Probability and Counting

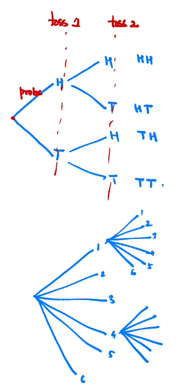
1) Coin toss of a fair coin $\rightsquigarrow \{H,T\} = P(H)=P(T)=\frac{1}{2}$

2) fair dice roll $\rightsquigarrow \{1,2,3,4,5,6\} = P(i)=\frac{1}{6}$

3) rolling two dice fair.
$\{(1,1),(1,2),\dots,(1,6),$
$(2,1),(2,2),\dots,(2,6),$
$\vdots\}$

$P(x,y)=\frac{1}{36}$

4) Toss two fair coin
$\{HH, HT, TH, TT\}$



Toss a coin twice: $\#S = 4$

$A = \{\text{atleast one } H\} = \{(H,H),(H,T),(T,H)\}$
$\Rightarrow n(A)=3$

$\therefore P(\text{at least one } H) = \frac{3}{4}$

$A^c = \{\text{no heads}\} = \{(T,T)\}$

$\boxed{P(A^c) = 1-P(A) = 1-\frac{3}{4} = \frac{1}{4}}$

Also
$n(A^c) \overset{?}{=} 1$
$\Rightarrow P(A^c) = \frac{1}{4}$

Suppose

a) $S \rightsquigarrow$ sample space of an exp with finitely many outcomes.

ie S is a finite set.

b) Every outcome in S is equally likely.

ie if $S=\{s_1,s_2,s_3,\dots,s_n\}$

$P(\{s_i\}) = \frac{1}{n}$ for all $i=1,2,3,\dots,n$.

Examples: ① Sample one element from $\{A,B,C\}$

$S=\{A,B,C\}$

$P(\{A\})=P(\{B\})=P(\{C\})=\frac{1}{3}$.

$\therefore$ Every outcome is equally likely.

② Sample one element from $\{A,A,B,C\}$

$S=\{A,B,C\}$

$P(\{A\})=\frac{2}{4}$, $P(\{B\})=\frac{1}{4}=P(\{C\})$.

Every outcome of this exp is not equally likely.

Suppose $S \rightsquigarrow$ finite sample space

$A \subset S \rightsquigarrow$ an event.

If every outcome in S is equally likely.

$$\boxed{P(A) = \text{"Probability that } A \text{ occurs"} = \frac{n(A)}{n(S)}.}$$

Where, $n(A) = \#$ of objects in A.

Note: In this scenario $\rightsquigarrow$ calculating probability

same as counting outcomes in an event.

Elementary but not easy!

$T: T_1 \to T_2 \to T_3 \to \cdots \to T_k$
$n_1 \times n_2 \times n_3 \cdots \times n_k$

Counting

"Fundamental Theorem of Counting"

Suppose there are k tasks: $T_1, T_2, \dots, T_k$ can be performed in $n_1, n_2, \dots, n_k$

Let T be the task of performing $T_1, T_2, \dots, T_k$ sequentially.

Then the total number of ways to perform T is

$$\boxed{\#\text{ways to perform } T = n_1 \times n_2 \times n_3 \times \cdots \times n_k = \binom{\#\text{ways to}}{\text{do } T_1} \times \binom{\#\text{ways to}}{\text{do } T_2} \times \cdots \times \binom{\#\text{ways to}}{\text{do } T_k}}$$

Typically we want

"To count the number of ways of selecting k objects from a set of n objects"

Example: Select 3 numbers from $\{0,1,2,3,4,5,6,7,8,9\}$

Four possibilities.

	Without Replacement	With Replacement
ordered	(1,1,2) not allowed. (1,2,3) different from (2,1,3).	(1,1,2) allowed. (1,1,2) different from (1,2,1).
unordered	(1,1,2) not allowed. (1,2,3) same as (2,1,3) same as (3,1,2).	(1,1,2) allowed. (1,1,2) same as (1,2,1) same as (2,1,1)

(2,2,3)
(1,1,1)
(5,5,1)
not allowed because no replacement.

$(1,4,7) \sim (4,1,7) \sim (7,1,4)$

Question: "$P_n = \#$ ways to permute n objects amongst themselves.

$n \times (n-1) \times (n-2)$ _ _ _ _ _ _ $\dots$ $3 \times 2 \times 1$

$= n!$

Number of possible arrangements of size k from n objects.

	Without replacement	With replacement
Ordered	${}^nP_k = \frac{n!}{(n-k)!}$ "n permute k"	n^k
Unordered	${}^nC_k = \frac{n!}{(n-k)!\,k!}$ "n choose k"	${}^{n+k-1}C_k$

For Proofs $\rightsquigarrow$ Use Fundamental Theorem.

(1) Ordered without Replacement

$n \times (n-1) \times (n-2) \cdots \times (n-k+1)$
Box 1, Box 2, Box 3, $\dots$, Box k.

$\rightsquigarrow$ total ways

$n \times (n-1) \times (n-2) \times \cdots (n-k+1) = \frac{n!}{(n-k)!} := {}^nP_k$

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

10 9 8

$8 = 10-3+1$

$10 \times 9 \times 8 = \frac{10!}{7!}$

Note: $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$.

② Unordered Without replacement

Step 1: ${}^nP_k = \frac{n!}{(n-k)!}$ ordered arrangements.

Step 2: Each ordered arrangement can be rearranged $k!$ times.

get rid of repeats by dividing by $k!$

$$\boxed{\therefore \text{total ways} = \frac{{}^nP_k}{k!} = \frac{n!}{(n-k)!\,k!} := {}^nC_k = \binom{n}{k}}$$

$\{(1,2,3) \sim (2,1,3) \sim (3,2,1) \sim \dots\}$

$3! = 3\times2\times1 = 6$

3) Ordered with replacement

total # of ways $= \underbrace{n \times n \times n \times n \cdots \times n}_{k\text{-boxes}}$

$= n^k$

④ Unordered with replacement

Want: number of unordered arrangements of size k from n objects with replacement

reformulate

ways to choose k "walls" from $(n+k-1)$ choices.

$$\boxed{\text{ie } {}^{n+k-1}C_k = \binom{n+k-1}{k} \text{ ways.}}$$

Example: $n=10$, $k=3$

$S=\{0,1,2,3,4,5,6,7,8,9\}$

$n+k-1=12 \rightsquigarrow$ _ _ _ _ _ _ _ _ _ _ _ _

Then,

(1,1,2) $\rightsquigarrow$ 1 X X 2 X 3 4 5 6 7 8 9

$(7,0,0) \sim (0,7,0) \sim$ (0,0,7) $\rightsquigarrow$ X X 1 2 3 4 5 6 7 X 8 9

(5,9,7) $\rightsquigarrow$ 1 2 3 4 5 X 6 7 X 8 9 X

(3,8,4) $\rightsquigarrow$ 1 2 3 X 4 X 5 6 7 8 X 9