Scattering Amplitudes and Color Ordering

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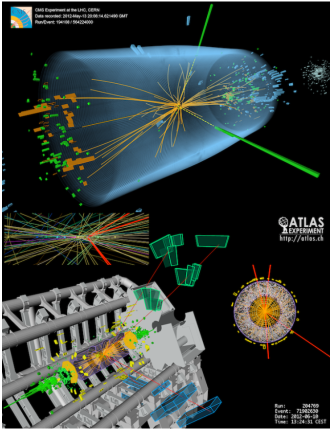
Table of Contents

1. Understanding Amplitudes
2. Computing Amplitudes
3. Geometry of Singularities
4. New Scattering Amplitudes
5. Conclusions and Future Directions

Introduction

Motivation: Scattering Amplitudes



Proton collisions at the LHC can be computed as collisions between gluons. We want to compute scattering amplitudes between these particles.

$$\mathcal{A}(gg \to gg)$$
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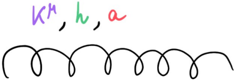
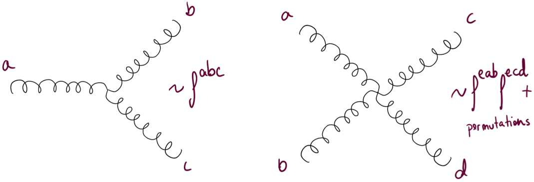
All About Gluons

The amplitudes are from Yang-Mills theory with Lie algebra $\mathfrak{u}(N)$.

Gluons are massless particles in the adjoint representation.



They carry momentum, helicity, and color labels. They interact as



Tree Level Amplitudes

We want to compute all scattering amplitudes.

We start with the simplest ones: tree level amplitudes.

This is still very difficult.

$g + g \to g + g$	4 diagrams
$g + g \to g + g + g$	25 diagrams
$g + g \to g + g + g + g$	220 diagrams
$g + g + g \to 7g$	more than one million!!!

The clever way of doing this is

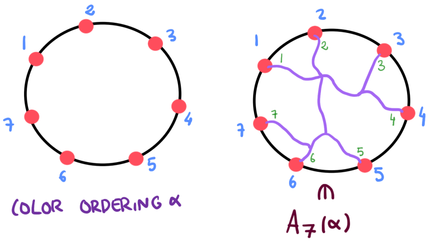
COLOR DECOMPOSITION

We separate the amplitude into a product

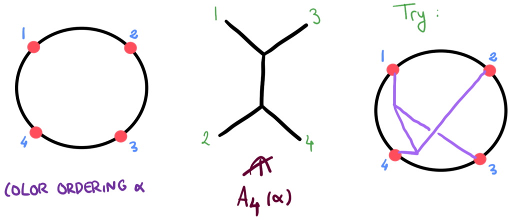
$$\mathcal{A}_n = \sum_{\alpha \in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\alpha_1}} T^{a_{\alpha_2}} \cdots T^{a_{\alpha_n}}\right) A_n(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

We refer to the quantity $A_n(\alpha_1, \alpha_2, \ldots, \alpha_n)$ as the **partial amplitude**.

Partial amplitudes have only the Feynman diagrams that can be put on the circle with no crossings.



This drastically reduces the number of diagrams. For example,



Computing Amplitudes

Parke-Taylor Amplitudes

In the 1980s, Parke and Taylor found

$$A_{n,0}(1^+, 2^+, \ldots, n^+) = 0$$
$$A_{n,1}(1^+, 2^+, \ldots, i^-, \ldots, n^+) = 0$$
$$A_{n,2}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle}$$
$$A_{n,n-2}(1^-, 2^-, \ldots, i^+, \ldots, j^+, \ldots, n^-) = \frac{[ij]^4}{[12][23][34]\cdots[n1]}$$

where

$$\langle ij\rangle = \det\begin{bmatrix}\lambda_{i1} & \lambda_{j1}\\ \lambda_{i2} & \lambda_{j2}\end{bmatrix}, \qquad [ij] = \det\begin{bmatrix}\tilde\lambda_{i1} & \tilde\lambda_{j1}\\ \tilde\lambda_{i2} & \tilde\lambda_{j2}\end{bmatrix}$$

Parke and Taylor (1986).

A Formula for Amplitudes

For n particles, k of which have negative helicity labeled by $i_1 \cdots i_k$, the partial amplitude can be written as

$$A_{k,n} = \int d^{k\times n}C \frac{|i_1 \cdots i_k|^4}{|1\cdots k||2\cdots k+1|\cdots|n\cdots k-1|} \delta\left(C\cdot\tilde{\Lambda}\right)\delta\left(C^{\perp}\cdot\Lambda\right)$$

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where we have defined

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{k1} & \dots & c_{kn} \end{pmatrix}, \qquad |1 \cdots k| = \det \begin{bmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{k1} & \dots & c_{kk} \end{bmatrix}$$

and $\Lambda, \tilde{\Lambda}$ are $2 \times n$ matrices containing all kinematic data.

Arkani-Hamed, Cachazo, Cheung, and Kaplan (2009). $\boxed{\text{0907.5418}}$

Geometry for $k = 2$

For $k = 2$, the singularity structure of the integrand is captured by six points on a circle. We think of this as six copies of $\mathbb{RP}^0$ inside $\mathbb{RP}^1$.

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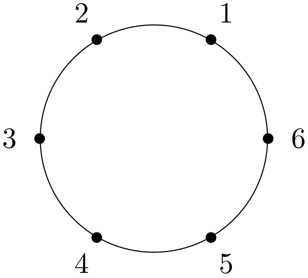
$$\frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} \quad =$$



Flipping two adjacent points gives a partial amplitude **for a different color ordering**.

3 Negative Helicity Particles

Geometry for $k = 3$

For n particles, **3 with negative helicity**, we consider

$$\mathcal{I} = \frac{|246|^4}{|123||234||345|\cdots|n12|}$$

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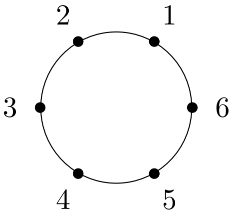
We want to represent the singularity structure of this object. Let's try the circle for $n = 6$.



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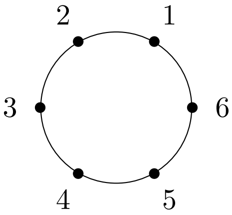

Unfortunately, **this does not work**.

We need to go one dimension higher!

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TRIANGLES

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The singularity structure of n particles, 3 with negative helicity, is captured by n **lines in the (projective) plane**.

Cachazo, Early, and Zhang (2022). $\boxed{2212.11243}$

5 Lines on the Plane

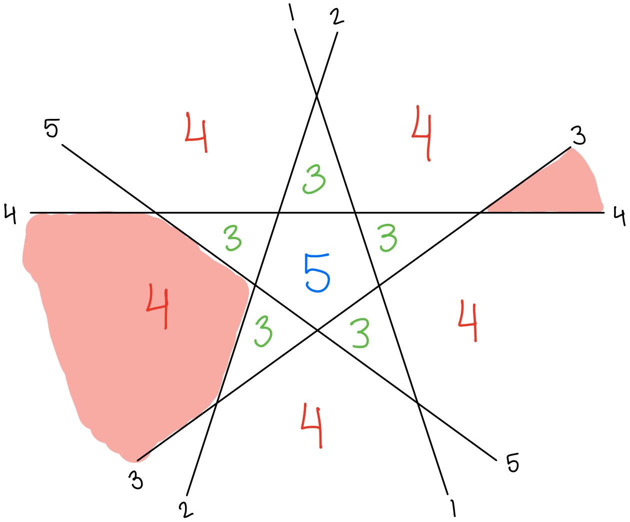
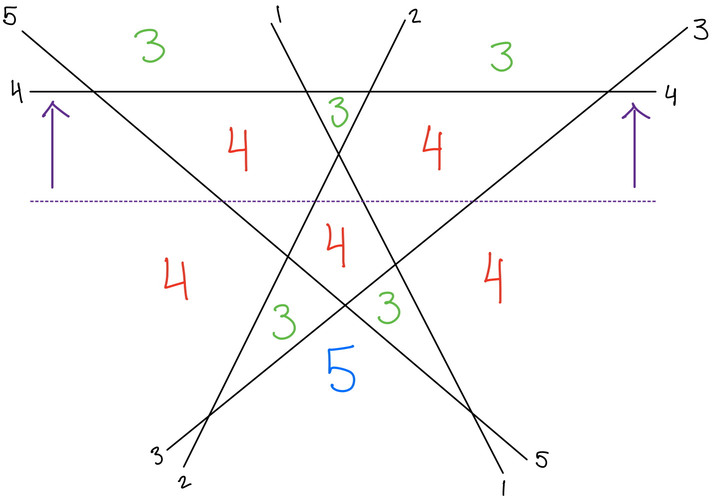


Figure: Five copies of $\mathbb{R}P^1$ in $\mathbb{R}P^2$.



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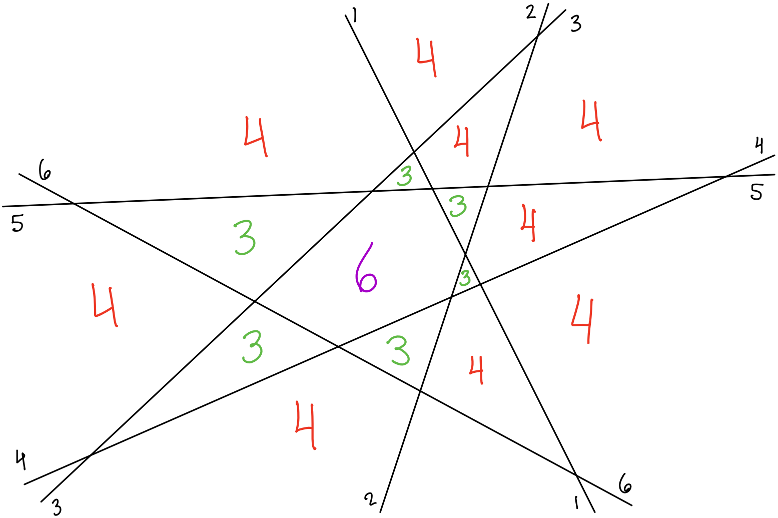

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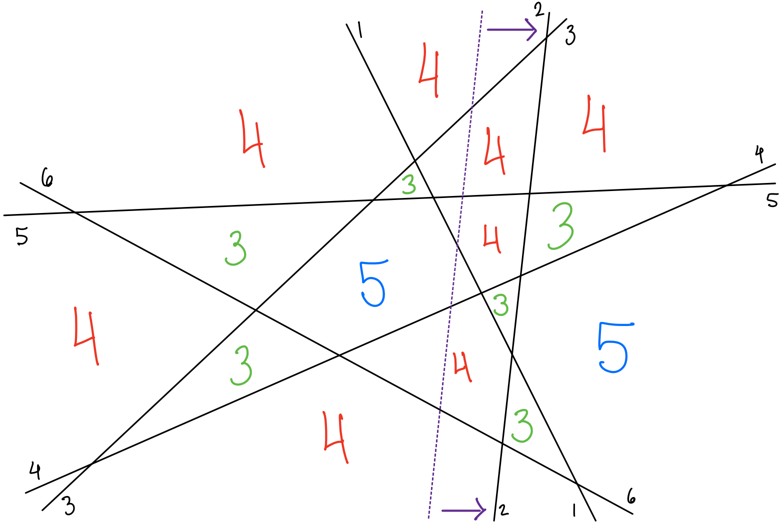
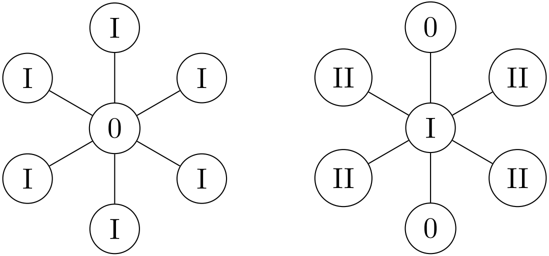


Figure: Six copies of $\mathbb{R}P^1$ in $\mathbb{R}P^2$.

Summary of 6 Line Configurations

Type	Triangles	Squares	Pentagons	Hexagons
0	6	9	0	1
I	6	8	2	0
II	7	6	3	0
III	10	0	6	0

Neighbours of Different Types



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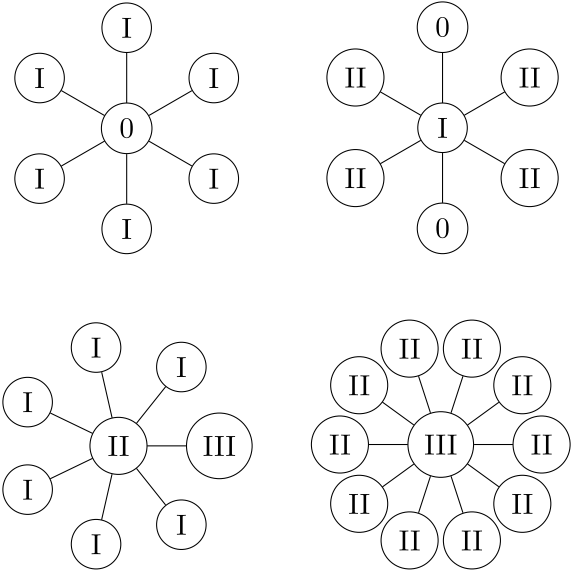

“Amplitudes” for Generalized Color Orderings

Computing Partial Amplitudes

Using the delta functions in the partial amplitude formula

$$A_{3,6} = \int d^{3\times 6}C \frac{|246|^4}{|123||234||345||456||561||612|} \delta\left(C \cdot \tilde{\Lambda}\right) \delta\left(C^{\perp} \cdot \Lambda\right)$$

we can reduce this to a single contour integral in $\mathbb{C}$.

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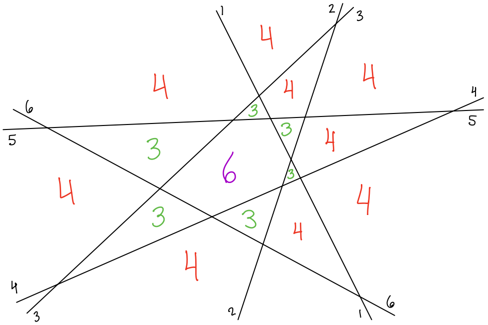
Therefore, we can compute partial amplitudes by computing residues!

From Pictures to Integrands

$$\frac{|246|^4}{|124||125||135||246||346||356|} =$$



Integrands for Different Types

$$\mathcal{I}_0 = \frac{|246|^4}{|123||234||345||456||561||612|},$$

$$\mathcal{I}_I = \frac{|246|^4}{|126||123||234||456||356||451|},$$

$$\mathcal{I}_{II} = \frac{|246|^4|156|}{|125||126||136||145||234||356||456|},$$

$$\mathcal{I}_{III} = \frac{|246|^4(|124||345||136||256| - |245||356||126||134|)}{|145||136||234||256||125||356||345||146||246||123|}.$$

Computing the Residues

We computed the residues of different integrand types as

$$\mathrm{Res}_{|456|=0}\,\mathcal{I}_0 = \frac{\langle 46\rangle^4[13]^4}{\langle 45\rangle\langle 56\rangle[12][23]\langle 4|P|1]\langle 6|P|3]P^2},\ P = 4+5+6$$

$$\mathrm{Res}_{|126|=0}\,\mathcal{I}_I = \frac{\langle 26\rangle^4[35]^4}{\langle 12\rangle[45]\langle 1|P|3]\langle 2|P|5]\langle 6|P|3]\langle 6|P|4]},\ P = 1+2+6$$

$$\mathrm{Res}_{|234|=0}\,\mathcal{I}_{II} = \frac{\langle 24\rangle^4[15]^4P^2}{\langle 4|P|1]\langle 3|P|1]\langle 2|P|5]\langle 3|P|5]\langle 2|P|6]\langle 4|P|6]},\ P = 2+3+4.$$

Conclusions

Future Directions

(1) We would like to understand the physical significance of the generalized color orderings.

(2) One possibility is to interpret them in terms of on-shell diagrams.

(3) We would also like to understand the correspondence between our results and the Grassmannian formulation of scattering amplitudes.

Thank Yous

Prof. Freddy Cachazo

Yong Zhang

Winter School Organizers

All of you!

Secret Slides

Geometric Interpretation of the Delta Functions

The delta functions in the formula

$$\delta\left(C \cdot \tilde{\Lambda}\right), \qquad \delta\left(C^{\perp} \cdot \Lambda\right)$$

have a very natural geometric interpretation.