Appendices

A Proofs

A.1 Solution for deterministic model

$$\begin{aligned}
\text{Young users:} \quad & c_t^u = e_t^u - s_t - (1+\varphi_t^b+\varphi_t^v)q_t p_t - \hat{q}_t\hat{p}_t && (1)\\
\text{Old users:} \quad & c_{t+1}^u = s_t(1+r_t) + (1-h_{t+1})(1+\theta_{t+1})q_t p_{t+1} + \hat{q}_t\hat{p}_{t+1} && (2)\\
\text{Young validators:} \quad & c_t^v = e_t^v + \varphi_t^v q_t p_t - L_t p_t && (3)\\
\text{Old validators:} \quad & c_{t+1}^v = (1+\delta_t)L_t p_{t+1} && (4)\\
\text{Hackers:} \quad & c_{t+1}^h = h_{t+1} q_t p_{t+1} && (5)
\end{aligned}$$

User's problem:

$$\begin{aligned}
\max_{s_t, q_t, \hat{q}_t} \quad & u\left(e_t^u - s_t - (1+\varphi_t^b+\varphi_t^v)q_t p_t - \hat{q}_t\hat{p}_t\right) \\
& + \beta u\left(s_t(1+r_t) + (1-h_{t+1})q_t p_{t+1} + \hat{q}_t\hat{p}_{t+1}\right)
\end{aligned} \tag{6}$$

FOCs:

$$\frac{\partial}{\partial s_t} : -u'(c_t^u) + \beta(1+r_t)u'(c_{t+1}^u) = 0 \tag{7}$$

$$\frac{\partial}{\partial q_t} : -(1+\varphi_t^b+\varphi_t^v)p_t u'(c_t^u) + \beta(1-h_{t+1})(1+\theta_{t+1})p_{t+1}u'(c_{t+1}^u) = 0 \tag{8}$$

$$\frac{\partial}{\partial \hat{q}_t} : -\hat{p}_t u'(c_t^u) + \beta\hat{p}_{t+1}u'(c_{t+1}^u) = 0 \tag{9}$$

Result:

$$\Rightarrow \frac{u'(c_t^u)}{u'(c_{t+1}^u)} = \beta(1+r_t) = \beta\frac{(1+\theta_{t+1})(1-h_{t+1})}{1+\varphi_t^b+\varphi_t^v}\frac{p_{t+1}}{p_t} = \beta\frac{\hat{p}_{t+1}}{\hat{p}_t} \tag{10}$$

Validator's problem:

$$\max_{L_t} \quad u(e_t^v + \varphi_t^v q_t p_t - L_t p_t) + \beta u((1+\delta_t)L_t p_{t+1}) \tag{11}$$

FOC:

$$-p_t u'(c_t^v) + \beta p_{t+1}(1+\delta_t)u'(c_{t+1}^v) = 0 \tag{12}$$

Result:

$$\Rightarrow \frac{u'(c_t^v)}{u'(c_{t+1}^v)} = \beta(1+\delta_t)\frac{p_{t+1}}{p_t} \tag{13}$$

With log utility:

$$u(c_t) = \ln c_t \Rightarrow \frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} \tag{14}$$

User (Solve for token demand q_t):

Combine the consumer budget equations by substituting s_t:

$$\begin{aligned}
c_{t+1}^u = &(1+r_t)\left[e_t^u - c_t^u - (1+\varphi_t^b+\varphi_t^v)q_t p_t - \hat{q}_t\hat{p}_t\right] \\
&+ (1+\theta_{t+1})(1-h_{t+1})q_t p_{t+1} + \hat{q}_t\hat{p}_{t+1}
\end{aligned} \tag{15}$$

To be continued...

Validator (solve for amount staked $L_t \rightarrow$ token supply $X_t = M_t - L_t$):

$$\frac{c^v_{t+1}}{c^v_t} = \frac{(1+\delta_t)L_t p_{t+1}}{e^v_t + \varphi^v_t q_t p_t - L_t p_t} = \beta(1+\delta_t)\frac{p_{t+1}}{p_t} \tag{16}$$

$$L_t = \beta(\frac{e^v_t}{p_t} + \varphi^v_t q_t - L_t) \tag{17}$$

$$L_t = \frac{\beta}{1+\beta}(\frac{e^v_t}{p_t} + \varphi^v_t q_t) \tag{18}$$