From Everyday to Extraordinary:
The Astroid Revealed in a Glass of Water

M. Ryu

mingshey@hafs.hs.kr

February 2, 2025

1 Introduction

A pencil partially immersed in water, appearing bent, is a common phenomenon encountered when first learning about refraction (Figure 1). However, the apparent position of the pencil tip varies depending on the viewing angle, a phenomenon that warrants further investigation.
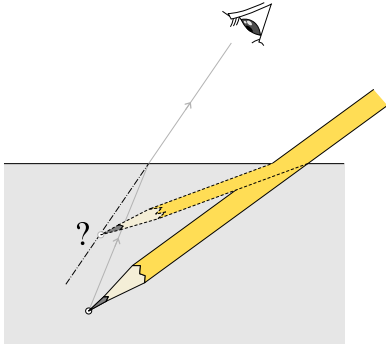

Figure 1: A pencil appearing bent in water

Typically, introductory physics courses cover the apparent depth when viewed vertically. However, a detailed analysis of how the apparent position changes when viewed obliquely is often omitted. Even advanced optics textbooks tend to gloss over this phenomenon, focusing instead on more fundamental topics such as lenses and mirrors.

This is perhaps because a rigorous analysis of this phenomenon requires a level of mathematical sophistication that may be beyond the scope of introductory courses, and its practical applications may be considered less significant compared to the workings of optical instruments.

Nevertheless, this seemingly simple yet intriguing phenomenon continues to spark curiosity. The author believes that many others have pondered this question as well. This paper aims to provide an answer.

2 Quick Answer

For those who are curious but short on time, here's the bottom line: When viewing a point object submerged
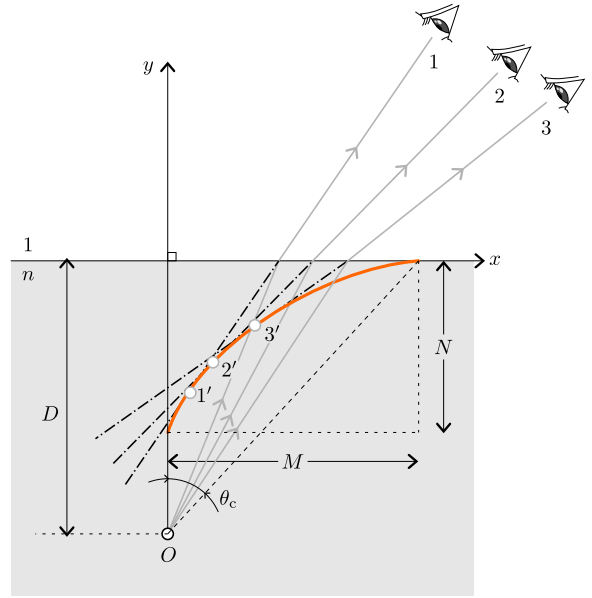

Figure 2: Locus of the image as the viewpoint moves

in water from above, the apparent position of the image varies depending on the point of view. As the viewing point moves, the locus of the observed image on the normal plane forms a type of caustic[1], which in this case is a curve known as a *squashed astroid*.

Considering the normal plane containing both the object and the viewer, with the intersection of the plane and the water surface as the x-axis and the normal passing through the object as the y-axis, as shown in Figure 2, the locus of the image can be described by the following equation:

$$\left|\frac{x}{M}\right|^{2/3} + \left|\frac{y}{N}\right|^{2/3} = 1,$$

where $M = D/\sqrt{n^2 - 1}$ is the maximum incident distance determined by the critical angle of total internal reflection (θ_c), $N = D/n$ is the apparent depth when viewed directly from above, D is the actual depth of the object, and n is the refractive index of water relative to air.

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[1]In this case you could call it a *virtual* caustic, since it's the locus of virtual images
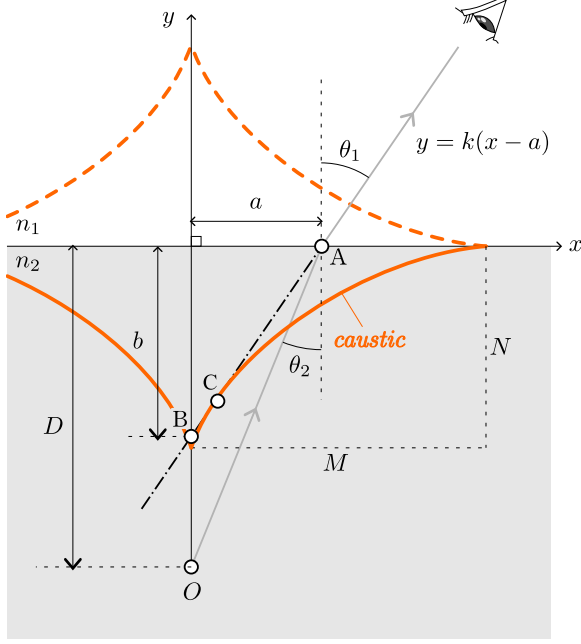
Figure 3: Light rays and the caustic.

3 Derivation of the Equation

Consider a point object O submerged at a depth D below the interface between air and water with refractive indices n_1 and n_2, respectively. A ray of light emanating from the object intersects the interface at point $A(a, 0)$. The ray makes an angle θ_2 with the normal at point A and is refracted into the air at an angle θ_1 with respect to the same normal. The extension of the refracted ray intersects the y-axis at $B(0, b)$.

As the angle of incidence θ_2 varies, the line segment AB envelop a virtual caustic curve. The image of the object is formed at the point of tangency, C, as it represents the local point of divergence for the neighboring pencil of rays.

Snell's law says that

$$\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 = n\sin\theta_2.$$

The line AB can be expressed by the equation

$$y = k(x - a)$$

with the distance of incidence a as a parameter and with the slope k, where, considering the Snell's law,

$$k = \frac{1}{\tan\theta_1} = \frac{\cos\theta_1}{\sin\theta_1} = \frac{\sqrt{1 - n^2\sin^2\theta_2}}{n\sin\theta_2}.$$

From the geometry, we have

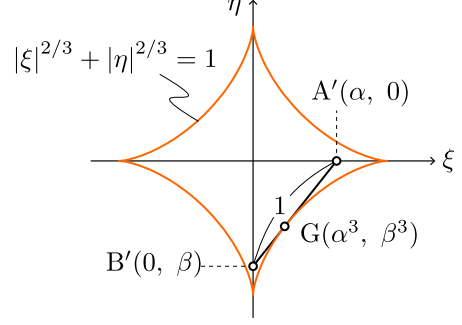$$a = D\tan\theta_2 = \frac{D\sin\theta_2}{\cos\theta_2},$$

Figure 4: Projection of the caustic onto the $\xi\eta$ plane, revealing an astroid shape.

and

$$b = -ka$$
$$= -\frac{D\sin\theta_2}{\cos\theta_2}\frac{\sqrt{1 - n^2\sin^2\theta_2}}{n\sin\theta_2}$$
$$= -\frac{D\sqrt{1 - n^2\sin^2\theta_2}}{n\cos\theta_2}.$$

Introducing dimensionless parameters $\alpha = a/M$ and $\beta = b/N$, we have

$$\alpha^2 + \beta^2 = \frac{a^2}{M^2} + \frac{b^2}{N^2}$$
$$= \frac{n^2 - 1}{D^2}\frac{D^2\sin^2\theta_2}{\cos^2\theta_2} + \frac{n^2}{D^2}\frac{D^2(1 - n^2\sin^2\theta_2)}{n^2\cos^2\theta_2}$$
$$= \frac{(n^2 - 1)\sin^2\theta_2 + 1 - n^2\sin^2\theta_2}{\cos^2\theta_2}$$
$$= \frac{1 - \sin^2\theta_2}{\cos^2\theta_2}$$
$$= 1.$$

Now, introducing dimensionless coordinates $\xi = x/M$ and $\eta = y/N$, as the point of observation moves in the xy-plane, the corresponding points $A'(\alpha, 0)$ and $B'(0, \beta)$ in the $\xi\eta$-plane also move, maintaining a constant distance of 1.

When a family of line segments $A'B'$ is overlaid, where each segment is defined by the parameter α, the envelope formed is a well-known curve called an *astroid*[2], described by the equation:

$$|\xi|^{2/3} + |\eta|^{2/3} = 1.$$

The point of tangency between the line segment $A'B'$ and the astroid is denoted by $G(\alpha^3, \beta^3)$, and this point corresponds to the image point C.

Therefore, the coordinates of the image point (x_C, y_C) can be obtained from the following relations:

$$\begin{cases} \xi_G = \dfrac{x_C}{M} = \alpha^3 = \dfrac{a^3}{M^3}, \\ \eta_G = \dfrac{y_C}{N} = \beta^3 = \dfrac{b^3}{N^3}, \end{cases}$$
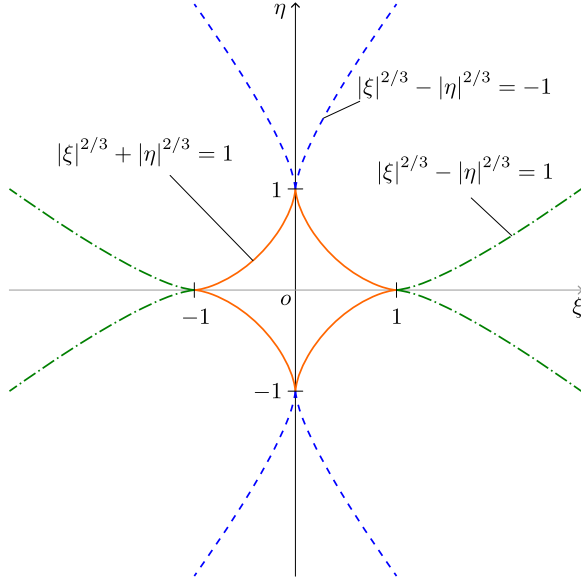
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[2]not to be confused with an asteroid

Figure 5: Astroid and '*hyperastroids*'

that is,

$$\begin{cases} x_{\mathrm{C}} = \dfrac{a^3}{M^2}, \\[2mm] y_{\mathrm{C}} = \dfrac{b^3}{N^2} = -\dfrac{k^3 a^3}{N^2}. \end{cases}$$

Using $\sin\theta_2 = a/\sqrt{D^2 + a^2}$ we obtain

$$k = \frac{\sqrt{D^2 - (n^2 - 1)a^2}}{na},$$

and the position of the image can be expressed as parametric functions of a:

$$\begin{cases} x_{\mathrm{C}} = (n^2 - 1)\dfrac{a^3}{D^2}, \\[3mm] y_{\mathrm{C}} = -\dfrac{D}{n}\left\{ 1 - (n^2 - 1)\dfrac{a^2}{D^2} \right\}^{3/2}. \end{cases}$$

4   When the viewpoint is submerged in water

Consider an object located at a height D above the boundary between air and water. When the observer is submerged in water, the relative refractive index is less than 1, i.e., $1/n < 1$. Through similar reasoning, we can derive the following equation for the caustic:

$$|\xi|^{2/3} - |\eta|^{2/3} = -1,$$

where $\xi = \dfrac{x}{W}$, $\eta = \dfrac{y}{Z}$, $W = \dfrac{nD}{\sqrt{n^2 - 1}}$, and $Z = nD$. While the curve does not have asymptotes, the slope approaches $\pm Z/W = \pm\sqrt{n^2 - 1}$ as $x \to \pm\infty$.

Consequently, the view of the sky observed from underwater is compressed into a circular region (or more accurately, a cone) bounded by the critical angle of to-

tal internal reflection, commonly known as Snell's window. This circular, wide-angle view, as perceived by an underwater observer, resembles the field of view of a fisheye lens.

A more general form of this curve is given by

$$|\xi|^r - |\eta|^r = \pm 1,$$

which is referred to as a *super hyperbola* in DML[3]. In the National Curve Bank[4], these curves, along with the *super-ellipses* defined by

$$|\xi|^r + |\eta|^r = 1,$$

are collectively termed *superconics*. (An astroid belongs to the super-ellipses.)

The specific case where

$$|\xi|^{2/3} - |\eta|^{2/3} = \pm 1,$$

has physical significance as the caustic of rays emanating from a point light source above water and refracted into water. Given its relationship to the astroid, which is analogous to the relationship between a hyperbola and an ellipse, we propose the term *hyperastroid* for this curve.

5   Finding the Image Location

The caustic curve can be utilized to determine the location of an image given the position of an object and the viewpoint.
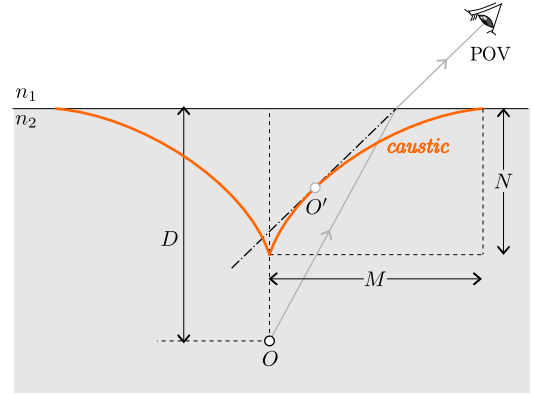

Figure 6: Construction of image utilizing the caustic curve

A tangent is drawn from the viewpoint to the caustic curve, as shown in Figure 6. The point of tangency represents the location of the image, and the intersection of the tangent with the water surface corresponds to the point where the light ray from the object strikes the surface. For instance, as shown in Figure 7, the image of a partially submerged pencil appears differently

[3]dynamicmathematicslearning.com/super-ellipse.html
[4]old.nationalcurvebank.org/superconicncb/superconicncb.htm

depending on the viewing angle. The image appears at positions $1'$ and $2'$ when viewed from points 1 and 2, respectively.
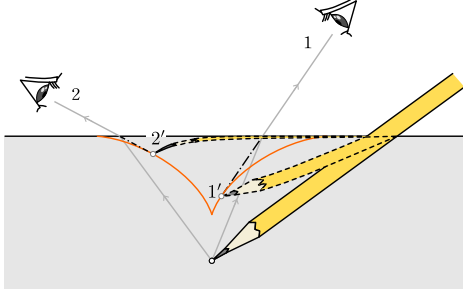

Figure 7: Location of image depends on POV.

For a continuous object submerged in water, we can find the image of each point on the object by repeating this process. The locus of these image points will form the image of the entire object.
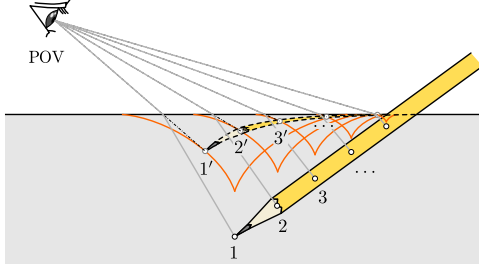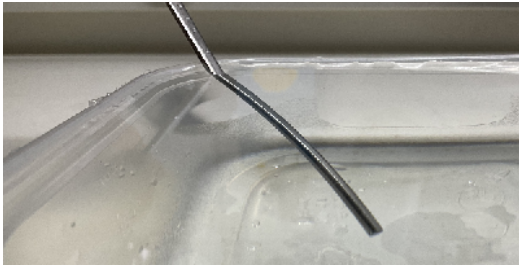

Figure 8: Image of a continuous object

Figure 9: A chopstick half-submerged in water:photo taken from low angle close to the surface.

Another approach is to numerically determine the intersection point of the ray with the water surface using Fermat's principle, and then determine the image position using the equation of the tangent point of the astroid.

Either method requires solving a quartic equation, which is generally difficult to solve manually. Although analytical solutions can be obtained using a computer algebra system(CAS), there is no significant advantage. Therefore, it is more efficient to find approximate values using numerical methods.

A Python example for finding the image position can be found at `github`[5]

Appendix: The Astroid as an Envelope

Consider a Cartesian coordinate system with a point $(a, 0)$ on the x-axis and a point $(0, b)$ on the y-axis, moving such that the distance between them remains constant at c. Thus, $a^2 + b^2 = c^2$. The equation of the line connecting these two points at any given time can be expressed as:

$$y = -\frac{b}{a}(x - a)$$

By substituting $b = \pm\sqrt{c^2 - a^2}$, we obtain

$$y(x, a) = \mp\frac{\sqrt{c^2 - a^2}}{a}(x - a)$$

As the value of a varies, the line connecting the two points also changes. The envelope of this family of lines is defined as the locus of points that are stationary as a varies infinitesimally, that is, the points where $\partial y/\partial a = 0$. To find these points, we differentiate y with respect to a:

$$
\begin{aligned}
\frac{\partial y}{\partial a} &= \pm\left[\left(\frac{1}{\sqrt{c^2 - a^2}} + \frac{\sqrt{c^2 - a^2}}{a^2}\right)(x - a) + \frac{\sqrt{c^2 - a^2}}{a}\right] \\
&= \pm\frac{(a^2 + c^2 - a^2)(x - a) + a(c^2 - a^2)}{a^2\sqrt{c^2 - a^2}} \\
&= \pm\frac{c^2 x - a^3}{a^2\sqrt{c^2 - a^2}} \\
&= 0.
\end{aligned}
$$

Therefore, the x-coordinate of the stationary point is $x = a^3/c^2$, and substituting this into the equation for y, we find the y-coordinate to be

$$
\begin{aligned}
y(x, a) &= \mp\frac{\sqrt{c^2 - a^2}}{a}\left(\frac{a^3}{c^2} - a\right) \\
&= \pm\frac{\left(c^2 - a^2\right)^{3/2}}{c^2} \\
&= \frac{b^3}{c^2}
\end{aligned}
$$

Thus, the coordinates (x, y) of the stationary points satisfy the equation of astroid:

$$\left|\frac{x}{c}\right|^{2/3} + \left|\frac{y}{c}\right|^{2/3} = 1.$$

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This document is translated from Korean with the assistance of Google's Gemini language model.

[5]`https://github.com/mingshey/python_projects/blob/main/Refraction_Image_en.ipynb`