Rediscovery of astroid from the refraction image by a flat boundary

M. Ryu
mingshey@hafs.hs.kr

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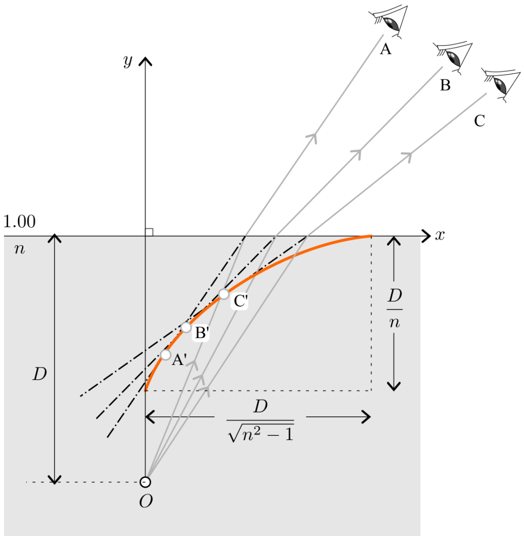
1 Introduction

If you see a point object under water from above the flat surface of water, the position of image depends on the point of view(POV). The image seen within a normal plane traces out a specific curve that is called a *squashed astroid* as the POV moves around.



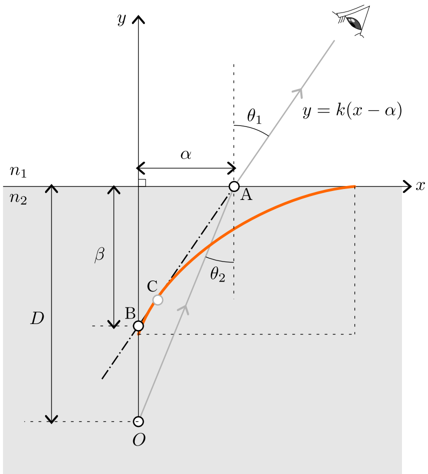
Let the normal plane that contains the object and POV be the xy-plane, and let the intersection of normal plane and the surface of water be the x-axis, and the normal through object y-axis. Then the trace of image is part of the curve

$$\left|\frac{x}{M}\right|^{2/3} + \left|\frac{y}{N}\right|^{2/3} = 1,$$

where $M = D/\sqrt{n^2-1}$ and $N = D/n$, and D is the depth of object and n is the index of refraction of the water relative to the air.

2 Derivation of the formula

Let the indices of refraction of air and of water be n_1 and n_2, respectively. The point object O is at the depth D below the boundary of air and water. A ray starts off the object and enters the boundary of media at α away from the y-axis with angle θ_2 from the normal at that point and then refracts into air with angle θ_1 from the same normal.



From Snell's law we have

$$\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 = n\sin\theta_2.$$

The extension of refracted ray is described by the equation

$$y = k(x-\alpha),$$

where

$$k = \frac{1}{\tan\theta_1} = \frac{\cos\theta_1}{\sin\theta_1},$$

and considering the Snell's law,

$$k = \frac{\sqrt{1-n^2\sin^2\theta_2}}{n\sin\theta_2}.$$

This line meets the y-axis at $y=\beta$, thus

$$\beta = -k\alpha.$$

By the geometry we have

$$\alpha = D\tan\theta_2 = \frac{D\sin\theta_2}{\cos\theta_2},$$

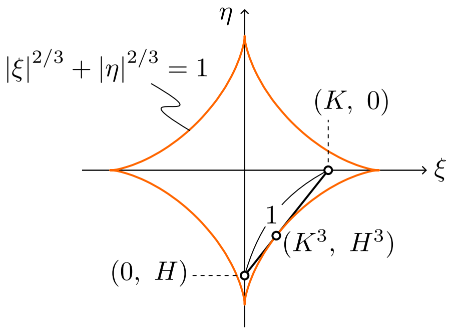
and

$$\begin{aligned}\beta &= -k\alpha \\ &= -\frac{D\sin\theta_2}{\cos\theta_2}\frac{\sqrt{1-n^2\sin\theta_2}}{n\sin\theta_2} \\ &= -\frac{D\sqrt{1-n^2\sin\theta_2}}{n\cos\theta_2}.\end{aligned}$$

Now, let $K = \alpha/M$ and $H = \beta/N$, then

$$\begin{aligned} K^2 + H^2 &= \frac{\alpha^2}{M^2} + \frac{\beta^2}{N^2} \\ &= \frac{(n^2-1)\sin^2\theta_2 + 1 - n^2\sin^2\theta_2}{\cos^2\theta_2} \\ &= \frac{1-\sin^2\theta_2}{\cos^2\theta_2} \\ &= 1 \end{aligned}$$

Let $\xi = x/M$ and $\eta = y/N$, then as the POV moves around in the xy-plane, the points $\mathrm{A}(\alpha, 0)$, and $\mathrm{B}(0, \beta)$ move accordingly, and the points $(K, 0)$ and $(0, H)$ in the $\xi\eta$-plane follow suite, while keeping the distance between them a constant; namely 1.



The envelope of such a segment is well-known as an *astroid*[1], which is described by the equation

$$|\xi|^{2/3} + |\eta|^{2/3} = 1.$$

The image is at the point of tangency C of the segment $\overline{\mathrm{AB}}$ and the envelope of the moving segment, for it is the instant point of divergence of the neighboring pencil of rays. Its corresponding point in the $\xi\eta$-plane is (K^3, H^3).

Thus we can obtain the coordinates of image $(x_\mathrm{C}, y_\mathrm{C})$ from the relation

$$\begin{cases} \xi_\mathrm{C} = \dfrac{x_\mathrm{C}}{M} = K^3 = \dfrac{\alpha^3}{M^3}, \\ \eta_\mathrm{C} = \dfrac{y_\mathrm{C}}{N} = H^3 = \dfrac{\beta^3}{N^3}. \end{cases}$$

That is

$$\begin{cases} x_\mathrm{C} = \dfrac{\alpha^3}{M^2}, \\ y_\mathrm{C} = \dfrac{\beta^3}{N^2} = -\dfrac{k^3\alpha^3}{N^2}. \end{cases}$$

Using

$$\sin\theta_2 = \frac{\alpha}{\sqrt{D^2+\alpha^2}},$$

we have

$$k = \frac{\sqrt{D^2 - (n^2-1)\alpha^2}}{n\alpha},$$

and we can derive the position of the image as parametric functions w.r.t. α:

$$\begin{cases} x_\mathrm{C} = (n^2-1)\dfrac{\alpha^3}{D^2}, \\ y_\mathrm{C} = -\dfrac{n^2}{D^2}\dfrac{\alpha^3}{n^3\alpha^3}\left\{D^2 - (n^2-1)\alpha^2\right\}^{3/2} \\ \quad\ = -\dfrac{D}{n}\left\{1 - (n^2-1)\dfrac{\alpha^2}{D^2}\right\}^{3/2}. \end{cases}$$

3 POV under water

If the object is in the air, at height of D above the boundary, and the POV is under water, the relative index of refraction is $1/n < 1$, and the similar reasoning leads to the equation

$$-|\xi|^{2/3} + |\eta|^{2/3} = 1,$$

with $\xi = \dfrac{x}{W}$ and $\eta = \dfrac{y}{Z}$, where $W = \dfrac{nD}{\sqrt{1-n^2}}$ and $Z = nD$.

I could not find the name for this shape of the curve. The astroid is a member of the family of curves named *superellipse*, which is defined by

$$|\xi|^n + |\eta|^n = 1.$$

Asteroid is the case where $n = 2/3$. But as far as I know there's no name for the family of curves in the form

$$|\xi|^n - |\eta|^n = \pm 1,$$

which could be named *super-hyperbola*, although the repeating cognates may bother you[2].

[1] Not to be confused with *asteroid*.

[2] I might suggest *superbola*.