

Streaming Facility Location in High Dimension via Geometric Hashing

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Abstract

In *Euclidean Uniform Facility Location*, the input is a set of clients in \mathbb{R}^d and the goal is to place facilities to serve them, so as to minimize the total cost of opening facilities plus connecting the clients. We study the classical setting of dynamic geometric streams, where the clients are presented as a sequence of insertions and deletions of points in the grid $\{1, \dots, \Delta\}^d$, and we focus on the *high-dimensional regime*, where the algorithm's space complexity must be polynomial (and certainly not exponential) in $d \cdot \log \Delta$.

We present a new algorithmic framework, based on importance sampling from the stream, for $O(1)$ -approximation of the optimal cost using only $\text{poly}(d \cdot \log \Delta)$ space. This framework is easy to implement in two passes, one for sampling points and the other for estimating their contribution. Over random-order streams, we can extend this to a one-pass algorithm by using the two halves of the stream separately. Our main result, for arbitrary-order streams, computes $O(d^{1.5})$ -approximation in one pass by using the new framework but combining the two passes differently. This improves upon previous algorithms that either need space exponential in d or only guarantee $O(d \cdot \log^2 \Delta)$ -approximation, and therefore our algorithms for high-dimensional streams are the first to avoid the $O(\log \Delta)$ -factor in approximation that is inherent to the widely-used quadtree decomposition. Our improvement is achieved by employing a geometric hashing scheme that maps points in \mathbb{R}^d into buckets of bounded diameter, with the key property that every point set of small-enough diameter is hashed into at most $\text{poly}(d)$ distinct buckets.

Finally, we complement our results with a proof that every streaming 1.085-approximation algorithm requires space exponential in $\text{poly}(d \cdot \log \Delta)$, even for insertion-only streams.

1 Introduction

Facility Location is a classical problem in combinatorial optimization and operations research, and models a scenario where one wishes to find a placement of facilities that optimizes the total service

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cost for a given set of customers. The cost has two different parts: opening a facility at a location incurs a so-called *opening cost*, and serving each customer incurs a so-called *connection cost*. The goal is to minimize the sum of both costs. Typical examples of applications include placement of servers in a network and location planning.

Euclidean Uniform Facility Location. In the *Uniform Facility Location (UFL)* problem, all possible facilities have the same opening cost $\mathfrak{f} > 0$. This is essentially a clustering problem; it is similar to k -median, except that the number of clusters is not prescribed in advance, but rather optimized by adding to the objective a regularization term (proportional to the number of clusters).

We consider the Euclidean version of UFL, where the data points and facilities all lie in \mathbb{R}^d . Formally, given as input a dataset $P \subseteq \mathbb{R}^d$ and $\mathfrak{f} > 0$, the goal is to *open* a set $F \subseteq \mathbb{R}^d$ of *facilities*, so as to minimize the total *connection cost* (the cost of connecting each data point to its nearest facility according to the Euclidean distance) plus the total *opening cost* (opening each facility costs \mathfrak{f}). That is, the goal is to find $F \subseteq \mathbb{R}^d$ that minimizes

$$\text{cost}(P, F) := \sum_{p \in P} \text{dist}(p, F) + \mathfrak{f} \cdot |F| ,$$

where we denote $\text{dist}(p, F) := \min_{q \in F} \text{dist}(p, q)$ and $\text{dist}(p, q) := \|p - q\|_2$ is the Euclidean distance. This problem has been studied extensively for decades in many algorithmic settings, including offline, online, dynamic, sublinear-time, streaming, and so forth; see Section 1.3 for an overview and references.

Streaming Setting. Euclidean UFL has also played a special role in the study of algorithms for dynamic geometric streams. Indeed, the seminal paper by Indyk [Ind04], which introduced this model, considered UFL as one of its benchmark problems, together with the minimum spanning tree, minimum-weight (bichromatic) matching, and k -median problems.

In the setting of *dynamic geometric streams*, the input P is presented as a stream of insertions and deletions of points from $[\Delta]^d := \{1, 2, \dots, \Delta\}^d$. The algorithms should be *space-efficient*, ideally using space that is polylogarithmic in Δ^d , which is a natural benchmark because it is polynomial in the representation of a single point using $O(d \cdot \log \Delta)$ bits. Due to this constraint, we cannot store the actual solution which could take space $\Omega(\Delta^d)$, hence the algorithms only approximate the optimal *cost* OPT , which in case of UFL is $\text{OPT} := \min_{F \subseteq \mathbb{R}^d} \text{cost}(P, F)$. We say that a randomized algorithm achieves α -*approximation*, for $\alpha \geq 1$, if it outputs an estimate E that with probability at least $2/3$ satisfies $\text{OPT} \leq E \leq \alpha \cdot \text{OPT}$. (In many cases, this success probability can be amplified using standard methods.)

In this dynamic geometric streaming model, algorithms generally exhibit a dichotomy, as their space complexity is either polynomial or exponential in the dimension d . Obviously, algorithms whose space complexity is exponential in d are applicable only in a low-dimensional setting, say $d = O(1)$ or $d = O(\log \log \Delta)$. This class contains many algorithms that achieve an $O(1)$ -approximation, or even $(1 + \varepsilon)$ -approximation for arbitrary fixed $\varepsilon > 0$, from facility location [LS08, CLMS13] to minimum spanning tree [FIS08], and several other basic geometric problems [FS05, ABIW09, CJKV22]. The list would extend even further if we included algorithms for the insertion-only model.

One would clearly prefer algorithms whose space complexity is polynomial in d . Such algorithms are known for facility location and several other problems, but unfortunately their approximation ratio is usually high, for example $O(d \cdot \log \Delta)$ or even higher; see, e.g., [Ind04, AIK08, CJLW22]. A major open problem in the area is to break this barrier and significantly improve the approximation

factor, say to $O(1)$ if not $1 + \varepsilon$, to match the ratios obtainable in low dimension, while using space polynomial in $d \cdot \log \Delta$. This question was partially resolved for k -median [BFL⁺17] and k -means [SYZ18], by designing coresets, to achieve $(1 + \varepsilon)$ -approximation using space $k \cdot \text{poly}(d \cdot \log \Delta)$ for a fixed $\varepsilon > 0$; however, these bounds are low-space only for small values of k .

As for UFL, the state-of-the-art streaming algorithms achieve only: $O(d \cdot \log^2 \Delta)$ -approximation using space $\text{poly}(d \cdot \log \Delta)$ [Ind04]; $O(1)$ -approximation for constant d [LS08] (which seems to generalize to $2^{O(d)}$ -approximation using $2^{O(d)}$ -space); and $(1 + \varepsilon)$ -approximation for $d = 2$ [CLMS13] (even if this approach could be extended to general d , its space bound seems to be $(\log \Delta)^{\Omega(d)}$). Since one can expect many applications of UFL (or clustering in general) to require high dimension d , the above bounds would not be satisfactory for either having large approximation ratio or using prohibitively large space.

The main barrier for closing this gap is the lack of suitable techniques for high-dimensional spaces. In particular, the quadtree subdivision is a natural geometric decomposition technique used in almost all previous geometric streaming algorithms, allowing for a lot of strong results in a low dimension [FS05, ABIW09, FIS08, CLMS13, CJKV22]. However, a quadtree in a high dimension, as used by Indyk [Ind04] and more recent follow-up papers [AIK08, CJLW22], gives a tree embedding, which is very useful algorithmically, but necessarily distorts distances by an $\Omega(d \cdot \log \Delta)$ -factor. It seems that new techniques must be developed in order to get a better approximation ratio in a high dimension, but surprisingly, very little, if at all, is known beyond the quadtree techniques. This technical barrier is not specific to UFL, but rather applies to many geometric problems.

1.1 Results

We break this barrier of existing techniques, and devise a geometric *importance-sampling* approach that is based on a *geometric hashing scheme*. Using this technique, we obtain improved approximation for UFL in high-dimensional streams, i.e., using $\text{poly}(d \cdot \log \Delta)$ space. We present our new streaming bounds for UFL below, and defer discussion of our technical contributions to Section 1.2. Throughout, we assume that all data points in P are distinct, and that the streaming algorithm knows in advance the parameters Δ and \mathfrak{f} . These are relatively standard assumptions in the model of dynamic geometric streams, and we omit them from the statements below.¹

We begin by presenting a *two-pass* streaming algorithm that computes an $O(1)$ -approximation to the cost of UFL. The underlying idea is to apply in the first pass the abovementioned importance-sampling method to select a sample of points from P , and then in the second pass to compute the contribution of each sampled point (for details see Section 1.2). This result demonstrates the significant advantage of our technique over the quadtree/tree-embedding based methods, which do not yield an $O(1)$ -approximation, even in two passes. This limitation of previous techniques can be seen also in recent work [CJLW22] that achieves $O(\log n)$ -approximation for bichromatic matching (earth mover distance) in two passes, where $n = |P|$ is the number of input points (their approach extends to one pass, albeit with an additive error).

Theorem 1.1 (*Two-Pass Algorithm*, see Theorem 4.1). *There is a two-pass randomized algorithm that computes an $O(1)$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a dynamic geometric stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

¹It suffices to know an $O(1)$ -approximation to $\log \Delta$ and to \mathfrak{f} for all our results. The assumption about distinct points was made also in [Ind04], and can sometimes be removed easily, e.g. in insertion-only streams using a random perturbation.

We can extend the approach from Theorem 1.1 to *random-order insertion-only streams* (where the stream is a uniformly random permutation of P and we assume $|P|$ is given), by using the first half of the stream to simulate the first pass of the algorithm from Theorem 1.1, and then using the second half to approximate the estimator used in the second pass of the algorithm. There is clearly correlation between the two halves of the stream, and neither half represents the full input faithfully, but we can circumvent these obstacles and obtain a one-pass $O(1)$ -approximation algorithm, as stated below. We are not aware of previous geometric streaming algorithms in high dimension, particularly those based on quadrees, that obtain $O(1)$ -approximation in random-order streams (apart from those for k -median [BFL⁺17] and k -means [SYZ18], though these bounds are low-space only for small k).

Theorem 1.2 (*Random-Order Algorithm*, see Theorem 4.2). *There is a one-pass randomized algorithm that computes an $O(1)$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a random-order insertion-only stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

Finally, to deal with arbitrary-order dynamic streams in one pass, we have to significantly extend our methods. Specifically, our previous scheme of first obtaining samples and then computing their estimations cannot work, and we need our importance sampling to provide additional structure. This more involved step crucially relies on a powerful property in our geometric hashing scheme, namely, that every point set of small-enough diameter is hashed to only $\text{poly}(d)$ distinct buckets. However, this also introduces the $O(d^{1.5})$ -factor in the approximation ratio, which corresponds to a key performance parameter of the hashing.

Theorem 1.3 (*One-Pass Algorithm*, see Theorem 4.5). *There is a one-pass randomized algorithm that computes an $O(d^{1.5})$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a dynamic geometric stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

While the approximation ratio of $O(d^{1.5})$ may look prohibitively large, it is useful to note that many geometric problems, including facility location, are reducible to the case $d = O(\log n)$, where $n = |P|$, by using standard techniques relying on the Johnson–Lindenstrauss lemma. Furthermore, the previously best streaming algorithm for UFL with $\text{poly}(d \cdot \log \Delta)$ space, by Indyk [Ind04], has an approximation factor of $O(d \cdot \log^2 \Delta)$ and thus, for $d = \Theta(\log n)$ and $\Delta = \text{poly}(n)$, we improve the approximation factor from $O(\log^3 n)$ to $O(\log^{1.5} n)$.

Other streaming algorithms for UFL in the literature focus solely on the case of constant d , or even just $d = 2$, and for general d seem to use space exponential in d . In particular, Lammersen and Sohler [LS08] gave an $\exp(O(d))$ -approximation using $\exp(\Omega(d)) \cdot \text{polylog } \Delta$ space (which thus has worse approximation ratio than our algorithm for $d = \omega(1)$), and Czumaj et al. [CLMS13] designed a $(1 + \varepsilon)$ -approximation for UFL in the Euclidean plane, which for a fixed $\varepsilon > 0$ seems to require space $(\log \Delta)^{\Omega(d)}$ in order to be extended to dimension d , meaning that the space would be superlogarithmic for $d = \omega(1)$.

The results above are complemented by the following lower bound, which follows by a reduction from the one-way communication complexity of the Boolean Hidden Matching problem.

Theorem 1.4 (*Streaming Lower Bound*; see Theorem 5.1). *For any dimension d , any one-pass randomized algorithm that for a given insertion-only stream of points from $[\Delta]^d$ for $\Delta = 2^{O(d)}$ approximates Uniform Facility Location within ratio better than 1.085, requires $2^{\text{poly}(d \cdot \log \Delta)}$ space.*

The lower bound in particular implies the impossibility of designing a streaming $(1 + \varepsilon)$ -approximation for any $\varepsilon > 0$ with $\text{poly}(d \cdot \log \Delta)$ space, even for insertion-only streams. In terms of $n = |P|$, the space lower bound is $\Omega(\sqrt{n})$ in the setting with $d = \Theta(\log n)$ and $\Delta = \text{poly}(n)$.

1.2 Technical Overview

For simplicity, in this section we consider the insertion-only setting, and assume that the instance is scaled so that the opening cost is $\mathfrak{f} = 1$. Then the input $P \subseteq \mathbb{R}^d$ of size $n = |P|$ is not restricted to a discrete grid. Our overall strategy is to design a streaming implementation of an estimator known to achieve an $O(1)$ -approximation. This estimator was proposed in [BCIS05] in the context of sublinear-time algorithms (based on an offline $O(1)$ -approximation algorithm in [MP03]). The idea is to associate to every data point $p \in P$ a value $r_p \in [1/n, 1]$ (formally defined in Definition 2.1), that satisfies two key properties (Fact 2.2): First, the sum of all r_p 's gives an $O(1)$ -approximation to UFL, i.e., $\sum_{p \in P} r_p = \Theta(\text{OPT})$. Second, r_p is roughly the inverse of the number of points inside the ball $B(p, r_p)$ centered at p with radius r_p , i.e., $|P \cap B(p, r_p)| = \Theta(1/r_p)$. It follows that r_p can be estimated (see Fact 2.3) by just counting the number of points in the balls $B(p, 2^{-j})$ for $j = 1, \dots, \log_2 n$, which is easy if p is known at the beginning of the stream. However, if p is given as a *query* at the end of the stream, then any finite approximation requires $\Omega(n)$ space, by a reduction from the communication complexity of indexing (see Appendix B).

1.2.1 Importance Sampling via Geometric Hashing

Naïve Approach: Uniform Sampling. Consider initially making two passes over the stream, the first one samples a few points, and the second pass estimates the r_p value for each sampled point p . Since all $r_p \in (0, 1]$, one immediate idea is to perform *uniform sampling*, and argue using Chernoff bounds that the resulting scaled estimate is likely to be $\Theta(\sum_{p \in P} r_p) = \Theta(\text{OPT})$. However, to obtain decent concentration, one needs the expectation $\sum_{p \in P} r_p$ to be large enough, which need not hold. Indeed, consider Example 1.5 below, where uniform sampling needs to draw $\Omega(\sqrt{n})$ samples to have a decent chance to see even one point p with $r_p = 1$, which is necessary for obtaining a nontrivial approximation.

Example 1.5. Let $P = P_1 \cup P_2$, where P_1 consists of \sqrt{n} points, whose pairwise distances are at least 1, and P_2 consists of $n - |P_1| = \Theta(n)$ points whose pairwise distances are (approximately) $1/n$; these two sets are at distance at least 1 from each other. (A realization of this point set is possible in dimension $\Theta(\log n)$.) One can easily verify from the definition that all $x \in P_1$ have $r_x = 1$, and all $x \in P_2$ have $r_x = \Theta(1/n)$, thus $\text{OPT} = \Theta(\sum_{p \in P} r_p) = \Theta(\sqrt{n})$.

To bypass the limitation of uniform sampling, we can employ *importance sampling*: sample one point $p^* \in P$, such that each $p \in P$ is picked with probability roughly proportional to r_p , and construct an unbiased estimator $\hat{Z} = 1/\Pr[p^*] \cdot r_{p^*}$. By a standard analysis, such an estimator has low variance, and thus averaging a few independent samples yields an accurate estimate. To get some intuition, in Example 1.5, when sampling proportionally to r_p , the total sampling probability of the points p with $r_p = 1$ is far larger than that of the remaining points.

However, this importance sampling idea is difficult to implement in streaming. At first glance, it is a chicken-and-egg problem: importance sampling aims to estimate $\sum_p r_p$, but it requires knowing the r_p values. The crux is that a coarse estimation for r_p suffices for importance sampling, but as noted above, computing r_p for point queries with any finite ratio requires $\Omega(n)$ space. Moreover, even if the r_p values of the sampled points could be estimated at the end of the stream, how would a streaming algorithm draw samples proportionally to these estimates?

New Idea: Geometric Importance Sampling. Instead of trying to estimate the value of r_p , we implement importance sampling indirectly (in Theorem 3.1), using a geometric hashing scheme $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that “isolates” points with large r_p . As usual in hashing, the codomain of φ is

somewhat arbitrary (e.g., in applications it could be $[\Delta]^d$), and a *bucket* refers to a preimage $\varphi^{-1}(z)$, i.e., the set of points mapped to the same image z . Ideally, we would like the aforementioned points (with large r_p) to each get its own bucket, and the others (small r_p) to collide, say, into one bucket per cluster, and thus take up only a few buckets. Given such a hashing scheme, we apply it to all the points in P and pick a non-empty bucket at random. This is equivalent to sampling uniformly from the hash values (of all points in P), and is easily implemented in streaming using a well-known tool called the ℓ_0 -sampler; see e.g. [CF14] (this tool produces a uniform sample from the *distinct elements* of a stream, and applying it here will sample uniformly from the distinct hash values).

The geometric hashing scheme, when combined with subsampling, guarantees that with high probability,

- a) the number of non-empty buckets is bounded by $\text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$, and
- b) every bucket with at least one point of large r_p contains at most $\text{poly}(d \cdot \log \Delta)$ points.

We may assume that points of large r_p constitute a significant fraction of $\sum_p r_p = \Theta(\text{OPT})$ (because we can anyway neglect a subset of points whose contribution is low), and employ the following *two-level uniform sampling*: First sample uniformly a non-empty bucket of the hashing scheme, and then sample uniformly a point from that bucket. Now, the probability of sampling a point p of large r_p is at least $1/\text{poly}(d \cdot \log \Delta)$ (this holds whenever $r_p > 1/\text{poly}(d \cdot \log \Delta)$). Thus, by taking $\text{poly}(d \cdot \log \Delta)$ samples we are likely to hit at least one point of large r_p , which in fact leads to a robust estimator. The aforementioned two-level uniform sampling is implemented by extending a standard construction of the ℓ_0 -sampler (in Lemma 3.3), inspired by a different extension in [FIS08].

This implementation of importance sampling bypasses the straightforward approach of first estimating the desired values (in our case r_p) and then sampling accordingly, as done previously in some fast algorithms, e.g., for counting [KL83] and for geometric problems [Ind07], and in “data-compression” algorithms, e.g., constructing graph sparsifiers [BK15, SS11] and geometric coresets [FL11, FSS20]. Previously, such a non-straightforward implementation of importance sampling was employed in streaming algorithms for matrices [LW16, BKKS20], for the same reason that the needed values are hard to compute in a streaming fashion.

Consistent Geometric Hashing with Bounded Gap. We now elaborate on the geometric hashing, which plays a central role in our importance-sampling algorithm and is formally defined as follows. Throughout, $\text{diam}(S)$ denotes the diameter of $S \subseteq \mathbb{R}^d$.

Definition 1.6 (Consistent Hashing). A mapping $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called a Γ -gap Λ -consistent hash with diameter bound $\ell > 0$, or simply (Γ, Λ) -hash,² if it satisfies:

1. Diameter: for every image $z \in \varphi(\mathbb{R}^d)$, we have $\text{diam}(\varphi^{-1}(z)) \leq \ell$; and
2. Consistency: for every $S \subseteq \mathbb{R}^d$ with $\text{diam}(S) \leq \ell/\Gamma$, we have $|\varphi(S)| \leq \Lambda$.

Intuitively, the first condition (diameter) requires that further apart points are never hashed (mapped) to the same bucket, and the second one (consistency) requires that highly-clustered points, even if their number is very large, are hashed to only a few different buckets.

We present this definition with a general parameter Λ , but in our application we always require $\Lambda := \text{poly}(d)$, which is sufficient as our algorithms have their approximation ratios independent of Λ and space only polynomially depending on Λ . However, the “gap” parameter Γ , equal to the ratio between the diameter bound ℓ and the consistency diameter ℓ/Γ , goes into the approximation

²While the parameter ℓ is important when applying the hashing, when constructing the hashing one can assume by scaling that $\ell = 1$.

factor of our one-pass streaming algorithm (Theorem 1.3), besides affecting the space complexity polynomially.

Comparison to Related Geometric Decompositions. Our definition of consistent hashing is essentially equivalent to the notion of sparse partitions that was introduced by Jia, Lin, Noubir, Rajaraman, and Sundaram [JLN⁺05]. Their definition concerns a *partition* of \mathbb{R}^d , if we view each part in that partition as a bucket in a hashing, then their definition is the same as Definition 1.6. However, in our setting it is more natural to think of a hash function, because the part/bucket that contains each input point $x \in \mathbb{R}^d$ must be computed *using a small amount of memory*, and a mere partition of \mathbb{R}^d may not suffice. The construction in [JLN⁺05] has consistency $\Lambda = 2^d$ (it is a straightforward partition into cubes), and is thus not useful in our context. Notably, Filtser [Fil20] designed a sparse partition with consistency $\Lambda = \text{poly}(d)$ and gap $\Gamma = O(d)$, however, in this partition the description of a part takes $\Omega(2^d)$ bits, and thus does not directly imply a hash function that can be evaluated on a point in small space.

We construct a consistent hashing scheme with gap $\Gamma = O(d^{1.5})$ (Lemma 3.6) that can be evaluated at a point $x \in \mathbb{R}^d$ in space and time $\text{poly}(d)$. We stress that the hash function is *data-oblivious*, because Definition 1.6 requires the function to be defined on the entire \mathbb{R}^d . This gap parameter Γ can potentially be improved to $O(d)$, to match the non-streaming construction in [Fil20]. We discuss other potential improvements regarding the hashing scheme in Section 6.

In fact, the same gap bound of $\Gamma = O(d^{1.5})$ was recently obtained by Dunkelman et al. [DGK⁺21]. It is not stated there explicitly (but can be verified by inspecting their analysis) because their construction is designed for a different notion, called *consistent rounding*, which is incomparable to our Definition 1.6 primarily because they require each bucket to have a bounded volume (instead of diameter), and their guarantee on the number of intersections is also slightly different. Their construction is somewhat similar to ours except that it works top-down, whereas ours works bottom-up.

We note that the second condition in Definition 1.6 does not easily follow from many known methods in the literature. Indeed, if we partition the space \mathbb{R}^d using standard methods, like hypercube subdivisions as in a quadtree (see, e.g., [Aro98]), it is hard to avoid clusters from intersecting $2^{\Omega(d)}$ buckets, instead of only $\text{poly}(d)$. Other geometric decompositions, such as padded decomposition [LS93, Bar96, CCG⁺98], and Locality-Sensitive Hashing [IM98], aim for different guarantees that are not directly comparable to those of Definition 1.6; see Section 1.3 for a broader comparison.

Construction of the Hashing. Our $O(d^{1.5})$ -gap construction first partitions the entire \mathbb{R}^d into unit hypercubes. Then we apply an identical decomposition/partition on every unit hypercube, and the partition of the entire space is the collection of all the parts (across all unit hypercubes). The partition into hypercubes immediately implies a diameter bound. Consider now the partition of an (arbitrary) unit hypercube H ; intuitively, we partition H into $d + 1$ *groups* of regions, such that regions in the same group are at distance $\geq \varepsilon := \ell/\Gamma$ of each other. This way, a subset T of diameter $< \varepsilon$ can only intersect one region from each of the $d + 1$ groups, which ensures consistency. Roughly, each of the $d + 1$ groups, say the i -th group, corresponds to all the i -dimensional faces of the hypercube. However, simply taking these i -dimensional faces cannot work, since they intersect and the minimum distance is 0. To make them separated, we employ a sequential process, where iteration $i = 0, 1, 2, \dots$ takes as our next group the i -dimensional faces and their close ℓ_∞ neighborhoods, excluding (i.e., removing) points that are sufficiently close (in the ℓ_∞ distance) to $(i - 1)$ -dimensional faces. In particular, we crucially use the following geometric

fact.

Fact 1.7 (See Lemma 3.11). *Consider two orthogonal i -dimensional subspaces S_1, S_2 and denote their intersection by I . Denoting by A^{+t} the ℓ_∞ neighborhood of A of radius t , define $S'_i := S_i^{+\varepsilon} \setminus I^{+2\varepsilon}$ for $i = 1, 2$. Then $\text{dist}(S'_1, S'_2) > \sqrt{2}\varepsilon$.*

Our construction iterates over $i = 0, \dots, d-1$ and repeatedly uses the above fact. Denoting by \mathcal{S}_i the union of the i -dimensional faces, the i -th group consists of the connected regions of $\mathcal{S}_i^{+(d-i)\varepsilon} \setminus \mathcal{S}_{i-1}^{+(d-i+1)\varepsilon}$. Finally, the d -th group consists of a single region corresponding to the hypercube interior after removing the previous groups.

To intuitively see why this geometric hashing helps, consider the instance in Example 1.5, and let us focus on estimating the number of points with $r_p = 1$ (for which uniform sampling does not work). In Example 1.5, points are grouped into natural clusters: the whole P_2 forms a cluster that consists of $O(n)$ points each with $r_p = O(1/n)$, and each point in P_1 forms a singleton cluster whose $r_p = 1$. We construct a $(\Gamma, \text{poly}(d))$ -hash with diameter bound $\ell = 1/2$ (as guaranteed in Lemma 3.6), and let $\varepsilon := \ell/\Gamma$. By the second guarantee, every small ball of radius $O(1/n) \ll \varepsilon$ is mapped to $\text{poly}(d)$ points/buckets, even if the ball originally could have $\Omega(n)$ points. Hence, after the hashing, the entire cluster P_2 with $O(n)$ points gets mapped to $\text{poly}(d)$ points, while points in P_1 are preserved because of the diameter bound of the buckets. Hence, applying the two-level uniform sampling, we hit one point in P_1 with at least $\text{poly}(d)$ samples on average.

Finally, we note that the actual implementation of this whole idea is more involved and requires additional steps. For instance, the nice cluster structure in Example 1.5 might not be present in a general input, and our analysis needs to explicitly define a clustering where a cluster containing p has diameter roughly r_p . Another issue is that our overview focused on $r_p = 1$. For general r_p , we use a subsampling at rate 2^{-i} to “reduce” the case $r_p = 2^{-i}$ to the case $r_p = 1$, which is conceptually similar to the subsampling used in the construction of ℓ_0 -samplers. At the end, this algorithm implements our desired importance sampling task, namely, it uses space $\text{poly}(d \cdot \log \Delta)$ and produces a sample $p^* \in P$ such that every $p \in P$ is picked with probability proportional to at least $r_p / \text{poly}(d \cdot \log \Delta)$.

1.2.2 Streaming Implementations

The above-mentioned geometric importance sampling (Theorem 3.1) can be implemented in one pass using small space. However, it only returns a set of *samples* $S \subseteq P$, and to actually estimate $\sum_{p \in P} r_p$ one still needs to estimate the value r_p for each $p \in S$. This limitation is a consequence of our importance-sampling approach, which bypasses estimating the r_p values on purpose.

Two Passes and Random-Order Streams. Our two-pass streaming algorithm (in Theorem 4.1) is quite simple: the first pass computes a sample $S \subseteq P$ using importance sampling (Theorem 3.1), and the second pass estimates r_p for each $p \in S$ using Fact 2.3 and straightforward counting. Its space complexity is $\text{poly}(d \cdot \log \Delta)$ per point $p \in S$, and we need $|S| = \text{poly}(d \cdot \log \Delta)$.

A similar approach can be applied also in the random-order model (i.e., the stream is a uniformly random permutation of P): the first half of the stream is used to generate a sample S , and the second half is used to estimate the r_p value of the points $p \in S$ sampled in the first half. However, more technical steps are needed in the analysis, due to the correlation between the two halves of the stream, and the fact that a random half does not represent the full stream accurately.

One-pass Implementation. The one-pass setting is significantly more difficult. We estimate $\sum_{p \in P} r_p$, by partitioning P into levels $i = 1, \dots, d \cdot \log_2 \Delta$, namely, we let $P_i := \{p \in P : r_p \in (2^{-i}, 2^{-i+1}]\}$ and $W_i := \sum_{p \in P_i} r_p$. We build an estimator for W_i separately for each i (recall that in this section we assume $f = 1$).

For simplicity, we focus here on $i = 0$, so we now only care about points $p \in P_0$, meaning that $r_p = \Theta(1)$. Then $W_0 = \sum_{p \in P_0} r_p = \Theta(|P_0|)$, and it suffices to estimate $|P_0|$. To this end, it is natural to use the estimator $1/\Pr[x] \cdot I(x \in P_0)$, where x is a sample generated by our importance sampling. However, the indicator $I(x \in P_0)$ turns out to be very sensitive, and it is difficult to estimate it in one pass even within a constant factor (e.g., distinguish between $r_x \geq 1/2$ and $r_x \leq 1/10$). Hence, we have to design a more relaxed tester and analyze how this affects the approximation ratio.

To implement this tester, we construct an $(\Gamma, \text{poly}(d))$ -hash with diameter bound $\ell = 1/10$. Recalling that the r_p values satisfy that $|P \cap B(p, r_p)| = \Theta(1/r_p)$, we observe that if a hashing bucket contains a point with $r_p \geq 1/2$, then the number of points in this bucket is bounded by $O(1)$. This means that a bucket either consists only of points with large r_p values, or no such point at all. Using this observation, we maintain a counter for the number of points mapped to every bucket, and when a point is sampled, we retrieve also the counter for its bucket, and if the counter is small, we use it as a proxy for the event that the bucket consists only of points with large r_p values, including in particular the sampled point.

However, a subtle technical issue is that some point y with $r_y \ll 1$ can possibly lie on the “boundary” of the bucket, and then the number of points in that bucket is small, while $P \cap B(y, r_y)$ contains many nearby points that lie in other buckets. Hence, we need to count the number of points in a slightly enlarged region, i.e., for a bucket $Q \subseteq \mathbb{R}^d$ we need to count the points in P that fall inside the r_y -neighborhood of Q , denoted here as $B(Q, r_y)$ and defined as the set of points in \mathbb{R}^d at distance at most r_y from Q . Hence, if we maintain the counter for $P \cap B(Q, \beta)$ for some $\beta > 0$ then the information of the counter suffices for rejecting y ’s whose $r_y \leq O(\beta)$.

Now, suppose we are to maintain the counter for some β . To implement this, whenever we see a data point x arrives, we should increase the counter for all buckets Q such that $x \in B(Q, \beta)$ (a similar trick appears e.g. in [FIS08]). However, this becomes challenging in the streaming setting, primarily due to the fact that the number of buckets Q such that $x \in B(Q, \beta)$ can be huge (e.g., 2^d), and more importantly, most of them may be “fake”, in that they do not contain any data point. Consequently, these fake nonempty buckets can enlarge the support of the ℓ_0 -sampler significantly, which makes it difficult to obtain a uniform sample.

To tackle this challenge, we make use of the guarantee from the geometric hashing, that any subset of \mathbb{R}^d with diameter less than $\varepsilon = O(1/\Gamma)$ is mapped to $\text{poly}(d)$ buckets (see Definition 1.6). Hence, if we choose $\beta = \varepsilon/2$, the effect of enlarging the buckets by an additive $\beta = \varepsilon/2$ is essentially making every data point x a ball $B(x, \varepsilon/2)$, and add the image of this ball to the buckets/counters. The $\text{poly}(d)$ intersection bound ensures that the number of fake buckets is still well bounded.

Finally, the approximate tester is off by a factor of Γ , which is the gap of the geometric hashing, and we show that this factor goes into the approximation ratio, which will thus be $O(\Gamma)$.

1.3 Related Work

Facility Location Problem. The facility location problem is one of the fundamental problems in operations research and combinatorial optimization, and has received extensive studies in the past. In the offline setting with uniform opening costs, the facility location problem has been proved to be NP-hard [MS84], and is hard to approximate within factor 1.463 [GK99] (unless $\text{NP} \subseteq \text{DTIME}[n^{O(\log \log n)}]$). For the upper bounds, the state of the art for general metrics is a 1.488-approximation by Li [Li13], and PTAS’s are known for special metric spaces, specifically,

doubling metrics [CFS21], minor-free graphs [CKM19], and near-linear time PTAS's for (bounded-dimensional) \mathbb{R}^d [KR07] and planar graphs [CPP19]. For the online setting, Meyerson [Mey01] gave an $O(\log n)$ -competitive algorithm, and Fotakis [Fot08] proved that it has ratio $\Theta(\log n / \log \log n)$ and that this ratio is asymptotically optimal.

Geometric Decomposition. Geometric decomposition is a topic that was studied extensively, with many different definitions, even beyond \mathbb{R}^d , that are motivated by numerous applications. For brevity, we only mention a few that are closer to our work. One basic genre, often called *space partitioning*, refers to a partition of \mathbb{R}^d , perhaps using a variant of the standard grid (quadtrees) partition, e.g. [AMN⁺98]. Sometimes it is convenient to use multiple space partitions, or a probability distribution over space partitions (e.g., a few shifts or a random shift of the grid partition [FK97, Aro98, CHJ20]). Another standard requirement is that every part in the partition has a bounded diameter (e.g., padded and separating decomposition [LS93, Bar96, CCG⁺98]), or alternatively that every part has a bounded volume (e.g., [DGK⁺21, KORW08]). The above examples ask that nearby points lie in the same part, but another type of decomposition, called Locality-Sensitive Hashing (LSH) [IM98], only asks that close-by points fall in the same part with noticeably higher probability than far-away points.

Two-pass and Random-order Streaming Algorithms. Besides the most studied streaming model of one-pass algorithms over an arbitrary (non-random) order streams, algorithms requiring a few passes or assuming that the stream order is random have received significant attention as well. For graph streams, two-pass streaming algorithms have been designed, for example, for graph spanners [KW14, FKN21], maximum matching [KN21], and triangle counting [CJ17]. Apart from graph streams, other examples of two-pass algorithms include those for matrix norm estimation [BCK⁺18], set cover [Ass17], and geometric earth mover distance [CJLW22]. The one-pass random-order (insertion-only) setting was studied for problems including matching [KKS14, AB21], quantile estimation [GM09], graph connected components and minimum spanning tree [PS18], and frequency moment estimation [WZ21].

2 Preliminaries

Notation. We use the usual notation $[n] := \{1, \dots, n\}$, and for a function $\varphi : X \rightarrow Y$ and $y \in Y$, we denote $\varphi^{-1}(y) := \{x \in X : \varphi(x) = y\}$. The d -dimensional ball centered at $x \in \mathbb{R}^d$ with radius $r \geq 0$ is defined as $B(x, r) := \{y \in \mathbb{R}^d : \text{dist}(x, y) \leq r\}$.

Definitions and Facts from Mettu-Plaxton (MP) Algorithm. We will need some machinery from the MP algorithm [MP03, BCIS05]. We first introduce the definition of r_p , and then recall useful facts, particularly that it suffices to approximate $\sum_{p \in P} r_p$, because it $O(1)$ -approximates OPT.

Definition 2.1 ([MP03]). For every $p \in P$, let r_p be the number such that

$$\sum_{x \in P \cap B(p, r_p)} (r_p - \text{dist}(p, x)) = \mathfrak{f}. \quad (1)$$

It is easy to see that r_p is well-defined and $\frac{\mathfrak{f}}{|P|} \leq r_p \leq \mathfrak{f}$. Indeed, using the notation $z^+ = \max(z, 0)$, we can write the left-hand side of (1) as $\sum_{x \in P} (r_p - \text{dist}(p, x))^+$, which is easily seen to

be non-decreasing with r_p . For illustration, suppose p is one of k points whose pairwise distances are all equal to $a \in (0, \mathfrak{f})$, and all other points are at distance at least \mathfrak{f} from p ; then $r_p = \Theta(a + \mathfrak{f}/k)$.

Fact 2.2 (Lemmas 1 and 2 in [BCIS05]). *The following holds.*

- For every $p \in P$, it holds that $|P \cap B(p, r_p)| \geq \mathfrak{f}/r_p$ and $|P \cap B(p, r_p/2)| \leq 2\mathfrak{f}/r_p$.
- $\sum_{p \in P} r_p = \Theta(\text{OPT})$.

We assume without loss of generality (w.l.o.g.) that Δ is a power of two. Let $L := d \cdot \log_2 \Delta \geq \log_2 |P|$ as $|P| \leq \Delta^d$ by the assumption that points are distinct. The first point of Fact 2.2 implies that the r_p value can be approximated within a constant factor by counting the number of points in balls of geometrically increasing radii.

Fact 2.3. *Let j_0 be the maximum $j \in \{0, \dots, L\}$ such that $|P \cap B(p, 2^{-j}\mathfrak{f})| \geq 2^j$. Then*

$$r_p \in (2^{-j_0-1}\mathfrak{f}, 2^{-j_0+1}\mathfrak{f}]. \quad (2)$$

Moreover, there is a one-pass deterministic streaming algorithm that given the opening cost $\mathfrak{f} > 0$ and a point p in advance of the data set P presented as a dynamic stream, returns an estimate \hat{r}_p such that $r_p \leq \hat{r}_p \leq O(r_p)$ using space of $O(L^2)$ bits.

Proof. By Fact 2.2, if $r_p \leq 2^{-j_0-1}\mathfrak{f}$ then we arrive at the contradiction $|P \cap B(p, 2^{-j_0-1}\mathfrak{f})| \geq |P \cap B(p, r_p)| \geq \mathfrak{f}/r_p \geq 2^{j_0+1}$. And if $r_p > 2^{-j_0+1}\mathfrak{f}$ then we arrive at the contradiction $|P \cap B(p, 2^{-j_0}\mathfrak{f})| \leq |P \cap B(p, r_p/2)| \leq 2\mathfrak{f}/r_p < 2^{j_0}$. To implement the estimation in dynamic streams, we count the number of points in each of the balls $B(p, 2^{-j}\mathfrak{f})$ for $j = 0, \dots, L$ using an L -bit counter. \square

3 Importance Sampling via Geometric Hashing

In this section, we develop a streaming algorithm for importance sampling on P , where the probability to report each point $x \in P$ is (at least) proportional to its contribution to $\sum_{x \in P} r_x$. Similarly to other streaming algorithms for sampling (e.g., ℓ_p -samplers), our algorithm might fail with a tiny but non-zero probability, in our case $q_{\text{fail}} = 1/\text{poly}(\Delta^d)$, and the analysis can effectively ignore these events by a union bound.³ While the algorithm's goal is to sample from P , we also allow it to return \perp , which is *not* considered a failure, as long as it returns points from P with sufficiently large probability.⁴ The output \perp is useful in the algorithm's design, as it can replace the use of a fixed point from P , and also handle properly the corner case $P = \emptyset$.

Theorem 3.1. *There is a one-pass randomized algorithm that, given $P \subseteq [\Delta]^d$ presented as a dynamic geometric stream, samples a random point $p^* \in P \cup \{\perp\}$ such that*

$$\forall x \in P, \quad \Pr[p^* = x] \geq \Omega\left(\frac{1}{\text{poly}(d \cdot \log \Delta)}\right) \cdot \frac{r_x}{\sum_{y \in P} r_y},$$

and also reports a 2-approximation $\widehat{\Pr}[p^*]$ for the probability of sampling this point, i.e., $\Pr[p^* = x] \leq \widehat{\Pr}[p^*] \leq 2\Pr[p^* = x]$. This algorithm uses $\text{poly}(d \cdot \log \Delta)$ bits of space, and fails with probability at most $1/\text{poly}(\Delta^d)$.

³Failure in Theorem 3.1 or Lemma 3.3 means that the algorithm may behave arbitrarily, e.g., not return anything or even return a point outside P , and it is not easy to verify if the point is in P . Formally, having failure probability q_{fail} means that the total variation distance between the algorithm's output distribution and desired distribution (e.g., uniform over a certain set in the case of ℓ_0 -sampler) is at most q_{fail} .

⁴For example, an acceptable output distribution may be \perp with probability $\frac{1}{2}$, and every $x \in P$ with probability $\frac{1}{2} r_x / \sum_{y \in P} r_y$.

Proof of Theorem 3.1. We first provide an algorithm that samples points $x \in P$ at a given level $i \in \{1, \dots, L\}$, which refers to points with r_x value roughly 2^{-i} . We present it as an offline algorithm in Algorithm 1, and discuss below how to implement it as a streaming algorithm. The main guarantee about its output is given in Lemma 3.2 below, whose proof appears in Section 3.1. We remark that the algorithm returns \perp in case $\text{sub}_i(P)$ is empty, i.e., no point survives the subsampling. (We will see later that Theorem 3.1 follows by simply executing a streaming implementation of this algorithm with a random level i .)

Lemma 3.2. *Algorithm 1 returns a random point $p^* \in P \cup \{\perp\}$ such that*

$$\forall x \in P_i, \quad \Pr[p^* = x] \geq \Omega\left(\frac{1}{\text{poly}(d \cdot \log \Delta)}\right) \cdot 2^{-i} \cdot \frac{f}{\text{OPT}},$$

where $P_i := \{x \in P : 2^{-i} \leq r_x \leq 2^{-i+1}\}$.

Algorithm 1 Importance sampling for single level i

- 1: let $\ell \leftarrow 0.1 \cdot 2^{-i}$, let φ_i be a $(\Gamma, \text{poly}(d))$ -hash of \mathbb{R}^d with diameter bound ℓ \triangleright use Lemma 3.6
 - 2: subsample P with rate 2^{-i} \triangleright denote the subsampled subset of P by $\text{sub}_i(P)$
 - 3: sample uniformly $a \in \varphi_i(\text{sub}_i(P))$, then uniformly $p^* \in \varphi_i^{-1}(a) \cap \text{sub}_i(P)$
 - 4: return p^* \triangleright if such p^* does not exist (e.g., if $\text{sub}_i(P) = \emptyset$), return \perp .
-

Streaming Implementation of Algorithm 1. Line 1 uses Lemma 3.6 to get a data-oblivious function φ_i , hence this step can be executed before the algorithm starts to process the stream.

Subsampling in Dynamic Streams. Line 2 performs subsampling with rate 2^{-i} , that is, each point in P is independently sampled with probability 2^{-i} . If the stream is insertion-only we just sample each newly added point independently with probability 2^{-i} . However, in dynamic streams we need consistency between insertions and deletions of a point, and we thus apply a random hash function $h : [\Delta]^d \rightarrow \{0, 1\}$ such that for every point p we have $\Pr[h(p) = 1] = 2^{-i}$. We draw this hash function at the beginning of the stream, and then for each insertion/deletion of a point p , we evaluate $h(p)$ to determine whether p is subsampled.

In the analysis, we assume that these subsampling events are independent for all points, i.e., that h is fully random. To deal with the fact that storing such a hash function takes Δ^d bits, we use Nisan’s pseudorandom generator (PRG) [Nis92] that has the following guarantee: For any parameters R and S , given a seed of $\Omega(S \cdot \log R)$ truly random bits, the PRG generates R bits that cannot be distinguished from truly random bits by any algorithm running in space S . Naturally, we use this PRG with S being the space cost of our algorithm and $R = i \cdot \Delta^d$ (which is the number of independent fair coin flips needed to generate h).

Two-level Uniform Distinct Sampling. To implement the final sampling step of Algorithm 1 (in line 3), we present in Lemma 3.3 a two-level ℓ_0 -sampler, which is more convenient to describe as sampling from a frequency matrix. The proof of this lemma, provided in Appendix A, is an extension of a standard ℓ_0 -sampler (from a frequency vector); see e.g. [CF14]. We are not aware of such a sampler in the literature, although similar extensions were devised before, e.g. in [FIS08].⁵

⁵The notion of ℓ_p -sampling with meta-data, which was recently introduced in [CJLW22], sounds related but is quite different, as each index i arrives with an associated value λ_i ; in fact, their approach builds on Precision

Lemma 3.3 (Two-Level ℓ_0 -Sampler). *There is a randomized algorithm, that given as input a matrix $M \in \mathbb{R}^{m \times n}$, with $m \leq n$ and integer entries bounded by $\text{poly}(n)$, that is presented as a stream of additive entry-wise updates, returns an index-pair (i, j) of M , where i is chosen uniformly at random (u.a.r.) from the non-zero rows, and then j is chosen u.a.r. from the non-zero columns in that row i . The algorithm uses space $\text{poly}(\log n)$, fails with probability at most $1/\text{poly}(n)$, and can further report the corresponding row-sum $\sum_{j'} M_{i,j'}$.*

It is straightforward to implement line 3 of Algorithm 1 using this sampler. Simply convert the updates to P , on the fly, into updates to a frequency matrix M , whose rows correspond to all hash buckets (images of φ_i) and columns correspond to all grid points $([\Delta]^d)$. This is clearly a huge matrix, but it is not maintained explicitly. The reported row-sum $\sum_{j'} M_{i,j'}$ corresponds to the number of points in P that are hashed (mapped) to the bucket returned by the sampler. Hence, we can implement Algorithm 1 in one pass over a dynamic geometric stream. The success probability depends on the two-level ℓ_0 -sampler, and is thus $1 - 1/\text{poly}(n) \geq 1 - 1/\text{poly}(\Delta^d)$, and assuming success, the output distribution is as described in Lemma 3.2.

We can now complete the proof of Theorem 3.1. The algorithm draws uniformly at random a level $i^* \in \{1, \dots, L\}$ and executes Algorithm 1 for this level i^* . Now consider a point $x \in P$, and let j be the level for which $x \in P_j$, i.e., $2^{-j}f < r_x \leq 2^{-j+1}f$. Then by Lemma 3.2, the probability to sample this point x is

$$\Pr[p^* = x] \geq \Pr[i^* = j] \cdot \frac{1}{\text{poly}(d \cdot \log \Delta)} \cdot 2^{-j} \cdot \frac{f}{\text{OPT}} \geq \frac{1}{L} \cdot \frac{1}{\text{poly}(d \cdot \log \Delta)} \cdot \Omega\left(\frac{r_x}{\sum_{y \in P} r_y}\right). \quad (3)$$

Recall that the algorithm needs to report also an estimate $\widehat{\Pr}[p^*]$ for the probability of sampling the specific point p^* that is reported. Given the randomly chosen level i^* (which might differ from the level j of x), for the algorithm to pick x , it must first subsample x , which happens with probability 2^{-i^*} , then pick the bucket of x under φ_{i^*} , while there are $|\varphi_{i^*}(\text{sub}_{i^*}(P))|$ non-empty buckets, and finally, it has to pick this point x from its bucket, which contains $|\varphi_{i^*}^{-1}(\varphi_{i^*}(x)) \cap \text{sub}_{i^*}(P)|$ subsampled points. Thus,

$$\Pr[p^* = x \mid i^*] = \frac{2^{-i^*}}{|\varphi_{i^*}(\text{sub}_{i^*}(P))| \cdot |\varphi_{i^*}^{-1}(\varphi_{i^*}(x)) \cap \text{sub}_{i^*}(P)|}.$$

Furthermore, the algorithm can accurately estimate all these quantities; indeed, the bucket size is known from the two-level ℓ_0 -sampler (recall that Lemma 3.3 reports also the corresponding row-sum), and to estimate the number of non-empty buckets the algorithm can run in parallel a standard streaming algorithm for counting distinct elements (see e.g. [KNW10]). \square

3.1 Proof of Lemma 3.2

Subsampling. The first step is to subsample every point in P independently with probability $1/2^i$. For every subset $S \subseteq P$, let $\text{sub}_i(S) \subseteq S$ be the random subset after the subsampling. The following describes several standard facts about the subsampling.

Fact 3.4. $\forall S \subseteq \mathbb{R}^d$ and $t \geq 2$, the following holds.

Sampling [AKO11] and is applicable only for $p > 0$. The use of a two-level structure and its representation as a matrix were introduced in [CM05, JW09] as cascaded aggregates/norms, however their algorithms estimate these norms, not sampling an index by the norm. A sampler for cascaded $\ell_{p,2}$ -norm was designed in [BKKS20], building on properties of the Gaussian distribution and ℓ_p -samplers.

- If $|S| \geq 2^i$, then $\Pr[|\text{sub}_i(S)| \geq t \cdot |S| \cdot 2^{-i}] \leq \exp(-\Theta(t))$.
- If $|S| \leq 2^i$, then $\Pr[|\text{sub}_i(S)| \geq t] \leq \exp(-\Theta(t))$.

Proof. For every $u \in S$, let $X_u \in \{0, 1\}$ be the indicator random variable such that $X_u = 1$ if and only if $u \in \text{sub}_i(S)$, so $\Pr[X_u = 1] = 2^{-i}$ for every $u \in S$. Then $|\text{sub}_i(S)| = \sum_{u \in S} X_u$, and $\mathbb{E}[|\text{sub}_i(S)|] = |S| \cdot 2^{-i}$. Let $\mu := \mathbb{E}[|\text{sub}_i(S)|]$.

- If $|S| \geq 2^i$, then $\mu \geq 1$. By Chernoff bound,

$$\begin{aligned} \Pr[|\text{sub}_i(S)| \geq t \cdot |S| \cdot 2^{-i}] &= \Pr[|\text{sub}_i(S)| - \mu \geq (t - 1) \cdot \mu] \\ &\leq \exp(-\Theta(t) \cdot \mu) \leq \exp(-\Theta(t)). \end{aligned}$$

- If $|S| \leq 2^i$, then $\mu \leq 1$. By Chernoff bound,

$$\begin{aligned} \Pr[|\text{sub}_i(S)| \geq t] &= \Pr[|\text{sub}_i(S)| - \mu \geq (t/\mu - 1) \cdot \mu] \\ &\leq \exp(-\Theta(t)). \end{aligned}$$

□

Geometric Hashing. We consider bounded consistent hashing schemes that do not map small “clusters” of points into too many buckets. We restate the definition below, and we prove the existence of such hashing schemes, with certain parameters, in Lemma 3.6. As mentioned in Section 1, Jia et al. [JLN⁺05] introduced an essentially equivalent notion called “sparse partitions”, although we further require that evaluating the hash function at a point is efficient.

Definition 3.5 (Restatement of Definition 1.6). A mapping $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called a Γ -gap Λ -consistent hash with diameter bound $\ell > 0$, or simply (Γ, Λ) -hash, if it satisfies:

1. Diameter: for every image $z \in \varphi(\mathbb{R}^d)$, we have $\text{diam}(\varphi^{-1}(z)) \leq \ell$; and
2. Consistency: for every $S \subseteq \mathbb{R}^d$ with $\text{diam}(S) \leq \ell/\Gamma$, we have $|\varphi(S)| \leq \Lambda$.

Lemma 3.6. *There exists a $(\Gamma, d+1)$ -hash $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ for $\Gamma = \Theta(d\sqrt{d})$. Furthermore, one can evaluate $\varphi(x)$ for input $x \in \mathbb{R}^d$ in time and space $\text{poly}(d)$.*

Proof. We construct a consistent hashing scheme with $\Gamma = \Theta(d\sqrt{d})$ and $\Lambda = d+1$ in Section 3.2. □

Sampling on $\varphi_i(\text{sub}_i(P))$. Suppose we apply Lemma 3.6 to find a $(\Gamma, \text{poly}(d))$ -hash φ_i with diameter bound $\ell = 2^{-i}\mathfrak{f}/10$, and define $\varepsilon := \ell/\Gamma$ which is the magnitude of the consistency guarantee. Then $\varphi_i(\text{sub}_i(P))$ essentially maps points in $\text{sub}_i(P)$ into buckets, and our plan is to sample from these buckets. Next, we wish to upper bound $|\varphi_i(\text{sub}_i(P))|$, which is the support of sampling, in terms of OPT. Since the guarantee on φ_i in Lemma 3.6 is about clusters/subsets, we need to first define a clustering of the point set (Lemma 3.7) such that the number of points in each cluster C is upper bounded by $O(\mathfrak{f}/\text{diam}(C))$. Then, in Lemma 3.8, we use the guarantee of the geometric hashing on the clusters resulting from Lemma 3.7 to bound $|\varphi_i(\text{sub}_i(P))|$. In general, there are two types of clusters according to the diameter: i) “small” with diameter at most $\varepsilon = \ell/\Gamma$, for which we use the consistency guarantee of our geometric hashing, i.e., the second point of Definition 1.6 (note that for small clusters, $\mathfrak{f}/\text{diam}(C)$ is not a useful bound on $|C|$), and ii) “large”, for which $O(1/\text{diam}(C))$ is not too large and the subsampling leaves only $\text{poly}(d \cdot \log \Delta)$ points for each “large” cluster with high probability. (In the lemma below, if $\text{diam}(C) = 0$, the bound $O(\mathfrak{f}/\text{diam}(C))$ is defined to be Δ^d .)

Lemma 3.7 (Extended MP-clustering). *There exists a partition \mathcal{C} of P such that $\mathfrak{f} \cdot |\mathcal{C}| \leq O(d \cdot \log \Delta) \cdot \text{OPT}$ and for every $C \in \mathcal{C}$, $|C| \leq O(\mathfrak{f}/\text{diam}(C))$.*

Proof. The following algorithm from [MP03], called MP algorithm, finds a 3-approximation for UFL.

1. List P in non-decreasing order of r_p .
2. Examine $p \in P$ in order, and if there is no open facility in $B(p, 2r_p)$, then open the facility at p .

Denote the set of the facilities opened by MP algorithm as F^{MP} . We use the following steps to construct a partition of P .

1. For every $p \in P$, assign it to the nearest point in F^{MP} .
2. For every $p \in F^{\text{MP}}$, let $C(p) \subseteq P$ be the set of points that are assigned to p .
3. For every $p \in F^{\text{MP}}$ and every $j = 1, \dots, L$, let $C(p)_j := C(p) \cap P_j$.
4. For every $p \in F^{\text{MP}}$ and every j , arbitrarily divide $C(p)_j$ into subsets of size 2^j , possibly with a unique subset that has size $< 2^j$, and include these subsets into \mathcal{C} .

Now we show that \mathcal{C} is the collection satisfying Lemma 3.7. Clearly, \mathcal{C} covers P , since every point in p is assigned to some point in F^{MP} .

To upper bound the number of points in each cluster, suppose $C \in \mathcal{C}$ is created by dividing $C(p)_j$ for some p and j . Then $\forall q \in C$, $p \in B(q, 2r_q)$. To see this, suppose for the contrary that $p \notin B(q, 2r_q)$. By the construction of $C(p)$, $p \in F^{\text{MP}}$ is the closest to q , hence $p \notin B(q, 2r_q)$ implies that no point in F^{MP} belongs to $B(q, 2r_q)$. However, by the MP algorithm, this means q should have been added to F^{MP} , which is a contradiction. Hence for every $q_1, q_2 \in C \subseteq P_j$,

$$\text{dist}(q_1, q_2) \leq \text{dist}(q_1, p) + \text{dist}(p, q_2) \leq O(2^{-j}\mathfrak{f}) + O(2^{-j}\mathfrak{f}) = O(2^{-j}\mathfrak{f}),$$

which implies that $\text{diam}(C) = O(2^{-j}\mathfrak{f})$. By Step 4 of the construction, we know that $|C| \leq 2^j$. Therefore, when $\text{diam}(C) > 0$, $|C| \leq O(\mathfrak{f}/\text{diam}(C))$ holds.

Finally, we bound the number of clusters. For every $p \in F^{\text{MP}}$ and every j , the number of subsets that we obtain from the division is at most $\lceil |C(p)_j|/2^j \rceil \leq |C(p)_j|/2^j + 1$. Summing over p and j ,

$$\begin{aligned} \mathfrak{f} \cdot |\mathcal{C}| &\leq \mathfrak{f} \cdot \sum_{p \in F^{\text{MP}}} \sum_{j=1}^L 1 + |C(p)_j|/2^j \leq O(d \cdot \log \Delta) \cdot |F^{\text{MP}}| \cdot \mathfrak{f} + O(1) \cdot \sum_{p \in F^{\text{MP}}} \sum_j \sum_{q \in C(p)_j} r_q \\ &\leq O(d \cdot \log \Delta) \cdot |F^{\text{MP}}| \cdot \mathfrak{f} + O(\text{OPT}) \\ &\leq O(d \cdot \log \Delta) \cdot \text{OPT}, \end{aligned}$$

where the second inequality uses that $C(p)_j \subseteq P_j$ and $\sum_{p \in P} r_p = \Theta(1) \cdot \text{OPT}$. This completes the proof of Lemma 3.7. \square

Lemma 3.8. *With probability at least $1 - 1/\text{poly}(\Delta^d)$, $\mathfrak{f} \cdot |\varphi_i(\text{sub}_i(P))| \leq \text{poly}(d \cdot \log \Delta) \cdot \Gamma \cdot \text{OPT} \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$.*

Proof. Let \mathcal{C} be the collection of subsets guaranteed by Lemma 3.7. Then

$$|\varphi_i(\text{sub}_i(P))| \leq \sum_{C \in \mathcal{C}} |\varphi_i(\text{sub}_i(C))| \leq |\mathcal{C}| \cdot \max_{C \in \mathcal{C}} |\varphi_i(\text{sub}_i(C))|.$$

Since $\mathfrak{f} \cdot |\mathcal{C}| \leq O(d \cdot \log \Delta) \cdot \text{OPT}$, it suffices to prove that with probability at least $1 - 1/\text{poly}(\Delta^d)$, $|\varphi_i(\text{sub}_i(C))| \leq \text{poly}(d \cdot \log \Delta)$ for every $C \in \mathcal{C}$.

We wish to bound $|\varphi_i(\text{sub}_i(C))|$ for C with “small” diameter and “large” diameter separately. For C with $\text{diam}(C) \leq \varepsilon$, by Definition 1.6 and Lemma 3.6, we have $|\varphi_i(C)| \leq \text{poly}(d)$, which implies that

$$|\varphi_i(\text{sub}_i(C))| \leq |\varphi_i(C)| \leq \text{poly}(d).$$

For C with $\text{diam}(C) > \varepsilon = \Theta(\mathfrak{f}/(2^i \Gamma))$, by Lemma 3.7, we have $|C| \leq O(\mathfrak{f}/\text{diam}(C)) \leq O(2^i \cdot \Gamma)$. By Fact 3.4, with probability at least $1 - 1/\text{poly}(\Delta^d)$, $|\text{sub}_i(C)| \leq O(d \cdot \log \Delta) \cdot \Gamma$. Finally, applying the union bound to all subsets $C \in \mathcal{C}$ with $\text{diam}(C) > \varepsilon$ concludes the proof of Lemma 3.8. \square

The next lemma states that if the close neighborhood (at distance r_p) of any point p in P_i does not contain too many points after subsampling P , then for any subsampled point $p \in P_i$ there are not too many subsampled points mapped by φ_i into the same bucket as p .

Lemma 3.9. *With probability at least $1 - 1/\text{poly}(\Delta^d)$, for every point p such that $r_p \geq 2^{-i}\mathfrak{f}$, $|\varphi_i^{-1}(\varphi_i(p)) \cap \text{sub}_i(P)| \leq O(d \cdot \log \Delta)$.*

Proof. Observe that if $r_p \geq 2^{-i}\mathfrak{f}$, then $|\varphi_i^{-1}(\varphi_i(p)) \cap P| \leq O(2^i)$. This is a consequence of having $\ell = 2^{-i}\mathfrak{f}/10$, and the fact that $|B(p, r_p/2) \cap P| \leq O(2^i)$ (by Fact 2.2). Hence, Lemma 3.9 follows by applying Fact 3.4 on the set $\varphi_i^{-1}(\varphi_i(p)) \cap P$. \square

Proof of Lemma 3.2. If $P_i = \emptyset$, then Algorithm 1 always returns an arbitrary point of P or \perp , and the guarantee of the lemma trivially holds. Now suppose $P_i \neq \emptyset$ and fix $x \in P_i$. Let $P' = P \setminus \{x\}$. Let \mathcal{E} be the event that the following happens (over the randomness of sub_i).

1. $\mathfrak{f} \cdot |\varphi_i(\text{sub}_i(P'))| \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$.
2. for every point p with $r_p \geq 2^{-i}\mathfrak{f}$, $|\varphi_i^{-1}(\varphi_i(p)) \cap \text{sub}_i(P')| \leq O(d \cdot \log \Delta)$.

By Lemma 3.8 and Lemma 3.9, $\Pr[\mathcal{E}] \geq 1 - 1/\text{poly}(\Delta^d)$. Moreover, \mathcal{E} implies that

$$\mathfrak{f} \cdot |\varphi_i(\text{sub}_i(P))| \leq \mathfrak{f} \cdot |\varphi_i(\text{sub}_i(P'))| + \mathfrak{f} \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT},$$

and for every point p with $r_p \geq 2^{-i}\mathfrak{f}$,

$$|\varphi_i^{-1}(\varphi_i(p)) \cap \text{sub}_i(P)| \leq |\varphi_i^{-1}(\varphi_i(p)) \cap \text{sub}_i(P')| + 1 \leq O(d \cdot \log \Delta).$$

Note that \mathcal{E} is independent to whether or not x survives the subsampling. Let $A_x := \varphi_i^{-1}(\varphi_i(x)) \cap \text{sub}_i(P)$ be the set of points that lie in the same bucket of φ_i as x . Suppose Algorithm 1 returns a random point p^* . Then a point $x \in P_i$ is sampled with probability

$$\begin{aligned} \Pr[p^* = x] &\geq \Pr[p^* = x \mid \mathcal{E}] \cdot \Pr[\mathcal{E}] \\ &\geq \Omega(1) \cdot 2^{-i} \cdot \frac{1}{|\varphi_i(\text{sub}_i(P))| \cdot |A_x|} \\ &\geq \Omega(1) \cdot \frac{1}{\text{poly}(d \cdot \log \Delta)} \cdot 2^{-i} \cdot \frac{\mathfrak{f}}{\text{OPT}}. \end{aligned}$$

This completes the proof of Lemma 3.2. \square

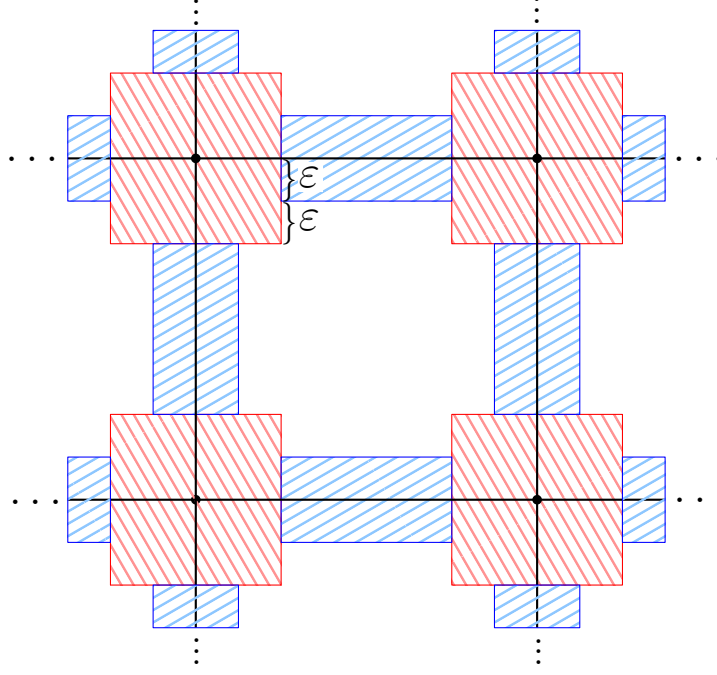


Figure 1: An illustration of the construction of a $(\Gamma, d+1)$ -hash in 2D, showing only a vicinity of a single square $[0, T]^2$ of the initial partition (in black). The $(l_0 = 2\varepsilon)$ -neighborhoods of vertices, which form \mathcal{B}_0 , are depicted in red (north-west shaded areas), and the $(l_1 = \varepsilon)$ -neighborhoods of edges, that is, set $\mathcal{B}_1 \setminus \mathcal{B}_0$, are depicted in blue (north-east shaded areas).

3.2 Construction of Geometric Hashing: Proof of Lemma 3.6

Let $\ell > 0$ and $\varepsilon := \ell/\Gamma$, where $\Gamma = \Theta(d\sqrt{d})$. We use the following construction to obtain a $(\Gamma, d+1)$ -hash $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with diameter bound ℓ . Let $T := \frac{\ell}{\sqrt{d}} = \Theta(d\varepsilon)$ and partition \mathbb{R}^d into hypercubes of side length T . Note that each of these hypercubes is of diameter ℓ . Let $l_i := (d-i)\varepsilon$ for $i = 0, 1, \dots, d-1$. For a point set $S \subseteq \mathbb{R}^d$, let $N_i^\infty(S)$ be the l_i -neighborhood of S under the ℓ_∞ distance, i.e.,

$$N_i^\infty(S) := \{x \in \mathbb{R}^d : \exists s \in S, \|x - s\|_\infty \leq l_i\}.$$

1. For $i = 0, \dots, d$, let \mathcal{A}_i be the set of i -dimensional faces of the hypercubes, and let $\mathcal{B}_i := N_i^\infty(\mathcal{A}_i)$. Define $\mathcal{B}_{-1} := \emptyset$ and $\mathcal{B}_{\leq i} := \bigcup_{j \leq i} \mathcal{B}_j$.
2. For every $i = 0, \dots, d-1$, for every i -dimensional face Q , let $\widehat{Q} := N_i^\infty(Q) \setminus \mathcal{B}_{\leq i-1}$ and for every $x \in \widehat{Q}$ assign $\varphi(x) = q$ where $q \in \widehat{Q}$ is an arbitrary but fixed point. Observe that $\widehat{Q} \subseteq \mathcal{B}_i$ and thus $x \in \widehat{Q}$ will *not* be assigned again in later iterations. (See Figure 1 for an illustration.)
3. For every d -dimensional face Q , i.e., a hypercube, let \widehat{Q} be the remaining part of Q whose $\varphi(x)$ has not been assigned, and assign $\varphi(x) = q$ for every $x \in \widehat{Q}$, where $q \in \widehat{Q}$ is arbitrary but fixed point.

In the following two lemmas, for any $0 \leq i \leq d$, we show that every two i -dimensional faces of the hypercubes, including the d -dimensional interiors, are at distance larger than ε after we remove

$\mathcal{B}_{\leq i-1}$, which is the union of the l_i -neighborhoods (under the ℓ_∞ distance) of j -dimensional faces for $j < i$. This implies the property that every point set of diameter at most $\varepsilon = \ell/\Gamma$ maps to at most $d+1$ buckets.

Lemma 3.10 (Analysis for $i \leq d-1$). *Suppose $0 \leq i \leq d-1$ is an integer. Let Q_1, Q_2 be two distinct i -dimensional faces of the hypercubes. Let $\widehat{Q}_1 := N_i^\infty(Q_1) \setminus \mathcal{B}_{\leq i-1}$, and $\widehat{Q}_2 := N_i^\infty(Q_2) \setminus \mathcal{B}_{\leq i-1}$ as in the construction. Then $\text{dist}(\widehat{Q}_1, \widehat{Q}_2) > \varepsilon$.*

Proof. We assume that the intersection of Q_1 and Q_2 is non-empty, as otherwise

$$\text{dist}(\widehat{Q}_1, \widehat{Q}_2) \geq T - 2l_i \geq T - 2d\varepsilon > \varepsilon.$$

Note that every i -dimensional face in the hypercube $[0, T]^d$ can be represented by fixing $d-i$ coordinates to be either 0 or T , and other i coordinates range over $[0, T]$. W.l.o.g., assume that Q_1 belongs to the hypercube $[0, T]^d$, and $Q_1 = [0, T]^i \times 0^{d-i}$. Let $Q = Q_1 \cap Q_2$ and then Q is a j -dimensional face of the hypercube, where $j < i$. W.l.o.g., assume $Q = [0, T]^j \times 0^{d-j}$, so $0 \in Q_1 \cap Q_2$. Since $Q \subseteq \mathcal{A}_{i-1}$, we have for all $t \in \{1, 2\}$

$$\widehat{Q}_t = N_i^\infty(Q_t) \setminus \mathcal{B}_{\leq i-1} \subseteq N_i^\infty(Q_t) \setminus N_{i-1}^\infty(Q). \quad (4)$$

Define $I_1 = \{1, \dots, j\}$, $I_2 = \{j+1, \dots, i\}$, $I_3 = \{i+1, \dots, d\}$. Then by the definition of N_i^∞ ,

$$x \in N_i^\infty(Q_1) \iff \begin{cases} x_k \in [-l_i, T + l_i] & \forall k \in I_1 \cup I_2 \\ |x_k| \leq l_i & \forall k \in I_3 \end{cases} \quad (5)$$

$$x \in N_{i-1}^\infty(Q) \iff \begin{cases} x_k \in [-l_{i-1}, T + l_{i-1}] & \forall k \in I_1 \\ |x_k| \leq l_{i-1} & \forall k \in I_2 \cup I_3 \end{cases}$$

Then by combining the above with the fact that $l_i < l_{i-1}$, we have

$$x \in N_i^\infty(Q_1) \cap N_{i-1}^\infty(Q) \iff \begin{cases} x_k \in [-l_i, T + l_i] & \forall k \in I_1 \\ x_k \in [-l_i, l_{i-1}] & \forall k \in I_2 \\ |x_k| \leq l_i & \forall k \in I_3 \end{cases} \quad (6)$$

Case I: Q_1 and Q_2 Belong to The Same Subspace. Since Q_1 and Q_2 are contained in the same linear subspace of dimension i and they intersect, we can assume that $Q_2 = [0, T]^j \times [-T, 0]^{i-j} \times 0^{d-i}$. By definition,

$$x \in N_i^\infty(Q_2) \iff \begin{cases} x_k \in [-l_i, T + l_i] & \forall k \in I_1 \\ x_k \in [-T - l_i, l_i] & \forall k \in I_2 \\ |x_k| \leq l_i & \forall k \in I_3 \end{cases}$$

By (5) and (6), we have that for every $x \in N_i^\infty(Q_1) \setminus N_{i-1}^\infty(Q)$, there exists $p \in I_2$ such that $x_p > l_{i-1}$. Moreover, for every $y \in N_i^\infty(Q_2)$, we have $y_p \leq l_i$. Thus

$$\text{dist}(x, y) = \|x - y\|_2 \geq |x_p - y_p| > l_{i-1} - l_i = \varepsilon$$

which indicates that

$$\text{dist}(N_i^\infty(Q_1) \setminus N_{i-1}^\infty(Q), N_i^\infty(Q_2) \setminus N_{i-1}^\infty(Q)) > \varepsilon.$$

By combining (4) and above, we have

$$\text{dist}(\widehat{Q}_1, \widehat{Q}_2) \geq \text{dist}(N_i^\infty(Q_1) \setminus N_{i-1}^\infty(Q), N_i^\infty(Q_2) \setminus N_{i-1}^\infty(Q)) > \varepsilon$$

Case II: Q_1 and Q_2 Belong to Different Subspaces. We then consider the case where Q_1 and Q_2 are contained in different subspaces. Let \mathcal{S}_1 and \mathcal{S}_2 be the i -dimensional linear subspaces containing Q_1 and Q_2 , respectively. By the definition of Q_1 and Q_2 , we know that \mathcal{S}_1 and \mathcal{S}_2 are not identical, and each of them can be spanned by i vectors from the orthonormal basis $\{e_1, e_2, \dots, e_d\}$ of the Euclidean space. We start with the following Lemma 3.11 which proves the distance lower bound between the whole *subspaces* (instead of bounded subsets that we need to deal with).

Lemma 3.11. *Suppose $0 \leq i \leq d-1$ is an integer. Let $\mathcal{S}_1, \mathcal{S}_2$ be two different i -dimensional linear subspaces, where both \mathcal{S}_1 and \mathcal{S}_2 can be spanned by i vectors from the orthonormal basis $\{e_1, e_2, \dots, e_d\}$ of the Euclidean space. Let $\mathcal{I} = \mathcal{S}_1 \cap \mathcal{S}_2$. Define $\mathcal{S}'_1 := N_i^\infty(\mathcal{S}_1) \setminus N_{i-1}^\infty(\mathcal{I})$, and define \mathcal{S}'_2 similarly. Then $\text{dist}(\mathcal{S}'_1, \mathcal{S}'_2) \geq \sqrt{2}\varepsilon$.*

Proof. W.l.o.g., suppose \mathcal{S}_1 is spanned by e_1, \dots, e_i and \mathcal{S}_2 is spanned by $e_1, \dots, e_j, e_{i+1}, \dots, e_d$, where $0 \leq j < i$, and $d = 2i - j$. Then $\mathcal{I} = \mathcal{S}_1 \cap \mathcal{S}_2$ is spanned by e_1, \dots, e_j .

Let $x = \sum_{k=1}^d x_k e_k$. Partition $\{1, \dots, d\}$ into three parts as follows: $I_1 = \{1, \dots, j\}$, $I_2 = \{j+1, \dots, i\}$, $I_3 = \{i+1, \dots, d\}$. Then

$$x \in N_{i-1}^\infty(\mathcal{I}) \iff |x_k| \leq l_{i-1}\varepsilon, \quad \forall k \in I_2 \cup I_3.$$

Similarly,

$$x \in N_i^\infty(\mathcal{S}_1) \iff |x_k| \leq l_i\varepsilon, \quad \forall k \in I_3,$$

and

$$x \in N_i^\infty(\mathcal{S}_2) \iff |x_k| \leq l_i\varepsilon, \quad \forall k \in I_2.$$

It follows that $x \in \mathcal{S}'_1 = N_i^\infty(\mathcal{S}_1) \setminus N_{i-1}^\infty(\mathcal{I})$ if and only if $|x_k| \leq l_i\varepsilon$ for every $k \in I_3$, and $\exists p \in I_2$ such that $|x_p| > l_{i-1}\varepsilon$. Also, $x \in \mathcal{S}'_2 = N_i^\infty(\mathcal{S}_2) \setminus N_{i-1}^\infty(\mathcal{I})$ if and only if $|x_k| \leq l_i\varepsilon$ for every $k \in I_2$, and $\exists q \in I_3$ such that $|x_q| > l_{i-1}\varepsilon$.

For every $x = \sum_{k=1}^d x_k e_k \in \mathcal{S}'_1$ and $y = \sum_{k=1}^d y_k e_k \in \mathcal{S}'_2$, there exists $p \in I_2$ such that $|x_p| > l_{i-1}\varepsilon$, and $q \in I_3$ such that $|y_q| > l_{i-1}\varepsilon$. Note that $|y_p| \leq l_i\varepsilon$, and $|x_q| \leq l_i\varepsilon$. Therefore, we have

$$\text{dist}(x, y) = \|x - y\|_2 \geq \|(x_p - y_p)e_p + (x_q - y_q)e_q\|_2 \geq \sqrt{2}\varepsilon$$

where the first and the second inequalities are derived from the fact that $\{e_1, \dots, e_d\}$ is orthonormal basis. This completes the proof of Lemma 3.11. \square

Let $\mathcal{I} := \mathcal{S}_1 \cap \mathcal{S}_2$. In order to apply Lemma 3.11, we need the next Claim 3.12 to relate the intersection of subspaces \mathcal{I} to Q .

Claim 3.12. *For all $t \in \{1, 2\}$,*

$$N_i^\infty(Q_t) \setminus N_{i-1}^\infty(Q) = N_i^\infty(Q_t) \setminus N_{i-1}^\infty(\mathcal{I}). \quad (7)$$

Proof. Since the cases for $t = 1$ and $t = 2$ are symmetric, we only prove for $t = 1$, and $t = 2$ follows similarly. Recall that $Q_1 = [0, T]^i \times 0^{d-i}$ and $Q = [0, T]^j \times 0^{d-j}$. Thus \mathcal{S}_1 is spanned by e_1, \dots, e_i and \mathcal{I} is spanned by e_1, \dots, e_j . By definition,

$$x \in N_{i-1}^\infty(\mathcal{I}) \iff \forall k \in I_2 \cup I_3, |x_k| \leq l_{i-1} \quad (8)$$

Then by (5), (8) and the fact that $l_i < l_{i-1}$, we have

$$x \in N_i^\infty(Q_1) \cap N_{i-1}^\infty(\mathcal{I}) \iff \begin{cases} x_k \in [-l_i, T + l_i] & \forall k \in I_1 \\ x_k \in [-l_i, l_{i-1}] & \forall k \in I_2 \\ |x_k| \leq l_i & \forall k \in I_3 \end{cases}$$

which is the same conditions for the set $N_i^\infty(Q_1) \cap N_{i-1}^\infty(Q)$ (see (6)). Thus $N_i^\infty(Q_1) \cap N_{i-1}^\infty(Q) = N_i^\infty(Q_1) \cap N_{i-1}^\infty(\mathcal{I})$, and $N_i^\infty(Q_1) \setminus N_{i-1}^\infty(Q) = N_i^\infty(Q_1) \setminus N_{i-1}^\infty(\mathcal{I})$. This concludes the proof of Claim 3.12. \square

Finally, by combining (4) and Claim 3.12, we have

$$\begin{aligned} \text{dist}(\widehat{Q}_1, \widehat{Q}_2) &\geq \text{dist}(N_i^\infty(Q_1) \setminus N_{i-1}^\infty(\mathcal{I}), N_i^\infty(Q_2) \setminus N_{i-1}^\infty(\mathcal{I})) \\ &\geq \text{dist}(N_i^\infty(\mathcal{S}_1) \setminus N_{i-1}^\infty(\mathcal{I}), N_i^\infty(\mathcal{S}_2) \setminus N_{i-1}^\infty(\mathcal{I})) \\ &> \varepsilon \end{aligned}$$

where the last inequality is by Lemma 3.11. This completes the proof of Lemma 3.10. \square

Lemma 3.13 (Analysis for $i = d$). *Let Q_1, Q_2 be two distinct hypercubes in the partition, and $\widehat{Q}_1, \widehat{Q}_2$ be the corresponding remaining parts constructed in step 3. Then $\text{dist}(\widehat{Q}_1, \widehat{Q}_2) > \varepsilon$.*

Proof. It suffices to show that for every $x \in \widehat{Q}_1$ and $y \in \widehat{Q}_2$, $\text{dist}(x, y) > \varepsilon$. By the construction, we know that $\text{dist}(x, \partial Q_1) \geq l_{d-1} = \varepsilon$, and similarly $\text{dist}(y, \partial Q_2) \geq \varepsilon$, where for a hypercube H , $\partial H \subseteq \mathbb{R}^d$ is the boundary of H .

Let $\text{seg}(x, y)$ be the segment with endpoints x, y . Pick $x' \in \partial Q_1 \cap \text{seg}(x, y)$ such that x' is closest to x , and pick $y' \in \partial Q_2 \cap \text{seg}(x, y)$ such that y' is closest to y . Then,

$$\text{dist}(x, y) \geq \text{dist}(x, x') + \text{dist}(y, y') \geq \text{dist}(x, \partial Q_1) + \text{dist}(y, \partial Q_2) \geq 2\varepsilon,$$

where the first inequality follows from the fact that Q_1 and Q_2 have disjoint interior. This completes the proof of Lemma 3.13. \square

Proof of Lemma 3.6. By Lemma 3.10, Lemma 3.13 and the structure of the hypercubes, we know that for every subset $S \subseteq \mathbb{R}^d$ with $\text{diam}(S) \leq \varepsilon$, $\varphi(S)$ can intersect at most one ε -neighborhood (which is defined with respect to ℓ_∞) of an i -dimensional face for every $0 \leq i \leq d$, since otherwise the diameter constraint of S would be violated. Therefore, $|\varphi(S)| \leq d+1$. To bound the diameter of the buckets, we note that for every \widehat{Q} in the construction, we have $\text{diam}(\widehat{Q}) < \sqrt{d} \cdot \max\{O(d\varepsilon), T\} \leq \ell$. Finally, it is immediate that one can evaluate $\varphi(x)$ at an input $x \in \mathbb{R}^d$ in time and space $\text{poly}(d)$. This completes the proof of Lemma 3.6. \square

4 Streaming Algorithms

4.1 A Two-Pass $O(1)$ -Approximation in Dynamic Streams

Theorem 4.1. *There is a two-pass randomized algorithm that computes an $O(1)$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a dynamic geometric stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

Proof sketch. Observe that Theorem 3.1 samples every point $p^* \in P$ with probability proportional to r_{p^*} up to a $\text{poly}(d \cdot \log \Delta)$ factor. Hence, the estimator $\widehat{Z} = 1/\widehat{\text{Pr}}[p^*] \cdot r_{p^*}$ has expectation $\mathbb{E}[\widehat{Z}] = \Theta(1) \cdot \sum_{p \in P} r_p$, noting that $\widehat{\text{Pr}}[p^*]$ returned by Theorem 3.1 is a 2-approximation to the true sampling probability $\text{Pr}[p^*]$. (If the algorithm of Theorem 3.1 returns \perp , we simply set $\widehat{Z} = 0$.) Moreover, a standard calculation of the variance of an importance sampler shows that $\text{Var}[\widehat{Z}] \leq \text{poly}(d \cdot \log \Delta) \cdot \mathbb{E}^2[\widehat{Z}]$. Therefore, averaging over $\text{poly}(d \cdot \log \Delta)$ independent samples of estimator \widehat{Z} , one obtains an $O(1)$ -approximate estimator with constant probability (by a standard application of Chebyshev's inequality).

Hence, the complete algorithm is straightforward: We draw $\text{poly}(d \cdot \log \Delta)$ independent samples S using Theorem 3.1 in the first pass, and in the second pass, we estimate the r_p value for every sampled $p \in S$ using Fact 2.3. \square

4.2 A One-pass $O(1)$ -Approximation in Random-order Streams

Theorem 4.2. *There is a one-pass randomized algorithm that computes an $O(1)$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a random-order insertion-only stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

Recall that in the proof of Theorem 4.1, we use the importance-sampler of Theorem 3.1 in the first pass, while in the second pass we estimate the r_p value for the sampled points p . We plan to adapt this two-pass algorithm to the random-order setting in a natural way. Roughly, we run the sampler in the first half of the stream, and we estimate the r_p values using the second half of the stream. We formalize this in Algorithm 2.

Algorithm 2 Importance sampling in a random-order stream

```

1: procedure ONE-SAMPLE
2:   let  $J$  denote the first half of the stream, and  $K$  the second half
3:   obtain a sample  $x^J \in J$  using Theorem 3.1 on  $J$  together with the corresponding estimated
     probability  $\widehat{\Pr}[x^J]$ 
4:   obtain an estimate  $\widehat{r}_{x^J}^K$  of  $r_{x^J}^K$  using Fact 2.3 on  $K \cup \{x\}$ 
5:   return  $1/\widehat{\Pr}[x^J] \cdot \widehat{r}_{x^J}^K$   $\triangleright$  if  $x^J = \perp$ , return 0
6: end procedure
7: procedure MAIN
8:    $m \leftarrow \text{poly}(d \cdot \log \Delta)$ 
9:   for  $i = 1, \dots, m$  do
10:    run ONE-SAMPLE to obtain  $\widehat{Z}_i$ 
11:   end for
12:   return  $\widehat{Z} = \frac{1}{m} \cdot \sum_i^m \widehat{Z}_i$ 
13: end procedure

```

Analysis. We give a constant probability upper and lower bounds on the estimator \widehat{Z} returned by Algorithm 2 separately. Let OPT^J and OPT^K denote the optimal UFL cost for point sets J and K , respectively; recall that OPT is the optimal cost for the whole point set P . Moreover, for any point x we define r_x^J and r_x^K similarly by restricting Definition 2.1 to sets J and K , respectively, with one adjustment: When computing r_x^K (resp. r_x^J) for $x \notin K$ (resp. $x \notin J$), we still take x into account. This is to ensure $r_x^K > 0$ is well defined (since otherwise $B(x, f) \cap K$ may be empty).

Before we proceed, we use the following interpretation of the randomness in the entire algorithm.

- We interpret the random permutation of the stream as first randomly partitioning P into the two halves J and K , and then applying a uniformly random permutation on each of J and K independently. Let JK denote the randomness of the partition, and let π_J and π_K denote the randomness of the permutations of J and K , respectively.
- We denote the randomness of the algorithm AG. Note that Algorithm 2 uses randomness independent to the random permutation of the stream, so AG is independent to JK, π_J, π_K . Since Fact 2.3 gives a deterministic estimation of the r_p values, the randomness of AG

comes solely from the importance-sampler (Theorem 3.1) running on J . In our analysis, we condition on the success of all m instances of Theorem 3.1, which happens with probability $1 - 1/\text{poly}(\Delta^d)$ by the union bound. Therefore, we have that for every point x^J sampled by Theorem 3.1, $\Pr[x^J] \leq \widehat{\Pr}[x^J] \leq 2\Pr[x^J]$. (Here, we write the notation $\Pr[p^* = x^J]$ from Theorem 3.1 as $\Pr[x^J]$ for short.)

Upper Bound. We upper bound the expectation of \widehat{Z} by $O(\text{OPT})$ in Lemma 4.3. Then a standard application of Markov's inequality implies

$$\Pr[\widehat{Z} \geq \Omega(\text{OPT})] \geq \frac{5}{6}.$$

Lemma 4.3. *Let $Z = 1/\widehat{\Pr}[x^J] \cdot \widehat{r}_{x^J}^K$ be an estimate returned by ONE-SAMPLE. Then $\mathbb{E}[Z] \leq O(1) \cdot \text{OPT}$.*

Proof. Using that $\widehat{r}_{x^J}^K \leq O(r_{x^J}^K)$ and $\widehat{\Pr}[x] \geq \Pr[x]$ (since we condition on the success of Theorem 3.1), we bound

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}_{\text{JK}} \left[\mathbb{E}_{\text{AG}} [1/\widehat{\Pr}[x^J] \cdot \widehat{r}_{x^J}^K \mid \text{JK}, \pi_J, \pi_K] \right] \\ &\leq O(1) \cdot \mathbb{E}_{\text{JK}} \left[\mathbb{E}_{\text{AG}} [1/\Pr[x^J] \cdot r_{x^J}^K \mid \text{JK}, \pi_J, \pi_K] \right] \\ &= O(1) \cdot \mathbb{E}_{\text{JK}} \left[\sum_{x \in J} r_x^K \right] \\ &\leq O(1) \cdot \mathbb{E}_{\text{JK}} \left[\sum_{x \in P} r_x^K \right] \\ &= O(1) \cdot \sum_{x \in P} \mathbb{E}_{\text{JK}} [r_x^K]. \end{aligned}$$

It is sufficient to upper bound $\mathbb{E}_{\text{JK}}[r_x^K]$ by $O(r_x)$ for every $x \in P$, since $\sum_{x \in P} r_x = O(\text{OPT})$ by Fact 2.2. Fix some $x \in P$. By Fact 2.2, we have $|B(x, r_x) \cap P| \geq \mathfrak{f}/r_x$. Let $Y = |(B(x, r_x) \cap K) \cup \{x\}|$ be the number of points in $B(x, r_x)$ that are in the second half of the stream (x is always included in Y). Note that $r_x^K \leq O(\mathfrak{f}/Y)$ (possibly less due to points outside $B(x, r_x)$, which we do not take into account). Thus, by bounding $\mathbb{E}[\mathfrak{f}/Y]$ we get the desired result. In the remaining part of the proof, the expectation is over the randomness of JK only.

To this end, for $u \in B(x, r_x) \cap P \setminus \{x\}$ let Y_u be the indicator random variable equal to 1 if and only if $u \in K$. We have $\mathbb{E}[Y_u] = 1/2$ and $Y = \sum_{u \in B(x, r_x) \cap P} Y_u$. Thus, by linearity of expectation $\mathbb{E}[Y] \geq |B(x, r_x) \cap P|/2 \geq \mathfrak{f}/(2r_x)$. Furthermore, variables Y_u are negatively associated, so Chernoff bound still applies [DR98], and we have

$$\Pr[Y \leq \mathfrak{f}/(4r_x)] \leq \Pr[Y \leq 0.5 \mathbb{E}[Y]] \leq \exp(-\Theta(\mathbb{E}[Y])) = \exp(-\Theta(\mathfrak{f}/r_x)).$$

We use this to bound the expectation of \mathfrak{f}/Y as follows:

$$\begin{aligned} \mathbb{E}[\mathfrak{f}/Y] &= \sum_{y \geq 1} \Pr[Y = y] \cdot \frac{\mathfrak{f}}{y} \leq O(r_x) + \Pr[Y \leq \mathfrak{f}/(4r_x)] \cdot \sum_{1 \leq y \leq \mathfrak{f}/(4r_x)} \frac{\mathfrak{f}}{y} \\ &\leq O(r_x) + \mathfrak{f} \cdot \exp(-\Theta(\mathfrak{f}/r_x)) \cdot O(\log(\mathfrak{f}/r_x)) = O(r_x). \end{aligned}$$

It follows that $\mathbb{E}_{\text{JK}}[r_x^K] = O(r_x)$, which concludes the proof. \square

Lower Bound. Consider $Z = 1/\widehat{\Pr}[x^J] \cdot \widehat{r}_{x^J}^K$ as returned by ONE-SAMPLE. Let

$$Z' := 1/\Pr[x^J] \cdot \min\{r_{x^J}^J, r_{x^J}^K\}.$$

Clearly, $Z \geq Z'/2$ with probability 1, since by Fact 2.3, $\widehat{r}_{x^J}^K \geq r_{x^J}^K \geq r_{x^J}$ and since we condition on the event that $\widehat{\Pr}[x^J] \leq 2\Pr[x^J]$. Therefore, we focus on showing that Z' is $\Omega(\text{OPT})$ with high constant probability, and this implies Z is also $\Omega(\text{OPT})$ with high constant probability.

The first step is to observe that by Definition 2.1,

$$\forall x \in P, \quad \min\{r_x^J, r_x^K\} \geq r_x. \quad (9)$$

We now show that OPT^J is not much smaller than OPT with high probability

Lemma 4.4. $\text{OPT}^J \geq \Omega(\text{OPT})$ with high constant probability (over the randomness of π_J, π_K, JK).

Proof. If $\text{OPT} \leq O(f)$, then the bound holds with probability 1, since any feasible solution of J must open at least one facility, which implies $\text{OPT}^J \geq f \geq \Omega(\text{OPT})$.

Now, assume $\text{OPT} \geq \Omega(f)$. It holds that $\text{OPT}^J = \Theta(1) \cdot \sum_{x \in J} r_x^J \geq \Theta(1) \cdot \sum_{x \in J} r_x$ by Fact 2.2 and (9). For $u \in P$, let X_u be the indicator variable such that $X_u = 1$ if and only if $u \in J$; by the definition of J , we have $\mathbb{E}[X_u] = 0.5$. Let $X := \sum_{u \in P} X_u \cdot r_u / f$. Then $\sum_{x \in J} r_x = f \cdot X$, and $f \cdot \mathbb{E}[X] = \Theta(1) \cdot \text{OPT}$. Observe that X_u 's are negatively associated, and we can still apply Chernoff bounds [DR98]. Therefore,

$$\begin{aligned} \Pr[\text{OPT}^J \leq O(1) \cdot \text{OPT}] &\leq \Pr[f \cdot X - f \cdot \mathbb{E}[X] < -0.5 \cdot f \cdot \mathbb{E}[X]] \\ &\leq \exp(-\Theta(1) \cdot \mathbb{E}[X]) \\ &\leq \exp(-\Theta(1) \cdot \text{OPT} / f) \\ &\leq O(1), \end{aligned}$$

where the first step holds for a small enough constant hidden in $O(1)$ and the last step follows from the assumption that $\text{OPT} \geq \Omega(f)$. \square

Now, we condition on the event \mathcal{E} that $\text{OPT}^J \geq \Omega(\text{OPT})$, which happens with high probability by Lemma 4.4. Thus, in the remaining part we fix the randomness of JK, π_J, π_K such that event \mathcal{E} happens. Then by (9) we get

$$\mathbb{E}_{\text{AG}}[Z'] \geq \mathbb{E}_{\text{AG}}[1/\Pr[x^J] \cdot r_{x^J}] = \mathbb{E}_{\text{AG}}[\text{OPT}^J] = \Omega(\text{OPT}).$$

By Theorem 3.1,

$$\begin{aligned} \mathbb{E}_{\text{AG}}[Z'^2] &= \mathbb{E}_{\text{AG}}[1/(\Pr[x^J])^2 \cdot (\min\{r_{x^J}^J, r_{x^J}^K\})^2] \\ &= \sum_{x \in J} 1/\Pr[x^J] \cdot (\min\{r_x^J, r_x^K\})^2 \\ &\leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}^J \cdot \sum_{x \in J} (\min\{r_x^J, r_x^K\})^2 / r_x^J \\ &\leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}^J \cdot \sum_{x \in J} r_x^J \\ &= \text{poly}(d \cdot \log \Delta) \cdot (\text{OPT}^J)^2 \\ &\leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}^2 \end{aligned}$$

Hence, conditioned on \mathcal{E} , ONE-SAMPLE is a low-variance sampler. A straightforward application of Chebyshev's inequality implies that Z' is at least $\Omega(\text{OPT})$ with probability 5/6. As $Z \geq Z'$ with probability 1, the same holds for the estimator \widehat{Z} returned by the final MAIN procedure in Algorithm 2. \square

4.3 A One-Pass $O(d^{1.5})$ -Approximation in Arbitrary-Order Streams

Theorem 4.5. *If for some $0 < \Gamma \leq \text{poly}(d)$ there exists a $(\Gamma, \text{poly}(d))$ -hash, then there is a one-pass randomized algorithm that computes an $O(\Gamma)$ -approximation to Uniform Facility Location of an input $P \subseteq [\Delta]^d$ presented as a dynamic geometric stream, using $\text{poly}(d \cdot \log \Delta)$ bits of space.*

Given the successful application of the importance sampling algorithm in two passes or in one pass over a random-order stream, a natural idea is to attempt to implement this approach in one pass over an arbitrary-order stream as well. However, recall that a crucial step in the importance sampling is to estimate the r_p value for some p sampled by the algorithm, and we managed to do so by using Fact 2.3 for the random-order and two-pass settings. However, Fact 2.3 does not apply to the one-pass arbitrary-order setting, since it only works in the case when p is given in advance, but in our case p is given only after processing the stream. In other words, after processing the stream one has to answer an estimate r_p for a *query* point p , with no foreknowledge of this point. Indeed, answering this query within any constant factor c requires $\Omega(n)$ space, even for 1D, which follows by a reduction from the one-way communication problem of INDEX; see Appendix B for details.

Hence, we turn to a more structured estimator. In particular, to estimate $\sum_{p \in P} r_p$ (which gives an $O(1)$ -approximation for Uniform Facility Location by Fact 2.2), we consider $i = 0, \dots, L$ and let $W_i := \sum_{p \in P_i} r_p$. We focus on estimating W_i separately for each i . Recall that $\text{sub}_i(P)$ samples each point in P independently with probability 2^{-i} . Since $W_i = \sum_{p \in P_i} r_p = \Theta(\mathfrak{f} \cdot |P_i|/2^i)$, the number of points $p \in \text{sub}_i(P)$ with $r_p \in P_i$ is a nearly-unbiased estimator for W_i/\mathfrak{f} . Therefore, our plan is to estimate $|\text{sub}_i(P_i)|$.

To this end, we use ideas from Algorithm 1 which defines an importance sampling on level i . We alternatively interpret it as sampling from $\text{sub}_i(P)$, and it can be seen that an estimator $Z = 1/\Pr[x] \cdot I(x \in P_i)$ is an unbiased estimator for $|\text{sub}_i(P_i)|$. However, the estimator may not be well-concentrated around the expectation when W_i is too small (compared with OPT). In case this happens, we argue that the estimator is also very small in the absolute sense (i.e., compared to OPT and not relative to the expectation), with high probability. This turns out to be sufficient for our purpose, since in such a case W_i contributes very little to OPT anyway (which means we can ignore it), and the small value of the estimator actually correctly reflects this. We formalize this idea in Lemma 4.6. Recall that $L := d \cdot \log_2 \Delta$.

Lemma 4.6. *Let κ be such that $\mathfrak{f} \cdot \kappa = \Theta(\text{OPT}/L^3)$ and suppose that $\text{OPT} = \Omega(L^4) \cdot \mathfrak{f}$. Then for every $0 \leq k \leq L$, each of the following holds with probability at least $1 - 1/\text{poly}(\Delta^d)$.*

- If $2^{k-i} \cdot W_k \geq \kappa \mathfrak{f}$, then $\mathfrak{f} \cdot |\text{sub}_i(P_k)| = \Theta(2^{k-i}) \cdot W_k$;
- Otherwise, $\mathfrak{f} \cdot |\text{sub}_i(P_k)| \leq O(\text{OPT}/L^2)$.

Proof. For $u \in P_k$, let $X_u \in \{0, 1\}$ be the indicator random variable such that $X_u = 1$ if and only if $u \in \text{sub}_i(P_k)$, so $\Pr[X_u = 1] = 2^{-i}$ for every $u \in P_k$. Then $|\text{sub}_i(P_k)| = \sum_{u \in P_k} X_u$, and $\mathbb{E}[|\text{sub}_i(P_k)|] = |P_k| \cdot 2^{-i}$.

Let $\mu := \mathbb{E}[|\text{sub}_i(P_k)|]$. By the definition of P_k , every $p \in P_k$ satisfies $2^{-k} \mathfrak{f} < r_p \leq 2^{-k+1} \mathfrak{f}$. Hence $\frac{|P_k|}{2^k} \mathfrak{f} < W_k = \sum_{p \in P_k} r_p \leq \frac{|P_k|}{2^{k-1}} \mathfrak{f}$, and this implies

$$\mu = \Theta(2^{k-i}) \cdot W_k / \mathfrak{f}. \quad (10)$$

- If $2^{k-i} \cdot W_k \geq \mathfrak{f} \cdot \kappa$, then $\mu \geq \Omega(\kappa) \geq \Omega(d \cdot \log \Delta)$ by (10) and $\text{OPT} = \Omega(L^4) \cdot \mathfrak{f}$. We apply the Chernoff bound to get,

$$\Pr[|\text{sub}_i(P_k)| \in (1 \pm 0.5) \cdot \mu] \geq 1 - \exp(-\Omega(\kappa)) = 1 - 1/\text{poly}(\Delta^d).$$

- Otherwise, $\mu \leq O(\kappa)$, so $|P_k| \leq O(\kappa) \cdot 2^i$. Using that $O(d \cdot \log \Delta) \cdot \mathfrak{f} \cdot \kappa = O(\text{OPT}/L^2)$, we apply Fact 3.4 to P_k and get,

$$\Pr[\mathfrak{f} \cdot |\text{sub}_i(P_k)| \leq O(\text{OPT}/L^2)] \geq 1 - 1/\text{poly}(\Delta^d).$$

This concludes the proof of Lemma 4.6. \square

Another more severe issue is that the estimator $I(x \in P_i)$ turns out to be very sensitive, and it seems difficult to estimate its value even up to a constant factor (say, test if $r_x \geq 2^{-i}\mathfrak{f}/100$), in one pass. To bypass this, we design an approximate tester for $x \in P_i$, and analyze how such an approximation affects the ratio.

Definition 4.7 (Approximate Tester). We say that $\hat{I}_{\leq i} : P \rightarrow \{0, 1\}$ is an α -approximate tester for level i if for all points $p \in P$ it holds that,

- if $p \in P_{\leq i}$, then $\hat{I}_{\leq i}(p) = 1$, and
- if $\hat{I}_{\leq i}(p) = 1$, then $p \in P_{\leq i+\alpha}$,

where $P_{\leq i} = \bigcup_{j \leq i} P_j$.

Note that $P_{\leq i}$ corresponds to points p with r_p value larger than $2^{-i}\mathfrak{f}$, and this approximate tester tolerates points with a not too large value of r_p . This is not a problem since for $j \geq i$, $\mathfrak{f} \cdot |\text{sub}_i(P_j)| \approx W_j/2^{i-j}$ and taking this into the estimation would result in an $O(1)$ times more contribution of W_j overall (due to summing the geometric sequence over $i \leq j$).

To implement this tester, we observe that by the definition of the r_p value (Definition 2.1), if a point x has $r_x \approx 2^{-i}\mathfrak{f}$, then a point y with $r_y \ll 2^{-i}\mathfrak{f}$ cannot appear in the ball $B(x, r_x/5)$, since otherwise by Fact 2.2, $|B(x, r_x/2) \cap P| \leq O(2^i)$ while $B(y, r_y) \subseteq B(x, r_x/5)$ and $|B(y, r_y) \cap P| \gg 2^i$. Therefore, we can assert that a bucket cannot contain any point p with a large enough r_p value if the number of points inside is too large (the buckets are those defined by the $(\Gamma, \text{poly}(d))$ -hash φ_i , where we set the diameter bound $\ell_i = 2^{-i}\mathfrak{f}/10$). This also means a bucket either consists of points with large r_p values only, or none at all. Using this observation, for each bucket we maintain a counter for points inside. When some point is sampled, we also retrieve this counter, and see if it is not too large, in order to test if the bucket consists of points with large enough r_p values.

However, it is possible that some point y with $r_y \ll 2^{-i}\mathfrak{f}$ lies inside the bucket, but the number of points inside the bucket is still small as the majority of the large number of points in $B(y, r_y) \cap P$ actually lie outside of the bucket. To resolve this issue, we observe that by a similar reasoning as in the previous paragraph, if a bucket contains a point x with $r_x \approx 2^{-i}\mathfrak{f}$, no point y with $r_y \ll 2^{-i}\mathfrak{f}$ may be *close* to the bucket with x . Hence, we actually need to count the number of points in a slightly enlarged area, i.e., for a bucket $Q \subseteq \mathbb{R}^d$ we need to count the points in the neighborhood $B(Q, r_y) \cap P$. Hence, if we maintain the counter for $B(Q, \beta) \cap P$ for some $\beta > 0$ then we can reject y 's with $r_y \ll O(\beta)$.

Now, suppose we are to maintain the counter for some β . To implement this, whenever we see a data point x arrives, we should increase the counter for all buckets Q such that $x \in B(Q, \beta)$. However, this becomes challenging in the streaming setting, primarily due to the fact that the number of buckets Q such that $x \in B(Q, \beta)$ can be huge, and more importantly, most of them may be “fake”, in that they do not contain any data point. Consequently, these fake nonempty buckets can enlarge the support of the ℓ_0 -sampler significantly, which makes it difficult to obtain a uniform sample from the non-empty buckets that are not “fake”.

To tackle this challenge, we make use of the guarantee from the geometric hashing, that any subset of \mathbb{R}^d with diameter less than $\varepsilon_i := \ell_i/\Gamma$ is mapped to $\text{poly}(d)$ buckets (see Definition 1.6). Hence, if we choose $\beta = \varepsilon_i/2$, the effect of enlarging the buckets by an additive $\beta = \varepsilon_i/2$ is essentially making every data point x a ball $B(x, \varepsilon_i/2)$, and add the image of this ball to the buckets/counters.

We implement these algorithmic steps using an extended ℓ_0 -sampler stated in Lemma 4.8 (this extension is somewhat simpler than that of the two-level ℓ_0 -sampler in Lemma 3.3). We show in Lemma 4.9 that we can still bound the number of nonempty buckets, even including the fake ones, by $\text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$, which is a strengthened version of Lemma 3.8. This is needed to have a bounded support for the ℓ_0 -samplers.

Lemma 4.8 (ℓ_0 -Sampler with Data Fields). *Let X be a length- n frequency vector, with the maximum frequency bounded by $\text{poly}(n)$ and with each index also associated with a data field of at most $\text{poly}(\log n)$ bits. Suppose X is presented in a dynamic data stream consisting of (positive or negative) updates to the frequency entries of X , and with updates to the associated data fields. Then, there exists a streaming algorithm that for every such X , returns an index i of X , together with its frequency and data field, such that i is chosen uniformly at random (u.a.r.) among indices with non-zero frequency or non-zero data field, using space $\text{poly}(\log n)$ and with success probability at least $1 - 1/\text{poly}(n)$.*

Proof sketch. The lemma follows directly from the guarantees of a standard ℓ_0 -sampler (see Lemma A.1) by only viewing the multiplicity of a point summarized by the ℓ_0 -sampler as a pair of frequency and data field. \square

With this, we state the complete algorithm in three parts: Algorithm 3 describes the initialization procedure before processing the stream and Algorithm 4 outlines the procedure to insert a new point (one can remove a point in the same way, just by decreasing instead of increasing the frequencies and counters). After the streams ends, we call Algorithm 5 that returns an estimate \hat{Z} for the Uniform Facility Location cost.

We note that $\{C_a^{(t)}\}_t$'s are independent counters for the points lying in the enlarged buckets. We show that each $\{C_a^{(t)}\}_t$ has a suitable expectation to yield an approximate tester, and we use T of them to implement the median trick to boost the success probability of an accurate estimation. A few more remarks regarding our algorithm are in order:

- In line 5 of Algorithm 3 we need to initialize T fully random hash functions $h_i^{(t)} : [\Delta]^d \rightarrow \{0, 1\}$. Each of them would require $[\Delta]^d$ bits to store, however, we use Nisan's pseudorandom generator (PRG) [Nis92] for space bounded computation, in the same way as described in Section 3.
- In Algorithm 5, we made one simplification: In line 8 we need $|\mathcal{L}_i^{(j)}|$, the size of the support of $\mathcal{L}_i^{(j)}$, which we cannot obtain exactly but an $O(1)$ -approximation can be computed in streaming (see e.g. [KNW10]) and is sufficient as it introduces only a small constant factor error in the analysis.

Support Size of ℓ_0 -Samplers at Level i . To analyze our algorithm, we first bound the number of buckets of φ_i that are either non-empty (i.e., have non-zero frequency) or have a non-zero counter.

Lemma 4.9. *For every level i and any $j = 1, \dots, m$, with probability at least $1 - 1/\text{poly}(\Delta^d)$, $\mathfrak{f} \cdot |\mathcal{L}_i^{(j)}| \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$.*

Algorithm 3 One-pass streaming algorithm: initialization

- 1: **for** $i = 0, \dots, L$ **do**
- 2: precompute $(\Gamma, \text{poly}(d))$ -hash $\varphi_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ using Lemma 3.6, with diameter bound $\ell_i = 2^{-i}\mathfrak{f}/10$ \triangleright define $\varepsilon_i := \ell/\Gamma$
- 3: set $m \leftarrow \text{poly}(d \cdot \log \Delta)$
- 4: set $T \leftarrow O(d \cdot \log \Delta)$
- 5: for $0 \leq t \leq T$, let $h_i^{(t)} : [\Delta]^d \rightarrow \{0, 1\}$ be a fully random hash function such that

$$\forall x \in [\Delta]^d, \quad \Pr[h_i^{(t)}(x) = 1] = 2^{-i}$$

- 6: for $0 \leq t \leq T$, $S \subseteq [\Delta]^d$, define $\text{sub}_i^{(t)}(S) \leftarrow \{x \in S : h_i^{(t)}(x) = 1\}$
 - 7: let $\text{sub}_i \leftarrow \text{sub}_i^{(0)}$
 - 8: initialize m extended ℓ_0 -samplers of Lemma 4.8 on domain $\varphi_i([\Delta]^d)$ and with T counters $\{C^{(t)}\}_{t=1}^T$ as the data field of each element; we denote them $\mathcal{L}_i^{(j)}$ for $j = 1, \dots, m$.
 - 9: **end for**
-

Algorithm 4 One-pass streaming algorithm: insert $p \in [\Delta]^d$

- 1: **for** $i = 0, \dots, L$ **do**
 - 2: **for** $j = 1, \dots, m$ **do**
 - 3: **if** p survives subsampling sub_i , i.e., $h_i^{(0)}(p) = 1$ **then** \triangleright sub_i subsamples with rate 2^{-i}
 - 4: increase the frequency of $\varphi_i(p)$ in $\mathcal{L}_i^{(j)}$
 - 5: **end if**
 - 6: **for** $u \in \varphi_i(B(p, \varepsilon_i/2))$ **do**
 - 7: for $1 \leq t \leq T$, increase the counter $C^{(t)}$ associated with u in $\mathcal{L}_i^{(j)}$ by $h_i^{(t)}(p)$
 - 8: \triangleright we increase $C^{(t)}$ by 1 iff $h_i^{(t)}(p) = 1$
 - 9: **end for**
 - 10: **end for**
 - 11: **end for**
-

Proof. Fix $t = 0, \dots, T$. Define an extended dataset $\tilde{P}^{(t)} := \bigcup_{p \in \text{sub}_i^{(t)}(P)} B(p, \varepsilon_i/2) \subseteq \mathbb{R}^d$ consisting of neighborhoods of points subsampled by $\text{sub}_i^{(t)}$. Apply Lemma 3.7 on P and obtain a partition \mathcal{C} of P . It suffices to upper bound $|\varphi_i(\bigcup_{C \in \mathcal{C}} B(\text{sub}_i^{(t)}(C), \varepsilon_i/2))|$, since $\tilde{P}^{(t)} \subseteq \bigcup_{C \in \mathcal{C}} B(\text{sub}_i^{(t)}(C), \varepsilon_i/2)$. We show this upper bound for the fixed t ; the final upper bound on $|\mathcal{L}_i^{(j)}|$ will be by only $T = O(d \cdot \log \Delta)$ larger.

Examine each $C \in \mathcal{C}$. If $\text{diam}(C) \leq \varepsilon_i/2$, then $\text{diam}(B(C, \varepsilon_i/2)) \leq \varepsilon_i$. Hence, by the guarantee of Definition 1.6, $|\varphi_i(B(\text{sub}_i^{(t)}(C), \varepsilon_i/2))| \leq \text{poly}(d)$. Otherwise, $\text{diam}(C) > \varepsilon_i/2$. In this case, we use the guarantee of \mathcal{C} that $|C| \leq O(\mathfrak{f}/\text{diam}(C))$, so $|C| \leq O(\varepsilon_i^{-1}\mathfrak{f}) \leq O(2^i \cdot \Gamma) \leq O(2^i \cdot \text{poly}(d))$. By Fact 3.4, with probability at least $1 - 1/\text{poly}(\Delta^d)$, $|\text{sub}_i^{(t)}(C)| \leq \text{poly}(d \cdot \log \Delta)$. We apply the union bound and condition on the success of this for every $C \in \mathcal{C}$. The guarantee of the geometric hashing implies that for every $p \in \text{sub}_i^{(t)}(C)$, $|\varphi_i(B(p, \varepsilon_i/2))| \leq \text{poly}(d)$. We thus have

$$|\varphi_i(B(\text{sub}_i^{(t)}(C), \varepsilon))| \leq \text{poly}(d) \cdot |\text{sub}_i^{(t)}(C)| \leq \text{poly}(d \cdot \log \Delta).$$

Algorithm 5 One-pass streaming algorithm: query procedure

```

1: for  $i = 0, \dots, L$  do
2:   for  $j = 1, \dots, m$  do ▷ recall that  $m = \text{poly}(d \cdot \log \Delta)$ 
3:     query  $\mathcal{L}_i^{(j)}$ , let  $a$  be the sampled non-empty bucket of  $\varphi_i$ , and let  $n_a$  and  $\{C_a^{(t)}\}_{t=1}^T$  be
       the frequency and counters of  $a$  returned by  $\mathcal{L}_i^{(j)}$ , respectively
4:     let  $\hat{C}_a$  be the median of  $\{C_a^{(1)}, \dots, C_a^{(T)}\}$  ▷ recall that  $T = O(d \cdot \log \Delta)$ 
5:     if  $n_a = 0$  or  $\hat{C}_a > c$  for some constant  $c$  (to be determined from the analysis) then
6:       let  $\hat{Z}_i^{(j)} \leftarrow 0$ 
7:     else
8:       let  $\hat{Z}_i^{(j)} \leftarrow |\mathcal{L}_i^{(j)}| \cdot n_a \cdot f$  ▷  $|\mathcal{L}_i^{(j)}|$  is the size of the support of  $\mathcal{L}_i^{(j)}$ 
9:     end if
10:   end for
11:   let  $\hat{Z}_i \leftarrow \frac{1}{m} \cdot \sum_{j=1}^m \hat{Z}_i^{(j)}$ 
12: end for
13: return  $\sum_{i=0}^L \hat{Z}_i$ 

```

Therefore,

$$|\mathcal{L}_i^{(j)}| \leq \sum_{t=1}^T \left| \varphi_i \left(\bigcup_{C \in \mathcal{C}} B(\text{sub}_i^{(t)}(C), \varepsilon_i/2) \right) \right| \leq T \cdot \text{poly}(d \cdot \log \Delta) \cdot |\mathcal{C}| \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT} / f.$$

This concludes the proof of Lemma 4.9. □

The Accuracy Bound for the Counters. Recall that we maintain multiple counters $\{C_a^{(t)}\}$ for the enlarged buckets, and we now analyze their accuracy. The fact that we need $T = \text{poly}(d \cdot \log \Delta)$ of them is used to ensure a success probability $1 - 1/\text{poly}(\Delta^d)$.

Lemma 4.10. *Let $a \in \mathbb{R}^d$ be a bucket sampled in Algorithm 5, and let $Q = \varphi_i^{-1}(a)$ be the point set corresponding to bucket a . Then with probability at least $1 - 1/\text{poly}(\Delta^d)$ (over the randomness of $\text{sub}_i^{(t)}$ for $1 \leq t \leq T$),*

- if $|B(Q, \varepsilon_i/2) \cap P|/2^i \leq O(1)$ then $\hat{C}_a \leq O(1)$, and
- otherwise $\hat{C}_a = \Theta(1) \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i$.

Proof. First consider the case of $|B(Q, \varepsilon_i/2) \cap P| \geq \Omega(2^i)$. Fix some $1 \leq t \leq T$. Observe that $C_a^{(t)} = |\text{sub}_i^{(t)}(B(Q, \varepsilon_i/2) \cap P)|$. Hence $\mathbb{E}[C_a^{(t)}] = |B(Q, \varepsilon_i/2) \cap P|/2^i$. By Markov's inequality,

$$\Pr[C_a^{(t)} \geq \Omega(1) \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i] \leq 0.1,$$

and using the case condition $|B(Q, \varepsilon_i/2) \cap P| \geq \Omega(2^i)$ and the full independence of subsampling points, by Chernoff bound

$$\Pr[C_a^{(t)} \leq 0.5 \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i] \leq 0.1.$$

Therefore,

$$\Pr[C_a^{(t)} = \Theta(1) \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i] \geq 0.8.$$

Since for $1 \leq t \leq T$, $C_a^{(t)}$'s are independent, then by the standard median trick (using Chernoff bound), we have that $\hat{C}_a = \Theta(1) \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i$ with high probability.

The other case of $|B(Q, \varepsilon_i/2) \cap P| \leq O(2^i)$ can be analyzed similarly (and is simpler since we only need an upper bound). □

Analysis of Level i . We now consider a level $i = 1, \dots, L$ and bound the estimator \widehat{Z}_i on this level. To this end, in Lemma 4.11 we first show that the counter \widehat{C}_a in our algorithm indeed serves as an adequate approximate tester, and then we bound the error of the algorithm at level i , particularly the effect of using the approximate tester according to Definition 4.7. Since $\varepsilon_i = \ell_i/\Gamma$, the approximate tester is off by a factor Γ which translates into $O(\Gamma)$ in the final ratio.

Lemma 4.11. *Given that $\text{OPT} = \Omega(L^4)$, for every i , with probability at least $2/3$ \widehat{Z}_i in Algorithm 5 satisfies*

- $\widehat{Z}_i \leq O(\text{OPT}/L) + \sum_{k \leq i+\alpha} \Theta(2^{k-i}) \cdot W_k$, and
- either $W_i \leq O(\text{OPT}/L^3)$, or $\widehat{Z}_i \geq \Omega(W_i)$,

where $\alpha := \lceil \log_2 \Gamma \rceil$.

Proof. First, condition on the success of the ℓ_0 -samplers. We apply Lemma 4.10, Lemma 4.9, Lemma 4.6, and the union bound to get that with probability at least $1 - 1/\text{poly}(\Delta^d)$, the following statements hold:

- (C1) $\mathfrak{f} \cdot |\mathcal{L}_i^{(j)}| \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT}$ for any $j = 1, \dots, m$.
- (C2) For every bucket $a \in \varphi_i([\Delta]^d)$ sampled in Algorithm 5, let $Q_a = \varphi_i^{-1}(a)$, then if $|B(Q_a, \varepsilon_i/2) \cap P| \leq O(1)$, we have $\widehat{C}_a \leq O(1)$; otherwise, $\widehat{C}_a = \Theta(1) \cdot |B(Q, \varepsilon_i/2) \cap P|/2^i$.
- (C3) Let κ be such that $\mathfrak{f} \cdot \kappa = \Theta(\text{OPT}/L^3)$ as in Lemma 4.6. For every $0 \leq k \leq L$, if $2^{k-i} \cdot W_k \geq \kappa \mathfrak{f}$ then $\mathfrak{f} \cdot |\text{sub}_i(P_k)| = \Theta(2^{k-i}) \cdot W_k$; otherwise $\mathfrak{f} \cdot |\text{sub}_i(P_k)| \leq O(\text{OPT}/L^2)$.

By condition (C2), we know that if for some sampled bucket a , $\widehat{C}_a \leq c$, then $|B(Q_a, \varepsilon_i/2) \cap P|/2^i \leq O(1)$, which implies $|Q_a \cap P|/2^i \leq O(1)$. Now consider the n_a defined in Algorithm 5. Since n_a counts $\text{sub}_i(Q_a \cap P)$, by Fact 3.4, $n_a \leq O(d \cdot \log \Delta)$ with probability at least $1 - 1/\text{poly}(\Delta^d)$ (over the randomness of $\text{sub}_i = \text{sub}_i^{(0)}$ which is independent of the randomness of \widehat{C}_a). Hence, we also condition on the success of this event, as follows.

- (C4) For every bucket $a \in \varphi_i([\Delta]^d)$ sampled in Algorithm 5, if $\widehat{C}_a \leq c$ then $n_a \leq \text{poly}(d \cdot \log \Delta)$.

Therefore, all the above conditions happen with probability at least $1 - 1/\text{poly}(\Delta^d)$, over the randomness of $\text{sub}_i^{(t)}$'s ($0 \leq j \leq T$).

Interpreting The If-else Branch as An Approximate Tester. We show that Algorithm 5 in fact simulates an approximate tester satisfying Definition 4.7. We define this tester $\widehat{I}_{\leq i}$ as follows: For every bucket a with non-zero frequency, we assign $\widehat{I}_{\leq i}(p) = 1$ for every p that belongs to a if a would reach line 8 (the else branch), i.e., if $\widehat{C}_a \leq c$ (we set $\widehat{I}_{\leq i}(p)$ for any p in any non-empty bucket a , no matter whether it is sampled or not). We then set all other values $\widehat{I}_{\leq i}(p)$ to be zero.

For a non-empty bucket a , let $Q_a := \varphi_i^{-1}(a) \cap P$ be its corresponding points in P . If a reaches line 8, then every point $p \in Q_a$ has $r_p \geq \Omega(\varepsilon_i) = \Omega(2^{-i}\mathfrak{f}/\Gamma)$, as otherwise $|B(Q_a, \varepsilon_i/2) \cap P| \geq \Omega(2^i \cdot \Gamma)$, which implies $\widehat{C}_a \geq \Omega(\Gamma) > c$ by condition (C2), giving us a contradiction.

On the other hand, if bucket a contains at least one point p (i.e., $p \in Q_a$) with $r_p \geq 2^{-i}\mathfrak{f}$, then as $\ell_i = 2^{-i}\mathfrak{f}/10$, the bucket a and its close neighborhood contain at most $O(2^i)$ points. More precisely, by triangle inequality and the fact that $\text{diam}(\varphi_i^{-1}(a)) \leq \ell_i$, we have

$$B(\varphi_i^{-1}(a), \varepsilon_i/2) \subseteq B(p, \ell_i + \varepsilon_i/2) \subseteq B(p, r_p/2),$$

hence by Fact 2.2

$$|B(\varphi_i^{-1}(a), \varepsilon_i/2) \cap P| \leq |B(p, r_p) \cap P| \leq O(2^i).$$

This implies that $\widehat{C}_a \leq c$ by condition (C2), so a would reach line 8 if it gets sampled (this case determines the constant c in line 5 of Algorithm 5). Therefore, $\widehat{I}_{\leq p}$ is an α -approximate tester for $\alpha := \lceil \log_2 \Gamma \rceil$.

Expectation Analysis. Now we analyze the expectation of $\widehat{Z}_i^{(j)}$ for every j . Observe that in Algorithm 5, $\widehat{Z}_i^{(j)} \neq 0$ only if the sampled a reaches line 8 of Algorithm 5, and by the interpretation of the approximate tester $I_{\leq i}$, we have for such a ,

$$n_a = |\{p \in \text{sub}_i(P) \cap \varphi_i^{-1}(a) : I_{\leq i}(p) = 1\}|. \quad (11)$$

Hence, by (11)

$$\begin{aligned} \mathbb{E}[\widehat{Z}_i^{(j)}] &= \mathfrak{f} \sum_{a \in \mathcal{L}_i^{(j)}} \frac{1}{|\mathcal{L}_i^{(j)}|} \cdot |\mathcal{L}_i^{(j)}| \cdot |\{p \in \text{sub}_i(P) \cap \varphi_i^{-1}(a) : I_{\leq i}(p) = 1\}| \\ &= \mathfrak{f} \cdot |\{p \in \text{sub}_i(P) : \widehat{I}_{\leq i}(p) = 1\}|. \end{aligned}$$

By the definition of the approximate tester (Definition 4.7), if $p \in P_{\leq i}$ then $\widehat{I}_{\leq i}(p) = 1$, which means the bucket that contains p can reach line 8. Hence

$$\mathbb{E}[\widehat{Z}_i^{(j)}] \geq \mathfrak{f} \cdot |\text{sub}_i(P_{\leq i})| \geq \mathfrak{f} \cdot |\text{sub}_i(P_i)|. \quad (12)$$

Similarly, by the guarantee that if $\widehat{I}_{\leq i}(p) = 1$ then $p \in P_{\leq i+\alpha}$, we have

$$\mathbb{E}[\widehat{Z}_i^{(j)}] \leq \mathfrak{f} \cdot \sum_{k \leq i+\alpha} |\text{sub}_i(P_k)|. \quad (13)$$

Since \widehat{Z}_i is the mean of $\widehat{Z}_i^{(j)}$ for $j = 1, \dots, m$, inequalities (12) and (13) hold for \widehat{Z}_i as well.

Upper Bound. We show that with good probability, $\widehat{Z}_i \leq O(\text{OPT} / (d \cdot \log \Delta)) + \sum_{k \leq i+\alpha} \Theta(2^{k-i}) \cdot W_k$. Let $K_l := \{k \leq i + \alpha : 2^{k-i} \cdot W_k \geq \mathfrak{f}\kappa\}$ and $K_s := \{k \leq i + \alpha : 2^{k-i} \cdot W_k < \mathfrak{f}\kappa\}$. Then by condition (C3),

$$\begin{aligned} \mathfrak{f} \cdot \sum_{k \leq i+\alpha} |\text{sub}_i(P_k)| &= \mathfrak{f} \cdot \sum_{k \in K_s} |\text{sub}_i(P_k)| + \mathfrak{f} \cdot \sum_{k \in K_l} |\text{sub}_i(P_k)| \\ &\leq |K_s| \cdot O(\text{OPT} / (d \cdot \log \Delta)^2) + \sum_{k \in K_l} \Theta(2^{k-i}) \cdot W_k \\ &\leq O(\text{OPT} / (d \cdot \log \Delta)) + \sum_{k \leq i+\alpha} \Theta(2^{k-i}) \cdot W_k. \end{aligned}$$

Combining with (13) and Markov's inequality,

$$\begin{aligned} \Pr \left[\widehat{Z}_i \geq \Omega(\text{OPT} / (d \cdot \log \Delta)) + \sum_{k \leq i+\alpha} \Theta(2^{k-i}) \cdot W_k \right] &\leq \Pr \left[\widehat{Z}_i \geq \Omega(\mathfrak{f}) \cdot \sum_{k \leq i+\alpha} |\text{sub}_i(P_k)| \right] \\ &\leq \Pr[\widehat{Z}_i \geq 6 \cdot \mathbb{E}[\widehat{Z}_i]] \leq 1/6. \end{aligned}$$

Lower Bound. It suffices to prove that if $W_i \geq \mathfrak{f} \cdot \kappa$, then $\Pr[\widehat{Z}_i \leq O(W_i)] \leq 1/6$. By condition (C3) and (12), $\mathbb{E}[\widehat{Z}_i^{(j)}] \geq \mathfrak{f} \cdot |\text{sub}_i(P_i)| \geq \Theta(W_i)$, thus also $\mathbb{E}[\widehat{Z}_i^{(j)}] \geq \mathfrak{f} \cdot \kappa = \Theta(\text{OPT}/L^3)$. By condition (C1), condition (C4) and the fact that $\widehat{Z}_i^{(j)} \neq 0$ only if the sampled a reaches line 8 of Algorithm 5, with probability 1,

$$\widehat{Z}_i^{(j)} \leq \mathfrak{f} \cdot |\mathcal{L}_i^{(j)}| \cdot n_a \leq \text{poly}(d \cdot \log \Delta) \cdot \text{OPT} \leq \text{poly}(d \cdot \log \Delta) \cdot \mathbb{E}[\widehat{Z}_i^{(j)}]. \quad (14)$$

Therefore, by Hoeffding's inequality and the definition of m ,

$$\begin{aligned} \Pr[\widehat{Z}_i \leq O(W_i)] &\leq \Pr[\widehat{Z}_i \leq 0.5 \cdot \mathbb{E}[\widehat{Z}_i]] \leq \Pr[\widehat{Z}_i - \mathbb{E}[\widehat{Z}_i] \leq -0.5 \cdot \mathbb{E}[\widehat{Z}_i]] \\ &\leq \exp\left(-\Theta(1) \cdot \frac{m^2 \mathbb{E}^2[\widehat{Z}_i]}{m \cdot \text{poly}(d \cdot \log \Delta) \mathbb{E}^2[\widehat{Z}_i]}\right) \\ &\leq 1/6. \end{aligned}$$

□

Proof of Theorem 4.5. We make use of known results in a black-box way to detect and solve the case when $\text{OPT} \leq \text{poly}(d \cdot \log \Delta) \cdot \mathfrak{f}$. Namely, we run an $\text{poly}(d \cdot \log \Delta)$ -space, $O(d \log^2 \Delta)$ -approximation for UFL by [Ind04] in parallel to our algorithm (Algorithms 3, 4, 5), and denote the resulting estimate by Y .

If $Y/\mathfrak{f} \geq \Omega(\text{poly}(d \cdot \log \Delta))$, then we know there exists an $O(1)$ -approximation using at least $\text{poly}(d \cdot \log \Delta)$ facilities, which means $\text{OPT} \geq \Omega(\text{poly}(d \cdot \log \Delta))$. Hence, in this case we obtain the assumption that we need for the analysis and we use the estimate returned by Algorithm 5. Thus, by Lemma 4.11 for $i = 0, \dots, L$ (with success probability amplified with the median trick, that is, by running sufficiently many independent repetitions and taking the median estimate) we obtain an estimator \widehat{Z} such that with high probability

$$\begin{aligned} \widehat{Z} &\leq \sum_{i \in [L]} O(\text{OPT}/L) + \sum_{k \leq i + \alpha} \Theta(2^{k-i}) \cdot W_k \\ &\leq O(\text{OPT}) + O(2^\alpha) \cdot \sum_i W_i \\ &\leq O(2^\alpha) \text{OPT}, \end{aligned}$$

and for the lower bound,

$$Z \geq \sum_{i \in [L]} \Omega(W_i) - O(\text{OPT}/L^3) \geq \Omega(\text{OPT}).$$

Since $\alpha = \lceil \log_2 \Gamma \rceil$, we conclude that we obtain an $O(\Gamma)$ -approximation in this case.

Otherwise, if $W/\mathfrak{f} \leq \text{poly}(d \cdot \log \Delta)$, we know the number of open facilities cannot be more than $\text{poly}(d \cdot \log \Delta)$ in any constant-factor approximate solution. To deal with this case, we run several instances of the following algorithm for k -MEDIAN in parallel to our algorithm. Recall that in the k -MEDIAN problem, the goal is to find a subset $C \subseteq \mathbb{R}^d$ with $|C| \leq k$, such that for dataset P , the objective $\sum_{x \in P} \text{dist}(x, C)$ is minimized.

Theorem 4.12 ([BFL⁺17]). *There is an algorithm that for every $0 < \varepsilon < 0.5$, integer $\Delta, k, d \geq 1$, for any set of points $P \subseteq [\Delta]^d$ presented as a dynamic stream, computes a solution $C \subseteq \mathbb{R}^d$ with $|C| \leq k$ to k -MEDIAN on P as well as an estimate R of the objective function (the total connection cost), that is $(1 + \varepsilon)$ -approximate with constant probability, using space $O(k\varepsilon^{-2} \text{poly}(d \cdot \log \Delta))$.*

We run the algorithm of Theorem 4.12 (with a constant ε) in parallel for every $k = 1, \dots, \text{poly}(d \cdot \log \Delta)$ so that if an optimal solution uses at most $\text{poly}(d \cdot \log \Delta)$ facilities, we find an $O(1)$ -approximate solution.

The space bound follows from the fact that our Algorithm 3, 4, 5, the $O(d \log^2 \Delta)$ -approximation of [Ind04], and Theorem 4.12 all use at most $\text{poly}(d \cdot \log \Delta)$ space, and that we run no more than $\text{poly}(d \cdot \log \Delta)$ independent instances of every algorithm. \square

5 Lower Bound

In this part, we give an $\Omega(\sqrt{n})$ space complexity lower bound for one-pass streaming algorithm for Uniform Facility Location with a small constant approximation ratio, even in insertion-only streams with $\Delta = \text{poly}(n)$. This implies impossibility to design a streaming $1 + \varepsilon$ -approximation algorithm for arbitrary $\varepsilon > 0$ with space polynomially depending on the dimension d and polylogarithmic in n and Δ . With this result, the $\text{poly}(d \cdot \log \Delta)$ space complexity of our streaming algorithms presented before is justified to be basically optimal. We first state our lower bound result, and then present the proof relying on the reduction from the Boolean Hidden Matching communication problem.

Theorem 5.1. *For any dimension d and every one-pass randomized algorithm that, given an insertion-only stream of points from $[\Delta]^d$ for $\Delta = 2^{O(d)}$, approximates Uniform Facility Location within ratio better than 1.085, must use space $\exp(\text{poly}(d \cdot \log \Delta))$. More precisely, the space lower bound for better than 1.085-approximation is $\Omega(\sqrt{n})$ in the setting with $d = \Theta(\log n)$ and $\Delta = \text{poly}(n)$, where $n = |P|$ is the number of data points.*

Boolean Hidden Matching (BHM). Recall that in the BHM one-way communication problem Alice gets a binary string $x \in \{0, 1\}^{2n}$, while Bob receives a perfect matching M on $[2n]$ and a binary string $w \in \{0, 1\}^n$ with one bit $w_{i,j}$ for each $(i, j) \in M$. It is promised that either $x_i \oplus x_j = w_{i,j}$ for every pair $(i, j) \in M$ (YES instance), or $x_i \oplus x_j \neq w_{i,j}$ for any $(i, j) \in M$ (NO instance). The goal is for Alice to send a message to Bob so that he can distinguish the two cases with probability at least $2/3$. The randomized one-way communication complexity of BHM is $\Omega(\sqrt{n})$ (see e.g. [GKK⁺09, VY11]).

We use a reduction from BHM to rule out a streaming approximation scheme for the Uniform Facility Location problem, where the opening cost f is the same for all facilities. We show the lower bound in a $\Theta(n)$ -dimensional Euclidean space, and then use a more general form of the Johnson–Lindenstrauss lemma from [NN19] and the Kirschbraun theorem [Kir34] to induce the same result in a logarithmic dimension. While even storing a single point in a $\Theta(n)$ -dimensional takes $\Theta(n)$ space, the point is to show that there is a gap between the optimal costs for YES and NO instances. In dimension $\Theta(n)$, all input points for Facility Location will belong to $\{0, 1\}^d$, however, an optimal solution may use other points as facilities, too. Moreover, each input point will have two or four 1s in its coordinates, while the rest of the coordinates will be set to 0; that is, they can be interpreted as a pair or a four-tuple of coordinates set to 1. We use $f = 2$ for convenience (by scaling, the result holds for any $f > 0$).

We present an instance of Uniform Facility Location problem in the $8n$ -dimensional Euclidean space, with each $i \in [2n]$ independently assigned four coordinates, denoted as i_0, i_1, i_2, i_3 , respectively. For each Alice's bit x_i , we define two points at distance 2: s_i^0 with 1s in coordinates i_0, i_1 , and s_i^1 with 1s in coordinates i_2, i_3 . If $x_i = 0$, then Alice adds $100n$ clients to point s_i^0 and no client to point s_i^1 ; for $x_i = 1$, there are $100n$ clients at point s_i^1 and no client at point s_i^0 . (Recall that clients are the data points in P . For simplicity, we view P as a multiset in this section, i.e., allow duplicate points; however, by a small-enough perturbation of points, the lower bound holds even when points

in P are distinct. Note that out of points s_i^0 and s_i^1 , exactly one is in P with multiplicity $100n$, while the other one is not in P .)

For each edge (i, j) in the matching, Bob adds two clients: one client at $t_{i,j}^0$, with 1s in coordinates $i_0, i_1, j_2-2w_{i,j}$, and $j_3-2w_{i,j}$, and one client at $t_{i,j}^1$, with 1s in coordinates $i_2, i_3, j_{2w_{i,j}}$, and $j_{2w_{i,j}+1}$ (note that these points have four coordinates set to 1); see Figure 2 for an illustration. Observe that points $t_{i,j}^b$ have disjoint sets of coordinates set to 1 and thus pairwise distances equal to $\sqrt{8}$.

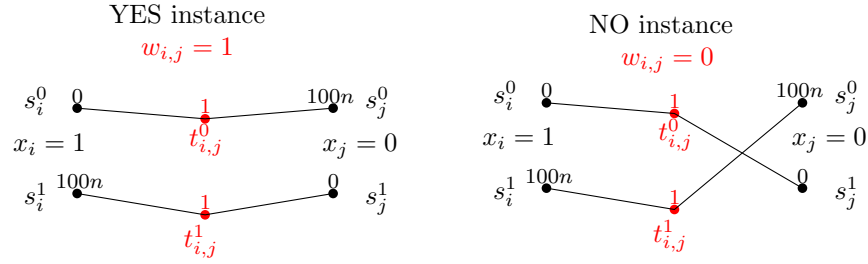


Figure 2: An illustration of the reduction from BHM to an instance of Facility Location. Alice's points are depicted in black, whereas Bob's points are in red. The number above each point represents the number of clients in that point. The edge depicts that the connected points are at distance $\sqrt{2}$, whereas black and red points not connected by an edge are at distance $\sqrt{6}$. The distance between any two black points is always 2 and between any two red points always $\sqrt{8}$.

It is crucial to observe that in either YES or NO case, the optimal solution contains a facility at $s_i^{x_i}$, for each $i = 1, \dots, 2n$, where the notation $s_i^{x_i}$ represents point s_i^0 if bit $x_i = 0$ and point s_i^1 otherwise; see Lemma 5.3 for a detailed argument. We show a separation between the costs in the YES and NO cases by the following lemmas.

Lemma 5.2. *In the YES case, the optimal cost is at most $2n \cdot (f + \sqrt{2})$.*

Proof. In the optimal solution, we build a facility in all $2n$ Alice's points $s_i^{x_i}$ with many clients, and connect each client in $t_{i,j}^b$ to the closest facility, which is at distance at most $\sqrt{2}$ as this is a YES instance. Then the total opening cost is $2n \cdot f$ and the total connection cost equals $2n \cdot \sqrt{2}$, implying that the optimal cost is at most $2n \cdot (f + \sqrt{2})$. \square

Lemma 5.3. *In the NO case, the optimal cost is $n \cdot (3f + \sqrt{2})$.*

Proof. We first observe that in any optimal solution, a facility is built at $s_i^{x_i}$ for each $i = 1, \dots, 2n$. Indeed, if there is no facility in $B(s_i^{x_i}, 0.1)$ for some i , then the connection cost for the clients at $s_i^{x_i}$ is at least $10n$, which means that we can decrease the total cost by building a facility at $s_i^{x_i}$. Thus, ball $B(s_i^{x_i}, 0.1)$ must contain a facility for any i . Furthermore, if there is a facility in $B(s_i^{x_i}, 0.1)$ but not at $s_i^{x_i}$, then all clients at $s_i^{x_i}$ are connected to this facility (to the closest one if there are more), thus by moving the facility to $s_i^{x_i}$, we decrease the total connection cost (this could increase the connection cost for Bob's clients, but there are only $2n$ of them).

Next, we argue about the cost for Bob's clients. One possible solution is the following: For each edge (i, j) in the matching, one of the clients sitting at $t_{i,j}^0$ and $t_{i,j}^1$ is at distance $\sqrt{2}$ from facilities $s_i^{x_i}$ and $s_j^{x_j}$, so we connect this client to either of these two facilities, while we build a facility at the location of the other client, which is at distance at least $\sqrt{6}$ from any facility built in such a way. Thus, as $f < \sqrt{6}$, it is cheaper to build a new facility for this client than to connect it to another

facility. This solution has opening cost of $3n \cdot f$ and connection cost of $n \cdot \sqrt{2}$, and we show that it is optimal. To this end, it suffices to rule out building a facility that serves more than one Bob's client.

Suppose for a contradiction that the optimal solution contains a facility such that $k > 1$ of Bob's clients are connected to it. Such a facility cannot be placed at $s_i^{x_i}$ for some i as the connection cost is then $k \cdot \sqrt{6} > k \cdot f$, i.e., it is cheaper to build one facility for each of the k Bob's clients. Recall that these Bob's clients sit on vertices of a $(k-1)$ -simplex with edge length $\sqrt{8}$. Since the sets of coordinates of 1s of different Bob's clients do not intersect, the best location for such a facility is to place it in the geometric median of the simplex, i.e., the point that minimizes the sum of distances from the vertices (which is the connection cost). Clearly, the geometric median has coordinate i equal to $1/k$ whenever there is a (single) vertex with 1 at coordinate i , and to 0 otherwise (thus, the geometric median is the average of the vertices). The distance from each of the k clients to this median is

$$\sqrt{4 \cdot (k-1) \cdot \left(\frac{1}{k}\right)^2 + 4 \cdot \left(1 - \frac{1}{k}\right)^2} = 2 \cdot \sqrt{\frac{1}{k} - \frac{1}{k^2} + 1 - \frac{2}{k} + \frac{1}{k^2}} = 2 \cdot \sqrt{1 - \frac{1}{k}}.$$

It follows that the connection cost for these k clients plus the opening cost of the facility is $k \cdot 2 \cdot \sqrt{1 - \frac{1}{k}} + f > k \cdot f$, which holds for any $f \leq 2$. Hence, it is cheaper to build k facilities for these k clients.

Concluding, the optimal cost for a NO instance is $n \cdot (3f + \sqrt{2})$. \square

Combining Lemma 5.2 and Lemma 5.3 and using $f = 2$, a one-pass streaming algorithm for Uniform Facility Location in an $\Theta(n)$ -dimensional space using space $o(\sqrt{n})$ with approximation ratio strictly better than $(3f + \sqrt{2})/(2f + \sqrt{8}) = (6 + \sqrt{2})/(4 + \sqrt{8}) \approx 1.085$ would solve the BHM problem.

In order to obtain the identical result in a logarithmic space, we introduce a more general form of Johnson-Lindenstrauss Lemma given by Narayanan and Nelson [NN19], as well as the Kirschbraun theorem given from [Kir34].

Theorem 5.4 ([NN19]). *For all $\varepsilon < 1$ and $x_1, \dots, x_n \in \mathbb{R}^d$, there exists $m = O(\varepsilon^{-2} \log n)$ and a (nonlinear) map $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that for all $1 \leq i \leq n$ and $u \in \mathbb{R}^d$,*

$$(1 - \varepsilon)\|u - x_i\|_2 \leq \|\pi(u) - \pi(x_i)\|_2 \leq (1 + \varepsilon)\|u - x_i\|_2.$$

Theorem 5.5 ([Kir34]). *For every subset $X \subset \mathbb{R}^d$ and L -Lipschitz map⁶ $\varphi : X \rightarrow \mathbb{R}^m$, there exists an L -Lipschitz extension $\tilde{\varphi}$ of φ from X to the entire space \mathbb{R}^d .*

Proof of Theorem 5.1. Given an instance of BHM, let $X = \{s_i^0, s_i^1 \mid i \in [2n]\} \cup \{t_{i,j}^0, t_{i,j}^1 \mid (i, j) \in M\}$ be the set of data points of the corresponding Uniform Facility Location instance constructed as above. By Theorem 5.4, there exists $\pi : \mathbb{R}^{8n} \rightarrow \mathbb{R}^d$ with $d = \Theta(\log n / \varepsilon^2)$ and a small enough constant $\varepsilon > 0$ such that for all $x \in X$ and $y \in \mathbb{R}^{8n}$,

$$(1 - \varepsilon)\|x - y\|_2 \leq \|\pi(x) - \pi(y)\|_2 \leq (1 + \varepsilon)\|x - y\|_2.$$

Thus, by Lemma 5.2, the optimal solution of Uniform Facility Location in the YES case after the dimension reduction by π is at most $(1 + \varepsilon) \cdot 2n \cdot (f + \sqrt{2})$.

⁶Recall that φ is L -Lipschitz if for all $x', x'' \in X$, we have $\|\varphi(x') - \varphi(x'')\|_2 \leq L\|x' - x''\|_2$.

It remains to show an upper bound of $n \cdot (3f + \sqrt{2})/(1 + \varepsilon)$ on the optimal cost in the NO case after the dimension reduction. Letting $Y = \pi(X)$, the inverse map $\pi^{-1} : Y \rightarrow \mathbb{R}^{8n}$, where $\pi^{-1}(y)$ is defined as arbitrary element in $\{x \mid \pi(x) = y\}$, is $(1 + \varepsilon)$ -Lipschitz, since for all $y', y'' \in Y$,

$$\|\pi^{-1}(y') - \pi^{-1}(y'')\|_2 \leq \frac{1}{1 - \varepsilon} \|y' - y''\|_2 \leq (1 + O(\varepsilon)) \|y' - y''\|_2.$$

Using Theorem 5.5, we extend the map $\pi^{-1} : Y \rightarrow \mathbb{R}^{8n}$ to $\tilde{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}^{8n}$ while preserving the $(1 + \varepsilon)$ -Lipschitz property. Suppose, for the sake of contradiction, there is a solution in the NO case with cost strictly less than $n \cdot (3f + \sqrt{2})/(1 + O(\varepsilon))$. Mapping the facilities opened in this solution back into \mathbb{R}^{8n} by $\tilde{\varphi}$, we obtain a solution in \mathbb{R}^{8n} with cost strictly less than $n \cdot (3f + \sqrt{2})$, which contradicts Lemma 5.3.

Thus a one-pass streaming algorithm for Uniform Facility Location in a $\Theta(\log n)$ -dimensional space using memory of $o(\sqrt{n})$ with approximation ratio strictly better than

$$\frac{3f + \sqrt{2}}{(1 + O(\varepsilon))^2(2f + \sqrt{8})} \approx 1.085$$

would solve the BHM problem, which contradicts the communication complexity of BHM.

Finally, while the dimension reduction of Theorem 5.4 yields an instance where points have real coordinates, we can scale the instance by a $\text{poly}(n)$ factor so that by rounding the coordinates to integers after this scaling we only change the optimal cost by an arbitrarily small factor in both YES and NO cases. Therefore, the lower bound holds even if the point set is restricted to $[\Delta]^d$ for $\Delta = \text{poly}(n)$. \square

6 Future Directions

As mentioned in Section 1, it is an interesting open question to design an $O(d)$ -gap $\text{poly}(d)$ -consistent hashing that is computable in $\text{poly}(d)$ space, that will match the non-streaming bound of Filtser [Fil20]. In fact, Filtser [Fil20] gave also a lower bound of $\Gamma = \Omega(d)$, and thus $\Gamma = O(d)$ is the best bound one can possibly obtain for Definition 1.6.

Relaxed Notion of Hashing. However, the abovementioned gap lower bound $\Gamma = \Omega(d)$ might be bypassed, since all algorithms presented in this paper actually work (with the same ratio and space) even with a suitably relaxed notion of hashing. In particular, the hashing may be randomized, and the consistency guarantee may be relaxed to hold only in expectation, as follows: for every $S \subseteq \mathbb{R}^d$ with $\text{diam}(S) \leq \ell/\Gamma$, we have $\mathbb{E}[\|\varphi(S)\|] \leq \Lambda$, where the randomness is over the choice of φ . The proof in [Fil20] that $\Gamma = \Omega(d)$ does not apply to this relaxed notion, and we conjecture that under this relaxed definition, the gap Γ may be improved significantly, say to $O(\log d)$ or even $O(1)$. This suggests a concrete direction to improve our $O(d^{1.5})$ -approximation for the one-pass setting.

Multiple Passes. It is also interesting to explore the power of multiple passes. In particular, is it possible to achieve $(1 + \varepsilon)$ -approximation using, e.g., $\text{poly}(d \cdot \log \Delta)$ passes? Our lower bound works only for one pass, and it would be interesting to strengthen it to $O(1)$ -passes using $\text{poly}(d \cdot \log \Delta)$ space.

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A Proof of Lemma 3.3

We restate the lemma for convenience.

Lemma 3.3 (Two-Level ℓ_0 -Sampler). *There is a randomized algorithm, that given as input a matrix $M \in \mathbb{R}^{m \times n}$, with $m \leq n$ and integer entries bounded by $\text{poly}(n)$, that is presented as a stream of additive entry-wise updates, returns an index-pair (i, j) of M , where i is chosen uniformly at random (u.a.r.) from the non-zero rows, and then j is chosen u.a.r. from the non-zero columns in that row i . The algorithm uses space $\text{poly}(\log n)$, fails with probability at most $1/\text{poly}(n)$, and can further report the corresponding row-sum $\sum_{j'} M_{i,j'}$.*

We show that this lemma can be proven using ℓ_0 -samplers together with the simple but powerful fact that ℓ_0 -samplers can be obtained by linear sketching. We state a lemma about the existence of such an ℓ_0 -sampler.

Lemma A.1 (ℓ_0 -Sampler; see e.g. [CF14]). *There is an algorithm that for integer $n \geq 1$, and every multiset S of elements of $[n]$ with multiplicities bounded by $\exp(\text{poly}(\log n))$ that is presented as a dynamic stream, succeeds with probability at least $1 - 1/\text{poly}(n)$ and, conditioned on it succeeding, returns an element $j \in S$ together with its multiplicity m_j in S such that for every $s \in S$ it holds that $\Pr[j = s] = 1/|S|$, where $|S|$ is the number of distinct elements in S . The algorithm has space and both update and query times bounded by $\text{poly}(\log n)$, and its memory contents is a linear sketch of the frequency vector \mathbf{m} of S .*

Proof sketch of Lemma 3.3. The idea is to use the ℓ_0 -sampler of Lemma A.1 in two levels, one for rows and one for columns of M , with the column level being encoded in the multiplicity of the row level (a similar trick has been used e.g. in [FIS08]). In more detail, we instantiate an ℓ_0 -sampler for rows of M such that the (binary representation of) multiplicity of any row i additionally encodes the memory contents of the ℓ_0 -sampler for the columns of row i (the true multiplicity of a row is also stored, which is needed to return the row sum). As an ℓ_0 -sampler takes $\text{poly}(\log n)$, the multiplicity of any row will be bounded by $\exp(\text{poly}(\log n))$, as desired. Since the memory contents of the column ℓ_0 -sampler is a linear sketch of row i , its representation is an all 0s vector if and only if row i is an all 0s vector.

The randomness (namely, hash functions) for column ℓ_0 -samplers of all rows will be the same. Using this and the linearity of ℓ_0 -samplers, an update to a column ℓ_0 -sampler in row i can be turned into an update of the multiplicity of row i in the row ℓ_0 -sampler. Thus, upon an update a to $M_{i,j}$, we first compute the update to the column ℓ_0 -sampler from j and a (this update is independent of i). Then we view this update as a change of the multiplicity of row i , and feed it to the entry for row i in the row ℓ_0 -sampler.

To sample a desired cell of M , we query the row ℓ_0 -sampler to obtain a u.a.r. non-zero row i , together with the multiplicity of row i , which encodes the memory contents of the column ℓ_0 -sampler for row i . We then sample from this column ℓ_0 -sampler to obtain a u.a.r. column j such that $M_{i,j} \neq 0$. We get the desired row sum from the row ℓ_0 -sampler. \square

B Lower Bounds Based on Indexing

In this appendix, we give details of our lower bound arguments that are based on the hardness of indexing in one-way communication. Recall that the one-way communication problem of INDEX is defined as follows: Alice holds a set of bits x_1, \dots, x_k , and Bob has an index $j \in [k]$. The goal is for Alice to send a message to Bob so that Bob can determine x_j ; crucially, Bob cannot send any information to Alice. It is well-known that the message of Alice needs to have $\Omega(k)$ bits for Bob to succeed with constant probability [KNR99] (see also [KN97, JKS08]).

Lower Bound for k -Median. We show that an algorithm with space $o(k)$ that determines whether the k -MEDIAN cost is 0 or 1 would solve the INDEX problem. This holds even in the one-dimensional case and also for related types of clustering, such as k -MEANS. Indeed, given such an algorithm, Alice inserts k points, one point $2 \cdot i + x_i$ for each of bit x_i . Then she sends the memory contents of this algorithm to Bob who inserts point $2 \cdot j$. Thus the instance has $k + 1$ distinct points if $x_j = 1$ and otherwise, it has k distinct points. It follows that the k -MEDIAN cost equals 1 iff $x_j = 1$ and otherwise, it equals 0.

Lower Bound for Answering Queries about The r_p Values. In Section 4.3 we discussed a straightforward one-pass implementation of our approach in arbitrary-order streams that would be similar to the implementation in two passes or in random-order streams. However, as we only get the sampled points at the end of the stream, we would need to estimate the r_p value for each of the sampled points *a posteriori*, that is, after processing the stream.

We now argue that an algorithm giving a c -approximation of r_p for any query point p would solve the INDEX problem. Indeed, given such an algorithm, we set $\mathbf{f} = 1$, and for every $i = 1, \dots, n$, Alice inserts 2 points at position i if $x_i = 0$ and $\lceil 2c + 1 \rceil$ points at position i otherwise. After that, for a point p at position i , we have that $r_p = 0.5$ if $x_i = 0$ and $r_p \leq 1/(2c)$ if $x_i = 1$. Then Bob queries the r_p value at position j . Since the returned value is a c -approximation of r_p , Bob can distinguish between $r_p = 0.5$ and $r_p \leq 1/(2c)$ and hence, determine the value of x_j .