Math 164: Homework #5, due on Tuesday, July 29
No late homework accepted.

Reading: Chapter 6 and Chapter 11.

[**1**](P187, Exercise 2.13) Prove that if the system

$$\mathbf{A}\mathbf{x} \le \mathbf{b}$$

has a solution, then the system

$$\begin{aligned} \mathbf{A}^T\mathbf{y} &= 0 \\ \mathbf{b}^T\mathbf{y} &< 0 \\ \mathbf{y} &\ge 0 \end{aligned}$$

has no solution.

[**2**](P188, Exercise 2.19) Consider the primal linear programming problem

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0. \end{aligned}$$

Assume that this problem and its dual are both feasible. Let $\mathbf{x}_*$ be an optimal solution vector to the primal, let z_* be its associated objective value, and let $\mathbf{y}_*$ be an optimal solution vector to the dual problem. Show that

$$z_* = \mathbf{y}_*^T\mathbf{A}\mathbf{x}_*.$$

[**3**](P194, Exercise 3.1) Use the dual simplex method to solve

$$\begin{aligned} \text{minimize} \quad & z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 4x_1 + 3x_2 \ge 10 \\ & 3x_1 - 5x_2 \ge 12 \\ & x_1, x_2 \ge 0. \end{aligned}$$

[**4**](P202, Exercise 4.3) The following questions below apply to the linear program

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 13x_2 + 13x_3 \\ \text{subject to} \quad & x_1 + x_2 \le 7 \\ & x_1 + 3x_2 + 2x_3 \le 15 \\ & 2x_2 + 3x_3 \le 9 \\ & x_1, x_2, x_3 \ge 0, \end{aligned}$$

with optimal basis $\{x_1, x_2, x_3\}$ and

$$\mathbf{B}^{-1} = \begin{pmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

All of the questions are independent.

(i) What is the solution to the problem? What are the optimal dual variables?

(iii) By how much can the right-hand side of the first constraint increase and decrease without changing the optimal basis?

(iv) By how much can the objective coefficient of x_1 increase and decrease without changing the optimal basis?

(v) Determine the solution of the linear program obtained by adding the constraint

$$x_1 - x_2 + 2x_3 \leq 10.$$

[5](P362, Exercise 2.7) Consider the problem

$$\text{minimize } f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^2).$$

(i) Show that the first- and second-order necessary conditions for optimality are satisfied at $(0, 0)^T$.

(ii) Show that the origin is a local minimizer of f along any line passing through the origin (that is, $x_2 = mx_1$).

(iii) Show that the origin is not a local minimizer of f (consider, for example, curves of the form $x_2 = kx_1^2$). What conclusions can you draw from this?