Bayes' Theorem

Recall: Given events $A, B \subseteq S$, $S \rightsquigarrow$ sample space and if $P(B) > 0$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \rightsquigarrow \text{"Probability that } A \text{ occurs given that } B \text{ has occurred"}$$

Let $\{A_1, A_2, A_3, \ldots, A_n\}$ be a partition for the sample space S

that is:

i) $A_1, \ldots, A_n$ are mutually exclusive $\rightsquigarrow A_i \cap A_j = \phi \ \forall\ i \neq j$

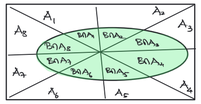
ii) $A_1, \ldots, A_n$ are exhaustive $\rightsquigarrow \bigcup_{i=1}^{n} A_i = S$

Given any event $B \subseteq S$, we have

$$B = B \cap S = B \cap \left(\bigcup_{i=1}^{n} A_i\right)$$

ie

$$\boxed{B = \bigcup_{i=1}^{n} (B \cap A_i)} \rightsquigarrow \text{disjoint union.}$$



Then

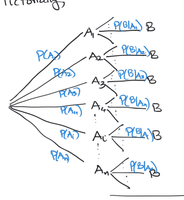
$$P(B) = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) = \sum_{i=1}^{n} P(B \cap A_i)$$

"Law of Total Probability"

Since $P(B \cap A_i) = P(B|A_i) \cdot P(A_i)$

$$\boxed{P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}$$

Pictorially,



Events	Probability
$B \cap A_1$	$\rightsquigarrow P(B\|A_1) \cdot P(A_1)$
$B \cap A_2$	$\rightsquigarrow P(B\|A_2) \cdot P(A_2)$
$B \cap A_3$	$\rightsquigarrow P(B\|A_3) \cdot P(A_3)$
$B \cap A_4$	$\rightsquigarrow P(B\|A_4) \cdot P(A_4)$
$\vdots$	$\vdots$
$B \cap A_i$	$\rightsquigarrow P(B\|A_i) \cdot P(A_i)$
$\vdots$	$\vdots$
$B \cap A_n$	$\rightsquigarrow P(B\|A_n) \cdot P(A_n)$

$$= \bigcup_{i=1}^{n} (B \cap A_i) \qquad P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

The probability information in the above picture ie $P(A_i)$ and $P(B|A_i)$ for $i = 1, 2, 3, \ldots, n$

Called "Prior probabilities"

Bayes' Theorem $\rightsquigarrow$ use prior probabilities to calculate "posterior probabilities"

want: $P(A_j|B)$ $\quad j = 1, 2, 3, \ldots, n$

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}$$

$$\boxed{P(A_j|B) = \frac{P(B|A_j) \cdot P(A_j)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}} \rightsquigarrow \text{Bayes' Theorem}$$

All probabilities on the RHS are prior probabilities calculable from the tree diagram!

Example:

City has two types of cabs: Blue, Green

$P(B) \rightsquigarrow$ proportion of blue cabs in the city

$P(G) = 1 - P(B) \rightsquigarrow$ proportion of Green cabs in the city.

Suppose there was a hit and run incident involving a cab in the city.

With no additional information available

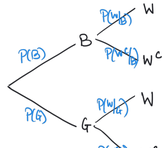
$P(\text{involved cab was a blue cab}) = $ proportion of blue cabs in the city $= P(B)$

Now, Suppose we have a witness.

Let $W \rightsquigarrow$ event that witness says that the involved cab was a Blue cab.

$W^c \rightsquigarrow$ event that witness says the involved cab was a Green cab.

We have the following tree diagram



$P(W|B) = P(\text{witness says involved cab is blue} \mid \text{involved cab is Blue})$

$P(W|G) = P(\text{witness says involved cab is Blue} \mid \text{involved cab is Green})$

$P(W^c|B) = P(\text{witness says involved cab is Green} \mid \text{involved cab is Blue})$

$P(W^c|G) = P(\text{witness says involved cab is Green} \mid \text{involved cab is Green})$

Note that witness is correct only in the first and last cases.

Question: "What is the probability that the involved cab was a Blue cab given that the witness says it was a Blue cab?"

Want the "posterior probability"

$$\boxed{P(B|W) = ?}$$

Use Bayes' Theorem to get

$$\boxed{P(B|W) = \frac{P(W|B) \cdot P(B)}{P(W|B) \cdot P(B) + P(W|G) \cdot P(G)}}$$

We now make "prior" choices for witness reliability.

Let $p \in (0,1) \rightsquigarrow$ probability that witness makes a correct observation

Note:

$W|_B$, $W^c|_G \rightsquigarrow$ events corresponding to correct observations by witness

$$\boxed{\therefore P(W|B) = P(W^c|G) = p}$$

Similarly,

$W|_G$, $W^c|_B \rightsquigarrow$ events corresponding to incorrect observations by witness

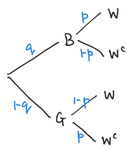
$$\boxed{P(W|G) = P(W^c|B) = 1 - p}$$

Let

$P(B) = $ proportion of Blue cabs in the city $=$ Probability that involved cab is Blue under no additional information

$q \in (0,1)$

We have the following updated tree diagram



$$P(B|W) = \frac{P(W|B) \cdot P(B)}{P(W|B) \cdot P(B) + P(W|G) \cdot P(G)}$$

$$\therefore \boxed{P(B|W) = \frac{p \cdot q}{p \cdot q + (1-p)(1-q)}}$$

Beautiful Equation!

Leads to three important cases depending on "prior" assumptions on "witness reliability" ie the choice of p.

Case I) $P(B|W) = 0 \rightsquigarrow$ Witness testimony exonerates the blue cab.

$$\therefore P(B|W) = 0 \Rightarrow \frac{pq}{pq + (1-p)(1-q)} = 0 \Rightarrow pq = 0$$

$$\Rightarrow p = 0 \quad \text{or} \quad q = 0$$

$p = 0$: witness reliability is zero

$q = 0$: no blue cabs in the city.

Case II) $P(B|W) = 1 \rightsquigarrow$ Witness testimony results in Blue cab involved with complete confidence!

$$P(B|W) = 1 \Rightarrow \frac{pq}{pq + (1-p)(1-q)} = 1 \Rightarrow \cancel{pq} = \cancel{pq} + (1-p)(1-q)$$

$$\Rightarrow (1-p)(1-q) = 0$$

$$\Rightarrow 1 - p = 0 \quad \text{or} \quad 1 - q = 0$$

$p = 1$, ie witness is completely reliable

$q = 1$; ie only Blue cabs in the city.

Case III) $P(B|W) = P(B) \rightsquigarrow$ Witness testimony does not change our perception of the blue cab being involved.

$$P(B|W) = P(B) \Rightarrow \frac{pq}{pq + (1-p)(1-q)} = q$$

$$q \neq 0 \Rightarrow p = pq + (1-p)(1-q) = pq + 1 - p - q + pq$$

$$\Rightarrow 2p - 2pq = 1 - q \Rightarrow 2p(1-q) = (1-q)$$

$$\because (1-q) \neq 0 \Rightarrow 2p = 1 \Rightarrow \boxed{p = \tfrac{1}{2}}$$

$\therefore P(B|W) = P(B)$, ie witness testimony independent of Blue car being at fault corresponds to setting the "prior" witness reliability to the probability of a fair coin toss, ie $p = \frac{1}{2}$!!!