All Students' Unbirthday

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Do you remember the Mad Hatter and the March Hare were celebrating their unbirthday[1] when Alice met the two M.H.'s at the Mad Tea-Party?[1]

It is not very likely that you have a common birthday with your closest friend. But you can share an unbirthday with almost anyone. Well, then can you have a common unbirthday with all the other students in your school? How many common unbirthdays of all students in your school, do you expect, are there in a year? Well, birthdays tend to be different from person to person. So you might expect there are almost no day left that is nobody's birthday in case it is a biggish school where the number of students is much larger than the number of days in a year. Is it really the case?

There is one birthday every year for a student. So you have $D-1$ unbirthdays in a year[2], where D is the number of days in a year. Therefore the probability that a day is an unbirthday of a certain student is

$$\begin{aligned} P_1 &= (D-1)/D \\ &= 1-\frac{1}{D}, \end{aligned}$$

If there are N students, where N is α times as large as the number of days in a year (i.e., $N=\alpha D$), in the school, the probability that a day is no student's birthday is

$$\begin{aligned} P_N = P_1^N &= (1-\frac{1}{D})^N \\ &= (1-\frac{1}{D})^{\alpha D} \end{aligned}$$

Since D is much bigger than 1 ($D \gg 1$), $(1-\frac{1}{D})^D \approx \frac{1}{e}$, and[3]

$$P_N \to e^{-\alpha},$$

And the expected number of "unbirthdays" of all students is, since there are D

[1] i.e., "when it isn't your birthday" – Humpty Dumpty[2]

[2] from Alice's formula[2]:

$$\begin{array}{r} 365 \\ -\quad 1 \\ \hline 364 \end{array}$$

[3] Since $\lim_{x\to 0}(1+x)^{1/x} = e$,
substituting with $y=-x$,
$\lim_{x\to 0}(1-x)^{1/x} = \lim_{y\to 0}(1+y)^{-1/y} = e^{-1}$.
Actually, for $D=365$, $(1-\frac{1}{D})^D \approx 0.367375$, while $1/e \approx 0.367879$. Thus the relative error is 0.137%

days in a year,

$$\begin{aligned} U &= D \times P_N \\ &\approx De^{-\alpha}. \end{aligned}$$

For example, in an ordinary year(in contrast to the leap year) $D = 365$, and in a school where there are 30 classes and in average 36.5 students every class, $N = 1095$ students, and $\alpha = 3$, therefore

$$\begin{aligned} P_N &= e^{-3} \approx \frac{1}{20}, \\ U &\approx 365 \times e^{-3} \\ &\approx 18.17\,\text{days}. \end{aligned}$$

So even in an ordinary 'biggish' school it si quite probabble that there are quite a few all-students' unbirthdays left over.

If the expected number of all students' unbirthday should be under d,

$$\begin{aligned} U &\approx De^{-\alpha} < d, \\ D/d &< e^{\alpha}, \\ \ln\frac{D}{d} &< \alpha = \frac{N}{D}, \\ \therefore N &> D \ln\frac{D}{d} \end{aligned}$$

That is, for example, if the expected number of all students' unbirthdays should be less than 1, then $\alpha > 5.9$ and the student number N should be larger than 2153. ■

References

[1] Disney, "*Alice in Wonderland*", 1951 film

[2] Carroll, Lewis, "*Through the Looking Glass*", 1871