Continuous Random Variables (III)

Goal: Study the Gamma Distribution

Useful when modelling → component lifetimes
→ modelling waiting times

Gamma Distribution

The gamma function is defined as

$$\boxed{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt \quad , \text{ for } \alpha>0}$$

Properties of Gamma Function

① $\boxed{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \quad \text{for } \alpha>0}$

Can check that $\Gamma(1)=1$ and using induction we have $\boxed{\Gamma(n) = (n-1)!}$ for $n \in \mathbb{N}$.

② $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$.

For $\alpha, \beta > 0$, let

$$\boxed{f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha\,\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}}$$

Note: ① $f(x;\alpha,\beta) \geq 0 \quad \forall x \in \mathbb{R}$

② $\int_{-\infty}^{\infty} f(x;\alpha,\beta)\,dx = 1$ (do a change of variables $y = \frac{x}{\beta}$)

We say X has the Gamma Distribution with shape parameter α, and scale parameter β

$\boxed{\text{if the pdf of } X \text{ is } f(x;\alpha,\beta)}$.

Note:

① $\boxed{\text{When } \beta=1}$ → $\boxed{\text{"Standard Gamma distribution"}}$ with shape param: α.

pdf: $$\boxed{f(x;\alpha) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}}$$

② $X \sim \text{Gamma}(\alpha,\beta)$ then

a) $\boxed{E(X) = \mu_X = \alpha \cdot \beta}$

b) $\boxed{V(X) = \sigma_X^2 = \alpha\beta^2}$

c) $\boxed{\sigma_X = \sqrt{\alpha} \cdot \beta}$

③ The cdf of $X \sim \text{Gamma}(\alpha,\beta)$

$$F_X(x;\alpha,\beta) = \begin{cases} \int_0^x f(t;\alpha,\beta)\,dt & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{P(X \leq x) = F_X(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right)}$$

cdf of Standard Gamma with param: α.

ie $$\boxed{F(x;\alpha) = \begin{cases} \int_0^x \dfrac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\,dt & \text{if } x>0 \\ 0 & \text{otherwise.} \end{cases}}$$

If T has Standard Gamma with shape param: α

$X = \beta T$ has Gamma distribution with shape: α, scale: β.

Special Cases of Gamma Distribution

① Exponential Distribution → set $\alpha = 1$, $\beta = \frac{1}{\lambda}$

Get Exponential Distribution with param $\lambda > 0$.

pdf: $$\boxed{f(x;\lambda) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

cdf: $$\boxed{F_X(x;\lambda) = \begin{cases} 1-e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}}$$

Note:

① If X has exponential Distribution with param $\lambda > 0$ → $\boxed{E(X) = \mu_X = \frac{1}{\lambda}}$

$\boxed{V(X) = \sigma_X^2 = \frac{1}{\lambda^2}}$

$\boxed{\sigma_X = \frac{1}{\lambda}}$

② If we have a "Poisson Process" with rate α → The exponential distribution with $\lambda = \alpha$ models the distribution of "elapsed time" between the occurrence of two successive events.

③ Also, if $X \sim \text{Exp}(\lambda)$

$$P(X \geq t+t_0 \mid X \geq t_0) = \frac{P[\{X \geq t+t_0\} \cap \{X \geq t_0\}]}{P(X \geq t_0)}$$

$$= \frac{P(X \geq t+t_0)}{P(X \geq t_0)} = \frac{1-F(t+t_0;\lambda)}{1-F(t_0;\lambda)} = e^{-\lambda t} = P(X \geq t)$$

That is, if X was modelling the lifetime of a component

The distribution of additional lifetime is exactly the same as the original distribution of lifetime.

The exp distribution has "memoryless property".

② Chi-Squared Distribution

X is said to have Chi-squared dist with param: ν (degrees of freedom) → $\boxed{X \sim \text{Gamma}(\alpha,\beta) \text{ with } \alpha = \frac{\nu}{2},\ \beta = 2.}$

pdf: $$\boxed{f(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \cdot x^{\nu/2-1} \cdot e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}}$$

$\boxed{E(X) = \mu_X = \alpha\beta = \nu}$

$\boxed{V(X) = \sigma_X^2 = \alpha\beta^2 = 2\nu}$

$\boxed{\sigma_X = \sqrt{2\nu}}$

Note: ① Chi-square distribution → plays important role in statistical inference

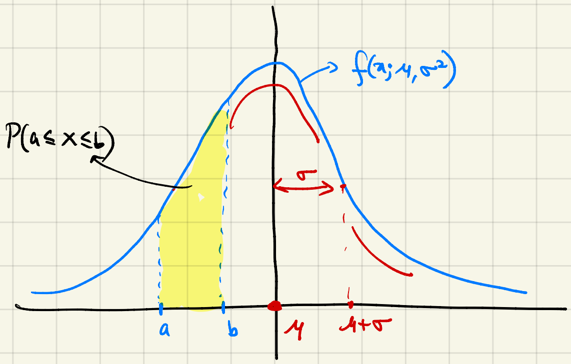
② If $X \sim N(\mu,\sigma^2)$ then $\left(\frac{X-\mu}{\sigma}\right)^2$ has chi-sq distribution with $\nu = 1$.

Recall:

$X \sim N(\mu, \sigma^2)$

X takes values $(-\infty, \infty)$

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$



$a, b \in \mathbb{R}$

$$P(a \leq X \leq b) = \int_a^b f(x;\mu,\sigma^2)\,dx$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= P(X \leq b) - P(X \leq a)$$

Example:

$P(|X-\mu| < \sigma) = ?$

$P(|X-\mu| < 2\sigma) = ?$

$P(|X-\mu| < 3\sigma) = ?$

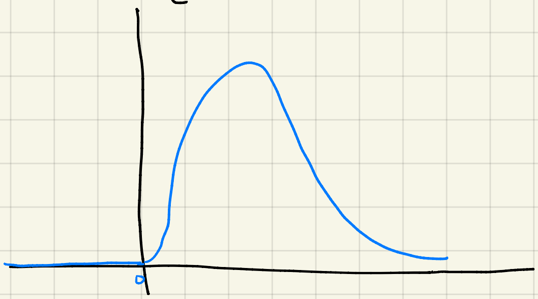
$\overset{?}{64}$ - 95 - 99.7

Gamma Distribution

$$\Gamma(\alpha) = \underbrace{\int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx}_{\text{gamma function}} \qquad \underline{\alpha > 0}$$

The gamma distribution has the pdf: $\alpha, \beta > 0$

$$f(x; \underset{\text{shape}}{\alpha}, \underset{\text{scale}}{\beta}) = \begin{cases} \dfrac{1}{\beta^\alpha \cdot \Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



To check that this is a pdf:

(i) $f(x;\alpha,\beta) \geq 0 \quad \forall x$ ✓

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x;\alpha,\beta)\, dx = 1$

$$\int_{-\infty}^{\infty} f(x;\alpha,\beta)\,dx = \cancel{\int_{-\infty}^{0} 0 \cdot dx} + \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta}\, dx$$

$$= \lim_{c\to\infty} \int_0^c \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta}\, dx$$

set $y = \frac{x}{\beta}$ $\qquad dy = \frac{dx}{\beta}$

$x = 0 \Rightarrow y = 0$

$x = c \Rightarrow y = \frac{c}{\beta}$ $\quad ; \quad \beta y = x$

$$= \lim_{c\to\infty} \int_0^c \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha-1}} \cdot \underset{\beta y}{x}^{\alpha-1} \cdot e^{-\overset{y}{x/\beta}} \cdot \overset{dy}{\frac{dx}{\beta}}$$

$$= \lim_{c\to\infty} \int_0^{c/\beta} \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha-1}} \cdot (\beta y)^{\alpha-1} \cdot e^{-y} \cdot dy$$

$$= \lim_{c\to\infty} \int_0^{c/\beta} \frac{1}{\Gamma(\alpha) \cdot \cancel{\beta^{\alpha-1}}} \cdot \cancel{\beta^{\alpha-1}} \cdot y^{\alpha-1} \cdot e^{-y} \cdot dy$$

$$= \lim_{c\to\infty} \int_0^{c/\beta} \frac{1}{\Gamma(\alpha)} \cdot y^{\alpha-1} \cdot e^{-y} dy = \frac{1}{\Gamma(\alpha)} \cdot \lim_{c\to\infty} \int_0^{c/\beta} y^{\alpha-1} \cdot e^{-y} dy$$

$$= \frac{1}{\Gamma(\alpha)} \overset{\Gamma(\alpha)}{\int_0^\infty y^{\alpha-1} \cdot e^{-y} \cdot dy} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = \underline{\underline{1}}$$

Properties of $\Gamma(\alpha)$

① $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \forall \alpha > 0$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(1) = 1 \longrightarrow$

$$\int_0^\infty x^{1-1} \cdot e^{-x} dx = \int_0^\infty e^{-x} dx = \lim_{c\to\infty} \int_0^c e^{-x} dx$$

$$= \lim_{c\to\infty} \left. \frac{e^{-x}}{-1} \right|_0^c$$

$$= \lim_{c\to\infty} \left(-e^{-c} + 1\right)$$

$$= \underline{\underline{1}} \Rightarrow \underline{\underline{\Gamma(1) = 1}}$$

② $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Standard Gamma Distribution

Gamma dist with parameter α, $\underline{\underline{\beta = 1}}$

$$f(x;\alpha) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If T has pdf $f(x;\alpha) \longrightarrow$ T is said to have the Standard Gamma dist with param: $\underbrace{\alpha}_{\text{shape param.}}$.

As α changes the shape of the standard gamma pdf is going to change.

$\Downarrow$

plot $f(x;\alpha)$ for diff values of $\underline{\alpha}$.

If T has shape param α, if we define

$X = \beta T \longrightarrow X \sim$ Gamma dist with shape: α, Scale: β.

How do we calculate the pdf of a r.v?

$T \sim \text{Gamma}(\alpha, \beta=1)$, $X = \beta T \sim \text{Gamma}(\alpha,\beta)$

$$f(t;\alpha) = \begin{cases} \frac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

pdf for $X = \beta \cdot f(t;\alpha) = ?$

$$\int_{-\infty}^{\infty} f(t,\alpha)\,dt = 1 \Rightarrow \int_{-\infty}^{\infty} \beta f(t;\alpha)\,dt = \beta$$

if $\beta \neq 1$

$\beta f(t;\alpha)$ is not a pdf.

How does one calculate the pdf of $X = \beta T$.

Use the fact: $f(x;\alpha,\beta) = \frac{d}{dx} F(x;\alpha,\beta)$

$$F(x;\alpha,\beta) = P(X \leq x) = P(\beta T \leq x)$$

$$= P\left(T \leq \frac{x}{\beta}\right)$$

$$= \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-t}\,dt$$

$$F(x;\alpha,\beta) = \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}\,dt$$

$$f(x;\alpha,\beta) = \frac{d}{dx}\left(F(x;\alpha,\beta)\right) = \frac{d}{dx}\left(\int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} \cdot t^{\alpha-1} e^{-t}\,dt\right)$$

Use the fact:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

$a(x) = 0$, $b(x) = \frac{x}{\beta}$

$a'(x) = 0$ $\quad b'(x) = \frac{1}{\beta}$

$$= f\left(\frac{x}{\beta}\right) \cdot \frac{1}{\beta} - \cancel{f(0) \cdot 0}$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \cdot e^{-x/\beta} \cdot \frac{1}{\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta^{\alpha-1}} \cdot \frac{1}{\beta} \cdot x^{\alpha-1} \cdot e^{-x/\beta}$$

$$= f(x;\alpha,\beta)$$

$X \sim \text{Gamma}(\alpha,\beta)$

$E(X) = \alpha \cdot \beta$

$V(X) = \alpha \cdot \beta^2$

Two special cases

Exponential

Chi-squared dist

$\text{Gamma}(\alpha=1, \beta=\frac{1}{\lambda})$

$\therefore X \sim \text{Exp}(\lambda)$

$X \sim \text{Gamma}(\alpha=1, \beta=1/\lambda)$

$$f(x;\lambda) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $X \sim \text{Exp}(\lambda)$, if $\gamma > 0$

$X^{1/\gamma} \sim$ Weibull Distribution

params: (γ, λ)

Use diff under int sign to calculate the pdf of $X^{1/\gamma}$ given pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwi} \end{cases}$$

Chi-squared Distribution

Special Case:

$Z \sim N(0,1)$

What is the distribution of Z^2?

Set

$Y = Z^2$

cdf of Y is given by

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y)$$
$$= P(-\sqrt{y} \le Z \le \sqrt{y})$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz$$

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz\right)$$

HW 6

= pdf of "Chi-squared" dist with one degree of freedom

$$\stackrel{?}{=} \frac{1}{\sqrt{2} \cdot \Gamma(1/2)} \cdot x^{-1/2} \cdot e^{-x/2}.$$

$\Gamma(1/2) \longrightarrow \sqrt{\pi}$

$\chi^2_v \sim \text{Gamma}(\alpha = \frac{v}{2}, \beta = 2)$

Chi-squared distribution

pdf of $\chi^2_v \rightsquigarrow \boxed{f(x; v) = \frac{1}{2^{v/2} \cdot \Gamma(v/2)} \cdot x^{v/2 - 1} \cdot e^{-x/2}}$

put $\alpha = \frac{v}{2}$ and $\beta = 2$ in the pdf of Gamma(α, β).

Recall: $\{x_1, x_2, \ldots, x_n\}$

Sample Variance:

$$S^2 = \frac{1}{(n-1)} \cdot \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$(n-1) S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\frac{(n-1) S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$
$$= \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$

= "sum of square standard normal" r.v's

distribution for $\frac{(n-1)S^2}{\sigma^2}$ is a sum of Chi-squared r.v's.

Chi-squared dist with $(n-1)$ degrees of freedom.

$X \sim \chi^2_v$; $E(X) = v$

$\text{Var}(X) = 2v.$

Suppose we know that a certain phenomena has distribution X.

Calculate: $\boxed{P(a \le X \le b) = F_X(b) - F_X(a)}$

$$= \int_a^b f_X(x)\, dx$$

p(name of dist) $\rightsquigarrow$ gives the cdf of the distribution R.