects of interest
typically the sample space S of an experiment.

Axioms of Probability

Let S be the sample space of an experiment.
Given $A \subset S$, an event ⇝ want to calculate the chance/probability that A occurs.
denote this number as $P(A)$.
the "probability of A"

Axioms for a Probability function

A probability function P satisfies:

Axiom 1: $P(A) \geq 0$ for any event $A \subset S$.

Axiom 2: $P(S) = 1$, S the sample space.

Axiom 3: For $A_1, A_2, A_3, \ldots$ collection of pair wise disjoint events, we must have

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + \cdots = \sum_{i=1}^{\infty} P(A_i).$$

Properties of a Probability Function P

① $P(\phi) = 0$ ⇝ Probability of no outcome is zero.

② Given an event A ⇝ A^c is set of all out comes not in A.

$A^c \cap A = \phi$ and $A^c \cup A = S$

$P(A) + P(A^c) = P(S) = 1.$

$P(A^c) = 1 - P(A).$

③ Note: Given two events $A, B \subset S$.

$A \setminus B$ ⇝ A minus B ⇝ $\{x \in A : x \notin B\}$

$A = (A \setminus B) \cup (A \cap B)$ disjoint union.

$P(A) = P(A \setminus B) + P(A \cap B).$

$P(A \setminus B) = P(A) - P(A \cap B)$

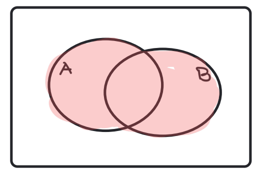
④ $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
pairwise disjoint union.

$$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$$
$$= (P(A) - P(A \cap B)) + P(A \cap B) + (P(B) - P(A \cap B))$$

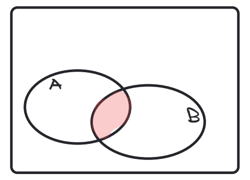
$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Venn Diagrams.

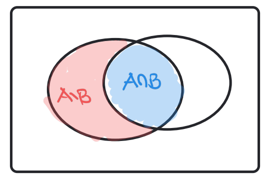
$A \cup B$



$A \cap B$



$A \setminus B$



$B \setminus A$