Joint Distributions (I)

Many experiments/applications require that we use more than one characteristic/variable related to the experimental units.

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① Studying Body mass index (BMI) → need height and weight.

② Predicting house prices → depends on many factors: location, house area, # bedrooms, bathrooms etc.

③ Predicting of the price of a stock → depends on many performance metrics

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perhaps how Elon tweets!

In all of these cases → need to calculate the probability of two or more events simultaneously.

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want the "Joint Distribution" of variables of interest.

Joint Distributions for two random Variables

Let X, Y be two r.vs → both discrete / or / both continuous.

(I) Two Discrete random Variables

Suppose X, Y are two discrete r.vs → $\mathcal{X}$ = values of X, $\mathcal{Y}$ = values of Y

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$\mathcal{X}\times\mathcal{Y} = \{(x,y) ; x\in\mathcal{X}, y\in\mathcal{Y}\}$

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"Cartesian Product" of $\mathcal{X}$ and $\mathcal{Y}$

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the "joint sample space"

The "joint probability mass function", i.e. "joint pmf" of X and Y → $\boxed{p(x,y) := P(X=x \text{ and } Y=y)}$

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$$\boxed{P\left(\begin{array}{l} X \text{ takes value } x \text{ and} \\ Y \text{ takes value } y \end{array}\right)}$$

Note: ① $p(x,y)$ satisfies

(i) $p(x,y) \geq 0 \quad \forall (x,y)\in\mathcal{X}\times\mathcal{Y}$

(ii) $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y) = 1$.

② If $A \subseteq \mathcal{X}\times\mathcal{Y}$ → an event in the joint dist

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$$\boxed{P(A) = \sum_{(x,y)\in A} p(x,y)}$$

③ Marginal and Conditional Distributions

(i) "Marginal probability mass function of X" → for fixed $x\in\mathcal{X}$, $p_X(x) := \sum_{y\in\mathcal{Y}} p(x,y)$

(ii) "Marginal probability mass function of Y" → for fixed $y\in\mathcal{Y}$, $p_Y(y) := \sum_{x\in\mathcal{X}} p(x,y)$

(iii) "Conditional Distributions"

$p_{X|Y}(x|y) := P\left(X=x \text{ given that } Y=y\right)$

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$:= \dfrac{p(x,y)}{p_Y(y)}$

$p_{Y|X}(y|x) := P\left(Y=y \text{ given that } X=x\right)$

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$:= \dfrac{p(x,y)}{p_X(x)}$

Important identity

$$\boxed{p(x,y) = p_{X|Y}(x|y)\cdot p_Y(y) = p_{Y|X}(y|x)\cdot p_Y(y)}$$

(II) Two Continuous random Variables

Let X, Y be two continuous r.vs → "joint sample space" is given by $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$.

The "joint probability function" i.e. "joint pdf" for the r.v's X and Y → function $f: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying

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(i) $f(x,y) \geq 0 \quad \forall (x,y)\in\mathbb{R}^2$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

Note:

① If $A \subseteq \mathbb{R}^2$,

$$\boxed{P((x,y)\in A) = \iint_A f(x,y)\,dx\,dy}$$

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If $A = \{(x,y) : x\in[a,b], y\in[c,d]\}$ = 2D rectangle $[a,b]\times[c,d]$

$$P((X,Y)\in A) = \int_a^b\int_c^d f(x,y)\,dx\,dy = P(a\leq X\leq b \text{ and } c\leq Y\leq d)$$

② Marginal and Conditional Distributions

(i) "Marginal probability density function for X" → $f_X(x) := \int_{-\infty}^{\infty} f(x,y)\,dy$ for $x\in(-\infty,\infty)$

(ii) "Marginal probability density function for Y" → $f_Y(y) := \int_{-\infty}^{\infty} f(x,y)\,dx$ for $y\in(-\infty,\infty)$

③ Conditional Probability distribution

The "conditional density function of Y given that $X=x$" → $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$, $y\in(-\infty,\infty)$

Similarly,

The "conditional density function of X given that $Y=y$" → $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ for $x\in(-\infty,\infty)$

Important identity:

$$\boxed{f(x,y) = f_{X|Y}(x|y)\cdot f_Y(y) = f_{Y|X}(y|x)\cdot f_X(x)}$$

Independent Random Variables

We say X and Y are independent if

$$p(x,y) = p_X(x)\cdot p_Y(y) \quad \forall (x,y)\in\mathcal{X}\times\mathcal{Y}$$

if X and Y are discrete

or

$$f(x,y) = f_X(x)\cdot f_Y(y) \quad \forall (x,y)\in\mathbb{R}\times\mathbb{R}$$

if X and Y are continuous.

if conditions for independence are not satisfied we say X and Y are dependent.

Note:

When X and Y are independent, can calculate the joint pdf by "multiplying" the pdf's of X and Y

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In general, joint pdf/pmf calculation is not so easy!!

Need joint distributions $\rightsquigarrow$ want to calculate probabilities of joint events.

(Case I) Work with two rand vars.

(Case II) Generalize to n-rand vars.

Working with two r.v's $\longrightarrow$ both discrete.

$\longrightarrow$ both continuous.

(I)

Given X, Y discrete $\rightsquigarrow$ $\mathcal{X} :=$ values of X

$\mathcal{Y} :=$ values of Y.

$\mathcal{X} \times \mathcal{Y} := \{(x,y) ; x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$

= "joint sample space".

"joint probability mass function" for the pair r.v's X, Y. $\rightsquigarrow$ probability function on $\mathcal{X} \times \mathcal{Y}$

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$p : \mathcal{X} \times \mathcal{Y} \longrightarrow (0,1)$

(i) $0 \leq p(x,y) \leq 1 \quad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}$.

(ii) $\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x,y) = 1$.

Perform the following exp:

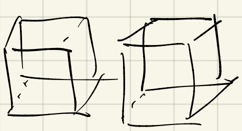
Bag that contains two 6-sided die: dice 1 $\rightsquigarrow$ fair.

dice 2 $\rightsquigarrow$ unfair.

$P(1) = P(2) = \cdots = P(5) = 0.1$

$P(6) = 0.5$

die 1 die 2.



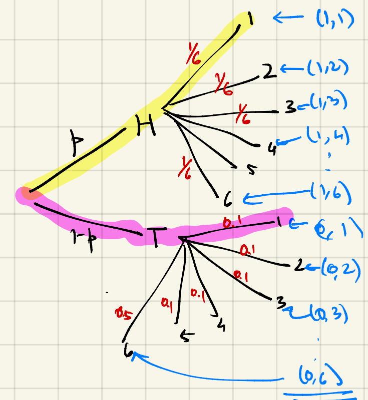
Experiment: Toss a coin, if H $\rightsquigarrow$ roll dice 1

T $\rightsquigarrow$ roll dice 2.

$P(H) = p$

$P(T) = 1-p$.

Tree Diagram for our exp.



X = outcome of the coin toss.

$= \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

Y = outcome of the roll of the die.

$\mathcal{X}$ = values of $X = \{0,1\}$

$\mathcal{Y}$ = values for $Y = \{1,2,3,4,5,6\}$

$\mathcal{X} \times \mathcal{Y} = \{ (0,1), (0,2), (0,3), \ldots, (0,6), (1,1), (1,2), (1,3), \ldots, (1,6) \}$

$(0,6) \rightsquigarrow \{X=0 \text{ and } Y=6\}$

$p(0,6) = P($coin landed T **and** 6 appeared on roll of die$)$

$\mathcal{X} \times \mathcal{Y}$	1	2	3	4	5	6
0	$p(0,1)$	$p(0,2)$				$p(0,6)$
1	$p(1,1)$					$p(1,6)$

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

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joint probability distribution table for this exp.

Given a joint pmf

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Three important distributions attached to a joint pmf.

Marginals:

(i) Marginal distribution for X $\rightsquigarrow$ for $x \in \mathcal{X}$

$$p_X(x) = \sum_{y \in \mathcal{Y}} p(x,y)$$

$$p_X : \mathcal{X} \longrightarrow (0,1)$$

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probability distribution on $\mathcal{X}$.

(ii) Marginal distribution for Y $\rightsquigarrow$ for fixed $y \in \mathcal{Y}$

$$p_Y(y) = \sum_{x \in \mathcal{X}} p(x,y)$$

$$p_Y : \mathcal{Y} \longrightarrow (0,1)$$

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Probability dist on $\mathcal{Y}$.

Example:

x \ y	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

(i) Marginal for X. $\rightsquigarrow$ $\mathcal{X} = \{0,1\}$

$$p_X(0) = \sum_{y \in \mathcal{Y}} p(0,y) = p(0,1) + p(0,2) + p(0,3) + \cdots + p(0,6)$$

$$= (1-p)(0.1) + (1-p)(0.1) + (1-p)(0.1) + \cdots + (1-p)\cdot 0.5$$

$$= (1-p)(0.1+0.1+0.1+0.1+0.1+0.5) = \underline{1-p}$$

$$p_X(1) = \sum_{y \in \mathcal{Y}} p(1,y) = p(1,1) + \cdots + p(1,6)$$

$$= p\left(\tfrac{1}{6} + \tfrac{1}{6} + \cdot + \tfrac{1}{6}\right) = p$$

x	0	1
$p_X(x)$	$1-p$	p

(ii) Marginal for Y.

x \ y	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

$$\mathcal{Y} = \{1,2,3,4,5,6\}$$

$$p_Y(1) = p(0,1) + p(1,1) = (1-p)\cdot 0.1 + p\cdot\tfrac{1}{6}$$

$$p_Y(2) = p(0,2) + p(1,2) = (1-p)(0.1) + p\cdot\tfrac{1}{6}$$

$$p_Y(3) = p(0,3) + p(1,3) = (1-p)(0.1) + p\cdot\tfrac{1}{6}$$

$$p_Y(4) = \qquad = $$

$$p_Y(5) = \qquad = $$

$$p_Y(6) = p(0,6) + p(1,6) = (1-p)(0.5) + p\cdot\tfrac{1}{6}.$$

y	1	2	3	4	5	6
$p_Y(y)$						

x \ y	1	2	3	4	5	6	
0	$(1-p)\cdot(0.1)$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$	$p_X(0)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$	$p_X(1)$
	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	$p_Y(4)$	$p_Y(5)$	$p_Y(6)$	1

Sum

Sum

(iii) Conditional Distribution

$$p_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$$:= \frac{p(x,y)}{p_X(x)}$$

$P_{Y|X}$ when $x = 0$?

To calculate the Conditional dist

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p\cdot(1/6)$

divide this row by $p_X(0)$

$$p_{Y|X}(y|_{x=0}) =$$

When $x=1$

$$p_{Y|X} = \frac{p(x,y)}{p_Y(1)} = P(Y=y\,|_{X=1})$$

$$\int_a^b \frac{k\theta^k}{x^{k+1}}\cdot dx = k\theta^k \cdot \int_a^b \frac{1}{x^{k+1}}\,dx$$