Discrete Random Variables (II)

Suppose X is a discrete random variable

$\mathcal{X}$ = set of values of X — finite / countably infinite.

Today's Goal:

Study some discrete r.v's which have $\mathcal{X}$ to be infinite.

① Poisson Distribution

$\boxed{X \text{ is said to have Poisson Distribution with parameter: } \lambda > 0}$ $\xrightarrow{\text{if}}$ $\mathcal{X} = \{0,1,2,3,\dots\}$ for $x \in \mathcal{X}$

$$\boxed{p_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$$

Note:

①
$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$\Rightarrow \boxed{1 = e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} p_X(x)}$$

Also, $\boxed{p_X(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \geq 0 \quad \forall x}$

$\therefore$ p_X is a legitimate probability Mass Function.

② $\boxed{\text{If } X \text{ has the Poisson Distribution with parameter } \lambda}$ $\longrightarrow$ $E(X) = \mu_X = \lambda$

$V(X) = \sigma_X^2 = \lambda$

$\therefore$ $\sigma_X = \sqrt{\lambda}$

③ The Poisson distribution is often used to model phenomena where we are waiting for events to happen.

For example: In a "Poisson Process" $\rightarrow$ the number of occurrences in a given time interval t.

② Negative Binomial Distribution

$\boxed{X \text{ has the Negative Binomial Distribution with parameters: } r, p}$ $\xrightarrow{\text{if}}$ $\mathcal{X} = \{0,1,2,3,\dots\}$ for $x \in \mathcal{X}$

$$\boxed{p_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x}$$

Note:

① The experiment for this distribution $\rightarrow$ Perform Bernoulli trials with $P(\text{success}) = p$ repeatedly until we get exactly r-successes.

$\boxed{X = \# \text{ of failures that precede the } r^{\text{th}} \text{ success.}}$

Then $P(X=x) = P(\text{exactly } x \text{ failures before the } r^{\text{th}} \text{ success})$

Total ways we can have exactly x failures before r^{th} success $\rightarrow$ Choosing x spots in $x+r-1$ possible places (since the last spot has to be a success)

can be done in $\binom{x+r-1}{x} = \binom{x+r-1}{r-1}$ ways.

$\therefore$ $P(X=x) = \binom{x+r-1}{r-1} \underbrace{p^r}_{\text{since } r \text{ successes}} \underbrace{(1-p)^x}_{\text{since } x \text{ failures}}$

② $\boxed{\text{If } X \text{ has negative Binomial Distribution with parameters: } r, p}$ $\longrightarrow$ $E(X) = \mu_X = \frac{r(1-p)}{p}$

$V(X) = \sigma_X^2 = \frac{r(1-p)}{p^2}$

③ Called negative binomial

$$\boxed{\binom{x+r-1}{x} = (-1)^x \binom{-r}{x} = (-1)^x \frac{(-r)(-r-1)(-r-2)\cdots(-r-x+1)}{x \cdot (x-1)(x-2)\cdots 3 \cdot 2 \cdot 1}}$$

So that

$$\boxed{p_X(x) = (-1)^x \binom{-r}{x} p^r (1-p)^x}$$

very similar to the pmf of Binomial Distribution.

③ Geometric Distribution

$\boxed{X \text{ has Geometric Distribution with parameter: } p}$ $\xrightarrow{\text{if}}$ $\mathcal{X} = \{1,2,3,\dots\}$ for $x \in \mathcal{X}$

$$\boxed{p_X(x) = p(1-p)^{x-1}}$$

Note:

① Geometric distribution $\rightarrow$ special case of Negative Binomial pmf

when $r=1$ and $x = 1,2,3,\dots$

② Is the simplest of the "waiting time" distributions

X can be interpreted as the trial at which first success occurs

"waiting for a success"

③ $\boxed{\text{If } X \text{ has Geometric distribution with "success" parameter: } p}$ $\longrightarrow$ $E(X) = \mu_X = \frac{1}{p}$

$V(X) = \sigma_X^2 = \frac{(1-p)}{p^2}$

④ Sometimes used to model "lifetime" or "time until failure" of components.

Relationships Between Discrete distributions (in the limit)

① Binomial ↔ Hypergeometric

Let $N, M, n \in \mathbb{N}$, $p = \frac{M}{N}$

(i) $bin(x; n, p)$ $\rightarrow$ pmf for binomial with params: n = sample size, p = prob of success

(ii) $hyper(x; N, M, n)$ $\rightarrow$ pmf for hypergeometric distribution with params: N = population size, M = # of successes, n = sample size

If $N, M \to \infty$ st $\frac{M}{N} \to p$ $\Big\}$ $\longrightarrow$ $\{hyper(x; N, M, n) \longrightarrow bin(x; n, p)\}$

if sample size n is small compared to population size N, can assume that samples are "approximately" independent.

② Binomial ↔ Poisson

Suppose

$bin(x; n, p)$ $\longrightarrow$ pmf of Binomial with params n = sample size, p = probability of success.

$pois(x; \lambda)$ $\longrightarrow$ pmf of Poisson Distribution with parameter $\lambda > 0$.

$\boxed{\text{If } n \to \infty \text{ and } p \to 0 \text{ such that } n \cdot p \to \lambda \;\Big\} \longrightarrow \{bin(x; n, p) \longrightarrow pois(x; \lambda)\}}$

If n is large, and p very small

$bin(x; n, p) \approx pois(x, \lambda = n \cdot p)$

"The Poisson Distribution is approximately Binomial for rare events."