Rediscovery of astroid from the refraction image by a flat boundary

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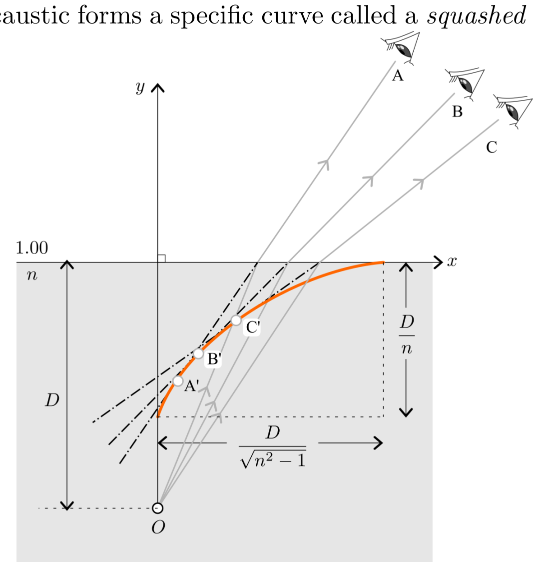
1 Introduction

When viewing a point object submerged in a flat water surface from above, the apparent position of the object varies depending on the point of view (POV). As the POV moves, the trace of the image observed within the normal plane is a kind of caustic[1], and in this case, the caustic forms a specific curve called a *squashed astroid*.



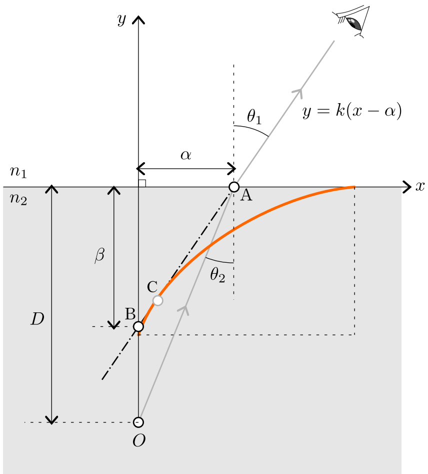
Let the normal plane that contains the object and POV be the xy-plane, and let the intersection of normal plane and the surface of water be the x-axis, and the normal through object y-axis. Then the trace of image is part of the curve

$$\left|\frac{x}{M}\right|^{2/3}+\left|\frac{y}{N}\right|^{2/3}=1,$$

where $M = D/\sqrt{n^2-1}$ is the maximum distance of incidence determined by the critical angle of total internal reflection and $N = D/n$ is the apparent depth of the object when observed from directly above, where D is the actual depth of the object, and n is the refractive index of water relative to the air.

2 Derivation of the formula

Let the refractive indices of air and water be n_1 and n_2, respectively. The point object O is at the depth D below the boundary of air and water. A ray starts off the object and enters the boundary of media at point A, which is a distance α away from the y-axis, with angle θ_2 from the normal at that point, and then refracts into air with angle θ_1 from the same normal.



From Snell's law we have

$$\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 = n\sin\theta_2.$$

The extension of refracted ray is described by the equation

$$y = k(x-\alpha),$$

where

$$k = \frac{1}{\tan\theta_1} = \frac{\cos\theta_1}{\sin\theta_1},$$

and considering the Snell's law,

$$k = \frac{\sqrt{1-n^2\sin^2\theta_2}}{n\sin\theta_2}.$$

This line meets the y-axis at B($y=\beta$), thus

$$\beta = -k\alpha.$$

By the geometry we have

$$\alpha = D\tan\theta_2 = \frac{D\sin\theta_2}{\cos\theta_2},$$

[1] Since it is a locus of virtual image, it can be called a *virtual caustic*.

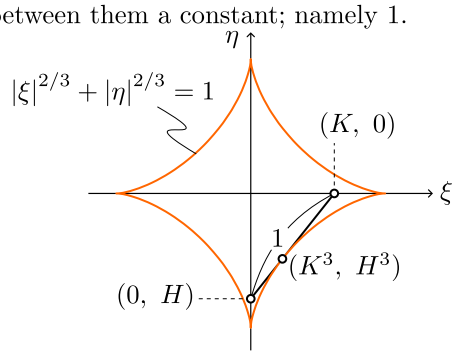
and

$$\begin{aligned}\beta &= -k\alpha\\ &= -\frac{D\sin\theta_2}{\cos\theta_2}\frac{\sqrt{1-n^2\sin^2\theta_2}}{n\sin\theta_2}\\ &= -\frac{D\sqrt{1-n^2\sin^2\theta_2}}{n\cos\theta_2}.\end{aligned}$$

Now, let $K=\alpha/M$ and $H=\beta/N$, then

$$\begin{aligned}K^2+H^2 &= \frac{\alpha^2}{M^2}+\frac{\beta^2}{N^2}\\ &= \frac{\left(n^2-1\right)\sin^2\theta_2+1-n^2\sin^2\theta_2}{\cos^2\theta_2}\\ &= \frac{1-\sin^2\theta_2}{\cos^2\theta_2}\\ &= 1\end{aligned}$$

Let $\xi=x/M$ and $\eta=y/N$, then as the POV moves around in the xy-plane, the points $\mathrm{A}(\alpha,0)$, and $\mathrm{B}(0,\beta)$ move accordingly, and the points $(K,0)$ and $(0,H)$ in the $\xi\eta$-plane follow suite, while keeping the distance between them a constant; namely 1.



The envelope of such a segment is well-known as an *astroid*[2], which is described by the equation

$$|\xi|^{2/3}+|\eta|^{2/3}=1.$$

The image is at the point of tangency C of the segment $\overline{\mathrm{AB}}$ and the envelope, for it is the instant point of divergence of the neighboring pencil of rays. Its corresponding point in the $\xi\eta$-plane is (K^3,H^3).

Thus we can obtain the coordinates of image $(x_\mathrm{C},y_\mathrm{C})$ from the relation

$$\begin{cases}\xi_\mathrm{C}=\dfrac{x_\mathrm{C}}{M}=K^3=\dfrac{\alpha^3}{M^3},\\ \eta_\mathrm{C}=\dfrac{y_\mathrm{C}}{N}=H^3=\dfrac{\beta^3}{N^3}.\end{cases}$$

That is

$$\begin{cases}x_\mathrm{C}=\dfrac{\alpha^3}{M^2},\\ y_\mathrm{C}=\dfrac{\beta^3}{N^2}=-\dfrac{k^3\alpha^3}{N^2}.\end{cases}$$

Using

$$\sin\theta_2=\frac{\alpha}{\sqrt{D^2+\alpha^2}},$$

we have

$$k=\frac{\sqrt{D^2-(n^2-1)\alpha^2}}{n\alpha},$$

and we can derive the position of the image as parametric functions w.r.t. α:

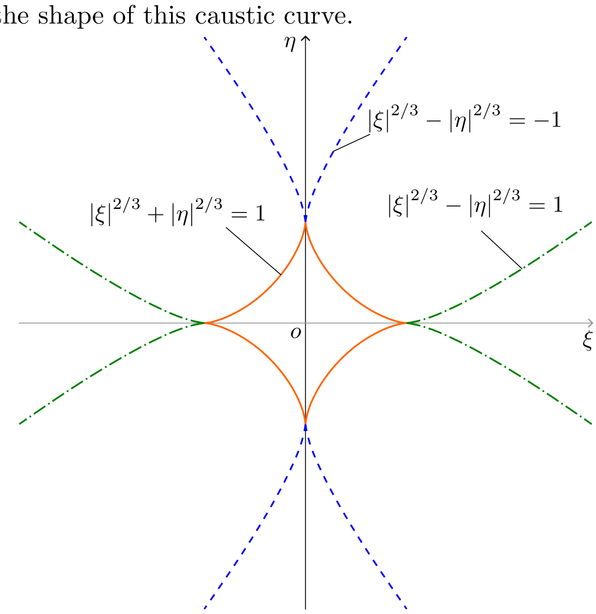
$$\begin{cases}x_\mathrm{C}=(n^2-1)\dfrac{\alpha^3}{D^2},\\ y_\mathrm{C}=-\dfrac{n^2}{D^2}\dfrac{\alpha^3}{n^3\alpha^3}\left\{D^2-(n^2-1)\alpha^2\right\}^{3/2}\\ \quad\ =-\dfrac{D}{n}\left\{1-(n^2-1)\dfrac{\alpha^2}{D^2}\right\}^{3/2}.\end{cases}$$

3 POV under water

If the object is located at a height D above a boundary surface in air and the POV is submerged in water, the relative refractive index is less than 1, i.e., $1/n<1$. Through similar reasoning, we can derive the following equation for the caustic:

$$|\xi|^{2/3}-|\eta|^{2/3}=-1,$$

where $\xi=\dfrac{x}{W}$, $\eta=\dfrac{y}{Z}$, $W=\dfrac{nD}{\sqrt{1-n^2}}$, and $Z=nD$. However, I have been unable to find a specific name for the shape of this caustic curve.



If this curve,

$$|\xi|^{2/3}-|\eta|^{2/3}=\pm 1,$$

which has physical significance and is also related to the astroid, does not yet have a name, how about calling it a *hyperastroid* because it has a relationship with the astroid similar to that of a hyperbola to an ellipse?

The astroid is a member of the family of curves named *superellipse*, which is defined by

$$|\xi|^r+|\eta|^r=1.$$

[2] Not to be confused with *asteroid*.

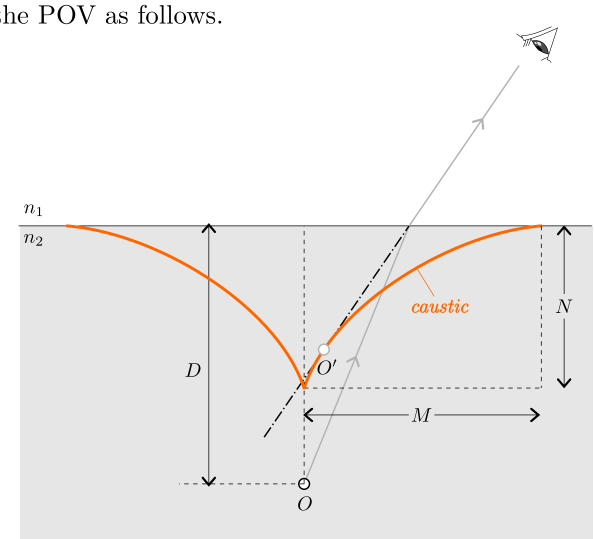
Asteroid is the case where $r = 2/3$. But as far as I know nor is there a name for the family of curves in the form

$$|\xi|^r - |\eta|^r = \pm 1,$$

which could be named *super-hyperbola*, although the repeating cognates may bother you[3].

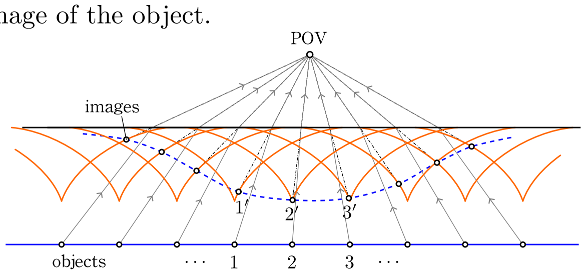
4 Finding the Image Location

Since we can obtain a closed form of the caustic, we can find the location of the image given the object and the POV as follows.



Draw a tangent line from the POV to the caustic. The point of tangency between the tangent line and the caustic is the location of the image, and the point where the tangent line intersects the water surface is the point where the light ray from the object is incident on the water surface.

If there is a continuous object in the water, we can find the image for each point $1, 2, 3, \ldots$ located along the surface of the object in the same way. The locus of the image as the point moves continuously will be the image of the object.



However, it is difficult, if not impossible, to find the tangent line to the caustic analytically, and in practice, we must be satisfied with finding an approximate value using numerical methods.

Another method is to numerically find the path of the light ray connecting the object and the viewpoint using Fermat's principle, and then find the location of the image by using the tangent point formula of the astroid based on the coordinates of the point where the light ray intersects the water surface. A Python example for this can be found at github.com/mingshey/python_projects.

[3] I might suggest *superbola*.

Note: Astroid as an envelope

Let's assume that a point $(K, 0)$ on the x-axis and a point $(0, H)$ on the y-axis in a Cartesian plane move while maintaining a constant distance a between them. Then, $K^2 + H^2 = a^2$, and the equation of the line containing the line segment at a certain moment can be written as

$$y = -\frac{H}{K}(x - K)$$

and using $H = \pm\sqrt{a^2 - K^2}$, we have

$$y(x, K) = \mp\frac{\sqrt{a^2 - K^2}}{K}(x - K)$$

As the value of K changes, the line segment or line connecting the two points changes, and the envelope drawn by it is the locus of stationary points at each moment, that is, the locus of points where $\partial y/\partial K = 0$. Let's find the (x, y) that satisfy this condition.

$$\begin{aligned}
\frac{\partial y}{\partial K} &= \pm\left[\left(\frac{1}{\sqrt{a^2 - K^2}} + \frac{\sqrt{a^2 - K^2}}{K^2}\right)(x - K) + \frac{\sqrt{a^2 - K^2}}{K}\right] \\
&= \pm\frac{(K^2 + a^2 - K^2)(x - K) + K(a^2 - K^2)}{K^2\sqrt{a^2 - K^2}} \\
&= \pm\frac{a^2 x - K^3}{K^2\sqrt{a^2 - K^2}} \\
&= 0.
\end{aligned}$$

Therefore the abscissa of the stationary point is $x = K^3/a^2$, and its ordinate is

$$\begin{aligned}
y(x, K) &= \mp\frac{\sqrt{a^2 - K^2}}{K}\left(\frac{K^3}{a^2} - K\right) \\
&= \pm\frac{(a^2 - K^2)^{3/2}}{a^2} \\
&= \frac{H^3}{a^2}
\end{aligned}$$

Therefore, the coordinate (x, y) of the stationary points satisfies the equation

$$\left|\frac{x}{a}\right|^{2/3} + \left|\frac{y}{a}\right|^{2/3} = 1.$$

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