MATH 164: Optimization, Summer 2014, Practice Midterm Exam
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NAME ______________________________

STUDENT ID # ______________________________

This is *open to all reference textbooks and class notes*. Calculators are NOT allowed. Please show all your work. Partial credit will be given to partial answers.
There are 4 questions of total 20 points.
Time: 60 minutes.

QUESTION	Maximum	SCORE
[1]	5	
[2]	5	
[3]	5	
[4]	5	
TOTAL	20	

[1] Foxconn accepts manufacturing orders of iPhone 6 from Apple. The orders from Apple are described in the following table:

Scheduled Delivery Month	Units Ordered
September	15K
October	20K
November	25K

However, Foxconn can produce at most 25K, 25K, and 15K units of iPhone 6 in September, October, and November, respectively (Because many Foxconn employees leave for home during the thanksgiving holiday). When Foxconn produces more units than ordered in this month, it has the option to deliver the addition units to a later month, with a storage fee of \$2 per phone per month. For example, if 10 iPhones produced in September are eventually delivered in November, then the storage fee is 10×(3 - 1)× \$2 = \$40. Assume that

- Foxcoon can not produce more than ordered; and Foxconn has to deliver enough units in each month.
- Foxconn starts producing iPhone 6 in September, i.e., there is no previously produced iPhone 6.

Formulate a linear program for deciding the amount of production in each month so that the total storage fee for Foxconn is minimized. (You do not need to give a standard form or solve this linear program. However, you must clearly explain each decision variable, each constraint and the objective function.)

Answer: Label September, October, and November as month 1, 2, 3, respectively. Define x_{ij} as the number of iPhone 6 produced in month i and delivered in month j, where $i \le j$. For simplicity, we can set $x_{ij} = 0$ if $i > j$ and set $1 \le i \le 3$, $1 \le j \le 3$. Then we have the following constraints:

- Production constraints. The total number of iPhone 6 produced in month i cannot exceed the production volume of month i.
  - $\sum_{j=1}^{3} x_{1j} \le 25K$ or $x_{11} + x_{12} + x_{13} \le 25K$
  - $\sum_{j=1}^{3} x_{2j} \le 25K$ or $x_{22} + x_{23} \le 25K$
  - $\sum_{j=1}^{3} x_{3j} \le 15K$ or $x_{33} \le 15K$
- Order balance constraints. The number of iPhone 6 delivered in each month is the same as the number of iPhone 6 ordered in the same month by Apple.
  - $\sum_{i=1}^{3} x_{i1} = 15K$ or $x_{11} = 15K$
  - $\sum_{i=1}^{3} x_{i2} = 20K$ or $x_{12} + x_{22} = 20K$
  - $\sum_{i=1}^{3} x_{i3} = 25K$ or $x_{13} + x_{23} + x_{33} = 25K$
- Some implicit constraints.

$$x_{ij} \ge 0(, \text{ and } x_{ij} \text{ are integers}). \text{ or } x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{33} \ge 0$$

The objective is to minimize the storage fee, and we can calculate the penalty function as following:

$$f(x) = (1 \times 2)x_{12} + (2 \times 2)x_{13} + (1 \times 2)x_{23}.$$

[2] Consider the linear programming

$$\begin{aligned} \text{minimize} \quad & z = 10x_1 - 2x_2 + 5x_3 \\ \text{subject to} \quad & x_1 + x_2 - 2x_3 \geq 1 \\ & 2x_1 + x_3 \geq 4 \\ & -x_2 + x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) Show that $\mathbf{x} = (2, 1, 1)^T$ is a feasible point and label each of the constraints as active or inactive.

(b) Find the set of all feasible directions $\mathbf{p} = (p_1, p_2, p_3)^T$ at $\mathbf{x}$.

(c) Using (b), verify that $\mathbf{p} = (-1, -1, -1)^T$ is a feasible direction at the feasible point $\mathbf{x}$, and determine the maximal step length $\alpha > 0$ such that $\mathbf{x} + \alpha\mathbf{p}$ remains feasible.

Answer: (a) Let $\mathbf{x} = (2, 1, 1)^T$

- $x_1 + x_2 - 2x_3 = 2 + 1 - 2 \times 1 = 1$ Active
- $2x_1 + x_3 = 2 \times 2 + 1 > 4$ Inactive
- $-x_2 + x_3 = -1 + 1 = 0$ Active
- $x_1 > 0$ Inactive
- $x_2 > 0$ Inactive
- $x_3 > 0$ Inactive

Therefore $\mathbf{x}$ is feasible.

(b) We have to only consider the active constraints. Therefore, feasible directions $\mathbf{p}$ has to satisfy

$$\begin{aligned} p_1 + p_2 - 2p_3 &\geq 0 \\ -p_2 + p_3 &\geq 0, \end{aligned}$$

i.e., $\{\mathbf{p} = (p_1, p_2, p_3)^T : p_1 + p_2 - 2p_3 \geq 0, -p_2 + p_3 \geq 0\}$ is the set of all feasible directions.

(c) Let $\mathbf{p} = (-1, -1, -1)^T$, we have $p_1 + p_2 - 2p_3 = -1 - 1 - 2 \times (-1) = 0 \geq 0$, $-p_2 + p_3 = -(-1) - 1 = 0 \geq 0$. Thus $\mathbf{p} = (-1, -1, -1)^T$ is a feasible direction at $\mathbf{x}$.

Let $\bar{\mathbf{x}} = \mathbf{x} + \alpha\mathbf{p} = (2 - \alpha, 1 - \alpha, 1 - \alpha)$, assume that $\bar{\mathbf{x}}$ is feasible, then we have

- $\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3 = 2 - \alpha + 1 - \alpha - 2 \times (1 - \alpha) = 1$ Always satisfied
- $2\bar{x}_1 + \bar{x}_3 = 2 \times (2 - \alpha) + 1 - \alpha = 5 - 3\alpha \geq 4 \Longrightarrow \alpha \leq 1/3$
- $-\bar{x}_2 + \bar{x}_3 = -(1 - \alpha) + 1 - \alpha = 0$ Always satisfied
- $\bar{x}_1 = 2 - \alpha \geq 0 \Longrightarrow \alpha \leq 2$
- $\bar{x}_2 = 1 - \alpha \geq 0 \Longrightarrow \alpha \leq 1$
- $\bar{x}_3 = 1 - \alpha \geq 0 \Longrightarrow \alpha \leq 1$

Therefore, the maximum step length is $\alpha = 1/3$.

[3] Consider the linear program.

$$\begin{aligned} \text{minimize} \quad & z = -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & 2x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 2 \\ & x_2 \geq 0 \end{aligned}$$

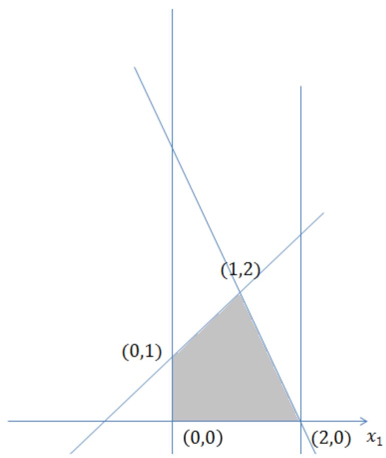
(a) Draw the feasible region of this linear problem in a 2D plane.

(b) Give **all** the basic solutions and identify **all** the extreme points of the feasible region.

(c) Is there a degenerate extreme point in the feasible region (yes/no)? If yes, specify it.

(d) Give the standard form of the linear program.

Answer:(a)



(b) Basic solutions are

- Feasible Basic solutions (extreme points): $(1,2)^T$, $(2,0)^T$, $(0,0)^T$, $(0,1)^T$,
- Infeasible Basis solutions: $(-1,0)^T$, $(2,3)^T$, $(0,4)^T$

(c) Yes, $(2,0)^T$ is degenerate because it has three active constraints:

- $2x_1 + x_2 = 4$;
- $x_1 = 2$;
- $x_2 = 0$.

(d)

$$\begin{aligned} \text{minimize} \quad & z = -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 + x_2 + s_1 = 1 \\ & 2x_1 + x_2 + s_2 = 4 \\ & x_1 + s_3 = 2 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

[4] Find the optimal solution of the following linear program problem in the standard form using the simplex method.

$$\begin{aligned} \text{minimize} \quad & z = -\frac{4}{3}x_3 + x_4 - 9 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Answer: The initial tableau

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	0	0	$-4/3$	1	0	9
x_3	-2	1	1	0	0	2
x_4	-1	2	0	1	0	7
x_5	1	0	0	0	1	3

Choose the initial base to be $\{x_3, x_4, x_5\}$, and modify the first

basic	$\Downarrow$ x_1	x_2	x_3	x_4	x_5	rhs
$-z$	$-5/3$	$-2/3$	0	0	0	$14/3$
x_3	-2	1	1	0	0	2
x_4	-1	2	0	1	0	7
x_5	$\boxed{1}$	0	0	0	1	3

After the first iteration

basic	x_1	$\Downarrow$ x_2	x_3	x_4	x_5	rhs
$-z$	0	$-2/3$	0	0	$5/3$	$29/3$
x_3	0	1	1	0	2	8
x_4	0	$\boxed{2}$	0	1	1	10
x_1	1	0	0	0	1	3

After the second iteration

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	0	0	0	$1/3$	2	13
x_3	0	0	1	$-1/2$	$3/2$	3
x_2	0	1	0	$1/2$	$1/2$	5
x_1	1	0	0	0	1	3

The optimal solution is $\mathbf{x} = (3, 5, 3, 0, 0)^T$, and the optimal objective value is -13.