Appendices

A Proofs

A.1 Solution for deterministic model

Budget constraints:

$$\begin{aligned}
\text{Young users:} \quad & c_t^u = e_t^u - s_t - (1+\varphi_t^b+\varphi_t^v)q_t p_t - \hat{q}_t\hat{p}_t && (1)\\
\text{Old users:} \quad & c_{t+1}^u = s_t(1+r_t) + (1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1} + \hat{q}_t\hat{p}_{t+1} && (2)\\
\text{Young validators:} \quad & c_t^v = e_t^v + \varphi_t^v q_t p_t - L_t p_t && (3)\\
\text{Old validators:} \quad & c_{t+1}^v = (1+\delta_t)L_t p_{t+1} && (4)\\
\text{Hackers:} \quad & c_{t+1}^h = h_{t+1} q_t p_{t+1} && (5)
\end{aligned}$$

Market clearing:

$$\begin{aligned}
\text{Risk-free assets:} \quad & s_t = 0 && (6)\\
\text{Tokens:} \quad & q_t = M_t - L_t && (7)\\
\text{Fiat:} \quad & \hat{q}_t = m_t && (8)
\end{aligned}$$

User's problem:

$$\begin{aligned}
\max_{s_t,q_t,\hat{q}_t} \quad & u\left(e_t^u - s_t - (1+\varphi_t^b+\varphi_t^v)q_t p_t - \hat{q}_t\hat{p}_t\right)\\
& + \beta u\left(s_t(1+r_t) + (1-h_{t+1})q_t p_{t+1} + \hat{q}_t\hat{p}_{t+1}\right)
\end{aligned} \tag{9}$$

FOCs:

$$\frac{\partial}{\partial s_t} : -u'(c_t^u) + \beta(1+r_t)u'(c_{t+1}^u) = 0 \tag{10}$$

$$\frac{\partial}{\partial q_t} : -(1+\varphi_t^b+\varphi_t^v)p_t u'(c_t^u) + \beta(1-h_{t+1})(1+\theta_{t+1})p_{t+1}u'(c_{t+1}^u) = 0 \tag{11}$$

$$\frac{\partial}{\partial \hat{q}_t} : -\hat{p}_t u'(c_t^u) + \beta\hat{p}_{t+1}u'(c_{t+1}^u) = 0 \tag{12}$$

Euler's equations:

$$\frac{u'(c_t^u)}{u'(c_{t+1}^u)} = \beta(1+r_t) = \beta\frac{(1+\theta_{t+1})(1-h_{t+1})}{1+\varphi_t^b+\varphi_t^v}\frac{p_{t+1}}{p_t} = \beta\frac{\hat{p}_{t+1}}{\hat{p}_t} \tag{13}$$

Validator's problem:

$$\max_{L_t} \quad u(e_t^v + \varphi_t^v q_t p_t - L_t p_t) + \beta u((1+\delta_t)L_t p_{t+1}) \tag{14}$$

FOC:

$$-p_t u'(c_t^v) + \beta p_{t+1}(1+\delta_t)u'(c_{t+1}^v) = 0 \tag{15}$$

Euler's equation:

$$\frac{u'(c_t^v)}{u'(c_{t+1}^v)} = \beta(1+\delta_t)\frac{p_{t+1}}{p_t} \tag{16}$$

A.1.1 Token market equilibrium with log utility

$$u(c_t) = \ln c_t \Rightarrow \frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} \tag{17}$$

Users' equilibrium token holding:

By users' BC②:

$$\hat{q}_t = \frac{1}{\hat{p}_{t+1}} \left[c^u_{t+1} - s_t(1+r_t) - (1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1}\right] \tag{18}$$

Substitute into users' BC①, eliminate s_t by market clearing:

$$c^u_t = e^u_t - (1+\varphi^b_t + \varphi^v_t)q_t p_t - \frac{\hat{p}_t}{\hat{p}_{t+1}} \left[c^u_{t+1} - (1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1}\right] \tag{19}$$

By users' FOC③:

$$\frac{c^u_{t+1}}{c^u_t} = \beta \frac{\hat{p}_{t+1}}{\hat{p}_t} \Rightarrow \frac{\hat{p}_t}{\hat{p}_{t+1}} c^u_{t+1} = \beta c^u_t \tag{20}$$

Eliminate c^u_{t+1} :

$$c^u_t(1+\beta) = e^u_t - (1+\varphi^b_t + \varphi^v_t)q_t p_t + \frac{\hat{p}_t}{\hat{p}_{t+1}}(1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1} \tag{21}$$

Substitute users' BC① and $s_t = 0, \hat{q}_t = m_t$ on LHS:

$$\begin{aligned}&(1+\beta)\left[e^u_t - (1+\varphi^b_t + \varphi^v_t)q_t p_t - m_t \hat{p}_t\right] \\ &= e^u_t - (1+\varphi^b_t + \varphi^v_t)q_t p_t + \frac{\hat{p}_t}{\hat{p}_{t+1}}(1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1}\end{aligned} \tag{22}$$

Merge terms:

$$\beta e^u_t - (1+\beta)m_t \hat{p}_t = q_t p_t \left[\beta(1+\varphi^b_t + \varphi^v_t) + \frac{\hat{p}_t}{\hat{p}_{t+1}}(1-h_{t+1})(1+\theta_{t+1})\frac{p_{t+1}}{p_t}\right] \tag{23}$$

Hence the inverse demand curve is

$$q_t = \left[\beta \frac{e^u_t}{p_t} - (1+\beta)m_t \frac{\hat{p}_t}{p_t}\right] \left[\beta(1+\varphi^b_t + \varphi^v_t) + \frac{(1-h_{t+1})(1+\theta_{t+1})}{1+r_t}\frac{p_{t+1}}{p_t}\right]^{-1} \tag{24}$$

Validators' equilibrium staking:

$$\frac{c^v_{t+1}}{c^v_t} = \frac{(1+\delta_t)L_t p_{t+1}}{e^v_t + \varphi^v_t q_t p_t - L_t p_t} = \beta(1+\delta_t)\frac{p_{t+1}}{p_t} \tag{25}$$

$$L_t = \beta\left(\frac{e^v_t}{p_t} + \varphi^v_t q_t - L_t\right) \tag{26}$$

$$L_t = \frac{\beta}{1+\beta}\left(\frac{e^v_t}{p_t} + \varphi^v_t q_t\right) \tag{27}$$

By market clearing $q_t + L_t = M_t$:

$$q_t + \frac{\beta}{1+\beta}\left(\frac{e_t^v}{p_t} + \varphi_t^v q_t\right) = M_t \tag{28}$$

$$q_t\left(1 + \frac{\beta}{1+\beta}\varphi_t^v\right) = M_t - \frac{\beta}{1+\beta}\frac{e_t^v}{p_t} \tag{29}$$

Hence the inverse supply curve is

$$q_t = \left(1 + \frac{\beta}{1+\beta}\varphi_t^v\right)^{-1}\left(M_t - \frac{\beta}{1+\beta}\frac{e_t^v}{p_t}\right) \tag{30}$$

Token market equilibrium:
Let p_t^* be the equilibrium price of crypto tokens. p_t^* must satisfy:

$$\begin{aligned}&\left[\beta\frac{e_t^u}{p_t^*} - (1+\beta)m_t\frac{\hat{p}_t}{p_t^*}\right]\left[\beta(1+\varphi_t^b+\varphi_t^v) + \frac{(1-h_{t+1})(1+\theta_{t+1})}{1+r_t}\frac{p_{t+1}}{p_t^*}\right]^{-1} \\ &= \left(1 + \frac{\beta}{1+\beta}\varphi_t^v\right)^{-1}\left(M_t - \frac{\beta}{1+\beta}\frac{e_t^v}{p_t^*}\right)\end{aligned} \tag{31}$$

which can be alternatively written as a quardatic function of $\frac{1}{p_t^*}$:

$$A_t\frac{1}{p_t^*} = (M_t + B_t\frac{1}{p_t^*})(C_t + D_t\frac{1}{p_t^*}) \tag{32}$$

where

$$A_t := [\beta e_t^u - (1+\beta)m_t\hat{p}_t]\left(1 + \frac{\beta}{1+\beta}\varphi_t^v\right); \tag{33}$$

$$B_t := -\frac{\beta}{1+\beta}e_t^v; \tag{34}$$

$$C_t := \beta(1+\varphi_t^b+\varphi_t^v); \tag{35}$$

$$D_t := \frac{(1-h_{t+1})(1+\theta_{t+1})}{1+r_t}p_{t+1} \tag{36}$$

Therefore the closed-form solution for equilibrium token price is given by

$$p_t^* = \frac{2B_tD_t}{A_t - M_tD_t - B_tC_t - \sqrt{(M_tD_t + B_tC_t - A_t)^2 - 4M_tB_tC_tD_t}} \tag{37}$$

under the following technical conditions:

$$4M_tB_tC_tD_t \leq (M_tD_t + B_tC_t - A_t)^2 \tag{38}$$

$$A_t - M_tD_t - B_tC_t \neq \sqrt{(M_tD_t + B_tC_t - A_t)^2 - 4M_tB_tC_tD_t} \tag{39}$$