Math 164: Homework #2, due on Tuesday, July 8
No late homework accepted.

Reading: Chapter 2 and Chapter 4

[**1**](P47, Exercise 2.1) Consider the feasible region defined by the constraints

$$1 - x_1^2 - x_2^2 \geq 0, \quad \sqrt{2} - x_1 - x_2 \geq 0, \text{ and } x_2 \geq 0.$$

For each of the following points, determine whether the point is feasible or infeasible, and (if it is feasible) whether it is interior to or on the boundary of each of the constraints: $\mathbf{x}_a = (\frac{1}{2}, \frac{1}{2})^T$, $\mathbf{x}_b = (1, 0)^T$, $\mathbf{x}_c = (-1, 0)^T$, $\mathbf{x}_d = (-\frac{1}{2}, 0)^T$, and $\mathbf{x}_e = (1/\sqrt{2}, 1/\sqrt{2})^T$.

[**2**](P47, Exercises 2.3) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = x_1 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 4 \\ & x_1^2 \geq 1. \end{aligned}$$

Graph the feasible set. Use the graph to find all local minimizers for the problem, and determine which of those are also global minimizers.
[**3**](P47, Exercise 2.4) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = x_1 \\ \text{subject to} \quad & (x_1 - 1)^2 + x_2^2 = 1 \\ & (x_1 + 1)^2 + x_2^2 \geq 1. \end{aligned}$$

Graph the feasible set. Are there local minimizers? Are there global minimizers?

[**4**](P47, Exercise 2.5) Give an example of a function that has no global minimizer and no global maximizer.

[**5**](P52, Exercise 3.3) Consider a feasible region S defined by a set of linear constraints

$$S = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

Prove that S is convex.

[**6**](P53, Exercise 3.13) Let f be a convex function on the convex set S. Prove that the level set

$$T = \{\mathbf{x} \in S : f(\mathbf{x}) \leq k\}$$

is convex for all real number k.

[**7**](P54, Exercise 3.19(i,iv,vii)) For each of the following functions, determine if it is convex, concave, both, or neither on the real line. If the function is convex or concave, indicate if it is strictly convex or strictly concave.

- (i) $f(x) = 3x^2 + 4x - 5$
- (iv) $f(x) = \sqrt{1 + x^2}$
- (vii) $f(x) = x/(1 + x^4)$.

[**8**](P54, Exercise 3.20) Determine if

$$f(x_1, x_2) = 2x_1^2 - 3x_1x_2 + 5x_2^2 - 2x_1 + 6x_2$$

is convex, concave, both, or neither for $\mathbf{x} \in \mathbf{R}^2$.

[**9**](P105, Exercise 2.2) Convert the following linear program to standard form:

$$\begin{array}{rl} \text{minimize} & z = x_1 - 5x_2 - 7x_3 \\ \text{subject to} & 5x_1 - 2x_2 + 6x_3 \geq 5 \\ & 3x_1 + 4x_2 - 9x_3 = 3 \\ & 7x_1 + 3x_2 + 5x_3 \leq 9 \\ & x1 \geq -2,\ x2,\ x3 \text{ free.} \end{array}$$

[**10**](P114, Exercise 3.1) Consider the system of linear constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(i) Write this system of constraints in standard form, and determine all the basic solutions (feasible and infeasible).
(ii) Determine the extreme points of the feasible region (corresponding to both the standard form of the constraints, as well as the original version).

[**11**](P114, Exercise 3.4) Consider the linear program

$$\begin{array}{rl} \text{minimize} & z = -5x_1 - 7x_2 \\ \text{subject to} & -3x_1 + 2x_2 \leq 30 \\ & -2x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{array}$$

(i) Draw a graph of the feasible region.
(ii) Determine the extreme points of the feasible region.
(iii) Determine two linearly independent directions of unboundedness.
(iv) Convert the linear program to standard form and determine the basic feasible solutions and two linearly independent directions of unboundedness for this version of the problem. Verify that the directions of unboundedness satisfy $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $\mathbf{d} \geq 0$.

[**12**](P114, Exercise 3.6) Consider a linear program with the constraints in standard form

$$\mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0.$$

Prove that if $\mathbf{d} \neq \mathbf{0}$ satisfies

$$\mathbf{A}\mathbf{d} = \mathbf{0} \text{ and } \mathbf{d} \geq 0,$$

then $\mathbf{d}$ is a direction of unboundedness.