Rediscovery of astroid from the refraction image by a flat boundary

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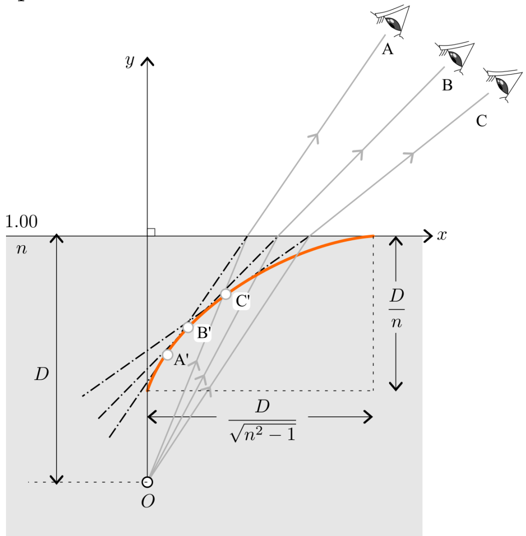
1 Introduction

If you see a point object under water from above the flat surface of water, the position of image depends on the point of view(POV). The image seen within a normal plane traces out a specific curve that is called a *squashed astroid* as the POV moves around.



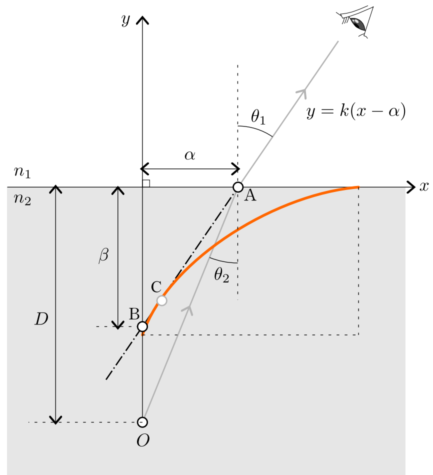
Let the normal plane that contains the object and POV be the xy-plane, and let the intersection of normal plane and the surface of water be the x-axis, and the normal through object y-axis. Then the trace of image is part of the curve

$$\left|\frac{x}{M}\right|^{2/3} + \left|\frac{y}{N}\right|^{2/3} = 1,$$

where $M = D/\sqrt{n^2-1}$ is the maximum distance of incidence determined by the critical angle of total internal reflection and $N = D/n$ is the apparent depth of the object when observed from directly above, where D is the actual depth of the object, and n is the refractive index of water relative to the air.

2 Derivation of the formula

Let the refractive indices of air and water be n_1 and n_2, respectively. The point object O is at the depth D below the boundary of air and water. A ray starts off the object and enters the boundary of media at α away from the y-axis with angle θ_2 from the normal at that point and then refracts into air with angle θ_1 from the same normal.



From Snell's law we have

$$\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 = n\sin\theta_2.$$

The extension of refracted ray is described by the equation

$$y = k(x-\alpha),$$

where

$$k = \frac{1}{\tan\theta_1} = \frac{\cos\theta_1}{\sin\theta_1},$$

and considering the Snell's law,

$$k = \frac{\sqrt{1-n^2\sin^2\theta_2}}{n\sin\theta_2}.$$

This line meets the y-axis at $y = \beta$, thus

$$\beta = -k\alpha.$$

By the geometry we have

$$\alpha = D\tan\theta_2 = \frac{D\sin\theta_2}{\cos\theta_2},$$

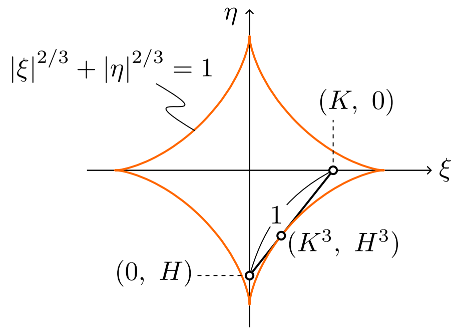
and

$$\begin{aligned}\beta &= -k\alpha \\ &= -\frac{D\sin\theta_2}{\cos\theta_2}\frac{\sqrt{1-n^2\sin\theta_2}}{n\sin\theta_2} \\ &= -\frac{D\sqrt{1-n^2\sin\theta_2}}{n\cos\theta_2}.\end{aligned}$$

Now, let $K = \alpha/M$ and $H = \beta/N$, then

$$\begin{aligned} K^2 + H^2 &= \frac{\alpha^2}{M^2} + \frac{\beta^2}{N^2} \\ &= \frac{\left(n^2-1\right)\sin^2\theta_2 + 1 - n^2\sin^2\theta_2}{\cos^2\theta_2} \\ &= \frac{1-\sin^2\theta_2}{\cos^2\theta_2} \\ &= 1 \end{aligned}$$

Let $\xi = x/M$ and $\eta = y/N$, then as the POV moves around in the xy-plane, the points $\mathrm{A}(\alpha, 0)$, and $\mathrm{B}(0, \beta)$ move accordingly, and the points $(K, 0)$ and $(0, H)$ in the $\xi\eta$-plane follow suite, while keeping the distance between them a constant; namely 1.



The envelope of such a segment is well-known as an *astroid*[1], which is described by the equation

$$|\xi|^{2/3} + |\eta|^{2/3} = 1.$$

The image is at the point of tangency C of the segment $\overline{\mathrm{AB}}$ and the envelope of the moving segment, for it is the instant point of divergence of the neighboring pencil of rays. Its corresponding point in the $\xi\eta$-plane is (K^3, H^3).

Thus we can obtain the coordinates of image $(x_\mathrm{C}, y_\mathrm{C})$ from the relation

$$\begin{cases} \xi_\mathrm{C} = \dfrac{x_\mathrm{C}}{M} = K^3 = \dfrac{\alpha^3}{M^3}, \\ \eta_\mathrm{C} = \dfrac{y_\mathrm{C}}{N} = H^3 = \dfrac{\beta^3}{N^3}. \end{cases}$$

That is

$$\begin{cases} x_\mathrm{C} = \dfrac{\alpha^3}{M^2}, \\ y_\mathrm{C} = \dfrac{\beta^3}{N^2} = -\dfrac{k^3\alpha^3}{N^2}. \end{cases}$$

Using

$$\sin\theta_2 = \frac{\alpha}{\sqrt{D^2+\alpha^2}},$$

we have

$$k = \frac{\sqrt{D^2-(n^2-1)\alpha^2}}{n\alpha},$$

and we can derive the position of the image as parametric functions w.r.t. α:

$$\begin{cases} x_\mathrm{C} = (n^2-1)\dfrac{\alpha^3}{D^2}, \\ y_\mathrm{C} = -\dfrac{n^2}{D^2}\dfrac{\alpha^3}{n^3\alpha^3}\left\{D^2-(n^2-1)\alpha^2\right\}^{3/2} \\ \quad\ = -\dfrac{D}{n}\left\{1-(n^2-1)\dfrac{\alpha^2}{D^2}\right\}^{3/2}. \end{cases}$$

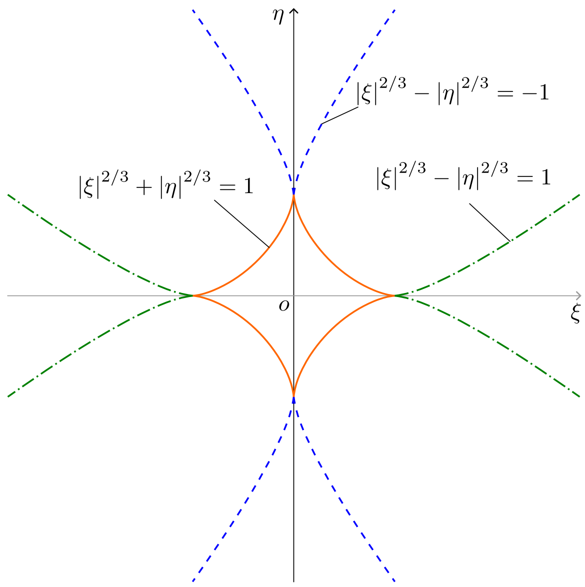
3 POV under water

If the object is in the air, at height of D above the boundary, and the POV is under water, the relative index of refraction is $1/n < 1$, and the similar reasoning leads to the equation

$$|\xi|^{2/3} - |\eta|^{2/3} = -1,$$

with $\xi = \dfrac{x}{W}$ and $\eta = \dfrac{y}{Z}$, where $W = \dfrac{nD}{\sqrt{1-n^2}}$ and $Z = nD$.

However, I could not find the name for this shape of the curve.



If this curve,

$$|\xi|^{2/3} - |\eta|^{2/3} = \pm 1,$$

which has physical significance and is also related to the astroid, does not yet have a name, how about calling it a *hyperastroid* because it has a relationship with the astroid similar to that of a hyperbola to an ellipse?

The astroid is a member of the family of curves named *superellipse*, which is defined by

$$|\xi|^r + |\eta|^r = 1.$$

Asteroid is the case where $r = 2/3$. But as far as I know nor is there a name for the family of curves in

[1]Not to be confused with *asteroid*.

the form

$$|\xi|^r - |\eta|^r = \pm 1,$$

which could be named *super-hyperbola*, although the repeating cognates may bother you[2].

Note: Astroid as an envelope

Let's assume that a point $(K, 0)$ on the x-axis and a point $(0, H)$ on the y-axis in a Cartesian plane move while maintaining a constant distance a between them. Then, $K^2 + H^2 = a^2$, and the equation of the line containing the line segment at a certain moment can be written as

$$y = -\frac{H}{K}(x - K)$$

and using $H = \pm\sqrt{a^2 - K^2}$, we have

$$y(x, K) = \mp\frac{\sqrt{a^2 - K^2}}{K}(x - K)$$

As the value of K changes, the line segment or line connecting the two points changes, and the envelope drawn by it is the locus of stationary points at each moment, that is, the locus of points where $\partial y/\partial K = 0$. Let's find the (x, y) that satisfy this condition.

$$\begin{aligned}
\frac{\partial y}{\partial K} &= \pm\left[\left(\frac{1}{\sqrt{a^2 - K^2}} + \frac{\sqrt{a^2 - K^2}}{K^2}\right)(x_0 - K) + \frac{\sqrt{a^2 - K^2}}{K}\right] \\
&= \pm\frac{(K^2 + a^2 - K^2)(x_0 - K) + K(a^2 - K^2)}{K^2\sqrt{a^2 - K^2}} \\
&= 0.
\end{aligned}$$

Therefore the abscissa of the stationary point is $x = K^3/a^2$, and its ordinate is

$$\begin{aligned}
y(x, K) &= \mp\frac{\sqrt{a^2 - K^2}}{K}\left(\frac{K^3}{a^2} - K\right) \\
&= \pm\frac{\left(a^2 - K^2\right)^{3/2}}{a^2} \\
&= \frac{H^3}{a^2}
\end{aligned}$$

Therefore, the coordinate (x, y) of the stationary points satisfies the equation

$$\left|\frac{x}{a}\right|^{2/3} + \left|\frac{y}{a}\right|^{2/3} = 1.$$

■

[2] I might suggest *superbola*.