Expectation Maximization and Total Variation Based Model for Computed Tomography Reconstruction from Undersampled Data

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ABSTRACT

Computerized tomography (CT) plays an important role in medical imaging, especially for diagnosis and therapy. However, higher radiation dose from CT will result in increasing of radiation exposure in the population. Therefore, the reduction of radiation from CT is an essential issue. Expectation maximization (EM) is an iterative method used for CT image reconstruction that maximizes the likelihood function under Poisson noise assumption. Total variation regularization is a technique used frequently in image restoration to preserve edges, given the assumption that most images are piecewise constant. Here, we propose a method combining expectation maximization and total variation regularization, called EM+TV. This method can reconstruct a better image using fewer views in the computed tomography setting, thus reducing the overall dose of radiation. The numerical results in two and three dimensions show the efficiency of the proposed EM+TV method by comparison with those obtained by filtered back projection (FBP) or by EM only.

Keywords: computerized tomography, image reconstruction, energy minimization, total variation, expectation maximization, Poisson noise, compressive sensing, Radon transform.

1. INTRODUCTION

As a group of methods for reconstructing two dimensional and three dimensional images from the projections of the object, iterative reconstruction has many applications such as in computerized tomography (CT), positron emission tomography (PET), and magnetic resonance imaging (MRI). This technique is quite different from the filtered back projection (FBP) method,[15] which is the most commonly used algorithm in practice by manufacturers. The main advantages of the iterative reconstruction technique over the filtered back projection are insensitivity to noise and flexibility.[8] The data can be collected over any set of lines, the projections do not have to be distributed uniformly in angle, and these can be even incomplete.

There are many available algorithms for iterative reconstruction. Most of these algorithms are based on the system of linear equations

$$Ax = b, \tag{1}$$

where $x = (x_1, \cdots, x_N)^T \in \mathbf{R}^N$ is the original unknown image represented as a vector, b is the given measurement with $b = (b_1, \cdots, b_M)^T \in \mathbf{R}^M$, and A is a $M \times N$ matrix describing the direct transformation from the original image to the measurements, which is different for different purposes. For example, in computerized tomography, A is the discrete Radon transform, with each row describing an integral along one straight line and all the elements are nonnegative.

We focus in this work on computerized tomography. We recall the continuous Radon transform in two dimensions, applied to an image function $f(x, y)$: assuming that a straight line in the plane can be represented by the equation depending on two parameters (θ, t),

$$x \cos\theta + y \sin\theta = t,$$

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the Radon transform $f \mapsto g$ (with $(\theta, t) \mapsto g(\theta, t)$ the image in the sinogram domain (θ, t) of the function $(x, y) \mapsto f(x, y)$) is defined by the projections along lines of f,

$$g(\theta, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - t)dxdy,$$

where δ is the one-dimensional Dirac delta function, with support on the line of equation $x\cos\theta + y\sin\theta = t$. In the two-dimensional discrete case, f represents a discrete image matrix and g (or a noisy version of g) is known only at a finite number of samples (θ_i, t_i). In computed tomography, we must recover f from its projections g. Although the Radon transform is an invertible operator, in the real discrete case only a small finite number of projections (θ_i, t_i) are given, which may be also noisy. Thus, in the real case, the recovery of f from projections g is a difficult inverse problem.

For the rest of the presentation, we will continue to use the notations x for the image to be recovered, b for the measurements or projections in the sinogram domain, and A for the discrete Radon transform with a finite number of samples.

One example of iterative reconstruction algorithm is expectation maximization (EM).[3,16] This is based on the assumption that the noise in b is Poisson noise. If x is given, the probability of obtaining b is

$$P(b|Ax) = \Pi_{i=1}^{M} \frac{e^{-(Ax)_i}((Ax)_i)^{b_i}}{b_i!}.$$

Therefore, given b and A, the objective is to find x such that the above probability is maximized. However, instead of maximizing the probability, we can minimize $-\log P(b|Ax) = (Ax)_i - b_i \log((Ax)_i) + C$, with C being a constant. Then the EM iteration is as follows: starting with an initial guess x^0, compute for $n \geq 0$,

$$x_j^{n+1} = \frac{\sum_{i=1}^{M} (a_{ij}(\frac{b_i}{(Ax^n)_i}))}{\sum_{i=1}^{M} a_{ij}} x_j^n. \tag{2}$$

The total-variation regularization method was proposed by Rudin, Osher and Fatemi[12,13] to remove noise and blur in an image, while preserving edges. This technique is widely used in image processing and amounts to minimize an energy functional of the form

$$\min_x \int_\Omega |\nabla x| + \alpha \|Ax - b\|_2^2,$$

where here x is viewed as a two or three-dimensional image with spatial domain Ω, A is usually a blurring operator, b is the given noisy-blurry image, and $\|Ax - b\|^2$ is the L^2 data-fidelity term imposed by the Gaussian noise assumption.

We propose in this work to combine the expectation maximization (EM) method using Poisson noise with the total variation TV regularization, when A is the discrete Radon transform and the data b represents a small number of projections in computerized tomography. The assumption is that the reconstructed image can not have too large total-variation (thus noise and reconstruction artifacts are removed). We will see that the proposed EM+TV method gives superior results to those obtained by filtered back projection, or by expectation maximization only.

For relevant prior work, we refer to the Compressive Sensing Resources.[20] Additionally, we refer the reader to Jia et al.,[6] Jung et al.,[7] Setzer et al.,[14] Le et al.,[11] Brune et al.,[1,2] Jafarpour et al.,[5] Harmany et al.,[4] Willet et al.,[18] among other work.

2. PROPOSED EM+TV RECONSTRUCTION METHOD

The objective is to reconstruct an image with both minimal total-variation and maximal probability, given fewer noisy projections in the sinogram domain. So we can consider finding a Pareto optimal point by solving a scalarization of these two objective functions and the problem is to solve

$$\begin{cases} \underset{x}{\text{minimize}} & \int_\Omega |\nabla x| + \alpha \sum_{i=1}^{M} \left((Ax)_i - b_i \log(Ax)_i\right), \\ \text{subject to} & x_j \geq 0, \quad j = 1, \cdots, N, \end{cases} \tag{3}$$

with $\alpha > 0$ a tuning parameter. This is a convex constrained optimization problem and we can find the optimal solution by solving the Karush-Kuhn-Tucker (KKT) conditions:[9, 10]

$$\begin{aligned} -\text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha \sum_{i=1}^{M} \Big(a_{ij}(1 - \frac{b_i}{(Ax)_i})\Big) - y_j = 0, &\qquad j = 1, \cdots, N, \\ y_j \geq 0, \quad x_j \geq 0, &\qquad j = 1, \cdots, N, \\ y^T x = 0. & \end{aligned}$$

By positivity of $\{x_j\}$, $\{y_j\}$ and the complementary slackness condition $y^T x = 0$, we have $x_j y_j = 0$ for every $j = 1, \cdots, N$. Thus, if we multiply the partial differential equation by x_j at pixel j, we obtain

$$-x_j \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha \sum_{i=1}^{M} \Big(a_{ij}(1 - \frac{b_i}{(Ax)_i})\Big) x_j = 0, \qquad j = 1, \cdots, N,$$

or equivalently

$$-\frac{x_j}{\sum\limits_{i=1}^{M} a_{ij}} \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha x_j - \alpha \frac{\sum\limits_{i=1}^{M} \Big(a_{ij}(\frac{b_i}{(Ax)_i})\Big)}{\sum\limits_{i=1}^{M} a_{ij}} x_j = 0, \qquad j = 1, \cdots, N.$$

After plugging the EM step from (2)

$$x_j^{EM} = \frac{\sum\limits_{i=1}^{M} (a_{ij}(\frac{b_i}{a_i x}))}{\sum\limits_{i=1}^{M} a_{ij}} x_j \tag{4}$$

into the last KKT condition, we obtain

$$-\frac{x_j}{\sum\limits_{i=1}^{M} a_{ij}} \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha x_j - \alpha x_j^{EM} = 0, \qquad j = 1, \cdots, N. \tag{5}$$

It is possible to show that the above equation (5) is the optimality condition for the following TV minimization problem

$$\underset{x}{\text{minimize}} \int_\Omega |\nabla x| + \alpha \sum_{j=1}^{N} \sum_{i=1}^{M} a_{ij} \left(x_j - x_j^{EM} \log x_j\right). \tag{6}$$

To solve the above EM+TV minimization problem, we use an iterative semi-implicit finite-differences scheme. Each iteration is called a TV step. Given x^{EM}, already computed from the EM step, we compute the new x

discretizing (5) (as a discrete minimizer of (6)), by the following simple iterative scheme, presented below in two spatial dimensions and assuming now that x is represented as a 2D matrix: from an initial guess x^0, we compute x^{n+1} with $n \geq 0$ from the following linearized discrete equation,

$$-\frac{1}{\alpha}\frac{x^n_{i,j}}{v_{i,j}}\frac{x^n_{i+1,j}-x^{n+1}_{i,j}}{\sqrt{\epsilon+(x^n_{i+1,j}-x^n_{i,j})^2+(x^n_{i,j+1}-x^n_{i,j})^2}}+\frac{1}{\alpha}\frac{x^n_{i,j}}{v_{i,j}}\frac{x^{n+1}_{i,j}-x^n_{i-1,j}}{\sqrt{\epsilon+(x^n_{i,j}-x^n_{i-1,j})^2+(x^n_{i-1,j+1}-x^n_{i-1,j})^2}}$$

$$-\frac{1}{\alpha}\frac{x^n_{i,j}}{v_{i,j}}\frac{x^n_{i,j+1}-x^{n+1}_{i,j}}{\sqrt{\epsilon+(x^n_{i+1,j}-x^n_{i,j})^2+(x^n_{i,j+1}-x^n_{i,j})^2}}+\frac{1}{\alpha}\frac{x^n_{i,j}}{v_{i,j}}\frac{x^{n+1}_{i,j}-x^n_{i,j-1}}{\sqrt{\epsilon+(x^n_{i+1,j-1}-x^n_{i,j-1})^2+(x^n_{i,j}-x^n_{i,j-1})^2}}$$

$$+x^{n+1}_{i,j}-x^{EM}_{i,j}=0,$$

where $v_{i,j}=\sum_{i=1}^{M} a_{ij}$.

Finally, the two steps (EM and TV) are solved in an alternating fashion. Usually, for each main iteration, we apply 2 or 3 EM steps, followed by 5 to 8 TV steps. For the TV step, the initial guess can be defined as the result from the previous EM update, or from the last TV update. The main algorithm is summarized as follows:

Input: $x^0 = 1$;
for *Out=1:1:IterMax* **do**
  $x^{0,0} = x^{Out-1}$;
  for k = *1:1:K* **do**
    $x^{k,0} = EM(x^{k-1,0})$;
  end
  for l = *1:1:L* **do**
    $x^{K,l} = TV(x^{K,l-1})$;
  end
  $x^{Out} = x^{K,L}$;
end

Algorithm 1: Proposed EM+TV algorithm.

3. NUMERICAL RESULTS

We present in this section reconstruction results in two dimensions using the fan-beam computed tomography geometry, and in three dimensions using the cone-beam computed tomography geometry, on artificially constructed experiments. Future work will include results on real data.

In two dimensions, we compare the reconstruction results obtained by the proposed EM+TV method with those obtained by filtered back projection (FBP). For the numerical experiments, we choose the two dimensional Shepp-Logan phantom of dimension 256x256. The projections are obtained using Siddon's algorithm.[17,19] We consider both the noise-free and noise cases. With the FBP method, we present results using 36 views (every 10 degrees), 180 views, and 360 views; for each view there are 301 measurements. In order to show that we can reduce the number of views by using EM+TV, we only use 36 views for the proposed method. We show in Figure 1 the sinogram data without noise corresponding to 360, 180 and 36 views respectively (the missing projection values are substituted by zero or black). The reconstruction results are shown in Figure 2. We notice the much improved results obtained with EM+TV using only 36 views (both visually and according to the root-mean-square-error between the original and reconstructed images, scaled between 0 and 255), by comparison with FBP using 36, 180 or even 360 views. Using the proposed EM+TV method, with only few samples we obtain sharp results and without artifacts.

We have seen that, in two dimensions, the proposed EM+TV method gives superior results over the standard filtered back projection. In three dimensions, we compare the reconstruction results obtained by the proposed

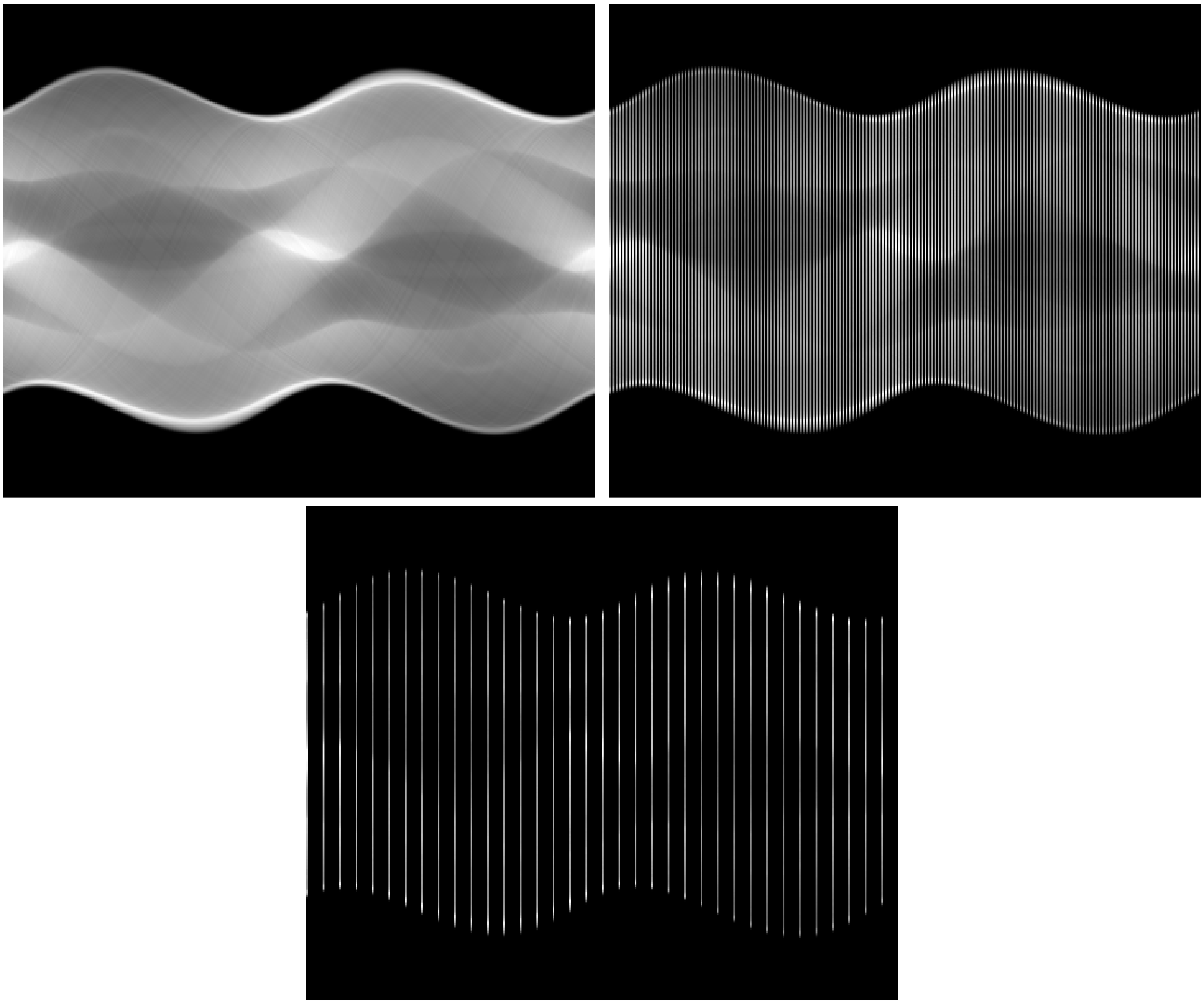

Figure 1. The sinogram data without noise in two dimensions, generated from the two-dimensional Shepp-Logan phantom. From left to right, top to bottom: 360 views, 180 views, 36 views. The x-scale is [0,360] in angle, and the y-scale is [1,301] for the second line parameter.

EM+TV method with those obtained by EM only, visually and according to the root-mean-square-error between original and reconstructed images after rescaling to [0,255]. For the numerical experiments, we similarly choose the three dimensional Shepp-Logan phantom of dimension 128x128x128. The projections are obtained using Siddon's algorithm.[17, 19] We consider only the noise-free case here. For both methods, we present results using 36 views, and for each view, there are 301x257 measurements. The reconstruction results are shown in Figure 3. We notice again the much improved results obtained using EM+TV, by comparison with EM only. The results obtained using EM only (without the total variation regularization) still have many very clear artifacts and needed many more EM steps.

4. CONCLUSION

We proposed a method combining expectation maximization and total variation minimization for image reconstruction in computerized tomography, in the presence of Poisson noise. This method provides comparable results when using very few views, comparing to filtered back projection with many more views; also, the proposed method provided much improved results without artifacts, by comparison to those obtained by the

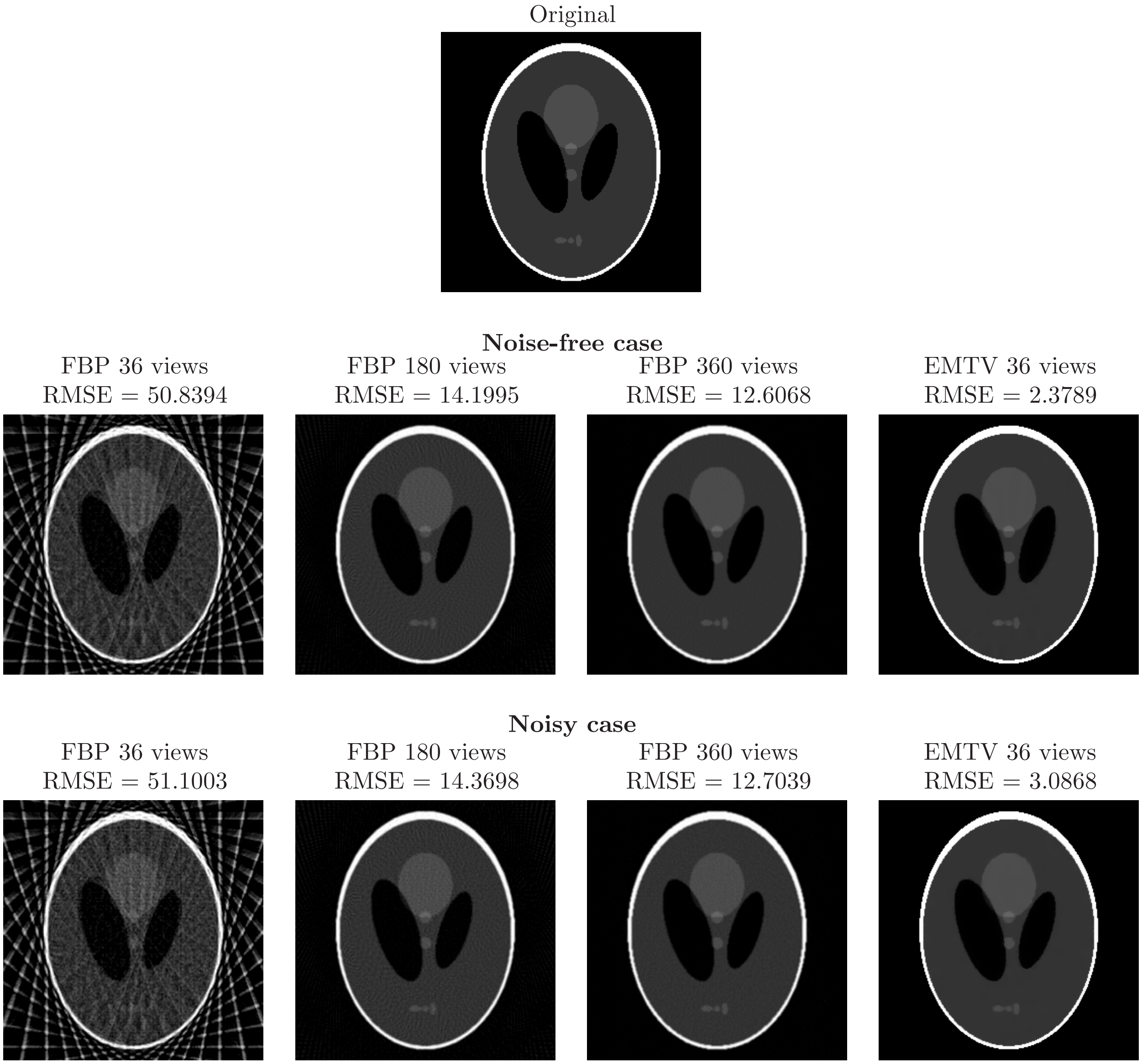

Figure 2. Reconstruction results in two dimensions. Top: original image (the Shepp-Logan phantom). Middle from left to right: reconstruction results in the noise-free case using FBP with 36, 180 and 360 views, and result using proposed EM+TV with 36 views. Bottom from left to right: reconstruction results in the noisy case using FBP with 36, 180 and 360 views, and result using proposed EM+TV with 36 views. The root mean square errors are also given.

expectation maximization only. In conclusion, the proposed method needs much fewer measurements to obtain a good quality image, which results in the decrease of the radiation dose. The method has been easily extended to three dimensions. Future work includes faster implementation using the advantage of graphics processing units (GPUs), parallel computing, and applications to real data.

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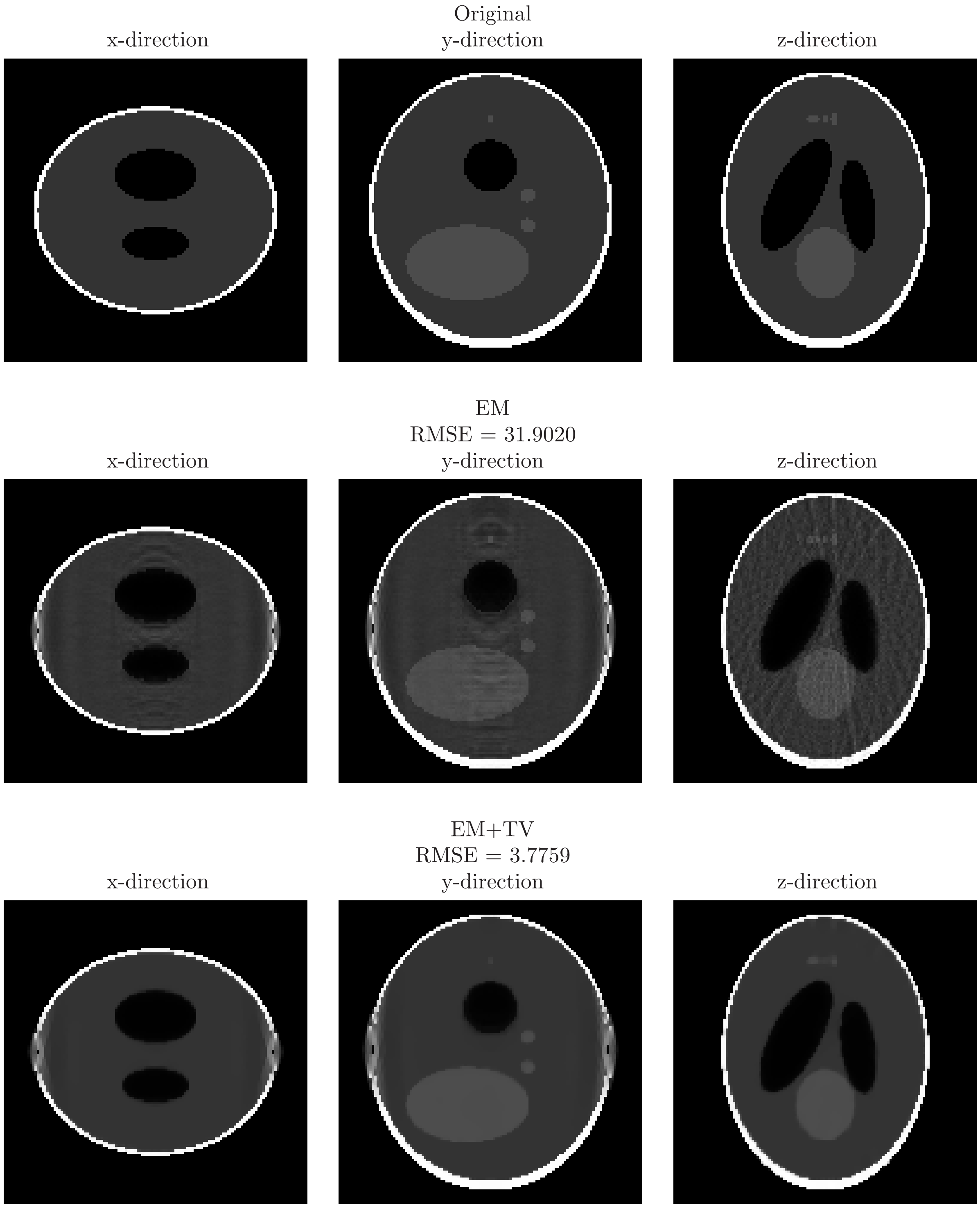

Figure 3. Reconstruction results in three dimensions in the noise-free case. Top row: two-dimensional views of the original three-dimensional Shepp-Logan phantom. Middle row: two-dimensional views of reconstruction results obtained using the EM method. Bottom row: two-dimensional views of reconstruction results obtained using the proposed EM+TV method. The root mean square errors are also given.

REFERENCES

[1] C. Brune, A. Sawatzky and M. Burger, *Bregman-EM-TV Methods with Application to Optical Nanoscopy*, LNCS, vol. 5567, pp. 235-246 (2009).

[2] C. Brune, A. Sawatzky and M. Burger, *Primal and Dual Bregman Methods with Application to Optical Nanoscopy*, UCLA C.A.M. Report 09-47 (2009).

[3] A. Dempster, N. Laird and D. Rubin,*Maximum likelihood from incomplete data via the EM algorithm*, Journal of the Royal Statistical Society Series B, vol. 39, pp. 1-38 (1977).

[4] Z.T. Harmany, R.F. Marcia, and R.M. Willett, *Sparse Poisson intensity reconstruction algorithms*, Proceedings of IEEE/SP 15th Workshop on Statistical Signal Processing, pp. 634-637 (2009).

[5] S. Jafarpour, R. Willett, M. Raginsky and R. Calderbank, *Performance bounds for expander-based compressed sensing in the presence of Poisson noise*, Proceedings of the IEEE Forty-Third Asilomar Conference on Signals, Systems and Computers, pp. 513-517 (2009).

[6] X. Jia, Y. Lou, R. Li, W.Y. Song, and S.B. Jiang, *GPU-based fast cone beam CT reconstruction from undersampled and noisy projection data via total variation*, Med Phys, vol. 37(4), pp. 1757-1760 (2010).

[7] M. Jung, E. Resmerita and L.A. Vese, *Dual norm based iterative methods for image restoration*, UCLA C.A.M. Report 09-88 (2009).

[8] A. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging*, Society of Industrial and Applied Mathematics (2001).

[9] W. Karush, *Minima of functions of several variables with inequalities as side constraints*, M.Sc. Dissertation Department of Mathematics, University of Chicago, Chicago, Illinois (1939).

[10] H. Kuhn and A. Tucker, *Nonlinear programming*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481-492 (1951).

[11] T. Le, R. Chartrand and T. Asaki, *A Variational Approach to Constructing Images Corrupted by Poisson Noise*, JMIV, vol. 27(3), pp. 257-263 (2007).

[12] L. Rudin, S. Osher and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D, vol. 60, pp. 259-268 (1992).

[13] L. Rudin and S. Osher, *Total variation based image restoration with free local constraints*, Proceedings of IEEE International Conference on Image Processing (ICIP), vol. I, pp. 31-35 (1994).

[14] S. Setzer, G. Steidl and T. Teuber, *Deblurring Poissonian images by split Bregman techniques*, J. Visual Communication and Image Representation, vol. 21, pp. 193-199 (2010).

[15] L. Shepp and B. Logan, *The Fourier Reconstruction of a Head Section*, IEEE Transactions on Nuclear Science, vol. 21, pp. 21-34 (1974).

[16] L. Shepp and Y. Vardi, *Maximum likelihood reconstruction for emission tomography*, IEEE Transactions on Medical Imaging, vol. 1, pp. 113-122 (1982).

[17] R. Siddon, *Fast calculation of the exact radiological path for a three-dimensional CT array*, Medical Physics, vol. 12, pp. 252-255 (1986).

[18] R.M. Willett, Z.T. Harmany and R.F. Marcia, *Poisson image reconstruction with total variation regularization*, Proceedings of 17th IEEE International Conference on Image Processing (ICIP), pp. 4177-4180 (2010).

[19] H. Zhao and A.J. Reader, *Fast ray-tracing technique to calculate line integral paths in voxel arrays*, IEEE Nuclear Science Symposium Conference Record, M11-197 (2003).

[20] *Compressive Sensing Resources*, http://dsp.rice.edu/cs