Budget-Constrained Auctions with Unassured Priors

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July 11, 2022

Abstract

In today's online advertising markets, it is common for an advertiser to set a long-period budget. Correspondingly, advertising platforms adopt budget control methods to ensure that an advertiser's payment is within her budget. Most budget control methods rely on the value distributions of advertisers. However, the platform hardly learns their true priors due to the complex environment advertisers stand in and privacy issues. Therefore, it is essential to understand how budget control auction mechanisms perform under unassured priors.

This paper gives a two-fold answer. First, we introduce the bid-discount method into first-price auction. We show that the resulting auction mechanism exhibits desirable revenue maximization and computation properties. Second, we compare this mechanism with other four in the prior manipulation model, where an advertiser can arbitrarily report a value distribution to the platform. These four mechanisms include the optimal mechanism satisfying budget-constrained IC, first-price/second-price mechanisms with the widely-studied pacing method, and bid-discount second-price mechanism. We consider two different settings varying on whether the seller knows the reported value distributions before choosing the auction mechanism. When the reported priors are pre-known to the seller, we show that the bid-discount first-price auction we introduce dominates the other four mechanisms concerning the platform's revenue. On the other hand, when the seller has no information on the reported priors before committing to a mechanism, we show a surprising strategic-equivalence result between this mechanism and the optimal auction. When buyers are further symmetric, we establish a wide equivalence among variants of first-price and second-price auctions. Based on these findings, we provide a thorough understanding of prior dependency in repeated auctions with budgets. The bid-discount first-price auction itself may also be of further independent research interest.

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1 Introduction

We have seen extensive growth in the online advertising market in recent years. Billions of advertising positions are sold every day on various kinds of platforms, in which the most iconic are major search engines (e.g., Google [29]) and social media (e.g., Facebook [20]). According to statistics, the volume of online advertising worldwide is hopeful of reaching above 500 billion dollars in 2022 [35]. From a macro view, the contents of ads present a massive heterogeneity according to different types of ad queries. For example, an ad delivered to a new father is more likely on baby products, while a teenager tends to receive ads on games.

To deal with such heterogeneity, advertising platforms adopt auctions to allocate ad spaces. Each advertiser bids a value she would like to pay for each ad query satisfying certain conditions (e.g., from a new father or a teenager). When a real-time query arrives, the platform holds an auction among all advertisers who bid positively on the query. Since a substantial amount of the above process happens every day, an advertiser's payment can vary highly. Major platforms now inquire a long-time (e.g., a day, a week, or a month) budget from advertisers to lessen such uncertainty. Correspondingly, the auction mechanisms taken by the platforms guarantee that each advertiser's payment does not exceed her budget.

A large number of works have studied different budget control methods, from either a dynamic view [7,17,21] or an equilibrium view [4,5,9,12,13]. An essential assumption in these works is that the platform knows the prior value distributions or even the actual values of advertisers in advance. Nevertheless, such a hypothesis can be unattainable in practice. From the view of information structure, the platform can only obtain the *historical bids* of an advertiser, rather than the historical values. Consequently, the platform has no access to either advertisers' values or priors. Moreover, the classic methodology of incentive-compatibility (IC) embraced by massive works hardly fits with today's advertisers due to two main reasons: (1) The traditional definition of IC does not capture the various constraints faced by advertisers, including budget [5] and ROI [3]. We must carefully trim the concept to suit more complex circumstances, which always leads to an intricate result. The concept becomes even more powerless considering that advertisers would simultaneously cooperate with multiple platforms. (2) Advertisers have intrinsic incentives to hide their true values to cope with the learning behavior of the platform and protect their data privacy. Once the platform has complete knowledge of an advertiser's actual value distribution, price discrimination can inevitably occur, which would be a curse for the advertiser. With the emergence of the above two phenomena, market designers must face the fact that they may never be able to get advertisers' true values/value priors. Therefore, a natural problem arises:

How do unassured priors affect budget control methods in repeated auctions? Specifically, when priors are unassured, how are budget control methods related?

This paper answers the above problem synthetically. We study a range of five kinds of budget control methods, respectively Bayesian revenue-optimal budget-constrained IC auction [5], bid-discount first-price/second-price auction, and pacing first-price/second-price auction under the prior manipulation model. Here, the prior manipulation model captures the fact that any advertiser's prior known to the platform could deviate from her real prior. We mention that pacing (a.k.a. bid-shading) is one of the most extensively studied strategies to control advertisers' payments [9,12,13]. Moreover, bid-discount is an expenditure management skill that modifies the allocation without changing the payment. Such strategy has been adopted in sponsored search auction [1,18,28] and second-price auction [22,32]. Even though it is natural to introduce such strategy into first-price auction, such combination, to the best of our knowledge, does not emerge in previous literature.

For variants of first-price/second-price auctions, we define the notion of *efficiency* which ensures that each advertiser gets sufficient allocation within her budget. This term resembles the notion of system equilibrium defined in [5]. We comprehensively compare the five mechanisms (when the latter four are in efficiency) concerning the platform's revenue. An important conclusion is that bid-discount first-price auction always performs no worse than the other four kinds of auctions under the prior manipulation model in various settings, both theoretically and practically. This result could also show the clairvoyance of major platforms turning to first-price auctions [24] from another angle.

1.1 Our Contributions and Techniques

We first consider how budget-constraint buyers participate in repeated auctions. In this paper, we consider the stochastic setting, in which buyers' values in each auction are drawn from some distributions and are unknown to the seller *ex-ante*. Therefore, we can only guarantee that the *expected* payment of each buyer does not exceed the budget. Compared with other settings, the stochastic setting is much closer to actual practice. (See a brief discussion in Section 2.1.) Capturing the possibility that the seller does not know buyers' true value distributions, we consider the information structure in the prior manipulation model, which was first proposed by [36]. Under the model, the prior of a buyer known by the seller (i.e., the reported distribution), which is a part of her strategy, can deviate from the buyer's true prior. The other part of a buyer's strategy is a bidding function mapping $U[0,1]$ to itself, which outputs her bid according to the reported distribution with her drawn value as input. By applying a slightly modified version of Hardy–Littlewood inequality, we show that for mechanism families with monotone allocation rule (see [31] and Section 2.3), the optimal bidding function for each buyer is the identical function from $[0,1]$ to itself (Lemma 2.1). Therefore, a buyer's strategy degenerates to a single reported prior in this case. In literature, a similar argument of Lemma 2.1 has already occurred in [14,36]. Nevertheless, in these two papers, the proof of the counterpart is not rigorous. We provide a refined proof of the result in Appendix A.1. Since all mechanism families this work considers have a monotone allocation rule, we explicitly exclude the bidding function from a buyer's strategy. Readers can refer to a typical execution of repeated auctions with budget constraints in Section 2.4.

Bid-discount first-price auction and its properties. Under the prior manipulation model, we first focus on the computation issue and properties of a mechanism instance given buyers' reported distributions (Section 3). Specifically, we consider five budget control mechanism families. They include the Bayesian revenue-optimal BCIC auction (BROA) given in [5], two variants of first-price auctions (BDFPA, PFPA), and two variants of second-price auctions (BDSPA, PSPA). Here, BCIC means that the auction family is incentive-compatible under budget constraints (Definition 2.4). A central contribution here is that we introduce the bid-discount method [22, 28, 32] into first-price auctions in the budget-constrained setting, resulting in a variant with desirable properties. We call this variant bid-discount first-price auction (BDFPA). In the mechanism, each buyer is assigned a discount multiplier. The buyer with the highest discount bid wins in each auction and pays her original bid. Given buyers' reported distributions, we define the notion of efficiency for BDFPA similar to the notion of system equilibrium in [5], which requires that any buyer exhausts her budget unless her multiplier is 1. Such a definition captures the desire that the market should be saturated. In this part, two results make efficient BDFPA a full-fledged form of auction: (1) Efficient BDFPA is revenue-optimal for the seller among all BDFPAs (Theorem 3.2), and (2) An efficient tuple of multipliers in BDFPA can be computed by solving the minimum of a convex function (Theorem 3.4). The proof of these two results applies the duality and convex optimization

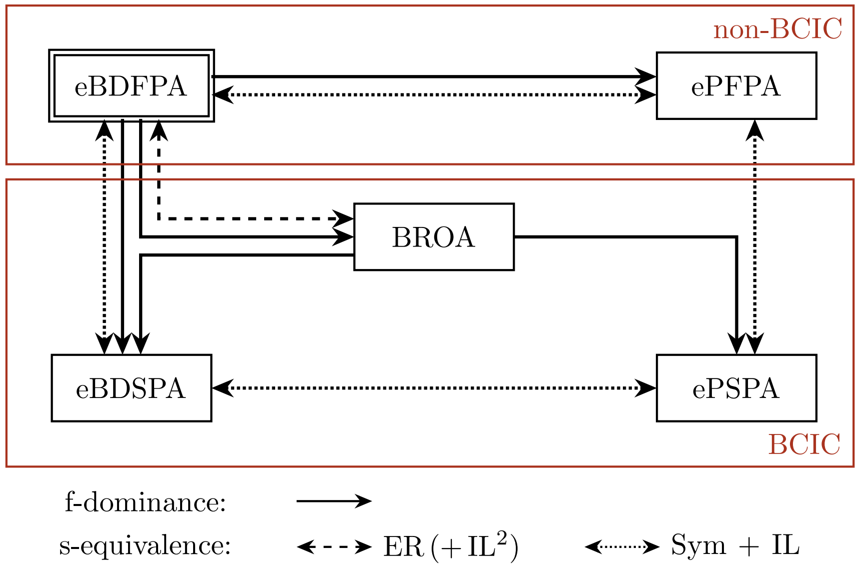

Figure 1: Summary of the results in Section 4 on the relationships of different auction types under the prior manipulation model. A solid arrow from A to B means that A f-dominates B (Definition 2.6), and a bidirectional dashed/dotted arrow implies s-equivalence (Definition 2.7) under different assumptions. Different line types indicate the assumptions. ER: When the virtual valuation is strictly increasing and differentiable for each buyer. IL: When the reported qf is inverse Lipschitz continuous for each buyer. IL^2: when both the reported qf and virtual valuation are inverse Lipschitz continuous for each buyer. Sym: When buyers are symmetric.

theories. Specifically, we formalize the seller's revenue-maximizing problem into a constrained optimization problem. We discover that the efficiency of BDFPA is equivalent to the optimality condition of the dual problem. Subsequently, we explore the conditions of efficiency under natural assumptions (Theorem 3.5). This result is proved majorly by a delicate continuity argument in which we analyze how a buyer's payment increases with her multiplier. A significant result is that for any buyer, her expected payments are identical in all efficient BDFPAs. We also revisit the pacing method [9, 12, 13] in first-price auctions and similarly define efficiency for this auction with knowledge of reported distributions. Under the stochastic model, we extend some of the results in [12] (Theorem 3.6), while [12] focuses on the deterministic setting where the seller knows the value of each buyer. We further clarify the notion of efficiency in two variants of second-price auctions adopting bid-discount and pacing.

Dominance and equivalence relationships among auction mechanisms. We further compare the five mechanism families under the prior manipulation model (Section 4). For this part, we consider two different settings, which differentiate on the time point when the buyer knows buyers' reported distributions (Section 2.4). On the one hand, when the seller already knows these distributions before committing to the mechanism family, we define the f-dominance relationship between two families (Definition 2.6), i.e., bringing a higher revenue for the seller. On the other hand, when the seller comes to know these priors after choosing the mechanism family, we consider the s-equivalence relationship among different auction forms (Definition 2.7), i.e., equivalence under a mapping of buyers' strategies. Subsequently, we exploit the relationships of all five kinds of auctions under these two different settings in Section 4, and plot our results in Figure 1. We point

2 buyers, 1 round of auction.
For both buyers: reported pdf = $U[0,1]$, budget = 0.312.
For the seller: opportunity cost = 0.1.

	eBDFPA	ePFPA [12]	BROA [5]	eBDSPA [22, 32]	ePSPA [9, 13]
Each buyer's payment	**0.312**	**0.312**	0.207	0.171	0.171
Seller's revenue	**0.54**	0.525	0.344	0.243	0.243
Budget sufficiently used?	**Yes**	**Yes**	No	No	No

Table 1: Summary of Example 4.1.

out three main threads in the figure for a better understanding.

1. eBDFPA $\succeq_{\mathrm{F}}$ ePFPA, eBDFPA $\succeq_{\mathrm{F}}$ BROA $\succeq_{\mathrm{F}}$ eBDSPA/ePSPA; (Theorem 4.1, Theorem 4.2, Lemma 4.3)
2. Under ER(+IL^2), eBDFPA $=_{\mathrm{S}}$ BROA; (Theorem 4.5)
3. Under Sym + IL, eBDFPA $=_{\mathrm{S}}$ ePFPA $=_{\mathrm{S}}$ eBDSPA $=_{\mathrm{S}}$ ePSPA. (Theorem 4.7, Theorem 4.8)

The first thread gives the domination relationships of different auctions when the reported priors are fixed and known to the seller before he commits to the auction mechanism. An essential conclusion is that BDFPA f-dominates all other four mechanisms when efficiency is met, that is, eBDFPA gets a higher revenue for the seller than the other four mechanisms. The proof of these lines is finished by using a duality argument. Such results further demonstrate the importance of this insufficiently studied mechanism. In a numerical example (Example 4.1), we illustrate that seller's revenue in efficient BDFPA could be 1.5 times as large as that in BROA and twice as large as that in efficient BDSPA/PSPA. The result is summarized in Table 1.

The other two threads consider the seller's scenario of not knowing the reported priors before choosing the auction mechanism. The second thread shows that efficient BDFPA is s-equivalent to BROA under this case. In other words, these two mechanisms have the same equilibrium outcome for the buyers. The main idea of the proof is to notice the intrinsic similarity between these two auction forms. However, there are many details to deal with. For cases when the virtual valuation can be negative, the proof is complemented by a technique we call "lifting". Also, we need to guarantee that the basic properties (e.g., inverse Lipschitz continuity) are kept under the mapping, which requires careful design. The last thread indicates the universal equivalence of first-price auctions and second-price auctions when buyers are symmetric. For the proof of this part, we explore that any possible prior is equivalent to a concatenation of linear function and exponential function in the symmetric setting. This observation solves the problem immediately.

1.2 Related Works

Prior manipulation model. In classic solutions, e.g., the seminal work of Myerson [30], a critical assumption is that the seller knows the distribution of buyers' values. In real life, however, from a buyer's view, when she takes some strategic behavior other than truthful bidding (e.g., when she is budget-constrained), the seller can never get the true distribution. A line of works captures such inconsistency between the ideal and real cases and focuses on how the auction market is affected when the seller wrongly estimates the buyers' value distributions. [36] studies the problem in general, and a surprising result is that Myerson auction, which is well-known for being revenue-optimal, is revenue-equivalent to first-price auction under such a model. Concurrently, [15, 16] consider specific distribution families from a statistical optimization view in this setting. [14] further studies the scenario in sponsored ad auctions and shows the general equivalence of different auction types under such setting. Compared with these works, our paper considers buyers' budgets and explores deep relationships among different auction forms.

Market equilibrium when buyers are budget-constrained. In real life, it is always the case that a buyer's affordability is small compared with the massive amount of auctions happening every day. Therefore, it is reasonable for a buyer to set a budget constraint. Many works consider the market equilibrium in this scenario.

The works most related with ours are [5, 6]. [5] surveys on various budget control methods in second-price auctions, and compares these methods from the aspects of seller's revenue and social welfare in equilibrium. Nevertheless, our work is not limited to second-price auctions. We also consider the optimal auction and variants of first-price auction, and further break the barrier among these three genres of auctions when buyers are budget-constrained in the prior manipulation model. [6] focuses on the contextual scenario in first-price auction where a buyer's value is decided collaboratively by a public item type and a private buyer type. The paper shows a revenue equivalence result across all standard auctions under symmetric Bayes-Nash equilibrium, which seems similar to one of our results. However, a crucial difference is that our paper considers the more general setting with prior manipulation and without any contextual information. Furthermore, our result is not limited to symmetric or equilibrium cases, which implies that our s-equivalence theorems differ from any classical revenue equivalence result.

Bid-discount method (in Section 3) has been adopted to generalized second-price auctions for sponsored search in early years [1, 18, 28], with the multipliers closely related to the click-through rates. In recent years, such a method was applied to second-price auctions [22,32], known as boosted second-price auction (BDSPA in our work). Experimental results show that such an auction form earns higher revenue for the seller than second-price auction and empirical Myerson auction without budget constraints. On the other hand, our work considers the scenario when buyers are budget-constrained. Further, we introduce the bid-discount method into first-price auctions and prove that the corresponding BDFPA mechanism has desirable properties compared to other auction forms.

Other works study specific strategies buyers and sellers adopt to maintain the budget. Pacing (a.k.a. bid-shading) is perhaps the most well-studied one among all those methods. In pacing, the seller would assign a multiplier to each buyer, and the multiplier would shade the bid of any buyer before being sorted. The payment of a buyer is also correspondingly scaled to control the budget. [12, 13] respectively consider the pacing equilibrium in first-price and second-price auctions. On top of [13], [9] shows that finding the (approximate) pacing equilibrium in second-price market is PPAD-hard. However, these papers focus on a discrete setting where buyers' values are known *a priori*. Instead, our work focuses on the stochastic setting where the value of each buyer is unsure *ex-ante*. Some works concentrate on the "throttling" strategy. The main idea of the strategy is to maintain a probability for each buyer to participate in an auction. [8, 26] consider the throttling strategy generally and propose algorithms satisfying good allocation properties. More in detail, [26] proposes a meta algorithm for buyers under various constraints while [8] focuses on having the buyers "regret-free" on their ROI under the allocation. [11] proves that in sufficiently large markets, there is a simple approximate equilibrium where the seller applies greedy throttling mechanisms with high probability. On the other hand, [10] studies the complexity issue of finding throttling equilibrium in first-price and second-price auctions. [2, 25] considers the behavior of a large number of revenue-maximizing buyers with budget constraints in second-price markets. They take the mean-field equilibrium as the central solution concept and prove that pacing is an approximately optimal strategy for these buyers. In comparison, the main goal of this paper is to provide general relationships among different auction types. Besides the above, [4] considers the dynamic environment in which each budget-constrained buyer takes a simple learning strategy similar to pacing. Our paper, nevertheless, does not explicitly involve any learning behavior.

Organization. The rest of this paper is organized as follows. Section 2 introduces how the prior manipulation model works in a budget-constrained setting. Section 3 considers five kinds of budget control methods and presents their corresponding properties given an instance. Section 4 compares these five kinds of auctions under the prior manipulation model under two different settings. Section 5 concludes the paper.

2 Model

In this section, we introduce how budget-constrained buyers would participate in repeated auctions when their value distributions are unassured to the seller.

2.1 On Buyers' Side

On values and budgets of buyers. In this work, we consider the stochastic setting, which has been adopted by lines of research works [2, 5, 13, 25]. In this setting, buyers' values for each auction are drawn from some distribution and *unknown* to the seller. As a result, we require that the *expected* payment of each buyer should not exceed her budget. We argue that such a model realistically captures the stationary behavior when buyers participate in a large number of auctions and the platform inquires for an "average budget constraint" (e.g., [23]).

In particular, we suppose there are n buyers in the market, and each buyer $1 \leq i \leq n$ has a value distribution F_i, which is independent of each other. We consider T auctions in a row. In each auction $1 \leq j \leq T$, a value $v_{ij} \in \mathcal{V}_i$ is drawn from F_i, which is buyer i's true value for auction j. Meanwhile, each buyer i has a budget $B_i > 0$ across all T auctions, which is known to the seller. We use $B = (B_1, \cdots, B_n)$ to write the budget profile. Further, we suppose the seller has a universal opportunity cost $\lambda > 0$ for the item in all auctions. Opportunity cost reflects the seller's unwillingness to sell the item. As long as the item is sold, the seller's revenue is the total payment of buyers minus the opportunity cost.

On prior manipulation model. In this work, we adopt the prior manipulation model to formalize a buyer's behavior. Such model was first proposed in [36] and also used in [14–16]. Under the model, we use a quantile q_{ij} to express the value of buyer i for auction j, where $q_{ij} = F_i(v_{ij})$, and q_{ij} is drawn uniformly from $[0, 1]$. Equivalently, the *quantile function* (abbreviated to qf) $v_i(q_i) = F_i^{-1}(q_i) \in \mathcal{F}_i$ maps a quantile to the true value of the buyer. [1] In scenarios where we do not care about the specific auction j, we use q_i to denote the quantile of buyer i, and $q = (q_1, \cdots, q_n)$ to be the true quantile profile. We suppose that the profile of *true* quantile functions for all n buyers is $\boldsymbol{v} = (v_1(\cdot), \cdots, v_n(\cdot)) \in \mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$. Meanwhile, each buyer i reports a distribution $\widetilde{v}_i(\cdot) \in \widetilde{\mathcal{F}}_i$ to the seller, which is called the *reported* quantile function. Here, the reported quantile

[1] In the original model of [36], the authors use the right quantile function, that is, $q_{ij} = 1 - F_i(v_{ij})$. In this paper, we adopt the left quantile function, which is a more popular way to define a quantile function.

function resembles the qf learned by the seller from a buyer's previous bids. From the seller's side, he knows that $\widetilde{v}_i(\cdot)$ could be different from $v_i(\cdot)$, which captures the terminology "unsureness". We assume that the reported qf for any buyer is bounded, strictly increasing and differentiable. The profile of reported quantile functions is denoted as $\widetilde{\boldsymbol{v}} = (\widetilde{v}_1(\cdot), \cdots, \widetilde{v}_n(\cdot)) \in \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \times \cdots \times \widetilde{\mathcal{F}}_n$.

We now define the virtual valuation of a reported quantile function. Suppose that for some buyer i, the cumulative distribution function (cdf) and probability distribution function (pdf) of $\widetilde{v}_i(\cdot)$ are $\widetilde{F}_i(\cdot)$ and $\widetilde{f}_i(\cdot)$, respectively. Then, the virtual valuation maps her value v to $v - \frac{1-\widetilde{F}_i(v)}{\widetilde{f}_i(v)}$. We can simply derive the equivalent representation with quantile as input by replacing v with $\widetilde{v}_i(x)$,

$$v - \frac{1-\widetilde{F}_i(v)}{\widetilde{f}_i(v)} = \widetilde{v}_i(x) - \frac{1-x}{\widetilde{f}_i(\widetilde{v}_i(x))} = \widetilde{v}_i(x) - (1-x)\widetilde{v}_i'(x).$$

As a result, we define the reported virtual valuation of buyer i as $\widetilde{\psi}_i(x) = \widetilde{v}_i(x) - (1-x)\widetilde{v}_i'(x)$. Notice that the quantile positively correlates with the value. Therefore, a buyer's reported qf is (strictly) *regular* if and only if the reported virtual valuation with quantile x as input is (strictly) increasing. In this work, the reported virtual valuation profile is written as $\widetilde{\boldsymbol{\psi}} = (\widetilde{\psi}_1(\cdot), \cdots, \widetilde{\psi}_n(\cdot))$.

We further suppose that each buyer i has a deterministic bidding strategy $b_i(\cdot)$, such that when the real quantile drawn is q_i, she bids $b_i(q_i)$ instead. Note that the quantile she bids, i.e. $b_i(q_i)$ should resemble a quantile from the seller's view in the prior manipulation model. Therefore, we require $b_i(q_i) \sim U[0,1]$, which is equivalent to requiring that $b_i(\cdot) \in \mathcal{S}$ where $\mathcal{S} = \{g(\cdot) \mid g(U[0,1]) = U[0,1]\}$. In other words, $b_i(\cdot)$ should map a uniform distribution on $[0,1]$ to itself (see also [14–16]).

With the above notations and reasoning, we can clarify a buyer's strategy in the prior manipulation model.

Definition 2.1 (A buyer's strategy)**.** In the prior manipulation model, for each buyer i, her strategy consists of the following two parts:

- $\widetilde{v}_i(\cdot) \in \widetilde{\mathcal{F}}_i$, which is the quantile function she would report to the seller.
- $b_i(\cdot) \in \mathcal{S}$, which maps a quantile to another. This is the bidding strategy she would take.

We need to mention here that in Section 2.3, we will further show that for auction mechanism families with monotone allocation rule (which include all mechanisms we discuss in this paper), any buyer i's best bidding strategy $b_i(\cdot) \in \mathcal{S}$ is always truthful-bidding. As a result, a buyer's strategy can be simplified to only include a reported qf.

2.2 On Seller's Auction Mechanism

We now come to model an auction mechanism. We use $\mathcal{M}^{\widetilde{\boldsymbol{v}}} = (X^{\widetilde{\boldsymbol{v}}}, P^{\widetilde{\boldsymbol{v}}})$ to denote a (direct) auction mechanism, which encodes the reported qf profile $\widetilde{\boldsymbol{v}}$. Here, $X^{\widetilde{\boldsymbol{v}}} : [0,1]^n \to \Delta^n$ is the allocation function and $P^{\widetilde{\boldsymbol{v}}} : [0,1]^n \to \mathbb{R}^n$ is the payment function. For succinctness, we suppose that instead of bidding a value for the item, each buyer submits a quantile to the seller. The seller can simply maps the quantile to the corresponding value with the reported qf.

We now write $b(q) = (b_1(q_1), \cdots, b_n(q_n))$ as buyers' bidding profile, and let $X_i(b(q))$ and $P_i(b(q))$ be the corresponding allocation and payment of buyer i. Given buyers' bidding strategy profile $\boldsymbol{b} = (b_1(\cdot), \cdots, b_n(\cdot)) \in \mathcal{S}^n$ and buyer i's quantile q_i, we define the *interim* allocation and payment of buyer i respectively as follows:

$$x_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}) = \mathbb{E}_{q_{-i}}\left[X_i^{\widetilde{\boldsymbol{v}}}(b(q))\right], \quad p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}) = \mathbb{E}_{q_{-i}}\left[P_i^{\widetilde{\boldsymbol{v}}}(b(q))\right].$$

Here, b_{-i} stands for the profile of *bidding strategies* of buyers other than i. As a consequence, the interim revenue of buyer i with quantile q_i is:

$$u_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}) = x_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}) \cdot v_i(q_i) - p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}).$$

In particular, when all buyers bid truthfully, we slightly abuse the notation, and use $x_i^{\widetilde{\boldsymbol{v}}}(q_i)$ and $p_i^{\widetilde{\boldsymbol{v}}}(q_i)$ to correspondingly represent the interim allocation and payment of buyer i with quantile q_i.

Meanwhile, in this paper, we suppose that all mechanisms guarantee that each buyer does not exceed her budget in expectation. We formalize this restriction in the following definition.

Definition 2.2 (Budget feasibility)**.** Given budget profile B, a mechanism family $\{\mathcal{M}^{\boldsymbol{g}} \mid \boldsymbol{g} \in \widetilde{\mathcal{F}}\}$ is *ex-ante budget-feasible* (or *budget-feasible* for short), if for any reported qf profile $\widetilde{\boldsymbol{v}} \in \widetilde{\mathcal{F}}$, bidding profile $\boldsymbol{b} \in \mathcal{S}^n$, the corresponding mechanism $\mathcal{M}^{\widetilde{\boldsymbol{v}}}$ satisfies:

$$T \int_0^1 p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}) \, \mathrm{d}q_i \le B_i, \quad \forall 1 \le i \le n. \tag{BF}$$

Here, it is essential to notice that under the definition of budget-feasibility, we implicitly suppose that buyers bid according to $\widetilde{\boldsymbol{v}}$ the reported qf profile (i.e., $\boldsymbol{b} \in \mathcal{S}^n$). The reason for such assumption is that, for almost all auction forms that control the budget of buyers, the reported qf profile and budget profile are encoded into the mechanism and help to compute critical parameters of the mechanism (e.g., γ for BROA, β for BDFPA, and α for PFPA, as we will discuss later). As a result, if a buyer does not bid according to her reported qf, the mechanism may no longer satisfy the budget constraint.

We now come to define other common properties that an auction mechanism should reach. These properties are defined under the *classic* setting that appears in economic literature, where the buyers' strategies are not confined, i.e., $b_i(\cdot) \in \mathcal{T} := \{g(\cdot) \mid g \colon [0,1] \to [0, +\infty)\}$.

We first describe the individual-rationality of the mechanism. Concretely,

Definition 2.3 (IR)**.** A mechanism family $\{\mathcal{M}^{\boldsymbol{g}} \mid \boldsymbol{g} \in \widetilde{\mathcal{F}}\}$ is *individual-rational*, if for any reported qf profile $\widetilde{\boldsymbol{v}} \in \mathcal{F}$, true qf profile $\boldsymbol{v} = \widetilde{\boldsymbol{v}}$, and bidding strategy profile $\boldsymbol{b} \in \mathcal{T}^n$, the corresponding mechanism $\mathcal{M}^{\widetilde{\boldsymbol{v}}}$ satisfies:

$$u_i^{\widetilde{\boldsymbol{v}}}(q_i, b_{-i}) \ge 0, \quad \forall q_i \in [0,1], \, 1 \le i \le n. \tag{IR}$$

We now want to characterize the incentive-compatibility of a mechanism when buyers are budget-constrained. To realize this part, we first use b^{TR} to represent the truthful bidding strategy, i.e., b^{TR} is the identical mapping from $[0,1]$ to $[0,1]$. We now give the following definition:

Definition 2.4 (BCIC [5])**.** A mechanism family $\{\mathcal{M}^{\boldsymbol{g}} \mid \boldsymbol{g} \in \widetilde{\mathcal{F}}\}$ is *budget-constraint incentive-compatible*, if for any reported qf profile $\widetilde{\boldsymbol{v}} \in \mathcal{F}$ and budget profile B, the following is satisfied by the corresponding mechanism $\mathcal{M}^{\widetilde{\boldsymbol{v}}}$:

$$\begin{aligned} b^{\mathrm{TR}} \in \arg\max_{b_i \in \mathcal{T}} & \, x_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}^{\mathrm{TR}}) \cdot \widetilde{v}_i(q_i) - p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}^{\mathrm{TR}}), \\ \text{s.t.} \quad & T \int_0^1 p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i), b_{-i}^{\mathrm{TR}}) \, \mathrm{d}q_i \le B_i, \quad \forall 1 \le i \le n. \end{aligned} \tag{BCIC}$$

In common words, a mechanism is BCIC if, for any buyer under the budget constraint, the best bidding strategy is bidding truthfully when other buyers do so.

Again, we mention that the seller has no idea whether he receives true quantile functions. Therefore, from a seller's view, when we define the above properties (BF), (IR) and (BCIC), it is necessary to let these properties hold concerning the reported quantile functions rather than the true ones.

2.3 Simplifying Buyers' Strategies

Although buyers' bidding strategy profile $\boldsymbol{b} := (b_1(\cdot), \cdots, b_n(\cdot))$ directly gets involved in the expression of buyers' revenue (as in Section 2.4), we give an essential lemma, showing that for auction mechanism families with monotone allocation rule [31], i.e., any buyer's probability of winning as a function of her bid is non-decreasing with others' bids fixed, we do not need to care about $\boldsymbol{b}$. [2] Specifically, we prove that, in this case, any buyer's best bidding strategy is truthful-bidding. We mention that a similar result was first presented in [36], and also used in [14]. However, their proofs of the result are not rigorous, and we give a refined proof in Appendix A.1.

Lemma 2.1. *For a mechanism family $\{\mathcal{M}^{\boldsymbol{g}} \mid \boldsymbol{g} \in \widetilde{\mathcal{F}}\}$, suppose for any reported qf profile $\widetilde{\boldsymbol{v}}$, $\mathcal{M}^{\widetilde{\boldsymbol{v}}} = (X^{\widetilde{\boldsymbol{v}}}, P^{\widetilde{\boldsymbol{v}}})$ satisfies that $X^{\widetilde{\boldsymbol{v}}}$ is monotone, i.e., for any buyer $1 \leq i \leq n$, $X_i^{\widetilde{\boldsymbol{v}}}(b(q))$ is non-decreasing with $b_i(q_i)$. Given $\widetilde{\boldsymbol{v}}$, for any buyer $1 \leq i \leq n$, her revenue in the prior manipulation model is maximized if her bidding strategy $b_i(\cdot)$ is the identical mapping from $[0, 1]$ to itself.*

All the mechanism families we discuss in this work have a monotone allocation rule. As a consequence of Lemma 2.1, the best bidding strategy for each buyer is truthful-bidding. Therefore, in the rest of this paper, we suppose that $b_i(\cdot)$ is the identical mapping from $[0, 1]$ to itself, and the strategy of each buyer i *degenerates* to only reporting a quantile function $\widetilde{v}_i(\cdot) \in \widetilde{\mathcal{F}}_i$. Concretely, we explicitly define the simplified strategy of a buyer in correspondence to Definition 2.1.

Definition 2.5 (A buyer's simplified strategy for mechanism families with monotone allocation rule)**.** In the prior manipulation model, when the auction mechanism family has a monotone allocation rule, then for each buyer i, her strategy is $\widetilde{v}_i(\cdot) \in \widetilde{\mathcal{F}}_i$, which is the quantile function she would report to the seller.

2.4 Execution of Repeated Auctions under Budget Constraints

With the above reasoning, we now give a typical procedure for executing a repeated auction instance under the prior manipulation model for a better understanding. Here, as we already presented in Section 2.3, for mechanism families with monotone allocation rule, we can degenerate a buyer's strategy to a single reported qf. Since in this paper, we only consider such mechanism families, therefore, we explicitly do the simplification in the following procedure.

1. Buyers truthfully submit the budget profile $B = (B_1, \cdots, B_n)$ to the seller, then the seller commits to a mechanism family *with monotone allocation rule* $\{\mathcal{M}^{\boldsymbol{g}} \mid \boldsymbol{g} \in \widetilde{\mathcal{F}}\}$, with the reported qf profile as an input to the mechanism. We implicitly suppose that the mechanism family is budget-feasible (Definition 2.2).

2. For each buyer $1 \leq i \leq n$, her true quantile profile is $v_i(\cdot)$, which is private. Further, she chooses a strategy $\widetilde{v}_i(\cdot) \in \widetilde{\mathcal{F}}_i$. She then reports $\widetilde{v}_i(\cdot)$ to the seller. Here, we should mention that it is also *possible* for the seller to already know $\widetilde{\boldsymbol{v}} = (\widetilde{v}_1(\cdot), \cdots, \widetilde{v}_n(\cdot))$ before he commits to the mechanism family.

3. The mechanism $\mathcal{M}^{\widetilde{\boldsymbol{v}}} = (X^{\widetilde{\boldsymbol{v}}}, P^{\widetilde{\boldsymbol{v}}})$ is run. The revenue of the seller is $\sum_{i=1}^{n} \int_0^1 (P_i^{\widetilde{\boldsymbol{v}}}(q) - \lambda) \cdot X_i^{\widetilde{\boldsymbol{v}}}(q)\,\mathrm{d}q$. The revenue of buyer $1 \leq i \leq n$ is $\int_0^1 u_i^{\widetilde{\boldsymbol{v}}}(q_i)\,\mathrm{d}q_i$.

[2] Here, we extend the definition of monotonicity of an allocation rule for a single mechanism in [32] to a mechanism family. That is, all instances in the mechanism family have a monotone allocation rule.

An important issue to mention here is the exact time point when the seller knows the reported qf profile. In real-life, both two cases given in the above procedure are possible. If the seller wants to turn to another auction form after interacting with buyers for a long time, then it is more likely that the seller knows the reported qf profile before executing the change. Meanwhile, if the seller just starts to trade with the buyers, then he does not know the bid distribution before choosing the auction form. In our paper, such a subtle difference leads to two sets of results, which are discussed correspondingly in Section 4.1 and Section 4.2. In the following part (Section 2.5), we will formally state the foundations of how such a detail induces different results.

2.5 Dominance and Equivalence of Auction Mechanisms

At the end of this section, we define a dominance and an equivalence relationship among auction mechanism families. This work mainly considers two relationships between two auction mechanism families: dominance under fixed reported qf profiles and equivalence under strategic reported qf profiles.

Dominance under fixed reported qf profiles captures the scene when the seller already knows the reported qf profile before choosing the mechanism. (See the execution of a repeated auction instance in Section 2.4.) In this case, we say mechanism family $\mathcal{M}_1$ f-dominates $\mathcal{M}_2$ if the former brings no lower revenue for the seller given any fixed reported qf profile $\widetilde{\boldsymbol{v}}$. Formally, we present the following definition.

Definition 2.6 (f-dominance)**.** We say mechanism family $\mathcal{M}_1$ dominates $\mathcal{M}_2$ under fixed reported qf profiles (or $\mathcal{M}_1$ *f-dominates* $\mathcal{M}_2$ for short), written as $\mathcal{M}_1 \succeq_{\mathrm{F}} \mathcal{M}_2$, if for any reported qf profile $\widetilde{\boldsymbol{v}}$, $\mathcal{M}_1^{\widetilde{\boldsymbol{v}}}$ brings *no less* revenue for the seller than $\mathcal{M}_2^{\widetilde{\boldsymbol{v}}}$.

We now define the equivalence relationship under strategic reported qf profiles, or s-equivalence for short. Unlike f-dominance, s-equivalence works when the seller does not know the reported qf profile in advance. Formally, we define s-equivalence as follows:

Definition 2.7 (s-equivalence)**.** For mechanism families $\mathcal{M}_1$ and $\mathcal{M}_2$, we write $\mathcal{M}_1 \succeq_{\mathrm{S}} \mathcal{M}_2$, if for any reported qf profile $\widetilde{\boldsymbol{v}}^{(2)}$, there is a reported qf profile $\widetilde{\boldsymbol{v}}^{(1)}$, such that for the seller and all buyers, $\mathcal{M}_1^{\widetilde{\boldsymbol{v}}^{(1)}}$ and $\mathcal{M}_2^{\widetilde{\boldsymbol{v}}^{(2)}}$ bring *the same* revenue. We say mechanism family $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent under strategic reported qf profiles (or $\mathcal{M}_1$ and $\mathcal{M}_2$ are *s-equivalent* for short), written as $\mathcal{M}_1 =_{\mathrm{S}} \mathcal{M}_2$, if $\mathcal{M}_1 \succeq_{\mathrm{S}} \mathcal{M}_2$ and $\mathcal{M}_2 \succeq_{\mathrm{S}} \mathcal{M}_1$.

Here, we can see that $\mathcal{M}_1 \succeq_{\mathrm{S}} \mathcal{M}_2$ stands for a higher "expressing ability" for $\mathcal{M}_1$, that is, for any specific instance in family $\mathcal{M}_2$, there is an instance in family $\mathcal{M}_1$ such that the two instances are equivalent for all participants. Naturally, we are more interested in the s-equivalence result between $\mathcal{M}_1$ and $\mathcal{M}_2$, which intuitively means that, when the seller has no information on the reported qf profile $\widetilde{\boldsymbol{v}}$ beforehand, $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same "express ability". More specifically, if we view the auction under the prior manipulation model as a game among the buyers and the seller, then $\mathcal{M}_1 =_{\mathrm{S}} \mathcal{M}_2$ states that the induced games by $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent within a mapping of buyers' strategies. Consequently, these two induced games own the same revenue space, and of more importance and interest, the same Bayesian Nash equilibria (BNE) outcome, if any exists.

3 Mechanisms

In this section, we will come into five types of specific auction forms and their features given the reported qf profile. One is the optimal BCIC auction, given by [5]. Two are variants of first-price auction, and the other two are variants of second-price auction. Most importantly, in Section 3.2, we introduce bid-discount first-price auction (BDFPA). We show that this mechanism has desirable properties, including that (1) seller gets optimal revenue when buyers get optimally allocated, (2) the revenue-optimal instance is computationally tractable. These results show that BDFPA is an essential-yet-ignored budget-constrained auction. All omitted proofs in this section can be found in Appendix B.

3.1 Bayesian Revenue-Optimal BCIC Auction

To start this section, we first review the optimal mechanism subject to budget constraints. The following proposition presented in [5] describes the Bayesian revenue-optimal BCIC auction. The mechanism is similar to Myerson auction to a large extent.

Proposition 3.1 (Theorem 3.5 in [5])**.** *Given buyers' profile of reported quantile functions* $\widetilde{\boldsymbol{v}}$ *and budgets* B*, suppose that the value function* $\widetilde{v}_i(\cdot)$ *is* strictly regular *for any buyer* $1 \le i \le n$*. Then, the Bayesian revenue-optimal BCIC auction (BROA for short)* $\mathcal{M}^{\mathrm{BROA}}$ *is characterized as follows:*

- *Let* γ^* *be the optimal solution of*

$$\min_{\gamma \in [0,1]^n} \left\{ T\mathbb{E}_q \left[\max_i \left\{ (1-\gamma_i)\widetilde{\psi}_i(q_i) - \lambda \right\}^+ \right] + \sum_{i=1}^n \gamma_i B_i \right\}; \tag{1}$$

- $X_i(q) \in \{0, 1\}$ *for any* $1 \le i \le n$*;*
- $X_i(q) = 1$ *only if* $(1-\gamma_i^*)\widetilde{\psi}_i(q_i) \ge \max_{i' \ne i} \left\{ (1-\gamma_{i'}^*)\widetilde{\psi}_i(q_{i'}), \lambda \right\}$*;*
- $\sum_{i=1}^n X_i(q) = 1$ *if and only if there exists* i *such that* $(1-\gamma_i^*)\widetilde{\psi}_i(q_i) \ge \lambda$*;*
- $P_i(q) = \min \left\{ \widetilde{v}_i(z) : (1-\gamma_i^*)\widetilde{\psi}_i(z) \ge \max_{i' \ne i} \{ (1-\gamma_{i'}^*)\widetilde{\psi}_i(q_{i'}), \lambda \} \right\}$ *if* $X_i(q) = 1$*, and* $P_i(q) = 0$ *otherwise.*

As we can see, the optimal solution γ^* of programming (1) plays an essential role in the mechanism. We will dig into the properties of the tuple γ^* in the proof of our major results. We further should mention that under the stochastic setting that we discuss in this work, the event that at least two buyers have the same highest effective bid has Lebesgue measure zero. Therefore, in the calculating part, we do not consider such a possibility (under which the exact winner of the auction is undefined). The same reasoning also holds for other auction types we discuss later.

3.2 Bid-Discount in First-Price Auction

In this part, we introduce an important variant of first-price auction, which we call bid-discount first-price auction (BDFPA). Although the bid-discount method has been considered in sponsored search auction [1, 18, 28] and second-price auction [22, 32], to the best of our knowledge, BDFPA under budget constraints has not been well-studied in previous works. It is similar to pacing in first-price auction (which is to be presented later), one of the most widely considered variants of first-price auction, but with a different payment rule. Under bid-discount first-price auction, the winner is asked to pay exactly her bid, rather than the paced bid for pacing first-price auction. Though it seemed unreasonable at first sight, our results show that bid-discount first-price auction is a powerful mechanism. Further, this mechanism connects the optimal auction BROA and other variants of first-price/second-price auction.

We now formally define the bid-discount method in repeated first-price auctions.

Definition 3.1 (Bid-discount in first-price auction)**.** Suppose buyers' profile of reported quantile functions $\widetilde{\boldsymbol{v}}$ and budgets B are given. We define $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ to be the mechanism (X, P) together with an auxiliary tuple of bid-discount multipliers $\beta = (\beta_i)_{1\le i\le n} \in [0,1]^n$ as follows.

- [binary allocation for each buyer.] $X_i(q) \in \{0,1\}$ for any $1 \le i \le n$;
- [buyer with highest discount-bid wins.] $X_i(q) = 1$ only if $\beta_i \widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}$;
- [item sold when demanded.] $\sum_{i=1}^n X_i(q) = 1$ if and only if there exists i such that $\beta_i \widetilde{v}_i(q_i) \ge \lambda$;
- [first-price.] $P_i(q) = \widetilde{v}_i(q_i)$ if and only if $X_i(q) = 1$, and $P_i(q) = 0$ otherwise;
- [budget-feasibility.] $T\int_0^1 p_i(q_i)\,\mathrm{d}q_i \le B_i$ holds for any $1 \le i \le n$.

Moreover, we say the bid-discount mechanism is *efficient*, if the following holds:

- [no over-discount.] If $T\int_0^1 p_i(q_i)\,\mathrm{d}q_i < B_i$ holds for some $1 \le i \le n$, then $\beta_i = 1$.

The first four points link first-price auction in conjunction with bid-discount multipliers in the above definition. The fifth point captures the necessity that a buyer's bid-discount multiplier should match her budget. Nevertheless, such a constraint alone may be too loose, as when the bid-discount multipliers are too low, the market is shrunk, the buyers do not get sufficient allocation, and the seller's revenue is cut. On the other side, the last point guarantees that no over-discount occurs. In other words, if the seller chooses to lessen some buyer's bid, then the buyer will exactly exhaust her total budget in expectation. We should mention here that the definition of efficiency is already adopted by multiple previous works (e.g., [5,9,10,12,13]) when analyzing the equilibrium of different budget control auctions. However, these papers do not use the notion of "efficiency".

We first show in Theorem 3.2 that for BDFPA, efficiency directly leads to revenue-maximizing for the seller. In fact, efficiency implies that each buyer sufficiently exhausts her budget and gets as much allocation as possible. In the meantime, the seller's goal is to maximize his revenue. Therefore, this theorem is essential in the sense that the seller reaches optimality as long as all the buyers are efficiently allocated in BDFPA. This result indicates that BDFPA, though not well-studied in the past, is a variant of repeated first-price auctions worthy of serious attention.

Theorem 3.2. *For BDFPA, efficiency implies revenue maximization for the seller.*

In other words, in the prior manipulation model, if a seller chooses a BDFPA scheme, his best strategy is to run an efficient BDFPA scheme. This observation is given in the following corollary.

Corollary 3.3. *Under the prior manipulation model, if the seller's strategy is confined to a BDFPA scheme, then committing to an efficient BDFPA scheme is a dominating strategy.*

From the computation side, we further show that an efficient tuple of bid-discount multipliers can be achieved by finding the global minimum of a convex function.

Theorem 3.4. *For BDFPA, an efficient tuple of multipliers can be computed by solving the global minimum of a convex function.*

This result, combined with Theorem 3.2, demonstrates that BDFPA is a full-fledged auction mechanism facing budget-constrained buyers on the following two sides: (1) All the buyers and the seller simultaneously achieve optimality (Theorem 3.2), and (2) such optimality can be efficiently computed (Theorem 3.4).

We now dig into more features of efficient BDFPA. For the following properties, we further suppose that the reported qf $\widetilde{v}_i(\cdot)$ of each buyer $1 \leq i \leq n$ is *inverse Lipschitz continuous.* Here, inverse Lipschitz continuity means that the inverse of the reported distribution function (i.e., the cdf of value, $\widetilde{F}_i(\cdot)$) is Lipschitz continuous. We remark that as the seller usually learns the reported quantile functions with parameterized models, the above assumptions are natural considering that the seller will adopt a rather simple model, e.g., truncated power-law distribution or Gaussian distribution for the pdf.

We now give the following theorem, which characterizes efficient BDFPA mechanisms and corresponding bid-discount multiplier tuples from multiple aspects.

Theorem 3.5. *Given buyers' profile of reported quantile functions $\widetilde{\boldsymbol{v}}$ and budgets B, if every buyer $1 \leq i \leq n$'s reported qf $\widetilde{v}_i(\cdot)$ is inverse Lipschitz continuous, then the following statements hold:*

- *There exists a* maximum *tuple of bid-discount multipliers $\beta^{\max}$, i.e., for any feasible tuple of bid-discount multipliers β, $\beta_i^{\max} \geq \beta_i$ for any $1 \leq i \leq n$.*
- *$\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ is efficient.*
- *For any efficient bid-discount multiplier tuple β^{e}, the following two conditions are satisfied:*
  - *There exists some $\nu \leq 1$, such that for any $1 \leq i \leq n$ satisfying $p_i^{\max}$ (buyer i's expected payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$) is positive, $\beta_i^{\mathrm{e}}/\beta_i^{\max} = \nu$;*
  - *For any $1 \leq i \leq n$ satisfying $p_i^{\max} = 0$, $p_i^{\mathrm{e}} = 0$ (buyer i never wins in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$) and $\beta_i^{\mathrm{e}} = \beta_i^{\max} = 1$.*
- *All efficient BDFPAs bring the* same *payment for each buyer.*
- *$\beta^{\max}$ is the unique efficient tuple of bid-discount multipliers if and only if one of the following two conditions is satisfied: (1) $\max_{i \in \mathcal{I}_1} \beta_i^{\max} \widetilde{v}_i(0) \leq \max_{i \in \mathcal{I}_2}\{\beta_i^{\max} \widetilde{v}_i(1), \lambda\}$, where $\mathcal{I}_1 = \{i \mid p_i^{\max} > 0\}$ and $\mathcal{I}_2 = [n] \setminus \mathcal{I}_1$, or (2) there exists $i \in \mathcal{I}_1$ such that $p_i^{\max} < B_i$.*

Here, we once again point out that the inverse Lipschitz continuity assumption on $\widetilde{v}_i(\cdot)$ is a rather weak one, which is always regarded as default in previous works, e.g., [5,19,27]. In the proof of Theorem 3.5, this assumption guarantees that when a buyer's bid-discount multiplier increases a small amount, her payment does not change too much as well. This continuity property indicates that the budget-feasible tuples form a closed set, and serves as a key lemma for us to prove the first two statements of the theorem. As for characterizing the set of efficient multiplier tuples, we observe that efficiency is in fact a much tight restriction. Specifically, for any buyer, her payment should be the same across all efficient mechanisms, or else at least one buyer would face a difference in payment between two efficient BDFPAs, contradicting the definition. This essential observation helps with the remaining three statements. On top of the theorem, the existence and efficiency of the maximum tuple of multipliers given in Theorem 3.5 allow us to restrict our attention to this special tuple, as long as the inverse Lipschitz continuity assumption on $\widetilde{v}_i(\cdot)$ establishes.

3.3 Pacing in First-Price Auction

A well-studied budget management strategy is pacing. Resembling bid-discount introduced above, pacing assigns a multiplier to each buyer, which decreases the chance of winning the item and payment. We mention here that pacing is extensively studied in previous works either for first-price auction [5, 12] or for second-price auction [13]. We note that in the past literature, pacing is also known as bid-shading [5]. Here we introduce the formal definition of pacing in first-price auction under the stochastic setting.

Definition 3.2 (Pacing in first-price auction)**.** Suppose buyers' profile of reported quantile functions $\widetilde{\boldsymbol{v}}$ and budgets B are given. We define $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ to be the mechanism (X, P) together with an auxiliary tuple of pacing multipliers $\alpha = (\alpha_i)_{1\le i\le n} \in [0,1]^n$ as follows.

- $X_i(b(q)) \in \{0,1\}$ for any $1 \le i \le n$;
- $X_i(q) = 1$ only if $\alpha_i\widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\alpha_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}$;
- $\sum_{i=1}^{n} X_i(q) = 1$ if and only if there exists i such that $\alpha_i\widetilde{v}_i(q_i) \ge \lambda$;
- $P_i(q) = \alpha_i\widetilde{v}_i(q_i)$ if and only if $X_i(q) = 1$, and $P_i(q) = 0$ otherwise;
- $T\int_0^1 p_i(q_i)\,\mathrm{d}q_i \le B_i$ holds for any $1 \le i \le n$.

Moreover, we say the pacing mechanism is *efficient*, if the following holds:

- If $T\int_0^1 p_i(q_i)\,\mathrm{d}q_i < B_i$ holds for some $1 \le i \le n$, then $\alpha_i = 1$.

We see that PFPA shares a similar allocation/payment function with BDFPA, except that for PFPA, the multiplier remains in the payment, which is not the case for BDFPA. As a result, PFPA exhibits a similar property as BDFPA. With the technique and preparations for Theorem 3.2 in hand, we now prove that an efficient maximum multiplier also exists for PFPA. Further, since a buyer's payment in PFPA is explicitly related to her multiplier, the efficient tuple of pacing multipliers is unique under weak assumptions.

Theorem 3.6. *Given buyers' profile of reported quantile functions $\widetilde{\boldsymbol{v}}$ and budgets B, if every buyer $1 \le i \le n$'s reported qf $\widetilde{v}_i(\cdot)$ is inverse Lipschitz continuous, then:*

- *There exists a* maximum *tuple of pacing multipliers $\alpha^{\max}$, i.e., for any feasible tuple of pacing multipliers α, $\alpha_i^{\max} \ge \alpha_i$ for any $1 \le i \le n$.*
- *$\alpha^{\max}$ is the* unique *efficient tuple of pacing multipliers.*
- *$\alpha^{\max}$ maximizes seller's revenue among all feasible tuples of pacing multipliers, i.e., $\alpha^{\max}$ is the optimal solution of the following programming:*

$$\max_{\alpha\in[0,1]^n} T\cdot\int_q \max_i\{\alpha_i\widetilde{v}_i(q_i)-\lambda\}^+\,\mathrm{d}q,$$
$$\text{s.t.}\quad T\cdot\int_0^1 \alpha_i\widetilde{v}_i(q_i)\cdot\left(\int_{q_{-i}} I\left[\alpha_i\widetilde{v}_i(q_i)\ge\max_{i'\neq i}\{\alpha_{i'}\widetilde{v}_{i'}(q_{i'}),\lambda\}\right]\mathrm{d}q_{-i}\right)\mathrm{d}q_i \le B_i,\quad \forall 1\le i\le n. \tag{2}$$

Both two variants of first-price auctions we cover in this section have simple forms and precise computation-wise characterization concerning efficiency. Nevertheless, they are non-examples of incentive-compatible mechanisms, even in the efficient case.

Lemma 3.7. *BDFPA and PFPA are not BCIC, even in the efficient case.*

3.4 Variants of Second-Price Auction

We further consider two variants of second-price auction, which correspond to BDFPA and PFPA, respectively. We refer them to BDSPA and PSPA. We here mention that both two variants have been already studied in literature. (See Section 1.2.) We now define these two types of auction.

Definition 3.3 (Bid-discount in second-price auction)**.** With bid-discount multipliers $(\beta_i)_{1\leq i\leq n}$ and quantile profile q, the allocation rule of BDSPA is the same with BDFPA. However, the payment of winner i is $\max_{i'\neq i}\{\beta_{i'}v_{i'}(q_{i'}),\lambda\}/\beta_i$. Define the efficiency of BDSPA similarly with the efficiency of BDFPA.

Definition 3.4 (Pacing in second-price auction)**.** With pacing multipliers $(\alpha_i)_{1\leq i\leq n}$ and quantile profile q, the allocation rule of PSPA is the same with PFPA. However, the payment of winner i is $\max_{i'\neq i}\{\alpha_{i'}v_{i'}(q_{i'}),\lambda\}$. Define the efficiency of PSPA similarly with the efficiency of PFPA.

The following lemma depicts the incentive compatibility of the above two variants of second-price auctions facing budget constraints.

Lemma 3.8. *BDSPA is unconditionally incentive-compatible, and therefore, efficient BDSPA is BCIC. Efficient PSPA is BCIC. However, PSPA is not IC unconditionally.*

4 Relationships under Prior Manipulation Model

In this section, we will discuss how the five mechanisms are f-dominated by (Definition 2.6) and s-equivalent to (Definition 2.7) each other under the prior manipulation model. Specifically, we will show the f-dominance relationships in Section 4.1 and provide the s-equivalence relationships in Section 4.2.

4.1 Dominance Relations with Fixed Priors

To start with, we consider the case when the seller knows the reported qf profile before he commits to an auction mechanism family. (See Section 2.4.) We now show that when buyers' quantile functions are fixed, the seller always gains a higher revenue by choosing an efficient BDFPA scheme than the BROA scheme. We remind the readers that BROA is the revenue-optimal BCIC auction when buyers cannot manipulate their distributions. Efficient BDFPA, however, is a non-BCIC first-price auction with naturally defined multipliers. Formally, we have the following important theorem.

Theorem 4.1. *eBDFPA* $\succeq_{\mathrm{F}}$ *BROA.*

This result might be anti-intuition at first sight. However, concerning that BROA is BCIC while BDFPA is non-IC, it is not strange that the proposition establishes. We will give an instance in Example 4.1 for a better understanding of the result. Here, we would emphasize again how this theorem is closely related to actual circumstances. The theorem shows that choosing efficient BDFPA is always better than choosing BROA for any given reported qf profile. This condition fits the real world satisfactorily. As is often the case, the seller holds a series of distributions from which buyers' bids are drawn. Consequently, the theorem further implies that the seller should commit to an efficient BDFPA scheme rather than BROA under such an information structure.

We continue to show that when buyers' strategies are fixed, an efficient BDFPA scheme is also better than an efficient PFPA scheme for the seller.

Theorem 4.2. *eBDFPA* $\succeq_{\mathrm{F}}$ *ePFPA.*

Intuitively, when the distributions are fixed, efficient BDFPA dominates efficient PFPA by noting the additional pacing parameter in the payment in PFPA. However, the existence of budgets complicates the issue, and a duality argument helps to prove the dominance result.

We now extend the f-dominance to second-price auctions, taking BROA as a bridge. Due to the budget-constrained incentive-compatibility of efficient BDSPA and efficient PSPA, these two types of auctions are dominated by BROA, which is the optimal BCIC auction.

Lemma 4.3 (Theorem 3.5 in [5])**.** *BROA* $\succeq_{\mathrm{F}}$ *eBDSPA, and BROA* $\succeq_{\mathrm{F}}$ *ePSPA.*

We achieve the following corollary when combining with Theorem 4.1.

Corollary 4.4. *eBDFPA* $\succeq_{\mathrm{F}}$ *eBDSPA, and eBDFPA* $\succeq_{\mathrm{F}}$ *ePSPA.*

The above result states that under the prior manipulation model, given buyers' strategies, efficient BDFPA always brings higher revenue for the seller than variants of second-price auctions. This result highly argues that from the seller's view, when he only sees buyers' bidding distributions, choosing efficient BDFPA as the auction mechanism is better than selecting second-price auctions.

We now give an example to illustrate Theorem 4.1, Theorem 4.2, Lemma 4.3 and Corollary 4.4.

Example 4.1. Now consider a symmetric scenario with $n = 2$ buyers and only $T = 1$ round of auction. Either buyer's true/reported pdf is a uniform distribution on $[0, 1]$ and either buyer's budget is $B_0 = 39/125 = 0.312$. Let the opportunity cost of the seller be $\lambda = 0.1$. Then, the true/reported qf of each buyer is $v_0(\cdot)$ with $v_0(x) = x$ on $[0, 1]$, and the true/reported virtual valuation $\psi_0(\cdot)$ satisfies $\psi_0(x) = 2x - 1$ on $[0, 1]$. Consequently, we have the following for eBDFPA, ePFPA, BROA, eBDSPA and ePSPA respectively:

- For efficient BDFPA, the maximum efficient multiplier tuple $\beta^{\max} = (1/4, 1/4)$. Both buyers exhaust their budgets in expectation, i.e., the expected payment is 0.312 for each buyer, and the seller's expected revenue equals to 0.54.
- For efficient PFPA, the maximum efficient multiplier $\alpha^{\max} = (\alpha_0, \alpha_0)$, where $\alpha_0 \approx 0.937$ is the solution to $1000\alpha^3 - 936\alpha^2 - 1 = 0$. Both buyers also exhaust their budget in expectation, and the seller's expected revenue is approximately 0.525.
- For BROA, the solution to programming (1) is $\gamma^* = (0, 0)$. Either buyer's expected payment is 0.207, and the seller's expected revenue equals to $1377/4000 \approx 0.344$.
- For efficient BDSPA, the efficient multiplier tuple is $\beta^{\mathrm{e}} = (1, 1)$. Either buyer's payment is 0.171, and the seller's expected revenue equals to 0.243.
- For efficient PSPA, the efficient multiplier tuple is also $\alpha^{\mathrm{e}} = (1, 1)$, and therefore, either buyer's payment is 0.171, and the seller's expected revenue equals to 0.243 as well.

4.2 Equivalence Relations with Strategic Priors

We now explore the scene in which the seller does not know the reported qf profile before he announces the auction mechanism family. (See Section 2.4). As a major result, we show in the following theorem that, under minor restrictions, BROA and an efficient BDFPA are equivalent to the seller in this setting.

Theorem 4.5. *When the virtual valuation is strictly increasing and differentiable for each buyer, BROA $=_{\mathrm{S}}$ eBDFPA. On top of the above condition, when both the reported qf and virtual valuation are further inverse Lipschitz continuous for each buyer, BROA $=_{\mathrm{S}}$ eBDFPA holds as well.*

Here, differentiability and inverse Lipschitz continuity of quantile functions and virtual valuations are only minor restrictions, concerning that the parametrized models adopted by the seller for buyers' pdfs usually satisfy high-order continuity. Theorem 4.5, in collaboration with Theorem 4.1, fully characterizes which one of the two mechanisms a seller should choose under different information structures. If the seller knows the reported qf profile in advance, efficient BDFPA would be a better choice than BROA. Otherwise, these two mechanisms are equivalent to the seller under minor restrictions.

From Theorem 4.5, we can derive an exciting result on the optimization side, which gives a natural sufficient condition for programming (1) to have a minimum optimal solution.

Corollary 4.6. *If for each buyer, her reported virtual valuation is further inverse Lipschitz continuous, then programming* (1) *has an* minimum *optimal solution $\gamma^{\min}$, which satisfies that for any optimal solution γ^*, $\gamma_i^{\min} \leq \gamma_i^*$ for any $1 \leq i \leq n$.*

Besides, when we suppose that all buyers are symmetric, we can show that with minor restrictions, eBDFPA and ePFPA are equivalent. This result is established by constructing a mapping between the reported qf profiles under two auction forms.

Theorem 4.7. *When all buyers are symmetric, and their common reported qf is inverse Lipschitz continuous, then eBDFPA $=_{\mathrm{S}}$ ePFPA.*

Resembling Theorem 4.5, Theorem 4.7 indicates that when the seller believes that all buyers are symmetric without having any further knowledge, then choosing efficient BDFPA and efficient PFPA are simply the same.

Surprisingly, we can even extend our s-equivalence results to the two variants of second-price auction, i.e., eBDSPA and ePSPA. In all, we have the following theorem.

Theorem 4.8. *When all buyers are symmetric and their common reported qf is inverse Lipschitz continuous, then eBDFPA $=_{\mathrm{S}}$ ePFPA $=_{\mathrm{S}}$ eBDSPA $=_{\mathrm{S}}$ ePSPA.*

Theorem 4.8 makes the last part of our results. This significant theorem gives the general equivalence of budget control first-price and second-price auctions in the symmetric case when buyers' common prior is unassured.

5 Concluding Remarks

Summary. In this paper, we study five types of auctions that control buyers' budgets: optimal auction satisfying budget-constrained incentive-compatibility, and first-price/second-price auctions adopting the popular pacing strategy and bid-discount strategy correspondingly. Here, the bid-discount strategy has been introduced to sponsored search auctions [1, 18, 28] and second-price auctions [22, 32] respectively, and this work introduces the method to first-price auctions in the budget-constrained setting. We show that bid-discount makes first-price auction a desirable mechanism in the aspects of revenue-maximization and computation. For bid-discount/pacing (BD/P) first-price/second-price auctions (FPA/SPA), we consider the notion of efficiency (similar to the notion of system equilibrium in [5]), which means that each buyer sufficiently spends her budget.

We compare these auctions in the prior manipulation model, in which buyers' reported priors can arbitrarily deviate from the truth. We consider two settings that differ on the time point the seller knows the reported priors of buyers. When the seller knows the reported priors before announcing the mechanism, we show that efficient BDFPA outperforms all other mechanisms regarding the seller's revenue. This setting resembles the scenario where the seller has already collected sufficient information on buyers' preferences. On the other hand, when the seller has no information on buyers' reported priors before deciding the mechanism, we prove that efficient BDFPA is strategically equivalent to the optimal auction. Such a setting captures the scene in which the seller has no prior knowledge of buyers. In this case, when buyers are symmetric, we further demonstrate the general equivalence of all four first-price/second-price auctions in the last setting. Our results provide practical suggestions for mechanism designers when seeking an appropriate auction form facing budget-constrained buyers.

Rethinking the prior manipulation model. From Lemma 2.1, readers may notice the deep relation between the prior manipulation model and the classic concept of incentive compatibility. We remark here that these two models describe the concept of "truthful-bidding" from two different perspectives. For the widely welcomed characterization, IC, the mechanism designer hopes to have buyers bid according to their true distributions. Nevertheless, from the angle of information structure, buyers' true distributions could be utterly blind to the seller if buyers choose to hide their true value. In this sense, all prior-dependent IC mechanisms lose their power in real life. Facing this, the prior manipulation model provides another view. When the designer gives up eliciting buyers' real thoughts but focuses on their historical behaviors, he learns how the buyers would bid, which is transparent and something buyers would never lie on. Such a step "back" shifts the designer's long-desired goal of "truthful-bidding" to the buyers' side, who should consider how to report the distribution more. As a result, designing a mechanism becomes much more accessible.

A Missing Proofs in Section 2

A.1 Proof of Lemma 2.1

Proof. With the other buyers' bidding strategy b_{-i} fixed, write out the revenue of buyer i as

$$\int_0^1 x_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i))v_i(q_i) - p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i))\,\mathrm{d}q_i.$$

Recall that for a uniform distribution preserving function $\phi(x)$ and any Riemann-integrable $g(x)$ we have $\int_0^1 g(\phi(x))\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$ (see e.g. [33] for a proof), hence $\int_0^1 p_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i))\,\mathrm{d}q_i$ is a constant. It suffices to consider the maximum of $\int_0^1 x_i^{\widetilde{\boldsymbol{v}}}(b_i(q_i))v_i(q_i)\,\mathrm{d}q_i$.

The result now follows from a slightly modified version of the Hardy–Littlewood inequality:

Lemma A.1. *Define the non-negative decreasing rearrangement of a non-negative measurable function $r(x)$ supported on $[0,1]$ as $r^*(x) = \inf\{t : \mu_r(t) \le x\}$ where $\mu_r(t) = \mu(\{x \in [0,1] : r(x) \ge t\})$ is the Lebesgue measure of $\{x \in [0,1] : r(x) \ge t\}$. Let the corresponding non-negative increasing rearrangement be $r_*(y) := r^*(1-y)$ for any $y \in [0,1]$. Suppose $r(x), s(x)$ are two non-negative measurable function supported on $[0,1]$, then*

$$\int_0^1 r(x)s(x)\,\mathrm{d}x \le \int_0^1 r^*(x)s^*(x)\,\mathrm{d}x = \int_0^1 r_*(y)s_*(y)\,\mathrm{d}y. \tag{3}$$

Proof of Lemma A.1. Note that for any sequence of non-negative Lebesgue measurable functions $\{r_n\}_{n=1}^{\infty}$, by monotone convergence theorem we have that $r \le \liminf_{n\to\infty} r_n$ (a.e.) implies $r^* \le \liminf_{n\to\infty} r_n^*$. Therefore, it suffices to consider the case where $r(x), s(x)$ are *simple* functions, which are finite linear combinations of indicator functions of measurable sets. Consider

$$r(x) = \sum_{i=1}^{n} r_i \mathbf{1}_{T_i}(x) \tag{4}$$

with $r_1 > r_2 > \cdots > r_n \ge 0 = r_{n+1}$. The corresponding decreasing rearrangement is

$$r^*(x) = \sum_{i=1}^{n} r_i \mathbf{1}_{[t_{i-1},t_i)}(x) \quad \text{where} \quad t_i = \sum_{k=1}^{i} \mu(T_k), t_0 = 0 \quad \text{for } 1 \le i \le n \tag{5}$$

by definition. Now rewrite $c_i = r_i - r_{i+1}$ and $R_i = \bigcup_{k=1}^{i} T_k$. Equation (4) and Equation (5) become

$$r(x) = \sum_{i=1}^{n} c_i \mathbf{1}_{R_i}(x), \quad r^*(x) = \sum_{i=1}^{n} c_i \mathbf{1}_{[0,\mu(R_i))}(x).$$

We also apply this transformation to $s(x)$ and obtain

$$s(x) = \sum_{j=1}^{m} d_j \mathbf{1}_{S_j}(x), \quad s^*(x) = \sum_{j=1}^{m} d_j \mathbf{1}_{[0,\mu(S_j))}(x).$$

Here $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n$ and $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m$ are Lebesgue measurable sets, and $c_i, d_j \ge 0$ for any $1 \le i \le n$ and $1 \le j \le m$.

Finally we bound as follows:

$$\begin{aligned}\int_0^1 r(x)s(x)\,\mathrm{d}x &= \sum_{i=1}^n c_i \int_{R_i} s(x)\,\mathrm{d}x = \sum_{i=1}^n c_i \sum_{j=1}^m d_j \mu(R_i \cap S_j)\\ &\le \sum_{i=1}^n c_i \sum_{j=1}^m d_j \min\{\mu(R_i), \mu(S_j)\}\\ &= \sum_{i=1}^n c_i \sum_{j=1}^m d_j \int_0^{\mu(R_i)} \mathbf{1}_{[0,\mu(S_j))}(x)\,\mathrm{d}x = \sum_{i=1}^n c_i \int_0^{\mu(R_i)} s^*(x)\,\mathrm{d}x\\ &= \int_0^1 \sum_{i=1} c_i \mathbf{1}_{[0,\mu(R_i))}(x) s^*(x)\,\mathrm{d}x = \int_0^1 r^*(x)s^*(x)\,\mathrm{d}x.\end{aligned}$$

This establishes the first inequality in (3), and the second equality follows from the change-of-variable $y := 1 - x$. □

Since $b_i(\cdot)$ maps $U[0,1]$ to $U[0,1]$ and $x_i^{\widetilde{\boldsymbol{v}}}(q_i), v_i(q_i)$ are both increasing in q_i, we have $\left(x_i^{\widetilde{\boldsymbol{v}}} \circ b_i\right)_* (q_i) = x_i^{\widetilde{\boldsymbol{v}}}(q_i)$ and $(v_i)_*(q_i) = v_i(q_i)$. By Lemma A.1 the buyer gets maximum revenue if $b_i(q_i)$ is an identity mapping. This concludes the proof of Lemma 2.1. □

B Missing Proofs in Section 3

B.1 Proof of Theorem 3.2

Proof. The proof of this theorem follows three steps. First, we give an upper bound on the Lagrangian dual of the revenue-maximizing problem, which happens to be similar to programming (1). Next, we characterize the optimality of the dual problem via a pair of equivalent conditions. With these conditions, we show that strong duality holds. Finally, we relate the above optimality conditions to the efficiency condition of BDFPA to prove the theorem.

For briefness, we write $\Phi_i(\tau, \widetilde{\boldsymbol{g}}, q) := I\left[\tau_i \widetilde{g}_i(q_i) \ge \max_{i' \ne i} \{\tau_i \widetilde{g}_i(q_{i'}), \lambda\}\right]$ for any $1 \le i \le n$ through the whole appendix, which describes whether i has the largest value $\tau_i \widetilde{g}_i(q_i)$ no less than λ among all buyers. This function is seen as a choice function, and fits with the allocation function of BROA, BDFPA, and PFPA.

We now formalize seller's revenue-maximizing problem in BDFPA as follows, and denote the optimal objective as $\mathrm{OPT}^{\mathrm{BD}}$:

$$\begin{aligned}&\max_{\beta \in [0,1]^n} T \cdot \sum_{i=1}^n \int_0^1 (\widetilde{v}_i(q_i) - \lambda) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q)\,\mathrm{d}q_{-i}\right) \mathrm{d}q_i,\\ \text{s.t.}\quad & T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q)\,\mathrm{d}q_{-i}\right) \mathrm{d}q_i \le B_i, \quad \forall 1 \le i \le n.\end{aligned} \tag{6}$$

Now we consider the Lagrangian dual problem of (6), which is as follows:

$$\chi^{\mathrm{BD}}(\tau) := \max_{\beta \in [0,1]^n} T \cdot \sum_{i=1}^n \int_0^1 ((1 - \tau_i)\widetilde{v}_i(q_i) - \lambda) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q)\,\mathrm{d}q_{-i}\right) \mathrm{d}q_i + \sum_{i=1}^n \tau_i B_i. \tag{7}$$

Here, $\tau_i \geq 0$ is the dual variable for the restriction on buyer i's payment. For a better understanding, we reorganize the integral part of $\chi^{\mathrm{BD}}(\tau)$ as the following:

$$T \cdot \sum_{i=1}^{n} \int_0^1 ((1-\tau_i)\widetilde{v}_i(q_i) - \lambda) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q) \,\mathrm{d}q_{-i} \right) \mathrm{d}q_i = T \cdot \mathbb{E}_q \left[(1-\tau_{i^*})\widetilde{v}_{i^*}(q_{i^*}) - \lambda \right],$$

where for any quantile profile q, i^* is defined as the buyer $0 \leq i \leq n$ with the highest value $\beta_i \widetilde{v}_i(q_i)$. Here, we involve a phantom buyer 0 with $\beta_0 = 1$, $\tau_0 = 0$ and $\widetilde{v}_0(q_0) = \lambda$ for all $q_i \in [0, 1]$.[3] Intuitively, the equation is based on the fact that when q is fixed, functions $\{\Phi_{1 \leq i \leq n}\}$ collaboratively act as a choice function to pick a buyer $0 \leq i^* \leq n$ with the highest discounted bid. Rigorously, the above equation establishes as the Lebesgue measure that at least two buyers share the same highest discounted bid is zero. With the equation, we can rewrite $\chi^{\mathrm{BD}}(\tau)$ as:

$$\chi^{\mathrm{BD}}(\tau) = \max_{\beta \in [0,1]^n} T \cdot \mathbb{E}_q \left[(1-\tau_{i^*})\widetilde{v}_{i^*}(q_{i^*}) - \lambda \right] + \sum_{i=1}^{n} \tau_i B_i. \tag{8}$$

Now, for fixed $\tau \in [0, 1]^n$, since i^* represents a specific buyer from 0 to n, we have

$$\chi^{\mathrm{BD}}(\tau) \leq T \cdot \mathbb{E}_q \left[\max_{1 \leq i \leq n} \{(1-\tau_i)\widetilde{v}_i(q_i) - \lambda\}^+ \right] + \sum_{i=1}^{n} \tau_i B_i,$$

and further we can see that the equality in fact holds when we take $\beta = 1^n - \tau$ in (8) and i^* is subsequently $\arg\max_{0 \leq i \leq n} \{(1-\tau_i)\widetilde{v}_i(q_i) - \lambda\}$. Now by weak duality, we have

$$\mathrm{OPT}^{\mathrm{BD}} \leq \min_{\tau \geq 0} \chi^{\mathrm{BD}}(\tau) \leq \min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau) = \min_{\tau \in [0,1]^n} T \cdot \mathbb{E}_q \left[\max_{1 \leq i \leq n} \{(1-\tau_i)\widetilde{v}_i(q_i) - \lambda\}^+ \right] + \sum_{i=1}^{n} \tau_i B_i. \tag{9}$$

We now characterize the optimal solution of the program via the following lemma.

Lemma B.1. *τ is a solution of $\min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau)$ if and only if for any $1 \leq i \leq n$:*

- $\tau_i \cdot \left(T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \tau, \widetilde{\boldsymbol{v}}, q) \,\mathrm{d}q_{-i} \right) \mathrm{d}q_i - B_i \right) = 0$, *and*
- $T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \tau, \widetilde{\boldsymbol{v}}, q) \,\mathrm{d}q_{-i} \right) \mathrm{d}q_i \leq B_i$.

Proof of Lemma B.1. We first give some temporary notations to ease the description. We let $p_i := \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \tau, \widetilde{\boldsymbol{v}}, q) \,\mathrm{d}q_{-i} \right) \mathrm{d}q_i$. We further write $y_0(\tau, q) := 0$ and $y_i(\tau, q) := (1-\tau_i)\widetilde{v}_i(q_i) - \lambda$ for $1 \leq i \leq n$. At last, we let $y(\tau, q) = \max_{0 \leq i \leq n} y_i(\tau, q)$. We now have

$$\chi^{\mathrm{BD}}(\tau) = T \cdot \mathbb{E}_q \left[y(\tau, q) \right] + \sum_{i=1}^{n} \tau_i B_i.$$

Note that for any $0 \leq i \leq n$, $y_i(\tau, q)$ is convex on τ. As a result, $y(\tau, q)$ is convex on τ as well. Meanwhile, let $\mathcal{J}(\tau, q) = \arg\max_{0 \leq i \leq n} y_i(\tau, q)$, then with probability 1, $|\mathcal{J}(\tau, q)| = 1$ when

[3]Involving the phantom buyer is only for the succinctness of writing, and does not matter with all the restrictions on buyers 1 to n.

q is chosen uniformly from $[0,1]^n$, and therefore, $y(\tau, q)$ is differentiable with probability 1. By Theorem 7.46 from [34], $\chi^{\mathrm{BD}}(\tau)$ is convex and differentiable. Further by Theorem 7.44 from [34],

$$\begin{aligned}
\frac{\partial}{\partial \tau_i}\chi^{\mathrm{BD}}(\tau) &= T \cdot \mathbb{E}_q\left[\frac{\partial}{\partial \tau_i} y(\tau, q)\right] + B_i \\
&= T \cdot \mathbb{E}_q\left[-\widetilde{v}_i(q_i) I\left[i \in \mathcal{J}(\tau, q)\right]\right] + B_i \\
&= -T \cdot p_i + B_i.
\end{aligned}$$

Therefore, $\nabla\chi^{\mathrm{BD}}(\tau) = (-T \cdot p_i + B_i)_{1 \le i \le n}$. As $\chi^{\mathrm{BD}}(\tau)$ is convex, τ is optimal if and only if for any $\tau' \in [0,1]^n$,

$$\nabla\chi^{\mathrm{BD}}(\tau) \cdot (\tau' - \tau) \geq 0. \tag{10}$$

We now finish the proof of the lemma.

"If" side. Suppose the given two conditions are satisfied. Let $\mathcal{K}(\tau) = \{i \mid \tau_i = 0\}$. For any $\tau' \in [0,1]^n$, if $i \notin \mathcal{K}(\tau)$, then $-T \cdot p_i + B_i = 0$, and $\nabla_i\chi^{\mathrm{BD}}(\tau) \cdot (\tau'_i - \tau_i) = 0$ holds. Otherwise, $-T \cdot p_i + B_i \le 0$ and $\tau'_i \ge \tau_i$, which leads to $\nabla_i\chi^{\mathrm{BD}}(\tau) \cdot (\tau'_i - \tau_i) \ge 0$. As a result, $\nabla\chi^{\mathrm{BD}}(\tau) \cdot (\tau' - \tau) \ge 0$ and τ is optimal.

"Only if" side. To start with, we claim that all entries of any optimal solution τ^* of the programming are strictly smaller than 1. To see this, counterfactually suppose $\tau_i^* = 1$, then $(1 - \tau_i^*)\widetilde{v}_i(q_i) - \lambda < 0$ holds for any q_i as $\lambda > 0$. Therefore, since $\widetilde{v}_i(\cdot)$ is bounded, subtracting τ_i^* by a small amount does not affect the expectation part of $\chi^{\mathrm{BD}}(\tau^*)$, but will strictly lessen the latter sum part $\sum_{i=1}^n \tau_i B_i$, contradicting the optimality.

Now, for some optimal τ^*, let $\mathcal{K}(\tau^*) = \{i \mid \tau_i^* = 0\}$. For $i \in \mathcal{K}(\tau^*)$, let $\tau' = \tau^* + \delta e_i$ for some $\delta \in (0,1]$, where e_i is the vector with the i-th entry one and all other entries zero. Plugging in (10), we derive that $T \cdot p_i^* \le B_i$. For $i \notin \mathcal{K}(\tau^*)$, we take $\tau' = \tau^* \pm \delta e_i$ in order for some small $\delta > 0$ satisfying $\tau_i^* \pm \delta \in [0,1]$. Such δ exists since $0 < \tau_i^* < 1$. Taking into (10) respectively, we derive that $T \cdot p_i^* - B_i = 0$. Lemma B.1 is proved. $\square$

With Lemma B.1, we show that strong duality holds. For the solution τ^* which minimizes $\chi^{\mathrm{BD}}(\tau)$ in $[0,1]^n$, we have

$$\begin{aligned}
\chi^{\mathrm{BD}}(\tau^*) &= T \cdot \mathbb{E}_q\left[\max_{1 \le i \le n}\{(1 - \tau_i^*)\widetilde{v}_i(q_i) - \lambda\}^+\right] + \sum_{i=1}^n \tau_i^* B_i \\
&= T \cdot \int_0^1 ((1 - \tau_i^*)\widetilde{v}_i(q_i) - \lambda) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \tau^*, \widetilde{\boldsymbol{v}}, q)\, \mathrm{d}q_{-i}\right) \mathrm{d}q_i + \sum_{i=1}^n \tau_i^* B_i \\
&= T \cdot \int_0^1 (\widetilde{v}_i(q_i) - \lambda) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \tau^*, \widetilde{\boldsymbol{v}}, q)\, \mathrm{d}q_{-i}\right) \mathrm{d}q_i.
\end{aligned}$$

Here, the first equation is by definition and the last equation is by the first condition in Lemma B.1. Let $\beta^* = 1^n - \tau^*$. Now by the second condition in Lemma B.1, β^* satisfies all budget constraints in (6), with the objective value $\chi^{\mathrm{BD}}(\tau^*)$. As a result, strong duality holds and β^* is the revenue-maximizing tuple of bid-discount multipliers.

Finally, we come to prove the theorem. For an efficient bid-discount multiplier tuple β^{e}, by definition, the following two groups of constraints hold:

$$T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta^{\mathrm{e}}, \widetilde{\boldsymbol{v}}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \le B_i, \quad \forall 1 \le i \le n.$$

$$\gamma_i \cdot \left(B_i - T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta^{\mathrm{e}}, \widetilde{\boldsymbol{v}}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \right) = 0, \quad \forall 1 \le i \le n. \tag{11}$$

By Lemma B.1, $\tau^* = 1^n - \beta^{\mathrm{e}}$ is an optimal solution of $\min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau)$, and by strong duality, $\beta^{\mathrm{e}} = 1^n - \tau^*$ is revenue-maximizing, which finishes the proof. ☐

B.2 Proof of Theorem 3.4

Proof. Notice that $\chi^{\mathrm{BD}}(\tau)$, which defined in the proof of Theorem 3.2 is a convex function by Theorem 7.46 from [34]. Now consider the optimal solution of $\min_{\gamma \in [0,1]^n} \chi^{\mathrm{BD}}(\tau)$, τ^*. By Lemma B.1, $\beta = 1 - \tau^*$ is an efficient tuple of multipliers for BDFPA. This concludes the proof. ☐

B.3 Proof of Theorem 3.5

Proof. The theorem is proved in steps. First, we characterize some essential properties of bid-discount in first-price auction. Then we prove the five statements in the theorem in order.

We come to some basic features of the bid-discount method in first-price auction. To start with, obviously, given buyers' profile of reported quantile functions $\widetilde{\boldsymbol{v}}$ and budget B, notice that $\mathcal{M}^{\mathrm{BDFPA}}(0)$ must be a feasible bid-discount mechanism, in which the item is never assigned and each buyer's payment is zero. Thus there exists a tuple of bid-discount multipliers β such that $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ is a feasible bid-discount mechanism.

We now show that when a buyer's bid-discount multiplier slightly increases, her expected payment does not increase too much.

Lemma B.2. *There exists a constant C which satisfies the following: For any $\beta = (\beta_1, \ldots, \beta_n)$ and $1 \le i \le n$ such that $\beta_i < 1$, let $\beta' = \beta + \delta e_i$ where $0 < \delta \le 1 - \beta_i$ and e_i is the vector with the i-th entry one and all other entries zero. Then the expected payment of buyer i in $\mathcal{M}^{\mathrm{BDFPA}}(\beta')$ is at most the expected payment of buyer i in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ plus $C\delta$.*

Before we prove the lemma, some preparations are required. We define $G_{i,\beta}$ be the cumulative distribution function of $\max_{i' \ne i} \{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}$ when q_{-i} is chosen uniformly in $[0,1]^{n-1}$. Then,

Lemma B.3. *For any $\beta \in [0,1]^n$, we have:*

- *$G_{i,\beta}(\cdot)$ is zero on $[0, \lambda)$.*
- *λ is the only possible discontinuous point of $G_{i,\beta}(\cdot)$.*
- *If $G_{i,\beta}(\lambda) < 1$, then $G_{i,\beta}(\cdot)$ is Lipschitz continuous on $[\lambda, +\infty)$.*

Proof of Lemma B.3. Let $\widehat{v} := \max_{i' \ne i} \{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}$ be a random variable when q_{-i} is uniformly drawn from $[0,1]^{n-1}$. The only non-trivial part is the third part, which is to show the Lipschitz continuity when $\widehat{v} \ge \lambda$. For any buyer $i' \ne i$, since her reported cdf $\widetilde{F}_{i'}(\cdot)$ is Lipschitz continuous, there exists a constant $C_{i'}$ such that for any $\widetilde{v}_{i'}^{(1)}$ and $\widetilde{v}_{i'}^{(2)}$, we have

$$\left| \widetilde{F}_{i'}(\widetilde{v}_{i'}^{(1)}) - \widetilde{F}_{i'}(\widetilde{v}_{i'}^{(2)}) \right| \le C_{i'} \left| \widetilde{v}_{i'}^{(1)} - \widetilde{v}_{i'}^{(2)} \right|.$$

Since $\widetilde{v}_{i'}(q_{i'})$ is upper bounded (say, by $\bar{v}_{i'}$) for any $1 \le i' \le n$, there exists a constant $\delta_{i'} > 0$ for each i' such that $\delta_{i'} \cdot \bar{v}_{i'} < \lambda$. Now let $\lambda \le \widehat{v}^{(1)} < \widehat{v}^{(2)}$. Clearly, since $G_{i,\beta}(\lambda) < 1$, there exists at least one $i' \neq i$ such that $\beta_{i'} > \delta_{i'}$. Then we have

$$\begin{aligned}
\left|G_{i,\beta}(\widehat{v}^{(1)}) - G_{i,\beta}(\widehat{v}^{(2)})\right| &= \left|\prod_{i'\neq i, \beta_{i'}>\delta_{i'}} \widetilde{F}_{i'}(\widehat{v}^{(1)}/\beta_{i'}) - \prod_{i'\neq i, \beta_{i'}>\delta_{i'}} \widetilde{F}_{i'}(\widehat{v}^{(2)}/\beta_{i'})\right| \\
&\le \sum_{i'\neq i, \beta_{i'}>\delta_{i'}} \left|\widetilde{F}_{i'}(\widehat{v}^{(1)}/\beta_{i'}) - \widetilde{F}_{i'}(\widehat{v}^{(2)}/\beta_{i'})\right| \\
&\le \left(\sum_{i'\neq i, \beta_{i'}>\delta_{i'}} C_{i'}/\beta_{i'}\right) \cdot \left|\widehat{v}^{(1)} - \widehat{v}^{(2)}\right| \\
&\le \left(\sum_{i'\neq i} C_{i'}/\delta_{i'}\right) \cdot \left|\widehat{v}^{(1)} - \widehat{v}^{(2)}\right|
\end{aligned}$$

Here, the first inequality is because of $\widetilde{F}_{i'}$ is no greater than 1 for any $i' \neq i$. This shows that $G_{i,\beta}$ is continuous with Lipschitz constant $\widehat{C}_i := \sum_{i'\neq i} C_{i'}/\delta_{i'}$ on the right side of λ, and the proof is finished. $\square$

Now, we come back to prove Lemma B.2.

Proof of Lemma B.2. Recall that by definition, the expected payment of buyer i in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$, which we denote as p_i is

$$\int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} I\left[\beta_i \widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}\right] \mathrm{d}q_{-i}\right) \mathrm{d}q_i.$$

Similarly, the expected payment of buyer i in $\mathcal{M}^{\mathrm{BDFPA}}(\beta')$, which we denote as p'_i is

$$\int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} I\left[(\beta_i + \delta)\widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}\right] \mathrm{d}q_{-i}\right) \mathrm{d}q_i.$$

Note that

$$\begin{aligned}
I\left[(\beta_i + \delta)\widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}\right] &= I\left[\beta_i\widetilde{v}_i(q_i) \ge \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\}\right] \\
&\quad + I\left[\beta_i\widetilde{v}_i(q_i) < \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\} \le (\beta_i + \delta)\widetilde{v}_i(q_i)\right].
\end{aligned} \tag{12}$$

Thus the increment of buyer i's expected payment after replacing β with β' is

$$\begin{aligned}
&\int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} I\left[\beta_i\widetilde{v}_i(q_i) < \max_{i'\neq i}\{\beta_{i'}\widetilde{v}_{i'}(q_{i'}), \lambda\} \le (\beta_i + \delta)\widetilde{v}_i(q_i)\right] \mathrm{d}q_{-i}\right) \mathrm{d}q_i. \\
&= \int_0^1 \widetilde{v}_i(q_i) \cdot \left(G_{i,\beta}\left((\beta_i + \delta)\widetilde{v}_i(q_i)\right) - G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right) \mathrm{d}q_i.
\end{aligned} \tag{13}$$

By Lemma B.3, $G_{i,\beta}(\delta_i\widetilde{v}_i(q_i)) = 0$ always holds for any $q_i \in [0, 1]$, where we recall that δ_i is defined as a constant such that $\delta_i \cdot \bar{v}_i < \lambda$. We use δ_i as a threshold to analyze the formula (13).

When $\beta_i \ge \delta_i$, there are three parts which we analyze correspondingly, depending on whether λ lies in $(\beta_i\widetilde{v}_i(q_i), (\beta_i + \delta)\widetilde{v}_i(q_i))$.

- $(\beta_i+\delta)\widetilde{v}_i(q_i)\le\lambda$. Let $\bar{q}$ be the minimum $q_i\le 1$ such that $(\beta_i+\delta)\widetilde{v}_i(q_i)\le\lambda$, if there exists, and $\bar{q}:=1$ otherwise. By monotonicity, $(\beta_i+\delta)\widetilde{v}_i(q_i)\le\lambda$ holds for any $\bar{q}<q_i\le 1$. By Lemma B.3, we have $\int_{\bar{q}}^1\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\mathrm{d}q_i=0$.

- $\beta_i\widetilde{v}_i(q_i)<\lambda<(\beta_i+\delta)\widetilde{v}_i(q_i)$. Let $\underline{q}$ be the maximum $q_i\ge 0$ such that $\beta_i\widetilde{v}_i(q_i)\ge\lambda$, if there exists, and $\underline{q}:=0$ otherwise. By monotonicity, $\beta_i\widetilde{v}_i(q_i)\le\lambda$ holds for any $\underline{q}\le q_i<1$. Now, since $\widetilde{v}_i$ is upper bounded by $\bar{v}_i$ and that $\widetilde{F}_i(\cdot)$ is continuous with Lipschitz constant C_i, we derive that

$$
\begin{aligned}
&\int_{\underline{q}}^{\bar{q}}\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\mathrm{d}q_i\le\bar{v}_i\cdot\left(\bar{q}-\underline{q}\right)\\
\le&\,\bar{v}_i\cdot\left(\widetilde{F}_i(\lambda/\beta_i)-\widetilde{F}_i(\lambda/(\beta_i+\delta))\right)\le\bar{v}_i\cdot C_i\cdot\left(\lambda/\beta_i-\lambda/(\beta_i+\delta)\right)\\
\le&\,\bar{v}_i\cdot C_i\cdot\frac{\lambda}{\beta_i^2}\cdot\delta.
\end{aligned}
$$

  Here, the first inequality holds since $G_{i,\beta}$ is bounded by 1.

- $\beta_i\widetilde{v}_i(q_i)\ge\lambda$. By monotonicity, $\beta_i\widetilde{v}_i(q_i)\ge\lambda$ holds for any $0\le q_i<\underline{q}$. By Lemma B.3, we have

$$
\begin{aligned}
&\int_0^{\underline{q}}\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\mathrm{d}q_i\\
\le&\int_0^{\underline{q}}\widetilde{v}_i^2(q_i)\cdot\widehat{C}_i\cdot\delta\,\mathrm{d}q_i\le\bar{v}_i^2\cdot\widehat{C}_i\cdot\delta.
\end{aligned}
$$

As a result, in this scenario, we have

$$
\begin{aligned}
&\int_0^1\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\mathrm{d}q_i\\
=&\left(\int_0^{\underline{q}}+\int_{\underline{q}}^{\bar{q}}+\int_{\bar{q}}^1\right)\left(\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\right)\mathrm{d}q_i\\
\le&\max\left\{\bar{v}_i\cdot C_i\cdot\frac{\lambda}{\beta_i^2},\bar{v}_i^2\cdot\widehat{C}_i\right\}\cdot\delta\le\max\left\{\bar{v}_i\cdot C_i\cdot\frac{\lambda}{\delta_i^2},\bar{v}_i^2\cdot\widehat{C}_i\right\}\cdot\delta.
\end{aligned}
$$

In the case that $\beta_i<\delta_i$, since $\delta_i\cdot\bar{v}_i<\lambda$, by Lemma B.3, $G_{i,\beta}(\delta_i\widetilde{v}_i(q_i))=G_{i,\beta}(\beta_i\widetilde{v}_i(q_i))$ always holds for any $q_i\in[0,1]$. Therefore, when $\delta\le\delta_i$, then the proof is finished, otherwise, notice that

$$
\begin{aligned}
&\int_0^1\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}\left(\beta_i\widetilde{v}_i(q_i)\right)\right)\mathrm{d}q_i\\
=&\int_0^1\widetilde{v}_i(q_i)\cdot\left(G_{i,\beta}\left((\beta_i+\delta)\widetilde{v}_i(q_i)\right)-G_{i,\beta}(\delta_i\widetilde{v}_i(q_i))\right)\mathrm{d}q_i\\
\le&\max\left\{\bar{v}_i\cdot C_i\cdot\frac{\lambda}{\delta_i^2},\bar{v}_i^2\cdot\widehat{C}_i\right\}\cdot(\beta_i+\delta-\delta_i)\le\max\left\{\bar{v}_i\cdot C_i\cdot\frac{\lambda}{\delta_i^2},\bar{v}_i^2\cdot\widehat{C}_i\right\}\cdot\delta.
\end{aligned}
$$

Now, we conclude the proof of Lemma B.2 by having $C:=\max_{1\le i\le n}\{\bar{v}_i\cdot C_i\cdot\lambda/\delta_i^2,\bar{v}_i^2\cdot\widehat{C}_i\}$. □

From Lemma B.2, we derive an essential property that the set of all feasible tuples of bid-discount multipliers is compact, which is given in the following lemma.

Lemma B.4. *Let $\mathcal{B}$ be the set of all β such that $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ is a feasible bid-discount mechanism. Then $\mathcal{B}$ is compact.*

Proof of Lemma B.4. It suffices to show that all βs satisfying the budget-feasible constraints form a closed set.

Define $\varphi : [0,1]^n \to \mathbb{R}^n$ to be the mapping from the tuple of multipliers β to the expected payment vector of all buyers when the quantile profile is uniformly distributed in $[0,1]^n$. By Lemma B.2 we know that φ is Lipschitz continuous, which implies that the pre-image of every closed set under φ is also closed. Since $\mathcal{B} = \{\beta : \varphi(\beta) \in \prod_i [0, B_i]\}$ is the pre-image of $\prod_i [0, B_i]$, which is apparently a closed set, $\mathcal{B}$ is closed as well. Moreover, $\mathcal{B} \subseteq [0,1]^n$ is apparently bounded. Therefore, $\mathcal{B}$ is a compact set. □

Now, we are ready to show that the maximum tuple of bid-discount multipliers exists by reasoning that a buyer's payment decreases when other buyers' discount multiplier increases and then applying Lemma B.4.

Lemma B.5. *There exists a maximum tuple of bid-discount multipliers $\beta^{\max}$, i.e., for any feasible tuple of bid-discount multipliers β, $\beta_i^{\max} \geq \beta_i$ for any $1 \leq i \leq n$.*

Proof of Lemma B.5. First, for any given $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$ and $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(2)})$, define β^{h} be the component-wise maximum of $\beta^{(1)}$ and $\beta^{(2)}$. We will show that β^{h} is also a feasible tuple of bid-discount multipliers.

We only need to verify that the budget-feasible constraint is met for any buyer, and we prove this by showing that a buyer's expected payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{h}})$ is no more than her expected payment in the higher of $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$ and $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(2)})$. For some buyer i, assume that $\beta_i^{\mathrm{h}} = \beta_i^{(1)}$ without loss of generality. Note that the payment is irrelevant with the discount multipliers given the allocation. Thus it suffices to show that when $q_1, \ldots, q_n$ are fixed, if buyer i does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$, she does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{h}})$ as well. Now that buyer i does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$, the highest discounted bid in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$ must be higher than the discounted bid of buyer i. Since buyer i's discounted bid in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{h}})$ is the same as in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$, and the highest discounted bid in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{h}})$ is no less than the highest discounted bid in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{(1)})$, we conclude that buyer i does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{h}})$.

We now complete the proof of the lemma. Let $\beta_i^{\max} = \sup\{\beta_i \mid \beta \text{ is a feasible tuple of multipliers}\}$. We will show that $\beta^{\max} = (\beta_i^{\max})_{1 \leq i \leq n}$ is also a feasible tuple of multipliers. For any $\epsilon > 0$ and any $1 \leq i \leq n$, there exists a feasible β such that $\beta_i > \beta_i^{\max} - \epsilon$. By repeatedly taking the component-wise maximum for all i, there is a feasible β^{ϵ} such that for every i, $\beta_i^{\epsilon} > \beta_i^{\max} - \epsilon$. Thus the sequence β^{ϵ} (as $\epsilon \to 0$) has a limit point $\beta^{\max}$. By Lemma B.4, this limit point is also a feasible tuple of multipliers. □

Next, We demonstrate the efficiency of the bid-discount mechanism induced by the maximum tuple of multipliers.

Lemma B.6. *$\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ is efficient.*

Proof of Lemma B.6. Prove by contradiction. Suppose $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ is not efficient, which means there is a buyer i such that $\beta_i^{\max} < 1$ and her budget is not binding, i.e., her expected payment is strictly less than her budget. By Lemma B.2, we can slightly increase $\beta_i^{\max}$ while buyer i's budget is still not binding. Note that when buyer i's multiplier increases, the payment of any other buyer does not increase. Hence the budget-feasible constraint is still met for all buyers, and we obtain a feasible tuple with a strictly larger component, which contradicts the assumption that $\beta^{\max}$ is the maximum feasible tuple of multipliers. $\square$

We have now already proved the first two statements, which claim that the maximum tuple of bid-discount multipliers $\beta^{\max}$ exists as well as its efficiency. Before proving the remaining statements, we present a critical observation in Lemma B.7. That is, given an efficient bid-discount multiplier tuple β^{e} and another feasible tuple β, if there is some buyer l with minimum $\beta_l/\beta^{\mathrm{e}}_l$ that satisfies $\beta_l < 1$, and another buyer k with a larger $\beta_k/\beta^{\mathrm{e}}_k$ that has a positive payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$, then $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ is not efficient. The insight of this observation is that for any quantile profile q, buyer l does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ as long as she does not get allocated in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$. Moreover, buyer k overbids buyer l in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ on some quantile profiles with positive measure on which buyer l wins in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$. Therefore, buyer l strictly pays less in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ than in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ in expectation, rendering the inefficiency of $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$.

Lemma B.7. *Let β^{e} be an efficient feasible tuple of bid-discount multipliers, and $\beta' \leq \beta^{\mathrm{e}}$ be another feasible one. Let $\mathcal{I} = \arg\min_i \beta'_i/\beta^{\mathrm{e}}_i$. If there is some $l \in \mathcal{I}$ such that $\beta'_l < 1$, and $k \notin \mathcal{I}$ such that the payment of buyer k in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ is positive, then β' is not efficient.*

Proof of Lemma B.7. Prove by contradiction. Suppose β' is efficient instead. Since buyer l's bid-discount multiplier is cut the most fraction from β^{e} to β', when she does not win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ with quantile profile q, she does not win the item in $\mathcal{M}^{\mathrm{BDFPA}}(\beta')$ as well. Thereby $p^{\mathrm{e}}_l \geq p'_l$, where p'_l and p^{e}_l denote buyer l's expected payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta')$ and $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ respectively. Now it suffices to show that $p'_l < p^{\mathrm{e}}_l$, which, combining $\beta_l < 1$, is inconsistent with the efficiency of β'.

By definition, we have

$$\begin{aligned}
p^{\mathrm{e}}_l - p'_l &= \int_0^1 \widetilde{v}_l(q_l) \left(\int_{q_{-l}} \left(\Phi_l(\beta^{\mathrm{e}}, \widetilde{\boldsymbol{v}}, q) - \Phi_l(\beta', \widetilde{\boldsymbol{v}}, q) \right) \mathrm{d}q_{-l} \right) \mathrm{d}q_l \\
&= \int_0^1 \widetilde{v}_l(q_l) \left(\int_0^1 \left(\Phi_{l>k}(\beta^{\mathrm{e}}, q_l, q_k) - \Phi_{l>k}(\beta', q_l, q_k) \right) \mathrm{d}q_k \right) \mathrm{d}q_l.
\end{aligned}$$

Here, $\Phi_{l>k}(\beta, q_l, q_k)$ is defined as $\int_{q_{-\{l,k\}}} \Phi_l(\beta, \widetilde{\boldsymbol{v}}, q) \, \mathrm{d}q_{-\{l,k\}}$, which represents the probability that buyer l wins the item given q_l and q_k. We implicitly take $\widetilde{\boldsymbol{v}}$ as fixed. Further, define

$$H(\beta, \underline{\eta}, \bar{\eta}, \underline{\theta}, \bar{\theta}) := \int_{\underline{\eta}}^{\bar{\eta}} \widetilde{v}_l(q_l) \left(\int_{\underline{\theta}}^{\bar{\theta}} \Phi_{l>k}(\beta, q_l, q_k) \, \mathrm{d}q_k \right) \mathrm{d}q_l$$

as the expected payment of buyer l in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ when q_l and q_k range from $[\underline{\eta}, \bar{\eta}]$ and $[\underline{\theta}, \bar{\theta}]$ respectively. Notice that for any β, $0 \leq \eta_1 \leq \eta_3 \leq \eta_2 \leq 1$ and $0 \leq \theta_1 \leq \theta_3 \leq \theta_2 \leq 1$,

$$\begin{aligned}
H(\beta, \eta_1, \eta_2, \theta_1, \theta_2) &= H(\beta, \eta_1, \eta_3, \theta_1, \theta_2) + H(\beta, \eta_3, \eta_2, \theta_1, \theta_2) \\
&= H(\beta, \eta_1, \eta_2, \theta_1, \theta_3) + H(\beta, \eta_1, \eta_2, \theta_3, \theta_2).
\end{aligned}$$

The remaining proof of Lemma B.7 is divided to two parts. We first demonstrate that for any $0 \leq \eta_1 \leq \eta_2 \leq 1$ and $0 \leq \theta_1 \leq \theta_2 \leq 1$, we have $H(\beta^{\mathrm{e}}, \eta_1, \eta_2, \theta_1, \theta_2) - H(\beta', \eta_1, \eta_2, \theta_1, \theta_2) \geq 0$. Then we find $\eta^0_1, \eta^0_2, \theta^0_1, \theta^0_2$ such that $H(\beta^{\mathrm{e}}, \eta^0_1, \eta^0_2, \theta^0_1, \theta^0_2) - H(\beta', \eta^0_1, \eta^0_2, \theta^0_1, \theta^0_2) > 0$. The above collaboratively implies that

$$p^{\mathrm{e}}_l - p'_l = H(\beta^{\mathrm{e}}, 0, 1, 0, 1) - H(\beta', 0, 1, 0, 1) \geq H(\beta^{\mathrm{e}}, \eta^0_1, \eta^0_2, \theta^0_1, \theta^0_2) - H(\beta', \eta^0_1, \eta^0_2, \theta^0_1, \theta^0_2) > 0,$$

which concludes the proof of Lemma B.7.

For the first part, with the observation that

$$H(\beta^{\mathrm{e}},\eta_1,\eta_2,\theta_1,\theta_2)-H(\beta',\eta_1,\eta_2,\theta_1,\theta_2)$$
$$=\int_{\eta_1}^{\eta_2}\widetilde{v}_l(q_l)\left(\int_{\theta_1}^{\theta_2}\left(\Phi_{l>k}(\beta^{\mathrm{e}},q_l,q_k)-\Phi_{l>k}(\beta',q_l,q_k)\right)\,\mathrm{d}q_k\right)\,\mathrm{d}q_l,$$

it suffices to prove for any $q_l\in[0,1], q_k\in[0,1]$,

$$\Phi_{l>k}(\beta^{\mathrm{e}},q_l,q_k)-\Phi_{l>k}(\beta',q_l,q_k)\geq 0. \tag{14}$$

Recall that the two terms in (14) are the probability that buyer l wins the item in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ and $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$ when q_k and q_l are fixed, respectively. Given quantile profile q, since $l\in\mathcal{I}=\arg\min_i\beta'_i/\beta^{\mathrm{e}}_i$ and $\beta'\leq\beta^{\mathrm{e}}$, then by the allocation rule, if l wins in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$, she wins in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ as well. As a result, when q_k and q_l are fixed, buyer l certainly does not have less probability to win in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ than in $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$.

Now we establish the existence of $\eta_1^0<\eta_2^0$ and $\theta_1^0<\theta_2^0$ such that $H(\beta^{\mathrm{e}},\eta_1^0,\eta_2^0,\theta_1^0,\theta_2^0)>H(\beta',\eta_1^0,\eta_2^0,\theta_1^0,\theta_2^0)$.

The efficiency of β' and that $\beta'_l<1$ in together imply $p'_l=B_l>0$. Since $p^{\mathrm{e}}_l\geq p'_l$, we have $p^{\mathrm{e}}_l=B_l>0$. As a result, there are $q_l^{(1)}<1$ and $q_k^{(1)}>0$ such that $\Phi_{l>k}(\beta^{\mathrm{e}},q_l^{(1)},q_k^{(1)})>0$.[4] Symmetrically, since p^{e}_k is positive as well, there are $q_k^{(2)}<1$ and $q_l^{(2)}>0$ such that $\Phi_{k>l}(\beta^{\mathrm{e}},q_k^{(2)},q_l^{(2)})>0$. We can further assume that $0<q_l^{(2)}\leq q_l^{(1)}<1$ and $0<q_k^{(1)}\leq q_k^{(2)}<1$, or else, we can swap $q_l^{(1)}$ and $q_l^{(2)}$ or $q_k^{(1)}$ and $q_k^{(2)}$ without breaking the above statements. We want to find $q_k^{(3)}$ and $q_l^{(3)}$ such that $\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(3)})=\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(3)})$, and the probability that l wins with $q_l^{(3)}$ under β^{e} is positive. We construct as follows:

- If $\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(2)})\geq\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(1)})$, let $q_l^{(3)}=q_l^{(1)}$, and there exists $q_k^{(3)}\in[q_k^{(1)},q_k^{(2)}]$ such that $\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(3)})=\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(3)})$ due to the continuity of $\widetilde{v}_k(q_k)$ and that $\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(1)})\geq\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(1)})$. l wins with positive probability with $q_l^{(3)}$ since $q_l^{(3)}=q_l^{(1)}$.

- If $\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(2)})<\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(1)})$, let $q_k^{(3)}=q_k^{(2)}$, and there exists $q_l^{(3)}\in[q_l^{(2)},q_l^{(1)}]$ such that $\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(3)})=\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(3)})$ due to the continuity of $\widetilde{v}_l(q_l)$ and that $\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(2)})\geq\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(2)})$. l wins with positive probability with $q_l^{(3)}$ since k wins with positive probability with $q_k^{(3)}=q_k^{(2)}$.

Moreover, as $\beta'_l/\beta^{\mathrm{e}}_l<\beta'_k/\beta^{\mathrm{e}}_k$, there exists a sufficiently small $\delta>0$ such that for any $q_k\in[q_k^{(3)}-\delta,q_k^{(3)}]$ and $q_l\in[q_l^{(3)},q_l^{(3)}+\delta]$ (note that $q_k^{(3)}<1$ and $q_l^{(3)}>0$), the probability that l wins with q_l under β^{e} is no less than a positive constant, and

$$\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k)\cdot\frac{\beta'_k}{\beta^{\mathrm{e}}_k}>\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l)\cdot\frac{\beta'_l}{\beta^{\mathrm{e}}_l},$$

that is, $\beta'_k\widetilde{v}_k(q_k)>\beta'_l\widetilde{v}_l(q_l)$. Moreover, for any $q_k\in[q_k^{(3)}-\delta,q_k^{(3)}], q_l\in[q_l^{(3)},q_l^{(3)}+\delta]$, we have $\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l)\geq\beta^{\mathrm{e}}_l\widetilde{v}_l(q_l^{(3)})=\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k^{(3)})\geq\beta^{\mathrm{e}}_k\widetilde{v}_k(q_k)$. Therefore,

$$H(\beta^{\mathrm{e}},q_l^{(3)},q_l^{(3)}+\delta,q_k^{(3)}-\delta,q_k^{(3)})=\int_{q_l^{(3)}}^{q_l^{(3)}+\delta}\widetilde{v}_l(q_l)\left(\int_{q_k^{(3)}-\delta}^{q_k^{(3)}}\Phi_{l>k}(\beta^{\mathrm{e}},q_l,q_k)\,\mathrm{d}q_k\right)\,\mathrm{d}q_l>0,$$
$$H(\beta',q_l^{(3)},q_l^{(3)}+\delta,q_k^{(3)}-\delta,q_k^{(3)})=\int_{q_l^{(3)}}^{q_l^{(3)}+\delta}\widetilde{v}_l(q_l)\left(\int_{q_k^{(3)}-\delta}^{q_k^{(3)}}\Phi_{l>k}(\beta',q_l,q_k)\,\mathrm{d}q_k\right)\,\mathrm{d}q_l=0.$$

[4]Otherwise, the Lebesgue measure of quantile profiles that l wins is zero, contradicting that the expected payment of l is positive.

Taking $\eta_1^0 = q_l^{(3)}, \eta_2^0 = q_l^{(3)} + \delta, \theta_1^0 = q_k^{(3)} - \delta, \theta_2^0 = q_k^{(3)}$ finishes this part, and the above collaboratively concludes the proof of Lemma B.7. $\square$

With the help of Lemma B.7, we can characterize an efficient bid-discount multiplier tuple by comparing it with the maximum tuple $\beta^{\max}$.

Lemma B.8. *For any efficient bid-discount multiplier tuple β^{e}, the following two conditions are satisfied:*

- *There exists some $\nu \le 1$, such that for any $1 \le i \le n$ satisfying $p_i^{\max}$ (buyer i's expected payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$) is positive, $\beta_i^{\mathrm{e}}/\beta_i^{\max} = \nu$;*
- *For any $1 \le i \le n$ satisfying $p_i^{\max} = 0$, $p_i^{\mathrm{e}} = 0$ (buyer i never wins in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$) and $\beta_i^{\mathrm{e}} = \beta_i^{\max} = 1$.*

Proof of Lemma B.8. Let $\mathcal{I}_1 = \{1 \le i \le n \mid p_i^{\max} > 0\}$ be the set of buyers whose payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ are positive, and $\mathcal{I}_2 = [n]\backslash\mathcal{I}_1$ be the set of buyers whose payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ are 0. For an efficient tuple of bid-discount multipliers β^{e} different from $\beta^{\max}$, we have $\beta_i^{\mathrm{e}} \le \beta_i^{\max}$ for all i, with the inequality holds for at least one buyer. Define $\mathcal{I} := \arg\min_i \beta_i^{\mathrm{e}}/\beta_i^{\max}$ as the set of buyers whose bid-discount multipliers are cut the most from $\beta^{\max}$ to β^{e}. Note that $\min_i \beta_i^{\mathrm{e}}/\beta_i^{\max} < 1$. Then we have $\mathcal{I}_2 \cap \mathcal{I} = \emptyset$, since otherwise every buyer in $\mathcal{I}_2 \cap \mathcal{I}$ has smaller bid-discount multiplier in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ than in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$, whereas her payment remains 0 in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$, contradicting the efficiency of β^{e}.

If $\mathcal{I}_1 \ne \mathcal{I}$, let $l \in \mathcal{I}$ and $k \in \mathcal{I}_1 \setminus \mathcal{I}$. Since $\beta_l^{\mathrm{e}} < \beta_l^{\max} \le 1$, $p_k^{\max} > 0$ and $\beta_l^{\mathrm{e}}/\beta_l^{\max} < \beta_k^{\mathrm{e}}/\beta_k^{\max}$, by Lemma B.7 we derive a contradiction that β^{e} is inefficient. Thus $\mathcal{I}_1 = \mathcal{I}$ must holds, which gives the first statement.

Moreover, if there exists $k \in \mathcal{I}_2$ such that $p_k^{\mathrm{e}} > 0$, let l be an arbitrary buyer in $\mathcal{I}_1$. Applying Lemma B.7, we conclude that $\beta^{\max}$ is not efficient, which contradicts the assumption. Hence $p_i^{\mathrm{e}} = 0$ for every $i \in \mathcal{I}_2$. This implies the second statement. $\square$

The properties of efficient BDFPA presented in Lemma B.8 are sufficient to show that all efficient BDFPAs bring the same payment for each buyer.

Lemma B.9. *All efficient BDFPAs bring the same payment for each buyer.*

Proof of Lemma B.9. Suppose β^{e} is an efficient bid-discount tuple different from $\beta^{\max}$. By Lemma B.8, we know that the buyers with payment 0 in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ have payment 0 in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ as well. As for those buyers with positive payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ (i.e., in $\mathcal{I}_1$), the corresponding ratios $\beta_i^{\mathrm{e}}/\beta_i^{\max}$ are identical, which are strictly less than 1. This indicates that these buyers' budgets are all binding in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ due to the efficiency of β^{e}. Meanwhile, we claim that for any buyer in $\mathcal{I}_1$, her payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ is no more than her payment in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$. In fact, any buyer in $\mathcal{I}_1$ cannot win on more quantile profiles in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$ than in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$. Therefore, buyers in $\mathcal{I}_1$ also exhaust their budgets in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$. $\square$

Finally, we present the proof of the last statement, which gives the necessary and sufficient conditions for the uniqueness of an efficient bid-discount multiplier tuple.

Lemma B.10. *$\beta^{\max}$ is the unique efficient tuple of bid-discount multipliers if and only if one of the following two conditions is satisfied: (1) $\max_{i\in\mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) \le \max_{i\in\mathcal{I}_2}\{\beta_i^{\max}\widetilde{v}_i(1), \lambda\}$, where $\mathcal{I}_1 = \{i \mid p_i^{\max} > 0\}$ and $\mathcal{I}_2 = [n] \setminus \mathcal{I}_1$, or (2) there exists $i \in \mathcal{I}_1$ such that $p_i^{\max} < B_i$.*

Proof of Lemma B.10.

"If" side. The proof of Lemma B.9 implies that if there is an efficient tuple other than $\beta^{\max}$, then the budgets of the buyers with positive payments in $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ are binding. In other words, if there exists $i \in \mathcal{I}_1$ such that $p_i^{\max} < B_i$ (which is the second condition), then $\beta^{\max}$ must be the unique efficient tuple.

Furthermore, if $\max_{i \in \mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) \leq \max_{i \in \mathcal{I}_2}\{\beta_i^{\max}\widetilde{v}_i(1), \lambda\}$, suppose there is another efficient tuple β^{e} different from $\beta^{\max}$. By Lemma B.8, for any $i \in \mathcal{I}_2$, we have $p_i^{\max} = p_i^{\mathrm{e}} = 0$ and $\beta_i^{\max} = \beta_i^{\mathrm{e}} = 1$. Also, there exists $0 < \nu < 1$ such that for any $i \in \mathcal{I}_1$, we have $\beta_i^{\mathrm{e}} = \nu\beta_i^{\max}$. Therefore, all buyers in $\mathcal{I}_1$ completely use their budgets. Now, since for any $i \in \mathcal{I}_1$, $\beta_i^{\mathrm{e}}\widetilde{v}_i(0) < \beta_i^{\max}\widetilde{v}_i(0)$, we derive that for any $i \in \mathcal{I}_1$,

$$\beta_i^{\mathrm{e}}\widetilde{v}_i(0) < \beta_i^{\max}\widetilde{v}_i(0) \leq \max_{i \in \mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) \leq \max_{i \in \mathcal{I}_2}\{\beta_i^{\max}\widetilde{v}_i(1), \lambda\} = \max_{i \in \mathcal{I}_2}\{\beta_i^{\mathrm{e}}\widetilde{v}_i(1), \lambda\}.$$

By continuity and strict monotonicity of quantile function, we state that under some quantile profiles with positive measure, no buyer in $\mathcal{I}_1$ wins when the multiplier tuple is β^{e}. This indicates that the total payment of buyers in $\mathcal{I}_1$ is strictly cut from $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ to $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\mathrm{e}})$, contradicting the efficiency of β^{e}, since the budget of at least one buyer in $\mathcal{I}_1$ is not binding.

"Only if" side. We prove by contradiction for this part. We suppose that $\max_{i \in \mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) > \max_{i \in \mathcal{I}_2}\{\beta_i^{\max}\widetilde{v}_i(1), \lambda\}$, $p_i^{\max} = B_i$ for any $i \in \mathcal{I}_1$, and $p_i^{\max} = 0$ for any $i \in \mathcal{I}_2$ reversely. As a result, there exists some $0 < \nu < 1$ such that $\nu \cdot \max_{i \in \mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) > \max_{i \in \mathcal{I}_2}\{\beta_i^{\max}\widetilde{v}_i(1), \lambda\}$. Define $\beta = (\beta_i)_{1 \leq i \leq n}$ as

$$\beta_i = \begin{cases} \nu\beta_i^{\max} & i \in \mathcal{I}_1 \\ 0 & i \in \mathcal{I}_2. \end{cases}$$

Now we show that β is efficient. On the one hand, for $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$, the maximum discounted bid of buyers in $\mathcal{I}_2$ is less than the minimum discounted bid of buyers in $\mathcal{I}_1$, therefore, any buyer in $\mathcal{I}_2$ does not win at all in all quantile profiles. Meanwhile, the item is always allocated as $\max_{i \in \mathcal{I}_1} \beta_i^{\max}\widetilde{v}_i(0) > \lambda$. On the other hand, the bid-discount multipliers of buyers in $\mathcal{I}_1$ are scaled by the same constant from $\beta^{\max}$ to β, thus the ordering of buyers in $\mathcal{I}_1$ remains unchanged in all quantile profiles from $\mathcal{M}^{\mathrm{BDFPA}}(\beta^{\max})$ to $\mathcal{M}^{\mathrm{BDFPA}}(\beta)$. This reasoning implies that payments of all buyers stay the same, and efficiency still holds for β. Therefore, $\beta^{\max}$ is not the unique efficient tuple. $\square$

The proof of Theorem 3.5 is finished by putting Lemma B.5, Lemma B.6, Lemma B.8, Lemma B.9, and Lemma B.10 together. $\square$

B.4 Proof of Theorem 3.6

Proof. We will adopt a similar methodology as in the proof of Theorem 3.5. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be pacing multipliers and define $G_{i,\alpha}$ as the cumulative distribution function of $\max_{i' \neq i}\{\alpha_{i'}\widetilde{v}_{i'}(q_i'), \lambda\}$ where q_{-i} is chosen uniformly in $[0,1]^{n-1}$. Then $G_{i,\alpha}(\cdot)$ has the properties stated in Lemma B.3. Next we prove an analogue of Lemma B.2.

Lemma B.11. *There exists a constant C which satisfies the following: For any $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $1 \leq i \leq n$ such that $\alpha_i < 1$, let $\alpha' = \alpha + \delta e_i$ where $0 < \delta \leq 1 - \alpha_i$ and e_i is the vector with the i-th entry one and all other entries zero. Then the expected payment of buyer i in $\mathcal{M}^{\mathrm{PFPA}}(\alpha')$ is at most the expected payment of buyer i in $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ plus $C\delta$.*

Proof of Lemma B.11. By definition, the expeceted payment of buyer i in $\mathcal{M}^{\mathrm{BDFPA}}(\alpha)$ is

$$\int_0^1 \alpha_i \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} I \left[\alpha_i \widetilde{v}_i(q_i) \geq \max_{i' \neq i} \{\alpha_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\} \right] \mathrm{d}q_{-i} \right) \mathrm{d}q_i.$$

Using the trick in (12) and (13) and noticing that $\alpha_i \in [0,1]$, we rewrite the increment of buyer i's expected payment after replacing α with $\alpha' = \alpha + \delta e_i$ as

$$\begin{aligned} &\int_0^1 \alpha_i \widetilde{v}_i(q_i) \cdot \left(G_{i,\alpha}\left((\alpha+\delta)\widetilde{v}_i(q_i)\right) - G_{i,\alpha}\left(\alpha_i \widetilde{v}_i(q_i)\right)\right) \mathrm{d}q_i \\ \leq &\int_0^1 \widetilde{v}_i(q_i) \cdot \left(G_{i,\alpha}\left((\alpha+\delta)\widetilde{v}_i(q_i)\right) - G_{i,\alpha}\left(\alpha_i \widetilde{v}_i(q_i)\right)\right) \mathrm{d}q_i. \end{aligned}$$

The conclusion now follows directly from the same case analysis as in the proof of Lemma B.2. $\square$

We then show that the set of budget feasible pacing multipliers is compact.

Lemma B.12. *Let $\mathcal{A}$ be the set of all α such that $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ is a feasible pacing mechanism. Then $\mathcal{A}$ is compact.*

Proof of Lemma B.12. Let $\varphi : [0,1]^n \to \mathbb{R}^n$ be defined as the map from the tuple of multipliers α to the expected payment vector of all buyers. Then by Lemma B.11, φ is Lipschitz continuous. Since $\mathcal{A} = \{\alpha : \varphi(\alpha) \in \prod_i [0, B_i]\}$ is the pre-image of $\prod_i [0, B_i]$ (which is certainly closed) under φ, $\mathcal{A}$ is closed. $\mathcal{A}$ is also bounded for $\mathcal{A} \subseteq [0,1]^n$. This concludes the proof of the Lemma. $\square$

We are now able to establish the existence of a maximum tuple of pacing multipliers $\alpha^{\max}$.

Lemma B.13. *There exists a maximum tuple of pacing multipliers $\alpha^{\max}$, i.e., for any feasible tuple of pacing multipliers α, $\alpha_i^{\max} \geq \alpha_i$, for any $1 \leq i \leq n$.*

Proof of Lemma B.13. We first show that given any feasible multipliers $\alpha^{(1)}, \alpha^{(2)}$, the element-wise maximum $\alpha^{\mathrm{h}} = \max(\alpha^{(1)}, \alpha^{(2)})$ is still feasible.

We need to show that budget feasibility is met for each buyer. For any buyer $1 \leq i \leq n$, without loss generality, assume $\alpha_i^{\mathrm{h}} = \alpha_i^{(1)}$, and we claim that buyer i's payment in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\mathrm{h}})$ is no more than her payment in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{(1)})$. Note that for any quantile profile q, if buyer i wins in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\mathrm{h}})$, she definitely wins in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{(1)})$ since $\alpha^{\mathrm{h}} \geq \alpha^{(1)}$, and her payment would be identical in these two auctions as her multipliers are the same in these two tuples. Therefore the claim is shown.

Now let $\alpha_i^{\max} = \sup\{\alpha_i \mid \alpha \text{ is a feasible tuple of multipliers}\}$ for each $1 \leq i \leq n$. Resembling the argument in the proof of Lemma B.6, $\alpha^{\max} = (\alpha_i^{\max})_{1 \leq i \leq n}$ is also a feasible tuple of multipliers. This concludes the proof of the lemma. $\square$

Now that we established the existence of maximum multipliers, we show that it is the unique efficient tuple. We first show it is efficient in the proceeding lemma.

Lemma B.14. *$\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$ is efficient.*

Proof of Lemma B.14. We prove the lemma by contradiction. Suppose $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$ is not efficient, then there is some buyer i such that $\alpha_i^{\max} < 1$ and her budget is not binding. By Lemma B.11, we can increase $\alpha_i^{\max}$ slightly so that buyer i's budget is still not binding. Note that other buyers' payments will not increase when only buyer i's multiplier increases. Therefore, this new tuple of multipliers is still feasible, which contradicts our definition of $\alpha^{\max}$ that it is the entry-wise supremum over all feasible α. $\square$

It remains to show that $\alpha^{\max}$ is the unique efficient tuple of multipliers.

Lemma B.15. $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$ *is the unique efficient tuple of multipliers.*

Proof of Lemma B.15. Let $\alpha \neq \alpha^{\max}$ be a feasible tuple of multipliers. We show that α is not efficient by contradiction.

Suppose α is efficient otherwise. Let $\mathcal{I}$ be the set of buyers such that for any $i \in \mathcal{I}$, $\alpha_i < \alpha_i^{\max}$, i.e., any buyer in $\mathcal{I}$ has a strictly smaller pacing multiplier in α than in $\alpha^{\max}$. Since for $i \in \mathcal{I}$, $\alpha_i < \alpha_i^{\max} \leq 1$, the expected payment of buyer i equals to her budget in $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ by the definition of efficiency. Hence, the Lebesgue measure of quantile profiles won by buyers in $\mathcal{I}$ is positive. Now consider buyers in $\mathcal{I}$ in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$. For any quantile profile won by some buyer in $\mathcal{I}$ in $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$, the quantile profile is also won by $\mathcal{I}$ in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$, as buyers outside $\mathcal{I}$ see no change in the paced bid from $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ to $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$. However, on these quantile profiles, buyers in $\mathcal{I}$ pay more in $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$ than in $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ with strictly higher pacing multipliers. Therefore the total payment of buyers in $\mathcal{I}$ strictly increases from $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ to $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$, and as a result, $\mathcal{M}^{\mathrm{PFPA}}(\alpha^{\max})$ is not budget-feasible. A contradiction. Hence α is not efficient, and $\alpha^{\max}$ is the unique efficient pacing multiplier tuple. $\square$

Finally we establish that $\alpha^{\max}$ maximizes seller's revenue among all feasible tuples of pacing multipliers.

Lemma B.16. $\alpha^{\max}$ *maximizes seller's revenue among all feasible tuples of pacing multipliers.*

Proof of Lemma B.16. Note that for PFPA, the revenue of the seller in $\mathcal{M}^{\mathrm{PFPA}}(\alpha)$ equals to $T \cdot \int_q \max_i \{\alpha_i \widetilde{v}_i(q_i) - \lambda\}^+ \,\mathrm{d}q$ by definition, which, increases with any entry of α. Now that $\alpha^{\max}$ is defined as the supremum over all feasible tuples, it extracts no lower revenue for the seller than any other feasible tuple. $\square$

Synthesizing Lemma B.13, Lemma B.14, Lemma B.15, and Lemma B.16, we conclue the proof of Theorem 3.6. $\square$

B.5 Proof of Lemma 3.7

Proof. We prove the lemma by an example, in which we consider two buyers with significant difference in their valuations.

Assume that $T = 1$ without loss of generality. Now suppose there are $n = 2$ buyers. Let $\lambda = 0.1$, $v_1(q) = \widetilde{v}_1(q) = q$ for any $q \in [0, 1]$ with $B_1 = 0.5$, and $v_2(q) = \widetilde{v}_2(q) = 10q + 10$ for any $q \in [0, 1]$ with $B_2 = 50$. That is, buyer 1's true/reported value distribution is $U[0, 1]$ and buyer 2's true/reported value distribution is $U[10, 20]$. Note that for both BDFPA and PFPA, buyer 2 never exhausts her budget, therefore $\beta_2^{\mathrm{e}} = \alpha_2^{\mathrm{e}} = 1$ when the corresponding mechanism is efficient. Hence, buyer 1 never wins in both auctions. Therefore, under the *classic* setting where buyers can arbitrarily bid, buyer 2 can clearly still wins in all cases and does not exceed her budget even when bidding smaller than her true value, e.g., cut her bid by a half. As a result, buyer 2 strictly pays less under strategically-bidding and gets higher revenue in efficient BDFPA and efficient PFPA. The above reasoning implies that these two auctions are non-BCIC. $\square$

B.6 Proof of Lemma 3.8

Proof. We only need to prove that BDSPA is unconditionally incentive-compatible, and the arguments on PSPA is already given in Theorem 3.4 of [5].

We come to the classic model and suppose the true and reported qf of any buyer $1 \leq i \leq n$ coincide on $\widetilde{v}_i(\cdot)$. Now, we fix some bid-discount multiplier tuple β and quantile profile q. We consider two cases for any buyer i.

- If $\beta_i \widetilde{v}_i(q_i) \geq \max_{i' \geq i}\{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}$, then $\max_{i' \geq i}\{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}/\beta_i \leq \widetilde{v}_i(q_i)$. For i, as long as she wins, her actual bid does not affect her payment, and her revenue remains unchanged at a non-negative value. If i cuts her bid to lose, then her revenue becomes zero, which is no better than winning the item.
- If $\beta_i \widetilde{v}_i(q_i) < \max_{i' \geq i}\{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}$, then $\max_{i' \geq i}\{\beta_{i'} \widetilde{v}_{i'}(q_{i'}), \lambda\}/\beta_i > \widetilde{v}_i(q_i)$. For i, as long as she loses, her revenue remains zero. On the other hand, if she raises her bid to win the item, then her payment becomes strictly larger than the value she receives, and i will get a negative revenue, which is worse than losing.

As a result, BDSPA is unconditionally incentive-compatible. □

C Missing Proofs in Section 4

C.1 Proof of Theorem 4.1

Proof. The proof is done by strong duality. Now when $\widetilde{\boldsymbol{v}}$ is fixed, by [5], the revenue of the seller in BROA is the value of programming (1), or

$$\min_{\gamma \in [0,1]^n} \chi^{\mathrm{BROA}}(\gamma) := \left\{ T\mathbb{E}_q \left[\max_i \left\{ (1-\gamma_i)\widetilde{\psi}_i(q_i) - \lambda \right\}^+ \right] + \sum_{i=1}^n \gamma_i B_i \right\}.$$

On the other side, by strong duality as we give in the proof of Theorem 3.2, the revenue of the seller in efficient BDFPA equals to

$$\min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau) = \left\{ T\mathbb{E}_q \left[\max_i \{(1-\tau_i)\widetilde{v}_i(q_i) - \lambda\}^+ \right] + \sum_{i=1}^n \tau_i B_i \right\}.$$

Notice that $\widetilde{v}_i(q_i) \geq \widetilde{\psi}_i(q_i)$ for any $1 \leq i \leq n$ and $q_i \in [0,1]$ by definition. Then for any $\tau \in [0,1]^n$, $\chi^{\mathrm{BD}}(\tau) \geq \chi^{\mathrm{BROA}}(\tau)$. As a result, $\min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau) \geq \min_{\gamma \in [0,1]^n} \chi^{\mathrm{BROA}}(\gamma)$, which finishes the proof. □

C.2 Proof of Theorem 4.2

Proof. The theorem follows by a duality argument. To start with, as we provide in the proof of Theorem 3.2, we have a strong duality result for BDFPA. In other words, given $\widetilde{\boldsymbol{v}}$ and B, the revenue of efficient BDFPA for the seller, or $\mathrm{OPT}^{\mathrm{BD}}$, equals to $\min_{\tau \in [0,1]^n} \chi^{\mathrm{BD}}(\tau)$, with $\chi^{\mathrm{BD}}(\tau)$ defined as:

$$\chi^{\mathrm{BD}}(\tau) := T \cdot \mathbb{E}_q \left[\max_{1 \leq i \leq n} \{(1-\tau_i)\widetilde{v}_i(q_i) - \lambda\}^+ \right] + \sum_{i=1}^n \tau_i B_i.$$

Now, the revenue of any efficient pacing mechanism is no larger than the value of the following programming, which represents the optimal revenue of any feasible (no requirement for efficiency) pacing first-price auction. Note that the above reasoning does not depend on Theorem 3.6, which demands that the quantile functions are bounded and inverse Lipschitz continuous.

$$\begin{aligned} &\max_{\alpha\in[0,1]^n} T\cdot\int_q \max_i \{\alpha_i\widetilde{v}_i(q_i)-\lambda\}^+ \,\mathrm{d}q, \\ \text{s.t.}\quad & T\cdot\int_0^1 \alpha_i\widetilde{v}_i(q_i)\cdot\left(\int_{q_{-i}}\Phi_i(\alpha,\widetilde{\boldsymbol{v}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i\le B_i,\quad \forall 1\le i\le n. \end{aligned}$$

Denote the optimal value of the above programming by $\mathrm{OPT}^{\mathrm{P}}$. We consider the Lagrangian dual χ^{P} of the above programming, with dual variables $\rho_{1\le i\le n}$:

$$\chi^{\mathrm{P}}(\rho):=\max_{\alpha\in[0,1]^n} T\cdot\sum_{i=1}^n\int_0^1((1-\rho_i)\alpha_i\widetilde{v}_i(q_i)-\lambda)\cdot\left(\int_{q_{-i}}\Phi_i(\alpha,\widetilde{\boldsymbol{v}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i+\sum_{i=1}^n\rho_iB_i. \tag{15}$$

By weak duality, we have

$$\begin{aligned} \mathrm{OPT}^{\mathrm{P}} &\le \min_{\rho\ge0}\chi^{\mathrm{P}}(\rho)\le\min_{\rho\in[0,1]^n}\chi^{\mathrm{P}}(\rho) \\ &=\min_{\rho\in[0,1]^n}\max_{\alpha\in[0,1]^n} T\cdot\sum_{i=1}^n\int_0^1((1-\rho_i)\alpha_i\widetilde{v}_i(q_i)-\lambda)\cdot\left(\int_{q_{-i}}\Phi_i(\alpha,\widetilde{\boldsymbol{v}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i+\sum_{i=1}^n\rho_iB_i \\ &\le\min_{\rho\in[0,1]^n}\max_{\alpha\in[0,1]^n} T\cdot\sum_{i=1}^n\int_0^1((1-\rho_i)\widetilde{v}_i(q_i)-\lambda)\cdot\left(\int_{q_{-i}}\Phi_i(\alpha,\widetilde{\boldsymbol{v}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i+\sum_{i=1}^n\rho_iB_i \\ &=\min_{\rho\in[0,1]^n} T\cdot\mathbb{E}_q\left[\max_{1\le i\le n}\{(1-\rho_i)\widetilde{v}_i(q_i)-\lambda\}^+\right]+\sum_{i=1}^n\rho_iB_i \\ &=\min_{\rho\in[0,1]^n}\chi^{\mathrm{BD}}(\rho)=\mathrm{OPT}^{\mathrm{BD}}. \end{aligned}$$

Here the third line is due to $\alpha_i\le1$. The fourth line follows a similar argument used when we prove Theorem 3.2, specifically (8) and (9). As a result, we have $\mathrm{OPT}^{\mathrm{P}}\le\mathrm{OPT}^{\mathrm{BD}}$. Since $\mathrm{OPT}^{\mathrm{P}}$ is no less than the revenue of any efficient PFPA, the theorem is proved. ☐

C.3 Proof of Theorem 4.5

Proof. We will make some preparations before we come to the main part of the proof. Notice that by Lemma 2.1, we can disregard the bidding strategy part in the proof.

We first discuss on the tuple γ^* in BROA (Proposition 3.1). We characterize any buyer's expected payment in BROA in the following lemma:

Lemma C.1 (in [5]). *Suppose the profile is $\widetilde{\boldsymbol{v}}$ for reported quantile functions, $\boldsymbol{b}$ for bidding strategies, and B for budgets. Then, in BROA given in Proposition 3.1, we have:*

$$\mathbb{E}_{q_i}\left[p_i^{\widetilde{\boldsymbol{v}}}(q_i)\right]=\mathbb{E}_{q_i}\left[\widetilde{\psi}_i(q_i)\cdot x_i^{\widetilde{\boldsymbol{v}}}(q_i)\right].$$

Therefore, by Lemma B.1, programming (1) is equivalent to the following conditions:

$$T\cdot\int_0^1\widetilde{\psi}_i(q_i)\cdot\left(\int_{q_{-i}}\Phi_i(1^n-\gamma,\widetilde{\boldsymbol{\psi}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i\le B_i,\quad\forall1\le i\le n.$$

$$\gamma_i\cdot\left(B_i-T\cdot\int_0^1\widetilde{\psi}_i(q_i)\cdot\left(\int_{q_{-i}}\Phi_i(1^n-\gamma,\widetilde{\boldsymbol{\psi}},q)\,\mathrm{d}q_{-i}\right)\mathrm{d}q_i\right)=0,\quad\forall1\le i\le n. \tag{16}$$

Note that the second multiplying term in the set of constraints represents the expected remaining budget of each buyer. Therefore, Lemma B.1 shows that as long as buyer i's budget is not binding, $\gamma_i^* = 0$ establishes.

Meanwhile, for a better look, we restate here the equivalence conditions of the multiplier tuple of the efficient BDFPA according to Definition 3.1 and Theorem 3.2:

$$T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \le B_i, \quad \forall 1 \le i \le n.$$

$$(1 - \beta_i) \cdot \left(B_i - T \cdot \int_0^1 \widetilde{v}_i(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta, \widetilde{\boldsymbol{v}}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \right) = 0, \quad \forall 1 \le i \le n. \tag{17}$$

We are now ready to prove our main theorem. We suppose that the reported quantile functions profile in BROA is $\widetilde{\boldsymbol{v}}^{(1)}$, while the counterpart in BDFPA is $\widetilde{\boldsymbol{v}}^{(2)}$. Meanwhile, we also give the revenue of any buyer $1 \le i \le n$ in the fake distribution model under both auction families in the following:

$$w_i^{(\mathrm{BROA}, \widetilde{\boldsymbol{v}}^{(1)})} = \int_0^1 \left(v_i(q_i) - \widetilde{\psi}_i^{(1)}(q_i) \right) \cdot \left(\int_{q_{-i}} \Phi_i(1^n - \gamma^*, \widetilde{\boldsymbol{\psi}}^{(1)}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i. \tag{18}$$

$$w_i^{(\mathrm{eBDFPA}, \widetilde{\boldsymbol{v}}^{(2)})} = \int_0^1 \left(v_i(q_i) - \widetilde{v}_i^{(2)}(q_i) \right) \cdot \left(\int_{q_{-i}} \Phi_i(\beta^{\mathrm{e}}, \widetilde{\boldsymbol{v}}^{(2)}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i. \tag{19}$$

From BROA to eBDFPA. In this part, we suppose that the virtual valuation $\widetilde{\psi}_i^{(1)}(0)$ is strictly increasing and differentiable for any $1 \le i \le n$. Without loss of generality, we suppose that $\widetilde{\psi}_i^{(1)}(0) \ge \lambda$ holds for any $1 \le i \le n$, or that the corresponding buyer has no chance to win any item at all. For the injective mapping, we let $\widetilde{v}_i^{(2)}(\cdot)$ to be a modification of $\widetilde{\psi}_i^{(1)}(\cdot)$, the virtual valuation of $\widetilde{v}_i^{(1)}(\cdot)$ for any $1 \le i \le n$. Intuitively, we will "lift" the negative part $\widetilde{\psi}_i^{(1)}(\cdot)$ so that the resulting function is non-negative, with no loss on other required properties. If $\widetilde{\psi}_i^{(1)}(1) \ge 0$, we easily define $\widetilde{v}_i^{(2)}(\cdot) = \widetilde{\psi}_i^{(1)}(\cdot)$. Otherwise, we construct $\widetilde{v}_i^{(2)}(\cdot)$ by replacing the tail part of $\widetilde{\psi}_i^{(1)}(\cdot)$ with an exponential function or a trigonometric function, depending on whether the slope of the joint of two functions is 0. More specifically, let $q_i^0 \in (0,1)$ be the point satisfying $\widetilde{\psi}_i^{(1)}(q_i^0) = \lambda/2$ (we will argue later that such q_i^0 exists in the proof of Lemma C.2), and $k := \left(\widetilde{\psi}_i^{(1)}\right)'(q_i^0) \ge 0$. If $k > 0$, define $\widetilde{v}_i^{(2)}(\cdot)$ as follows:

$$\widetilde{v}_i^{(2)}(q_i) = \begin{cases} a_1 \cdot \exp\{a_2 q_i\} & 0 \le q_i < q_i^0 \\ \widetilde{\psi}_i^{(1)}(q_i) & q_i^0 \le q_i \le 1, \end{cases}$$

where $a_1 = \lambda \cdot \exp\{-2kq_i^0/\lambda\}/2$ and $a_2 = 2k/\lambda$. Such "lifting" process is plotted in Figure 2. Otherwise when $k = 0$, we use a trigonometric function instead to "lift" the negative part and define $\widetilde{v}_i^{(2)}(\cdot)$ as follows:

$$\widetilde{v}_i^{(2)}(q_i) = \begin{cases} (\lambda/4) \cdot \sin(a_3 q_i + \pi/4) + \lambda/4 & 0 \le q_i < q_i^0 \\ \widetilde{\psi}_i^{(1)}(q_i) & q_i^0 \le q_i \le 1, \end{cases}$$

where $a_3 = \pi/(4q_i^0)$.

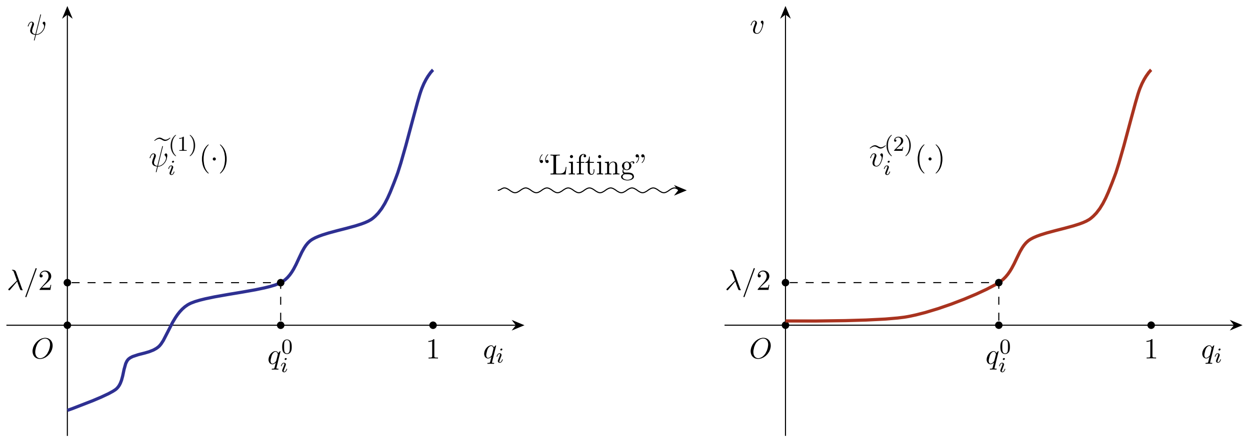

Figure 2: The "lifting" process, which is adopted to construct $\widetilde{v}_i^{(2)}(\cdot)$ when $\widetilde{\psi}_i^{(1)}(\cdot)$ has negative parts.

In the following lemma, we show that the above injective mapping guarantees that $\widetilde{v}_i^{(2)}$ is non-negative, strictly increasing and differentiable. Inverse Lipschitz continuity also remains after the mapping for any $1 \le i \le n$.

Lemma C.2. *Suppose that $\widetilde{\psi}_i^{(1)}$ is strictly increasing and differentiable for some i, then under the mapping given above, $\widetilde{v}_i^{(2)}$ also satisfies these properties, and is further non-negative. Meanwhile, inverse Lipschitz continuity is also kept under the mapping.*

Proof of Lemma C.2. In the first case when $\widetilde{\psi}_i^{(1)}(x)$ is non-negative on $[0,1]$, the lemma trivially holds. For the second case, by continuity and that $\widetilde{\psi}_i^{(1)}(0) < 0, \widetilde{\psi}_i^{(1)}(1) \ge \lambda$, the point $q_i^0 \in (0,1)$ such that $\widetilde{\psi}_i^{(1)}(q_i^0) = \lambda/2$ exists. By construction, $\widetilde{v}_i^{(2)}(x)$ is certainly non-negative and continuous. We further show that $\widetilde{v}_i^{(2)}(x)$ is differentiable, which clearly, reduces to demonstrate the function is differentiable at q_i^0. We can verify this by direct computation that $\left(\widetilde{v}_i^{(2)}\right)'(q_i^0) = k$ whether $k > 0$ or $k = 0$. Strictly increasing monotonicity follows from that $\widetilde{\psi}_i^{(1)}(x)$ is strictly increasing, and $a_1 \exp\{a_2 x\}$ (when $k > 0$) and $(\lambda/4)\sin(a_3 x + \pi/4) + \lambda/4$ (when $k = 0$) are both strictly increasing on $[0, q_i^0]$.

Furthermore, when inverse Lipschitz continuity holds for $\widetilde{\psi}_i^{(1)}$, then $\left(\widetilde{\psi}_i^{(1)}\right)'$ has an upper bound strictly higher than 0 by definition, which leads to $k > 0$. In this case, $a_1 \exp\{a_2 x\}$ is also inverse Lipschitz continuous on $[0, q_i^0]$ as a_1, a_2 are both constants. As a result, $\widetilde{v}_i^{(2)}(x)$ is inverse Lipschitz continuous as well. □

By Lemma C.2, $\widetilde{\boldsymbol{v}}^{(2)}$ is a valid reported quantile function profile. We now let $\beta_i^* = 1 - \gamma_i^*$ for any $1 \le i \le n$, and argue that β^* is feasible for (17).

In fact, noticing that under the injective mapping we give above, $\widetilde{v}_i^{(2)}(q_i) = \widetilde{\psi}_i^{(1)}(q_i)$ always holds when the value is above $\lambda/2$. At the same time, due to the threshold effect brought by the opportunity cost λ and that $\beta^* \le 1^n$, we have

$$\int_{q_{-i}} \Phi_i(1^n - \gamma^*, \widetilde{\boldsymbol{\psi}}^{(1)}, q)\, \mathrm{d}q_{-i} = \int_{q_{-i}} \Phi_i(\beta^*, \widetilde{\boldsymbol{v}}^{(2)}, q)\, \mathrm{d}q_{-i}$$

holds for any $1 \leq i \leq n$ and $q_i \in [0,1]$. Therefore, the programming (17) is equivalent to programming (16) with $\widetilde{\boldsymbol{\psi}}^{(1)} = \widetilde{\boldsymbol{v}}^{(2)}$ and $\gamma = 1^n - \beta$. Hence, by Lemma B.1, $\beta^* = 1^n - \gamma^*$ satisfies (17). Note that buyers' revenue function in BROA and efficient BDFPA are given by (18) and (19) respectively. Therefore, in any BROA with $\widetilde{\boldsymbol{v}}^{(1)}$, there is some efficient BDFPA with $\widetilde{\boldsymbol{v}}^{(2)}$, such that for any buyer, her utilities are the same in the two auctions.

From eBDFPA to BROA. In this part, we show that for any efficient BDFPA with $\widetilde{\boldsymbol{v}}^{(2)}$, there is some BROA with $\widetilde{\boldsymbol{v}}^{(1)}$, such that every buyer's revenue is identical in the two auctions.

For the mapping from $\widetilde{\boldsymbol{v}}^{(2)}$ to $\widetilde{\boldsymbol{v}}^{(1)}$, we hope to have $\widetilde{\psi}_i^{(1)}(\cdot) = \widetilde{v}_i^{(2)}(\cdot)$ for any i. In other words, we carefully pick $\widetilde{v}_i^{(1)}(\cdot)$ such that for any $q_i \in [0,1]$, $\widetilde{v}_i^{(1)}(q_i) - (1-q_i)\left(\widetilde{v}_i^{(1)}\right)'(q_i) = \widetilde{v}_i^{(2)}(q_i)$. The following lemma shows that such a function $\widetilde{v}_i^{(1)}(\cdot)$ exists.

Lemma C.3. *For each strictly increasing and differentiable function $r(x)$ on $[0,1]$ that satisfies $r(1) > 0$ and $\int_0^1 r(x)\,\mathrm{d}x \geq 0$, there exists a non-negative, strictly increasing and differentiable function $s(x)$ on $[0,1]$ such that $r(x) = s(x) - (1-x)s'(x)$ for any $x \in [0,1]$. Meanwhile, inverse Lipschitz continuity is also kept under the mapping.*

Proof of Lemma C.3. We define the function $s(x)$ as follows: $s(1) = r(1)$, and when $x < 1$, $s(x) = \frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x}$. We now show that $s(\cdot)$ satisfies that $r(x) = s(x) - (1-x)s'(x)$ for any $x \in [0,1]$. In fact, the equality obviously holds when $x = 1$, and when $x < 1$, we have

$$
\begin{aligned}
s(x) - (1-x)s'(x) &= \frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x} - (1-x)\left(\frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x}\right)' \\
&= \frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x} - (1-x)\left(\frac{-r(x)(1-x) + \int_x^1 r(t)\,\mathrm{d}t}{(1-x)^2}\right) = r(x).
\end{aligned}
$$

Differentiability holds naturally. We now show that $s(x) = \frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x}$ is non-negative and strictly increasing on $[0,1]$, which establishes as $r(t)$ is strictly increasing and $\int_0^1 r(x)\,\mathrm{d}x \geq 0$.

It remains to show that $s(\cdot)$ is inverse Lipschitz continuous when $r(\cdot)$ is. To see this, we do a derivation on $s(\cdot)$,

$$
s'(x) = \left(\frac{\int_x^1 r(t)\,\mathrm{d}t}{1-x}\right)' = \frac{-r(x)(1-x) + \int_x^1 r(t)\,\mathrm{d}t}{(1-x)^2} = \frac{\int_x^1 \left(r(t) - r(x)\right)\,\mathrm{d}t}{(1-x)^2}.
$$

By inverse Lipschitz continuity of $r(\cdot)$, there exists a positive constant C, such that $r(t) - r(x) \geq C(t-x)$ for any $0 \leq t < x \leq 1$. Therefore,

$$
s'(x) = \frac{\int_x^1 \left(r(t) - r(x)\right)\,\mathrm{d}t}{(1-x)^2} \geq C \cdot \frac{\int_x^1 (t-x)\,\mathrm{d}t}{(1-x)^2} = C/2 > 0,
$$

which indicates that $s(\cdot)$ is inverse Lipschitz continuous as well. $\square$

Lemma C.3 demonstrates the feasibility of $\widetilde{v}_i^{(1)}(\cdot)$. Now we let $\gamma_i = 1 - \beta_i$ for any i with slight abuse of notation. With a similar reasoning, γ is feasible for (16) if and only if β is efficient for BDFPA, by noticing that for any $1 \leq i \leq n$ and quantile profile q,

$$
\int_{q_{-i}} \Phi_i(1^n - \gamma^*, \widetilde{\boldsymbol{\psi}}^{(1)}, q)\,\mathrm{d}q_{-i} = \int_{q_{-i}} \Phi_i(\beta^*, \widetilde{\boldsymbol{v}}^{(2)}, q)\,\mathrm{d}q_{-i}.
$$

Therefore, by Lemma B.1, γ is optimal for programming (1) if and only if β is efficient for BDFPA. As a result, by (18) and (19), the proof of this direction is finished.

Synthesizing the above two parts, we finish the proof of the essential theorem. $\square$

C.4 Proof of Corollary 4.6

Proof. Now suppose that all functions we considered are inverse Lipschitz continuous. By the last part of the proof of Theorem 4.5, we see that under the mapping from BDFPA with $\widetilde{\boldsymbol{v}}^{(2)}$ to BROA with $\widetilde{\boldsymbol{v}}^{(1)}$, β is an efficient multiplier tuple for BDFPA if and only if $1^n - \beta$ is an optimal solution for programming (1) in BROA. By Theorem 3.5, there is an efficient multiplier tuple $\beta^{\max}$ for BDFPA that is maximum among all efficient tuples. Consequently, $\gamma^{\min} := 1 - \beta^{\max}$ is the minimum optimal solution of programming (1). $\square$

C.5 Proof of Theorem 4.7

Proof. The proof is done in two parts. To start with, we will show that under the conditions, the maximum efficient tuple induced by either BDFPA or PFPA is symmetric. Next, we give the equivalence mapping between eBDFPA and ePFPA.

Now, the following lemma solves first part.

Lemma C.4. *In the symmetric case where each buyer's reported qf is inverse Lipschitz continuous, the maximum efficient tuple induced by either BDFPA or PFPA is symmetric.*

Proof of Lemma C.4. We prove by contradiction. By Theorem 3.5, there exists a maximum efficient tuple $\beta^{\max}$ for BDFPA. Suppose otherwise $\beta^{\max}$ is not symmetric. Then by Lemma B.5, Lemma B.6 and symmetry, $\max(\beta^{\max})1^n$ is also efficient, which contradicts the optimality of $\beta^{\max}$. The case for PFPA is similar. $\square$

We come to the main part of the proof. We let the report qf profile of BDFPA be $\widetilde{\boldsymbol{v}}^{(2)}$, and the counterpart of PFPA be $\widetilde{\boldsymbol{v}}^{(3)}$. Meanwhile, we let some efficient tuple of BDFPA and PFPA be respectively $\beta^{\mathrm{e}} = \beta_0 1^n$ and $\alpha^{\mathrm{e}} = \alpha_0 1^n$.

From eBDFPA to ePFPA. For this part, if $\beta_0 = 1$, then by the budget constraints of BDFPA and PFPA, $\alpha = 1^n$ is also feasible for PFPA with $\widetilde{\boldsymbol{v}}^{(3)} = \widetilde{\boldsymbol{v}}^{(2)}$. As a result, under the identical mapping from $\widetilde{\boldsymbol{v}}^{(2)}$ to $\widetilde{\boldsymbol{v}}^{(3)}$, we have $\alpha_0 = 1$, and the two mechanisms are essentially the same. The revenue of the seller faces no change naturally under the mapping.

We now consider the more general case with $\beta_0 < 1$. Our intuition is to "raise" the right part of $\widetilde{v}_0^{(2)}(\cdot)$ to construct $\widetilde{v}_0^{(3)}(\cdot)$, and keep $\alpha_0 = \beta_0$, which eases our analysis. Here, $\widetilde{v}_0^{(2)}(\cdot)$ and $\widetilde{v}_0^{(3)}(\cdot)$ are the identical reported qf functions for all buyers respectively in $\widetilde{\boldsymbol{v}}^{(2)}$ and $\widetilde{\boldsymbol{v}}^{(3)}$.

By definition of efficiency, each buyer exhausts her budget under $\beta^{\max}$. Therefore, we can define $q_0 \in [0,1]$ as $q_0 := \inf\{x \in [0,1] \mid \widetilde{v}_0^{(2)}(x) \geq \lambda/\beta_0\}$.

Thus, the expected payment of any buyer i satisfies that

$$
\begin{aligned}
p^{\mathrm{BDF}}(\beta_0) &= T \cdot \int_0^1 \widetilde{v}_0^{(2)}(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\beta^{\mathrm{e}}, \widetilde{\boldsymbol{v}}^{(2)}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \\
&= T \cdot \int_{q_0}^1 \widetilde{v}_0^{(2)}(x) \cdot x^{n-1} \, \mathrm{d}x = B_0. \qquad (20)
\end{aligned}
$$

Here, B_0 is the universal budget for all buyers. We now let $k_0 > 0$ (which we will show later) be a constant satisfying

$$\beta_0 \cdot k_0 \cdot \int_{q_0}^{1} (x - q_0) \cdot x^{n-1} \, \mathrm{d}x = \frac{B_0}{T} - \lambda \cdot \frac{1 - q_0^n}{n},$$

and define $\widetilde{v}_0^{(3)}(\cdot)$ as:

$$\widetilde{v}_0^{(3)}(x) = \begin{cases} a_1 \cdot \exp\{a_2 x\} & 0 \le x < q_0 \\ k_0 \cdot (x - q_0) + \lambda/\beta_0 & q_0 \le x \le 1, \end{cases}$$

where $a_1 = \lambda/\beta_0 \cdot \exp\{-k_0\beta_0 q_0/\lambda\}$ and $a_2 = k_0\beta_0/\lambda$.

We now claim that ePFPA with $\widetilde{\boldsymbol{v}}^{(3)}$ brings the same revenue for the seller as eBDFPA with $\widetilde{\boldsymbol{v}}^{(2)}$. We first show that $\widetilde{v}_0^{(3)}(\cdot)$ is strictly increasing, differentiable, and inverse Lipshitz continuous, which reduces to $k_0 > 0$. Note that

$$B_0 = T \cdot \int_{q_0}^{1} \widetilde{v}_0^{(2)}(x) \cdot x^{n-1} \, \mathrm{d}x > T \cdot \frac{\lambda}{\beta_0} \int_{q_0}^{1} x^{n-1} \, \mathrm{d}x > T \cdot \lambda \int_{q_0}^{1} x^{n-1} \, \mathrm{d}x = T \cdot \lambda \cdot \frac{1 - q_0^n}{n}.$$

Here, the first inequality is because $\widetilde{v}_0^{(2)}(\cdot)$ is strictly increasing and $\widetilde{v}_0^{(2)}(q_0) = \lambda/\beta_0$. The second inequality is because $\beta_0 < 1$. As a result, $B_0/T - \lambda(1 - q_0^n)/n$ is positive and so is k_0 by definition.

We now show that $\alpha_0 = \beta_0$. To see this, we notice that since $q_0 = \inf\{x \in [0,1] \mid \widetilde{v}_0^{(3)}(x) \ge \lambda/\beta_0\}$,

$$\begin{aligned} p^{\mathrm{PF}}(\beta_0) &= T \cdot \int_{q_0}^{1} \beta_0 \widetilde{v}_0^{(3)}(x) \cdot x^{n-1} \, \mathrm{d}x \\ &= T \cdot \int_{q_0}^{1} \beta_0 \left(k_0(x - q_0) + \frac{\lambda}{\beta_0} \right) \cdot x^{n-1} \, \mathrm{d}x \\ &= T \cdot \left(\frac{B_0}{T} - \lambda \cdot \frac{1 - q_0^n}{n} \right) + T \cdot \lambda \cdot \frac{1 - q_0^n}{n} = B_0. \end{aligned}$$

Therefore, any buyer exhaust her budget when the common multiplier for everyone is β_0. By definition, we have $\alpha_0 = \beta_0$.

Now, we see that the revenue of the seller for eBDFPA with $\widetilde{\boldsymbol{v}}_0^{(2)}$ is

$$nB_0 - T \cdot \lambda \cdot (1 - q_0^n),$$

since the probability that each item is not allocated to any buyer is q_0^n. By definition and that $\widetilde{v}_0^{(3)}(q_0) = \lambda/\beta_0 = \lambda/\alpha_0$, the revenue of the seller for ePFPA with $\widetilde{\boldsymbol{v}}_0^{(3)}$ is also the above value. Therefore, the proof of this side is finished.

From ePFPA to eBDFPA. For this side, the case of $\alpha_0 = 1$ is similar with the other side we have already discussed, and we now suppose $\alpha_0 < 1$. Intuitively, the skill we apply in the previous part does not work here, as in that part, we "raise" the right side of $\widetilde{v}_0^{(2)}(\cdot)$ to construct $\widetilde{v}_0^{(3)}(\cdot)$, and increasing monotonicity still holds. However, in this part, we cannot directly "lower" the right side of $\widetilde{v}_0^{(3)}(\cdot)$ while guaranteeing increasing monotonicity since we have no prior knowledge on the shape of $\widetilde{v}_0^{(2)}(\cdot)$. Hence, we cannot take $\beta_0 = \alpha_0$ and must have β_0 sufficiently close to 1.

We now give the formal proof. Let $q_0 := \inf\{x \in [0,1] \mid \widetilde{v}_0^{(3)}(x) \geq \lambda/\alpha_0\} \in [0,1]$, which should not be confused with the q_0 defined in the previous part. Since $\alpha_0 < 1$ and every buyer's budget is binding, the expected payment of each buyer satisfies

$$\begin{aligned} p^{\mathrm{PF}}(\alpha_0) &= T \cdot \int_0^1 \alpha_0 \widetilde{v}_0^{(3)}(q_i) \cdot \left(\int_{q_{-i}} \Phi_i(\alpha^{\mathrm{e}}, \widetilde{\boldsymbol{v}}^{(3)}, q) \, \mathrm{d}q_{-i} \right) \mathrm{d}q_i \\ &= T \cdot \int_{q_0}^1 \alpha_0 \widetilde{v}_0^{(3)}(x) \cdot x^{n-1} \, \mathrm{d}x = B_0. \end{aligned} \tag{21}$$

By inverse Lipschitz continuity, there exists some constant $l_0 > 0$ such that for any $x \in [q_0, 1]$,

$$\widetilde{v}_0^{(3)}(x) - \widetilde{v}_0^{(3)}(q_0) \geq l_0(x - q_0).$$

We now let $\beta_0 \in (0,1)$ (which we will show later) be the solution to

$$\frac{\lambda}{\beta_0} \cdot \frac{1 - q_0^n}{n} = \frac{B_0}{T} - \frac{\alpha_0 l_0}{2} \int_{q_0}^1 (x - q_0) \cdot x^{n-1} \, \mathrm{d}x,$$

and define $\widetilde{v}_0^{(2)}(\cdot)$ as:

$$\widetilde{v}_0^{(2)}(x) = \begin{cases} a_3 \cdot \exp\{a_4 x\} & 0 \leq x < q_0 \\ (\alpha_0 l_0/2) \cdot (x - q_0) + \lambda/\beta_0 & q_0 \leq x \leq 1, \end{cases}$$

where $a_3 = \lambda/\beta_0 \cdot \exp\{-l_0\alpha_0\beta_0 q_0/(2\lambda)\}$ and $a_4 = l_0\alpha_0\beta_0/(2\lambda)$.

Again, the feasibility of $\widetilde{v}_i^{(2)}(\cdot)$ establishes as long as $\beta_0 \in (0,1)$. Note that

$$\begin{aligned} B_0 &= T \cdot \int_{q_0}^1 \alpha_0 \widetilde{v}_0^{(3)}(x) \cdot x^{n-1} \, \mathrm{d}x \\ &\geq T \cdot \int_{q_0}^1 \alpha_0 \left(l_0(x - q_0) + \frac{\lambda}{\alpha_0} \right) \cdot x^{n-1} \, \mathrm{d}x \\ &> T \cdot \left(\frac{\alpha_0 l_0}{2} \int_{q_0}^1 (x - q_0) \cdot x^{n-1} \, \mathrm{d}x + \lambda \cdot \frac{1 - q_0^n}{n} \right), \end{aligned}$$

which leads to

$$\frac{B_0}{T} - \frac{\alpha_0 l_0}{2} \int_{q_0}^1 (x - q_0) \cdot x^{n-1} \, \mathrm{d}x > \lambda \cdot \frac{1 - q_0^n}{n} > 0,$$

and therefore $\beta_0 \in (0,1)$.

We now verify that β_0 is indeed the maximum efficient multiplier for each buyer in BDFPA. To see this, we notice that $\widetilde{v}_0^{(2)}(\cdot)$ is strictly increasing, and that $\widetilde{v}_0^{(2)}(q_0) = \lambda/\beta_0$. Therefore,

$$\begin{aligned} p^{\mathrm{BDF}}(\beta_0) &= T \cdot \int_{q_0}^1 \widetilde{v}_0^{(2)}(x) \cdot x^{n-1} \, \mathrm{d}x \\ &= T \cdot \int_{q_0}^1 \left((\alpha_0 l_0/2)(x - q_0) + \frac{\lambda}{\beta_0} \right) \cdot x^{n-1} \, \mathrm{d}x \\ &= T \cdot \left(\frac{\alpha_0 l_0}{2} \int_{q_0}^1 (x - q_0) \cdot x^{n-1} + \frac{\lambda}{\beta_0} \cdot \frac{1 - q_0^n}{n} \right) \\ &= T \cdot \frac{B_0}{T} = B_0, \end{aligned}$$

Here, the very first equality holds since $q_0 = \inf\{x \in [0,1] \mid \widetilde{v}_0^{(2)}(x) \geq \lambda/\beta_0\}$. As a result, each buyer's budget under BDFPA is binding with multiplier β_0, and β_0 is an efficient multiplier. By a similar argument from the previous part, the seller's revenue in both ePFPA with $\widetilde{\boldsymbol{v}}^{(3)}$ and eBDFPA with $\widetilde{\boldsymbol{v}}^{(2)}$ equals to $nB_0 - T \cdot \lambda \cdot (1 - q_0^n)$, and the proof of this part is also done.

By combining the two directions, we finish the proof of the theorem. □

C.6 Proof of Theorem 4.8

Proof. The proof of this essential theorem mainly focuses on showing the equivalence of eBDFPA and eBDSPA. The equivalence of ePFPA and ePSPA is similar, which we omit. At last, by Theorem 4.7, the proof is done.

Before we come to the main part of the proof, we first show that resembling Lemma C.4, there exists a maximum symmetric efficient multiplier tuple for either BDSPA and PSPA. We let the reported qf of every buyer be $\widetilde{v}_0^{(4)}(\cdot)$ in BDSPA, and $\widetilde{v}_0^{(5)}(\cdot)$ in PSPA. Meanwhile, the budget of each buyer equals to B_0.

Lemma C.5. *In the symmetric case where each buyer's reported qf is inverse Lipschitz continuous, there is a symmetric efficient multiplier tuple for either BDSPA and PSPA.*

Proof of Lemma C.5. We first show that the lemma holds with BDSPA. Under the given conditions, when the multiplier tuple is symmetric (suppose, to be $\beta = \beta_0 1^n$), the payment of each buyer is the following:

$$p^{\mathrm{BDS}}(\beta_0) = T \cdot \int_{[0,1]^n} \max_{i' \neq i} \left\{ \widetilde{v}_0^{(4)}(q_{i'}), \frac{\lambda}{\beta_0} \right\} \cdot \Phi_i(\beta_0 1^n, \widetilde{\boldsymbol{v}}^{(4)}, q) \, \mathrm{d}q, \tag{22}$$

which certainly, is a Lipschitz continuous function[5] of β_0 when $\beta_0 > \lambda/\widetilde{v}_0^{(4)}(1) > 0$. Therefore, if $\sup\{p^{\mathrm{BDS}}(\beta_0) \mid \beta_0 \in [0,1]\} < B_0$, then by definition, 1^n is the unique symmetric efficient tuple. Otherwise, there exists some $0 < \beta_0^* < 1$ such that $p^{\mathrm{BDS}}(\beta_0^*) = B_0$ by continuity, from which we derive that $\beta_0^* 1^n$ is a symmetric efficient tuple.

For PSPA, we notice that with symmetric multipliers $\alpha_0 1^n$, the payment of each buyer is

$$p^{\mathrm{PS}}(\alpha_0) = T \cdot \int_{[0,1]^n} \max_{i' \neq i} \left\{ \alpha_0 \widetilde{v}_0^{(5)}(q_{i'}), \lambda \right\} \cdot \Phi_i(\alpha_0 1^n, \widetilde{\boldsymbol{v}}^{(5)}, q) \, \mathrm{d}q, \tag{23}$$

and is increasingly Lipschitz continuous on α_0. Therefore, if $p^{\mathrm{PS}}(1) < B_0$, then by definition, 1^n is the unique symmetric efficient tuple. Otherwise, there exists some $0 < \alpha_0^* < 1$ such that $p^{\mathrm{PS}}(\alpha_0^*) = B_0$ by continuity, which indicates that $\alpha_0^* 1^n$ is an efficient tuple. □

Now we are ready to show the equivalence of eBDFPA and eBDSPA. Following previous notations, we let the identical reported qf of all buyers in BDFPA be $\widetilde{v}_0^{(2)}$, and the maximum multiplier be β_0^{F}; for BDSPA, let some efficient multiplier be β_0^{S}.

From eBDSPA to eBDFPA. For this part, we will give a mapping from $\widetilde{v}_0^{(4)}(\cdot)$ to $\widetilde{v}_0^{(2)}(\cdot)$ meanwhile guaranteeing that $\beta_0^{\mathrm{F}} = \beta_0^{\mathrm{S}}$. Nevertheless, we first deal with extreme cases when $\beta_0^{\mathrm{S}} = 1$ and $\widetilde{v}_0^{(4)}(1) \leq \lambda$, which means that the item is never allocated. In this scenario, let $\widetilde{v}_0^{(2)}(\cdot) = \widetilde{v}_0^{(4)}(\cdot)$ suffices, as payment of any buyer in both auctions will always be zero, and the revenue of the seller is zero as well in either auction.

[5] Interestingly, $p^{\mathrm{BDS}}(\beta_0)$ could be decreasing on some interval(s).

Now we let $q_0 := \inf\{x \in [0,1] \mid \widetilde{v}_0^{(4)}(x) \geq \lambda/\beta_0^{\mathrm{S}}\} \in [0,1]$. Such q_0 exists since the payment of each buyer is non-zero by the definition of efficiency. We first write the payment of buyers in BDSPA, which is

$$
\begin{aligned}
p^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}) &= T \cdot \int_{[0,1]^n} \max_{i' \neq i} \left\{ \widetilde{v}_0^{(4)}(q_{i'}), \frac{\lambda}{\beta_0^{\mathrm{S}}} \right\} \cdot \Phi_i(\beta_0^{\mathrm{S}} 1^n, \widetilde{\boldsymbol{v}}^{(4)}, q) \,\mathrm{d}q \\
&= T \cdot \int_{q_0}^{1} \left(\frac{\lambda}{\beta_0^{\mathrm{S}}} q_0^{n-1} + \int_{q_0}^{x} \widetilde{v}_0^{(4)}(z)(n-1)z^{n-2} \,\mathrm{d}z \right) \mathrm{d}x \\
&> T \cdot \int_{q_0}^{1} \left(\frac{\lambda}{\beta_0^{\mathrm{S}}} q_0^{n-1} + \int_{q_0}^{x} \frac{\lambda}{\beta_0^{\mathrm{S}}}(n-1)z^{n-2} \,\mathrm{d}z \right) \mathrm{d}x \\
&= T \cdot \int_{q_0}^{1} \frac{\lambda}{\beta_0^{\mathrm{S}}} x^{n-1} \,\mathrm{d}x = T \cdot \frac{\lambda}{\beta_0^{\mathrm{S}}} \cdot \frac{1-q_0^n}{n}.
\end{aligned}
\tag{24}
$$

Here, the second equality follows by considering the second-max discounted value when the max value is fixed. The inequality holds since $\widetilde{v}_0^{(4)}$ is strictly increasing and inverse Lipschitz continuous. We should notice here that $p^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}) = B_0$ may not establish as it is possible that each buyer does not exhaust her budget with $\beta_0^{\mathrm{S}} = 1$.

Now we come to define $\widetilde{v}_0^{(2)}(\cdot)$. We let

$$
\widetilde{v}_0^{(2)}(x) = \begin{cases} a_1 \cdot \exp\{a_2 x\} & 0 \leq x < q_0 \\ k_0 \cdot (x - q_0) + \lambda/\beta_0^{\mathrm{S}} & q_0 \leq x \leq 1. \end{cases}
$$

Here $k_0 > 0$ is the solution to

$$
\frac{k_0}{n(n+1)} \cdot (n - (n+1)q_0 + q_0^{n+1}) = \frac{p^{\mathrm{BDS}}(\beta_0^{\mathrm{S}})}{T} - \frac{\lambda}{\beta_0^{\mathrm{S}}} \cdot \frac{1-q_0^n}{n}.
$$

Meanwhile, $a_1 = \lambda/\beta_0^{\mathrm{S}} \cdot \exp\{-k_0\beta_0^{\mathrm{S}} q_0/\lambda\}$ and $a_2 = k_0\beta_0^{\mathrm{S}}/\lambda$.

We can verify that under the construction, $\widetilde{v}_0^{(2)}(\cdot)$ is strictly increasing, differentiable and inverse Lipschitz continuous. Now we have

$$
\begin{aligned}
p^{\mathrm{BDF}}(\beta_0^{\mathrm{S}}) &= T \cdot \int_{q_0}^{1} \widetilde{v}_0^{(2)}(x) \cdot x^{n-1} \,\mathrm{d}x \\
&= T \cdot \int_{q_0}^{1} \left(k_0 \cdot (x - q_0) + \lambda/\beta_0^{\mathrm{S}} \right) \mathrm{d}x \\
&= T \cdot \left(\frac{k_0}{n(n+1)} \cdot (n - (n+1)q_0 + q_0^{n+1}) + \frac{\lambda}{\beta_0^{\mathrm{S}}} \cdot \frac{1-q_0^n}{n} \right) = p^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}).
\end{aligned}
$$

The very first equation holds because $q_0 = \inf\{x \in [0,1] \mid \widetilde{v}_0^{(2)}(x) \geq \lambda/\beta_0^{\mathrm{S}}\}$. If $p^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}) = B_0$, then by the monotonicity of $\widetilde{v}_0^{(2)}(\cdot)$, we see that $\beta_0^{\mathrm{F}} = \beta_0^{\mathrm{S}}$ and the budget is binding in BDFPA. Otherwise, we have $\beta_0^{\mathrm{S}} = 1$. We claim that $\beta_0^{\mathrm{F}} = \beta_0^{\mathrm{S}} = 1$ as well, or otherwise, $\beta_0^{\mathrm{F}} < 1$ and the payment of buyer will be strictly less than $p^{\mathrm{BDS}} < B_0$. In either cases, we show that: (1) the effective quantile for each buyer starts at q_0 for both auctions; (2) the payment of each buyer in BDFPA p^{BDF} equals to p^{BDS}. Now since the revenue of the seller is $np^{\mathrm{BDF}}(\beta_0^{\mathrm{F}}) - T \cdot \lambda \cdot (1-q_0^n) = np^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}) - T \cdot \lambda \cdot (1-q_0^n)$, the proof of this side is finished.

From eBDFPA to eBDSPA. Before we start this part, we mention that since the expected payment formula is fairly complex for BDSPA, the mapping for this part would be a bit uneasy to follow. Again, we first deal with the special case that $\beta_0^{\mathrm{F}} = 1$ and $\widetilde{v}_0^{(2)}(\cdot) \le \lambda$. Under this case, we take $\widetilde{v}_0^{(4)}(\cdot) = \widetilde{v}_0^{(2)}(\cdot)$, therefore, the expected payment of any buyer in either BDSPA or BDFPA is zero. As a result the revenue of the seller stays at zero as well.

In the more general case, we let $q_0 := \inf\{x \in [0,1] \mid \widetilde{v}_0^{(2)}(x) \ge \lambda/\beta_0^{\mathrm{F}}\} \in [0,1]$, which exists by the definition of efficiency. Therefore, we have

$$p^{\mathrm{BDF}}(\beta_0^{\mathrm{F}}) = T \cdot \int_{q_0}^{1} \widetilde{v}_0^{(2)}(x) \cdot x^{n-1}\,\mathrm{d}x > T \cdot \frac{\lambda}{\beta_0^{\mathrm{F}}} \cdot \frac{1-q_0^n}{n}.$$

Now we let $l_0 > 0$ be the solution to

$$T \cdot \frac{l_0}{n(n+1)} \cdot (1-q_0) \cdot \left(\sum_{i=1}^{n-1} (1-q_0^i)(1-q_0^{n-i})\right) = p^{\mathrm{BDF}}(\beta_0^{\mathrm{F}}) - T \cdot \frac{\lambda}{\beta_0^{\mathrm{F}}} \cdot \frac{1-q_0^n}{n},$$

and define $\widetilde{v}_0^{(4)}(\cdot)$ as:

$$\widetilde{v}_0^{(4)}(x) = \begin{cases} a_3 \cdot \exp\{a_4 x\} & 0 \le x < q_0 \\ l_0 \cdot (x - q_0) + \lambda/\beta_0^{\mathrm{F}} & q_0 \le x \le 1, \end{cases}$$

where $a_3 = \lambda/\beta_0^{\mathrm{F}} \cdot \exp\{-l_0 \beta_0^{\mathrm{F}} q_0/\lambda\}$ and $a_4 = l_0 \beta_0^{\mathrm{F}}/\lambda$.

It is easy to see that $\widetilde{v}_0^{(4)}(\cdot)$ is strictly increasing, differentiable and inverse Lipschitz continuous as $\widetilde{v}_0^{(2)}(\cdot)$. We further notice that

$$\begin{aligned}
p^{\mathrm{BDS}}(\beta_0^{\mathrm{F}}) &= T \cdot \int_{q_0}^{1} \left(\frac{\lambda}{\beta_0^{\mathrm{F}}} q_0^{n-1} + \int_{q_0}^{x} \widetilde{v}_0^{(4)}(z)(n-1)z^{n-2}\,\mathrm{d}z \right) \mathrm{d}x \\
&= T \cdot \int_{q_0}^{1} \left(\frac{\lambda}{\beta_0^{\mathrm{F}}} q_0^{n-1} + \int_{q_0}^{x} \left(l_0(z-q_0) + \frac{\lambda}{\beta_0^{\mathrm{F}}} \right)(n-1)z^{n-2}\,\mathrm{d}z \right) \mathrm{d}x \\
&= T \cdot \int_{q_0}^{1} \left(\frac{\lambda}{\beta_0^{\mathrm{F}}} q_0^{n-1} + \frac{\lambda}{\beta_0^{\mathrm{F}}}(x^{n-1} - q_0^{n-1}) + l_0 \int_{q_0}^{x} (z-q_0)(n-1)z^{n-2}\,\mathrm{d}z \right) \mathrm{d}x \\
&= T \cdot \left(\frac{\lambda}{\beta_0^{\mathrm{F}}} \cdot \frac{1-q_0^n}{n} + l_0 \int_{q_0}^{1} \int_{q_0}^{x} (z-q_0)(n-1)z^{n-2}\,\mathrm{d}z\,\mathrm{d}x \right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{q_0}^{1} \int_{q_0}^{x} (z-q_0)(n-1)z^{n-2}\,\mathrm{d}z\,\mathrm{d}x \\
=& \frac{1}{n} \int_{q_0}^{1} \left((n-1)x^n - n q_0 x^{n-1} + q_0^n \right) \mathrm{d}x \\
=& \frac{1}{n(n+1)} \left((n-1)(1-q_0^{n+1}) - (n+1)(q_0 - q_0^n) \right) \\
=& \frac{1}{n(n+1)} (1-q_0) \cdot \left(\sum_{i=1}^{n-1} (1-q_0^i)(1-q_0^{n-i}) \right).
\end{aligned}$$

Again, the very first equation holds as $q_0 = \inf\{x \in [0,1] \mid \widetilde{v}_0^{(4)}(x) \geq \lambda/\beta_0^{\mathrm{F}}\}$. As a result, we have

$$p^{\mathrm{BDS}}(\beta_0^{\mathrm{F}}) = T \cdot \left(\frac{\lambda}{\beta_0^{\mathrm{F}}} \cdot \frac{1-q_0^n}{n} + l_0 \int_{q_0}^{1}\int_{q_0}^{x} (z-q_0)(n-1)z^{n-2}\,\mathrm{d}z\,\mathrm{d}x\right) = p^{\mathrm{BDF}}(\beta_0^{\mathrm{F}}).$$

From the above formula, we see that $\beta_0^{\mathrm{S}} = \beta_0^{\mathrm{F}}$ is an efficient multiplier for BDSPA with $\widetilde{\boldsymbol{v}}_0^{(4)}$, under which the effective quantile starts at q_0 and $p^{\mathrm{BDS}} = p^{\mathrm{BDF}}$ under this efficient multiplier. Consequently, the revenue of the seller under both auctions equal to $np^{\mathrm{BDS}}(\beta_0^{\mathrm{S}}) - T \cdot \lambda \cdot (1-q_0^n) = np^{\mathrm{BDF}}(\beta_0^{\mathrm{F}}) - T \cdot \lambda \cdot (1-q_0^n)$. As a result, we have shown that efficient BDFPA is strategically-equivalent to efficient BDSPA.

The two-way constructions to show the equivalence between efficient PFPA and efficient PSPA is the same with what we just present, and we omit the details here. Combining with the equivalence of eBDFPA and ePFPA as we already shown in Theorem 4.7, the theorem is proved. □

Remark C.1. By comparing the expected payment of a buyer in BDFPA and BDSPA, an appealing approach to prove the theorem is to take $\beta_0^{\mathrm{F}} = \beta_0^{\mathrm{S}}$, the effective quantile of both auctions start at an identical q_0, and to have

$$\widetilde{v}_0^{(2)}(x) \cdot x^{n-1} = \frac{\lambda}{\beta_0^{\mathrm{S}}} q_0^{n-1} + \int_{q_0}^{x} \widetilde{v}_0^{(4)}(z)(n-1)z^{n-2}\,\mathrm{d}z$$

when $x \geq q_0$. This seems to be an elegant solution, with $\widetilde{v}_0^{(2)}(\cdot)$ being a continuously weighted average of $\widetilde{v}_0^{(4)}(\cdot)$. However, this idea does not work. The reason is that, the above mapping from $\widetilde{v}_0^{(4)}(\cdot)$ to $\widetilde{v}_0^{(2)}(\cdot)$ would lose the inverse Lipschitz continuity, as the derivative of $\widetilde{v}_0^{(2)}(\cdot)$ at q_0 would be zero. On the other side, the mapping from $\widetilde{v}_0^{(2)}(\cdot)$ to $\widetilde{v}_0^{(4)}(\cdot)$ would even lose the strict monotonicity. As a result, we have to adopt the methodology we use in the proof of Theorem 4.7.

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