Joint Distributions (I)

Many experiments/applications require that we use more than one characteristic/variable related to the experimental units.

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① Studying Body mass index (BMI) → need height and weight.

② Predicting house prices → depends on many factors: location, house area, # bedrooms, bathrooms etc.

③ Predicting if the price of a stock → depends on many performance metrics

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perhaps how Elon tweets!

In all of these cases → need to calculate the probability of two or more events simultaneously.

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want the "Joint Distribution" of variables of interest.

Joint Distributions for two random Variables

Let X, Y be two r.vs → both discrete / or / both continuous.

(I) Two Discrete random Variables

Suppose X, Y are two discrete r.vs → $\mathcal{X}$ = values of X, $\mathcal{Y}$ = values of Y

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$\mathcal{X}\times\mathcal{Y} = \{(x,y)\,;\, x\in\mathcal{X}, y\in\mathcal{Y}\}$

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"Cartesian Product" of $\mathcal{X}$ and $\mathcal{Y}$

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the "joint sample space"

The "joint probability mass function", i.e. "joint pmf" of X and Y → $p(x,y) := P(X=x \text{ and } Y=y)$

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$P\left(\begin{matrix}X \text{ takes value } x \text{ and} \\ Y \text{ takes value } y\end{matrix}\right)$

Note: ① $p(x,y)$ satisfies

(i) $p(x,y) \geq 0 \quad \forall (x,y)\in\mathcal{X}\times\mathcal{Y}$

(ii) $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y) = 1$.

② If $A \subseteq \mathcal{X}\times\mathcal{Y}$ → an event in the joint dist

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$$P(A) = \sum_{(x,y)\in A} p(x,y)$$

③ Marginal and Conditional Distributions

(i) "Marginal probability mass function of X" → for fixed $x\in\mathcal{X}$, $p_X(x) := \sum_{y\in\mathcal{Y}} p(x,y)$

(ii) "Marginal probability mass function of Y" → for fixed $y\in\mathcal{Y}$, $p_Y(y) := \sum_{x\in\mathcal{X}} p(x,y)$

(iii) "Conditional Distributions"

$p_{X|Y}(x|y) := P\left(X=x \text{ given that } Y=y\right)$ ⇕ $:= \dfrac{p(x,y)}{p_Y(y)}$

$p_{Y|X}(y|x) := P\left(Y=y \text{ given that } X=x\right)$ ⇕ $:= \dfrac{p(x,y)}{p_X(x)}$

Important identity:

$$p(x,y) = p_{X|Y}(x|y)\cdot p_Y(y) = p_{Y|X}(y|x)\cdot p_Y(y)$$

(II) Two Continuous random Variables

Let X, Y be two continuous r.vs → "joint sample space" is given by $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$.

The "joint probability function", i.e. "joint pdf" for the r.v's X and Y → function $f: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying

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(i) $f(x,y)\geq 0 \quad \forall (x,y)\in\mathbb{R}^2$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

Note: ① If $A\subseteq\mathbb{R}^2$,

$$P((x,y)\in A) = \iint_A f(x,y)\,dx\,dy$$

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If $A = \{(x,y) : x\in[a,b],\, y\in[c,d]\}$ = 2D rectangle $[a,b]\times[c,d]$

$$P((X,Y)\in A) = \int_a^b\int_c^d f(x,y)\,dx\,dy$$
‖
$P(a\leq X\leq b \text{ and } c\leq Y\leq d)$

② Marginal and Conditional Distributions

(i) "Marginal probability density function for X" → $f_X(x) := \int_{-\infty}^{\infty} f(x,y)\,dy$ for $x\in(-\infty,\infty)$

(ii) "Marginal probability density function for Y" → $f_Y(y) := \int_{-\infty}^{\infty} f(x,y)\,dx$ for $y\in(-\infty,\infty)$

③ Conditional Probability distribution

The "conditional density function of Y given that $X=x$" → $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$, $y\in(-\infty,\infty)$

Similarly,

The "conditional density function of X given that $Y=y$" → $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ for $x\in(-\infty,\infty)$.

Important identity:

$$f(x,y) = f_{X|Y}(x|y)\cdot f_Y(y) = f_{Y|X}(y|x)\cdot f_X(x)$$

Independent Random Variables

We say X and Y are independent if

$$p(x,y) = p_X(x)\cdot p_Y(y) \quad \forall (x,y)\in\mathcal{X}\times\mathcal{Y}$$

if X and Y are discrete

or

$$f(x,y) = f_X(x)\cdot f_Y(y) \quad \forall (x,y)\in\mathbb{R}\times\mathbb{R}$$

if X and Y are continuous.

if conditions for independence are not satisfied we say X and Y are dependent.

Note: When X and Y are independent, can calculate the joint pdf by "multiplying" the pdf's of X and Y

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In general, joint pdf/pmf calculation is not so easy!!

Need joint distributions $\rightsquigarrow$ want to calculate probabilities of joint events.

(Case I) Work with two rand vars.

(Case II) Generalize to n-rand vars.

Working with two r.v's $\longrightarrow$ both discrete.
$\longrightarrow$ both continuous.

(I)

Given X, Y discrete $\rightsquigarrow$ $\mathcal{X} :=$ values of X
$\mathcal{Y} :=$ values of Y.

$$\mathcal{X} \times \mathcal{Y} := \{(x,y)\,;\ x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$$
= "joint sample space".

"joint probability mass function" for the pair r.v's X, Y. $\rightsquigarrow$ probability function on $\mathcal{X} \times \mathcal{Y}$

$$p : \mathcal{X} \times \mathcal{Y} \longrightarrow (0,1)$$

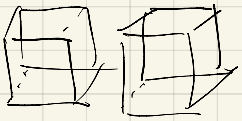
(i) $0 \le p(x,y) \le 1 \quad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}$.

(ii) $\displaystyle\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x,y) = 1$.

Perform the following exp:

Bag that contains two 6-sided die: dice 1 $\rightsquigarrow$ fair.
dice 2 $\rightsquigarrow$ unfair.
$P(1) = P(2) = \cdots = P(5) = 0.1$
$P(6) = 0.5$

die 1 die 2.

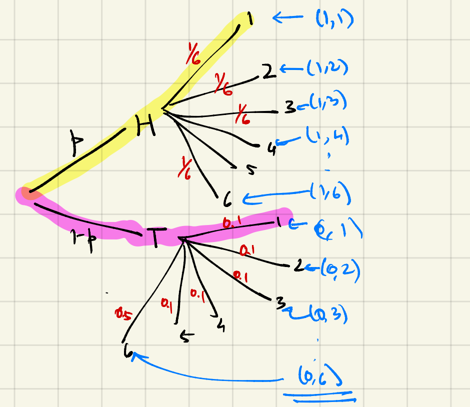


Experiment: Toss a coin, if H $\rightsquigarrow$ roll dice 1
T $\rightsquigarrow$ roll dice 2.

$P(H) = p$

$P(T) = 1-p$.

Tree Diagram for our exp.



X = out come of the coin toss.
$= \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

Y = outcome of the roll of the die.

$\mathcal{X}$ = Values of $X = \{0, 1\}$

$\mathcal{Y}$ = Values for $Y = \{1,2,3,4,5,6\}$

$$\mathcal{X} \times \mathcal{Y} = \{(0,1), (0,2), (0,3), \ldots, (0,6),\ (1,1), (1,2), (1,3), \ldots, (1,6)\}$$

$(0,6) \rightsquigarrow \{X = 0 \text{ and } Y = 6\}$

$p(0,6) = P($coin landed T **and** 6 appeared on roll of die$)$

$\mathcal{X} \times \mathcal{Y}$	1	2	3	4	5	6
0	$p(0,1)$	$p(0,2)$				$p(0,6)$
1	$p(1,1)$					$p(1,6)$

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot(0.5)$
1	$p \cdot 1/6$	$p \cdot 1/6$	$p \cdot 1/6$	$p \cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

joint probability distribution table for this exp.

Given a joint pmf

Three important distributions attached to a joint pmf.

Marginals:

(i) Marginal distribution for X $\rightsquigarrow$ for $x \in \mathcal{X}$

$$p_X(x) = \sum_{y \in \mathcal{Y}} p(x,y)$$

$p_X : \mathcal{X} \to (0,1)$

probability distribution on $\mathcal{X}$.

(ii) Marginal distribution for Y $\rightsquigarrow$ for fixed $y \in \mathcal{Y}$

$$p_Y(y) = \sum_{x \in \mathcal{X}} p(x,y)$$

$p_Y : \mathcal{Y} \to (0,1)$

Probability dist on $\mathcal{Y}$.

Example:

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

(i) Marginal for X. $\rightsquigarrow$ $\mathcal{X} = \{0,1\}$

$$p_X(0) = \sum_{y \in \mathcal{Y}} p(0,y) = p(0,1) + p(0,2) + p(0,3) + \cdots + p(0,6)$$

$$= (1-p)(0.1) + (1-p)(0.1) + (1-p)(0.1) + \cdots + (1-p)\cdot 0.5$$

$$= (1-p)(0.1+0.1+0.1+0.1+0.1+0.5) = \underline{1-p}$$

$$p_X(1) = \sum_{y \in \mathcal{Y}} p(1,y) = p(1,1) + \cdots + p(1,6)$$

$$= p\left(\tfrac{1}{6} + \tfrac{1}{6} + \cdots + \tfrac{1}{6}\right) = p$$

x	0	1
$p_X(x)$	$1-p$	p

(ii) Marginal for Y.

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$

$\mathcal{Y} = \{1,2,3,4,5,6\}$

$$p_Y(1) = p(0,1) + p(1,1) = (1-p)\cdot 0.1 + p\cdot\tfrac{1}{6}$$

$$p_Y(2) = p(0,2) + p(1,2) = (1-p)(0.1) + p\cdot\tfrac{1}{6}$$

$$p_Y(3) = p(0,3) + p(1,3) = (1-p)(0.1) + p\cdot\tfrac{1}{6}$$

$p_Y(4) = \quad\quad =$

$p_Y(5) = \quad\quad =$

$$p_Y(6) = p(0,6) + p(1,6) = (1-p)(0.5) + p\cdot\tfrac{1}{6}.$$

y	1	2	3	4	5	6
$p_Y(y)$						

$x \backslash y$	1	2	3	4	5	6	
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)(0.5)$	$p_X(0)$
1	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot 1/6$	$p\cdot(1/6)$	$p(1/6)$	$p_X(1)$
	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	$p_Y(4)$	$p_Y(5)$	$p_Y(6)$	1

Sum

Sum

(iii) Conditional Distribution

$$p_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$$:= \frac{p(x,y)}{p_X(x)}$$

$p_{Y|X}$ when $x = 0$?

To calculate the Conditional dist

$x \backslash y$	1	2	3	4	5	6
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot(0.5)$
1	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot(\frac{1}{6})$	$p\cdot(\frac{1}{6})$

$\rightsquigarrow$ divide this row by $p_X(0)$

$$p_{Y|X}(y|_{x=0}) = \frac{p(x,y)}{p_X(0)}$$

When $x=1$

$$p_{Y|X} = \frac{p(x,y)}{p_Y(1)} = P(Y=y\,|_{X=1})$$

$$\int_a^b \frac{k\theta^k}{x^{k+1}}\,dx = k\theta^k \cdot \int_a^b \frac{1}{x^{k+1}}\,dx$$

April 7th:

$p_{Y|X}(y|_{x=0})$

=

"$P($ $Y=y$ given that the coin lands T $)$"

$x \backslash y$	1	2	3	4	5	6	
0	$(1-p)\cdot(0.1)$	$(1-p)\,0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot 0.1$	$(1-p)\cdot(0.5)$	$p_X(0)$
1	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot \frac{1}{6}$	$p\cdot(\frac{1}{6})$	$p\cdot(\frac{1}{6})$	$p_X(1)$

$$p_{Y|X}(y|_{x=0}) = \frac{p(0,y)}{p_X(0)}$$

$y=1$, $p_{Y|X}(y=1|_{x=0}) = \frac{p(0,1)}{p_X(0)} = \frac{(1-\cancel{p})\cdot 0.1}{(1-\cancel{p})} = 0.1$

$p_{Y|X}(y=5|_{x=0}) = \frac{p(5,1)}{p_X(0)} = \cdots = 0.1$

$p_{Y|X}(y=6|_{x=0}) = \frac{p(0,6)}{p_X(0)} = \frac{(1-\cancel{p})\cdot 0.5}{(1-\cancel{p})} = 0.5$

Note $p_X(0) = (1-p)$

y	1	2	3	4	5	6
$p_{Y|X}(y|_{x=0})$	0.1	0.1	0.1	0.1	0.1	0.5

$\rightsquigarrow$ distribution of the unfair.

$\Updownarrow$

Conditional distribution of "$Y|_{X=0}$"

$p_{Y|X}(y|_{x=1})$

=

$P($ $Y=y$ given that the coin lands H $)$

$$p_{Y|X}(y|_{x=1}) = \frac{p(1,y)}{p_X(1)} \qquad p_X(1) = p$$

$$= \frac{\frac{1}{6}\cdot\cancel{p}}{\cancel{p}} \quad \text{for } y=1,2,3,4,5,6$$

y	1	2	3	4	5	6
$p_{Y|X}(y|_{x=1})$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\Updownarrow$

Conditional distribution of $Y|_{X=1}$

$\Updownarrow$

exactly the distribution of the fair dice.

Are the r.v's X and Y independent?

$\Updownarrow$

probability that the die rolls 6 depends on whether the coin lands H or T

$\Downarrow$

expect the r.v's to be dependent

Independent iff

$$p(x,y) = p_Y(y) \cdot p_X(x) \qquad \forall\ (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

$p(0,6) = P(\text{dice lands 6 given that coin lands T})$

$= (1-p)(0.5)$

$p_Y(6) = (1-p)(0.5) + p \cdot \frac{1}{6}.$ $\qquad p_X(0) = (1-p)$

$p(0,6) = p_Y(6) \cdot p_X(0)$

RHS: $\left((1-p)\cdot(0.5) + p\left(\frac{1}{6}\right)\right)(1-p) = (1-p)^2(0.5) + p(1-p)\frac{1}{6}$

LHS: $(1-p)\cdot(0.5)$

If $p = 0.4$; $\qquad p(0,6) \neq p_Y(6) \cdot p_X(0)$

$\therefore$ X and Y are dependent.