Analysis of Multiple Time Series

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This week's material

- Vector Autoregressions
- Basic examples
- Properties
  - Stationarity
- Revisiting univariate ARMA processes
- Forecasting
  - Granger Causality
  - Impulse Response functions
- Cointegration
  - Examining long-run relationships
  - Determining whether a VAR is cointegrated
  - Error Correction Models
  - Testing for Cointegration
    - Engle-Granger

Lots of revisiting univariate time series.

VAR

~~VaR~~

$y_t \sim I(1)$

Δy_t

Vector Autoregressions

Why VAR analysis?

- Stationary VARs
  - Determine whether variables feedback into one another
  - Improve forecasts
  - Model the effect of a shock in one series on another
  - Differentiate between short-run and long-run dynamics
- Cointegration
  - Link random walks
  - Uncover long run relationships
  - Can improve medium to long term forecasting a lot

VAR Defined

- P^{th} order autoregression, AR(P):

$$y_t = \phi_0 + \phi_1 y_{t-1} + \ldots + \phi_P y_{t-p} + \epsilon_t$$

- P^{th} order vector autoregression, VAR(P):

$k \times 1$ $\quad$ $k \times k$

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t$$

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{bmatrix}$$

where $\mathbf{y}_t$ and $\boldsymbol{\epsilon}_t$ are k by 1 vectors

- Bivariate VAR(1):

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Compactly expresses two linked models:

$$y_{1,t} = \phi_{01} + \phi_{11} y_{1,t-1} + \phi_{12} y_{2,t-1} + \epsilon_{1,t}$$

$$y_{2,t} = \phi_{02} + \phi_{21} y_{1,t-1} + \phi_{22} y_{2,t-1} + \epsilon_{2,t}$$

Stationarity Revisited

- Stationarity is a statistically meaningful form of regularity. A stochastic process $\{y_t\}$ is covariance stationary if

$$\begin{aligned} \mathrm{E}[y_t] &= \mu & \forall t \\ \mathrm{V}[y_t] &= \sigma^2 & \sigma^2 < \infty \forall t \\ \mathrm{E}[(y_t - \mu)(y_{t-s} - \mu)] &= \gamma_s & \forall t, s \end{aligned}$$

- AR(1) stationarity: $y_t = \phi y_{t-1} + \epsilon_t$
  - $|\phi| < 1$
  - ϵ_t is white noise
- AR(P) stationarity: $y_t = \phi_1 y_{t-1} + \ldots + \phi_P y_{t-P} + \epsilon_t$
  - Roots of $(z^P - \phi_1 z^{P-1} - \phi_2 z^{P-2} - \ldots - \phi_{P-1} z - \phi_P)$ less than 1
  - ϵ_t is white noise

in modulus

- No dependence on t

Relationship to AR

- AR(1)

$$
\begin{aligned}
y_t &= \phi_0 + \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1 \left(\phi_0 + \phi_1 y_{t-2} + \epsilon_{t-1} \right) + \epsilon_t \\
&= \phi_0 + \phi_1 \phi_0 + \phi_1^2 y_{t-2} + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1 \phi_0 + \phi_1^2 \left(\phi_0 + \phi_1 y_{t-3} + \epsilon_{t-2} \right) + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= \phi_0 \sum_{i=0}^{\infty} \phi_1^i + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} \\
&= (1 - \phi_1)^{-1} \phi_0 + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}
\end{aligned}
$$

$Y_{t-1} = \phi_0 + \phi_1 Y_{t-2} + \varepsilon_{t-1}$

$Y_{t-2} = \phi_0 + \phi_1 Y_{t-3} + \varepsilon_{t-2}$

$\frac{1}{1-\phi_1}$

Relationship to AR

- VAR(1)

$k=1$

$y_{t-1} = \Phi_0 + \Phi_1 y_{t-2} + \epsilon_{t-1}$

$$
\begin{aligned}
\mathbf{y}_t &= \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \left(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{y}_{t-2} + \boldsymbol{\epsilon}_{t-1} \right) + \boldsymbol{\epsilon}_t \\
&= \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1^2 \mathbf{y}_{t-2} + \boldsymbol{\Phi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1^2 \left(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{y}_{t-3} + \boldsymbol{\epsilon}_{t-2} \right) + \boldsymbol{\Phi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\epsilon}_t \\
&= \sum_{i=0}^{\infty} \boldsymbol{\Phi}_1^i \boldsymbol{\Phi}_0 + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_1^i \boldsymbol{\epsilon}_{t-i} \\
&= \left(\mathbf{I}_k - \boldsymbol{\Phi}_1 \right)^{-1} \boldsymbol{\Phi}_0 + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_1^i \boldsymbol{\epsilon}_{t-i}
\end{aligned}
$$

$\to y_{t-2}$

$|\lambda_1|, |\lambda_2| < 1$

"1"

$k=2 \to$

$$\Phi^i = U \begin{pmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{pmatrix} V$$

Properties of a VAR(1) and AR(1)

$$\text{AR(1)}: y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$
$$\text{VAR(1)}: \mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

	AR(1)	VAR(1)
Mean	$\phi_0/(1-\phi_1)$	$(\mathbf{I}_k - \mathbf{\Phi}_1)^{-1}\mathbf{\Phi}_0$
Variance	$\sigma^2/(1-\phi_1^2)$	$(\mathbf{I} - \mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1)^{-1}\text{vec}(\mathbf{\Sigma})$
s$^{\text{th}}$ Autocovariance	$\gamma_s = \phi_1^s \mathrm{V}[y_t]$	$\mathbf{\Gamma}_s = \mathbf{\Phi}_1^s \mathrm{V}[\mathbf{y}_t]$
-s$^{\text{th}}$ Autocovariance	$\gamma_{-s} = \phi_1^s \mathrm{V}[y_t]$	$\mathbf{\Gamma}_{-s} = \mathrm{V}[\mathbf{y}_t]\mathbf{\Phi}_1^{s\prime}$

Autocovariances of vector processes are not symmetric, but $\mathbf{\Gamma}_s = \mathbf{\Gamma}'_{-s}$

- Stationarity
  - AR(1): $|\phi_1| < 1$
  - VAR(1): $|\lambda_i| < 1$ where λ_i are the eigenvalues of $\mathbf{\Phi}_1$

Stock and Bond VAR

- VWM from CRSP
- TERM constructed from 10-year bond *return minus 1-year return* from FRED
- February 1962 until December 2018 (683 months)

$$\begin{bmatrix} VWM_t \\ TERM_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11,1} & \phi_{12,1} \\ \phi_{21,1} & \phi_{22,1} \end{bmatrix} \begin{bmatrix} VWM_{t-1} \\ TERM_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Market model:

$$VWM_t = \phi_{01} + \phi_{11,1} VWM_{t-1} + \phi_{12,1} 10YR_{t-1} + \epsilon_{1,t}$$

- Long bond model

$$TERM_t = \phi_{01} + \phi_{21,1} VWM_{t-1} + \phi_{22,1} TERM_{t-1} + \epsilon_{2,t}.$$

- Estimates

$$\begin{bmatrix} VWM_t \\ TERM_t \end{bmatrix} = \begin{bmatrix} \underset{(0.000)}{0.801} \\ \underset{(0.041)}{0.232} \end{bmatrix} + \begin{bmatrix} \underset{(0.122)}{0.059} & \underset{(0.004)}{0.166} \\ \underset{(0.000)}{-0.104} & \underset{(0.002)}{0.116} \end{bmatrix} \begin{bmatrix} VWM_{t-1} \\ TERM_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Stock and Bond VAR

- Estimates from VAR

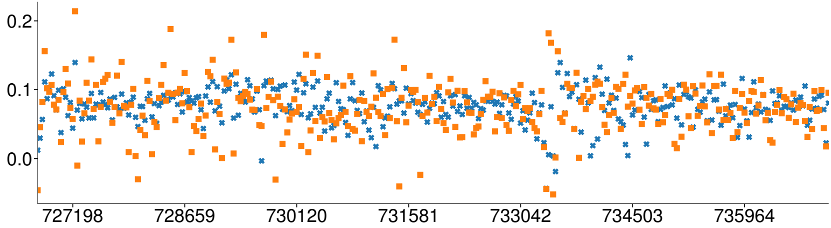
$$\begin{array}{rccccccc}
VWM_t = & \underset{(0.000)}{0.816} & + & \underset{(0.117)}{0.060} & VWM_{t-1} & + & \underset{(0.003)}{0.168} & TERM_{t-1} \\
TERM_t = & \underset{(0.045)}{0.228} & - & \underset{(0.000)}{0.104} & VWM_{t-1} & + & \underset{(0.002)}{0.115} & TERM_{t-1}
\end{array}$$

- Estimates from AR

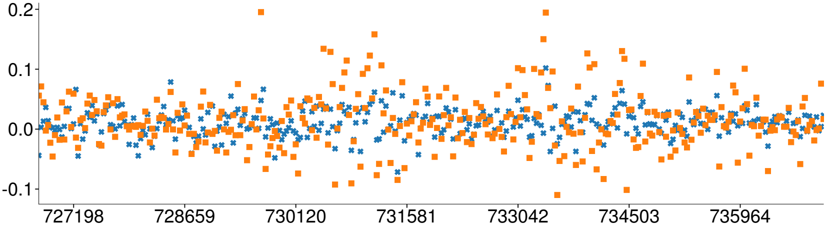
$$\begin{array}{rccccccc}
VWM_t = & \underset{(0.000)}{0.830} & + & \underset{(0.057)}{0.073} & VWM_{t-1} & & & \\
TERM_t = & \underset{(0.224)}{0.137} & & & & + & \underset{(0.011)}{0.098} & TERM_{t-1}
\end{array}$$

Comparing AR and VAR forecasts

1-month-ahead forecasts of the VWM returns



1-month-ahead forecasts of 10-year bond returns



Monetary Policy VAR

- Standard tool in monetary policy analysis
  - Unemployment rate (differenced)
  - Federal Funds rate
  - Inflation rate (differenced)

$$\begin{bmatrix} \Delta\mathrm{UNEMP}_t \\ \mathrm{FF}_t \\ \Delta\mathrm{INF}_t \end{bmatrix} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \begin{bmatrix} \Delta\mathrm{UNEMP}_{t-1} \\ \mathrm{FF}_{t-1} \\ \Delta\mathrm{INF}_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}.$$

	$\Delta \ln \mathrm{UNEMP}_{t-1}$	FF_{t-1}	$\Delta\mathrm{INF}_{t-1}$
$\Delta \ln \mathrm{UNEMP}_t$	0.624 (0.000)	0.015 (0.001)	0.016 (0.267)
FF_t	−0.816 (0.000)	0.979 (0.000)	−0.045 (0.317)
$\Delta\mathrm{INF}_t$	−0.501 (0.010)	−0.009 (0.626)	−0.401 (0.000)

Interpreting Estimates

- Variable scale affects cross-parameter estimates
  - Not an issue in ARMA analysis
- Standardizing data can improve interpretation when scales differ

	$\Delta \ln \mathrm{UNEMP}_{t-1}$	FF_{t-1}	$\Delta\mathrm{INF}_{t-1}$
$\Delta \ln \mathrm{UNEMP}_t$	0.624 (0.000)	0.153 (0.001)	0.053 (0.267)
FF_t	−0.080 (0.000)	0.979 (0.000)	−0.015 (0.317)
$\Delta\mathrm{INF}_t$	−0.151 (0.010)	−0.028 (0.626)	−0.401 (0.000)

- Other important measures – statistical significance, persistence, model selection – are unaffected by standardization

VAR(P) is really a VAR(1)

- Companion form:

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

- Reform into a single VAR(1) where $\boldsymbol{\mu} = \mathrm{E}[\mathbf{y}_t] = (\mathbf{I} - \mathbf{\Phi}_1 - \ldots - \mathbf{\Phi}_P)^{-1} \mathbf{\Phi}_0$

$$\mathbf{z}_t = \mathbf{\Upsilon} \mathbf{z}_{t-1} + \boldsymbol{\xi}_t$$

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{y}_t - \boldsymbol{\mu} \\ \mathbf{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \mathbf{y}_{t-P+1} - \boldsymbol{\mu} \end{bmatrix}, \quad \mathbf{\Upsilon} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \mathbf{\Phi}_3 & \ldots & \mathbf{\Phi}_{P-1} & \mathbf{\Phi}_P \\ \mathbf{I}_k & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_k & \mathbf{0} \end{bmatrix}$$

  - All results can be directly applied to the companion form.
  - Can also be used to transform AR(P) into VAR(1)

Forecasting

Revisiting Univariate Forecasting

- Consider standard AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

- Optimal 1-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+1}] &= \mathrm{E}_t[\phi_0] + \mathrm{E}_t[\phi_1 y_t] + \mathrm{E}_t[\epsilon_{t+1}] \\ &= \phi_0 + \phi_1 y_t + 0 \end{aligned}$$

- Optimal 2-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+2}] &= \mathrm{E}_t[\phi_0] + \mathrm{E}_t[\phi_1 y_{t+1}] + \mathrm{E}_t[\epsilon_{t+2}] \\ &= \phi_0 + \phi_1 \mathrm{E}_t[y_{t+1}] + 0 \\ &= \phi_0 + \phi_1(\phi_0 + \phi_1 y_t) \\ &= \phi_0 + \phi_1 \phi_0 + \phi_1^2 y_t \end{aligned}$$

- Optimal h-step ahead forecast:

$$\mathrm{E}_t[y_{t+h}] = \sum_{i=0}^{h-1} \phi_1^i \phi_0 + \phi_1^h y_t$$

Forecasting with VARs

- Identical to univariate case

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

- Optimal 1-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[\mathbf{y}_{t+1}] &= \mathrm{E}_t[\mathbf{\Phi}_0] + \mathrm{E}_t[\mathbf{\Phi}_1 \mathbf{y}_t] + \mathrm{E}_t[\boldsymbol{\epsilon}_{t+1}] \\ &= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_t + \mathbf{0} \end{aligned}$$

- Optimal h-step ahead forecast:

$$\begin{aligned} \mathrm{E}_t[y_{t+h}] &= \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{\Phi}_0 + \ldots + \mathbf{\Phi}_1^{h-1} \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t \\ &= \sum_{i=0}^{h-1} \mathbf{\Phi}_1^i \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t \end{aligned}$$

- Higher order forecast can be recursively computed

$$\mathrm{E}_t[\mathbf{y}_{t+h}] \quad = \quad \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathrm{E}_t[\mathbf{y}_{t+h-1}] + \ldots + \mathbf{\Phi}_P \mathrm{E}_t[\mathbf{y}_{t+h-P}]$$

What makes a good forecast?

- Forecast residuals

$$\hat{e}_{t+h|t} = y_{t+h} - \hat{y}_{t+h|t}$$

- Residuals are *not* white noise
- Can contain an MA($h-1$) component
  - Forecast error for $y_{t+1} - \hat{y}_{t+1|t-h+1}$ was not known at time t.
- Plot your residuals
- Residual ACF
- Mincer-Zarnowitz regressions
- Three period procedure
  - Training sample: Used to build model
  - Validation sample: Used to refine model
  - Evaluation sample: Ultimate test, ideally 1 shot

Multi-step Forecasting

- Two methods
- Iterative method
  - Build model for 1-step ahead forecasts

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

  - Iterate forecast out to period h

$$\hat{\mathbf{y}}_{t+h|t} = \sum_{i=0}^{h-1} \mathbf{\Phi}_1^i \mathbf{\Phi}_0 + \mathbf{\Phi}_1^h \mathbf{y}_t$$

  - Makes efficient use of information
  - Imposes a lot of structure on the problem
- Direct Method
  - Build model for h-step ahead forecasts

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_h \mathbf{y}_{t-h} + \boldsymbol{\epsilon}_t$$

  - Directly forecast using a pseudo 1-step ahead method

$$\hat{\mathbf{y}}_{t+h|t} = \mathbf{\Phi}_0 + \mathbf{\Phi}_h \mathbf{y}_t$$

  - Robust to some nonlinearities

Multi-step Forecast Evaluation

- Multistep forecast evaluation is identical to one-step ahead forecast evaluation with one caveat
- h-step ahead forecast errors may be correlated with any forecast error not known at time t

$$\hat{e}_{t+1|t-h+1}, \hat{e}_{t+2|t-h+2}, \ldots, \hat{e}_{t+h-1|t-1}$$

- Leads to a MA($h-1$) structure in the forecast errors
- Solutions:
  - Use regular GMZ regression with a Newey-West covariance estimator

$$y_{t+h} - \hat{y}_{t+h|t} = \beta_1 + \beta_2 \hat{y}_{t+h|t} + \boldsymbol{\gamma}\mathbf{x}_t + \eta_t$$

    $H_0 : \beta_1 = \beta_2 = \boldsymbol{\gamma} = 0$, $H_1 : \beta_1 \neq 0 \cup \beta_2 \neq 0 \cup \gamma_j \neq 0 \ \exists j$
  - Explicitly model the MA($h-1$) and use a standard covariance estimator

$$y_{t+h} - \hat{y}_{t+h|t} = \beta_1 + \beta_2 \hat{y}_{t+h|t} + \boldsymbol{\gamma}\mathbf{x}_t + \eta_t + \sum_{i=1}^{h-1} \theta_i \eta_{t-i}$$

    Note: Null is the same; does not impose a restriction on $\boldsymbol{\theta}$

Example: Monetary Policy VAR

- Forecasts produced iteratively for 1 to 8 quarters ahead
- Random walk (FF) or constant mean benchmark
- AR and VAR select lag length using BIC
- Restricted force reversion to in-sample mean using 2-step estimator
  1. Estimate sample mean, and subtract to produce $\tilde{\mathbf{y}}_t = \mathbf{y}_t - \hat{\boldsymbol{\mu}}$
  2. Estimate VAR *without* a constant

$$\tilde{\mathbf{y}}_t = \mathbf{\Phi}_1 \tilde{\mathbf{y}}_{t-1} + \ldots + \mathbf{\Phi}_P \tilde{\mathbf{y}}_{t-P} + \boldsymbol{\epsilon}_t$$

  3. Forecast and then add the in-sample mean

$$\mathrm{E}_t \left[\tilde{\mathbf{y}}_{t+h} \right] + \hat{\boldsymbol{\mu}}$$

- Evaluation based on relative MSE

$$\text{Rel. MSE} = \frac{\text{MSE}}{\text{MSE}_{bm}}, \; \text{MSE} = {}^{1}\!/_{T-h-R} \sum_{t=R}^{T-h} \left(y_{t+h} - \hat{y}_{t+h|t} \right)^2$$

Example: Monetary Policy VAR

		VAR		AR	
Horizon	Series	Restricted	Unrestricted	Restricted	Unrestricted
1	Unemployment	0.522	0.520	**0.507**	0.507
	Fed. Funds Rate	**0.887**	0.903	0.923	0.933
	Inflation	0.869	0.868	**0.839**	0.840
2	Unemployment	0.716	**0.710**	0.717	0.718
	Fed. Funds Rate	**0.923**	0.943	*1.112*	*1.130*
	Inflation	*1.082*	*1.081*	*1.031*	*1.030*
4	Unemployment	0.872	**0.861**	0.937	0.940
	Fed. Funds Rate	**0.952**	0.976	*1.082*	*1.109*
	Inflation	*1.000*	0.999	0.998	**0.998**
8	Unemployment	0.820	**0.806**	0.973	0.979
	Fed. Funds Rate	**0.974**	*1.007*	*1.062*	*1.110*
	Inflation	*1.001*	1.000	0.998	**0.997**

Estimation

Estimation and Identification

- Univariate Identification: Box-Jenkins
  - Use ACF and PACF to determine AR and MA lag order
  - Examine residuals
  - Parsimony principle
- The autocorrelation of a scalar process is defined

$$\rho_s = \frac{\gamma_s}{\gamma_0}$$

where γ_s is s^{th} the autocovariance
  - Regression coefficient:

$$y_t = \mu + \rho_s y_{t-s} + \epsilon_t$$

- Partial autocorrelation ψ_s
  - Regression interpretation of s^{th} partial autocorrelation:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_{s-1} y_{t-s+1} + \psi_s y_{t-s} + \epsilon_t$$

  - ψ is the s^{th} partial autocorrelation

CCF and Partial CCF

- Multivariate equivalents
  - ACF and PACF have same regression definitions
  - Cross-correlation function

$$\rho_{xy,s} = \frac{\mathrm{E}[(x_t - \mu_x)(y_{t-s} - \mu_y)]}{\sqrt{\mathrm{V}[x_t]\mathrm{V}[y_t]}}$$

$$\rho_{yx,s} = \frac{\mathrm{E}[(y_t - \mu_y)(x_{t-s} - \mu_x)]}{\sqrt{\mathrm{V}[x_t]\mathrm{V}[y_t]}}$$

  - Generally different
  - Cross-partial-correlation function $\psi_{xy,s}$

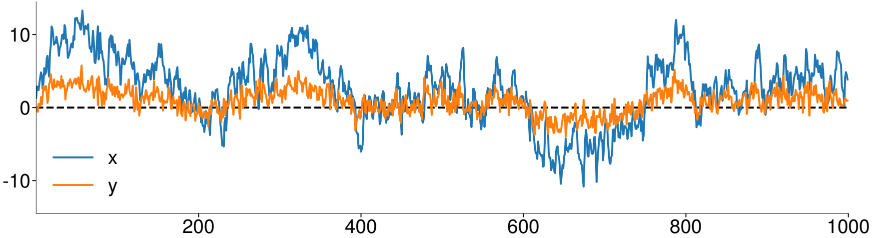
$$\begin{aligned} x_t =& \phi_0 + \phi_{x1} x_{t-1} + \ldots + \phi_{xs-1} x_{t-(s-1)} \\ & + \phi_{y1} y_{t-1} + \ldots + \phi_{ys-1} y_{t-(s-1)} + \varphi_{xy,s} y_{t-s} + \epsilon_{x,t} \end{aligned}$$

    - Can help identify VAR order
- Deeper issue: too many and too complicated
- Simple solution: Model selection

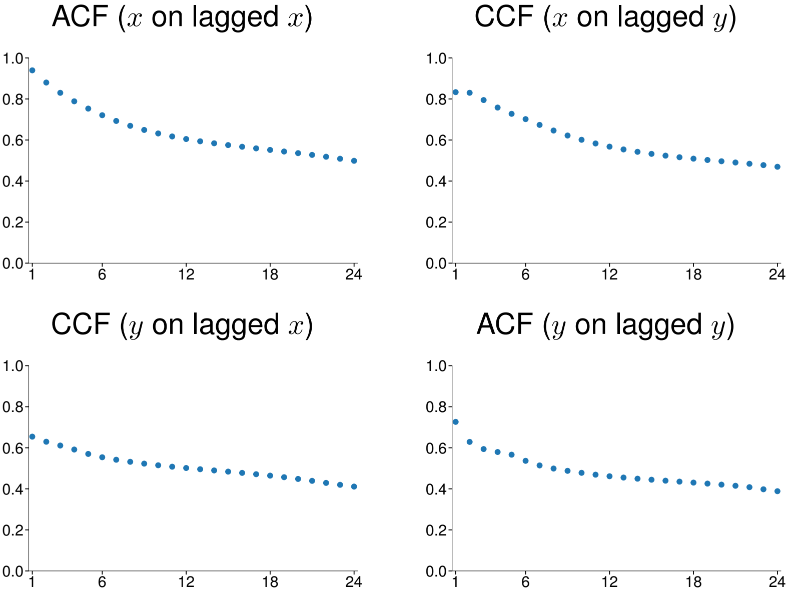
- y has HAR dynamics, spills over to x

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.9 \\ .0 & 0.47 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \sum_{i=2}^{5} \begin{bmatrix} 0 & 0 \\ 0 & 0.06 \end{bmatrix} \begin{bmatrix} x_{t-i} \\ y_{t-i} \end{bmatrix}$$
$$+ \sum_{j=6}^{22} \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} x_{t-j} \\ y_{t-j} \end{bmatrix} + \begin{bmatrix} \epsilon_{x,t} \\ \epsilon_{y,t} \end{bmatrix}$$

- Simulated data

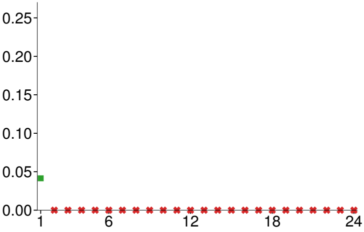


ACFs and CCFs

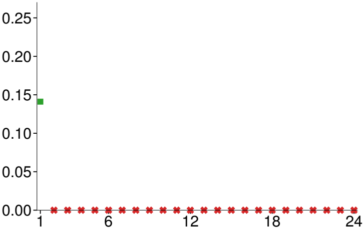


PACFs and Partial CCFs

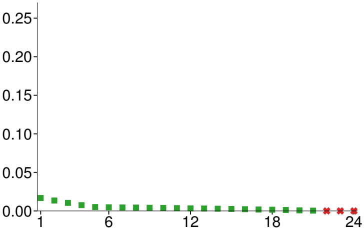
PACF (x on lagged x)



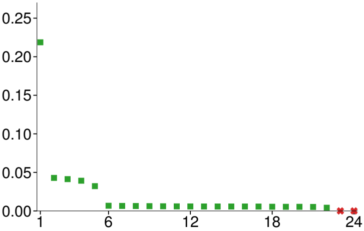
PCCF (x on lagged y)



PCCF (y on lagged x)



PACF (y on lagged y)



- Step 1: Pick maximum lag length
  - Information criteria

$$\text{AIC:} \quad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{2}{T}$$

$$\text{Hannan-Quinn IC (HQIC):} \quad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{\ln \ln T}{T}$$

$$\text{SIC:} \quad \ln|\mathbf{\Sigma}(P)| + k^2 P \frac{\ln T}{T}$$

    - $\Sigma(P)$ is the covariance of the residuals using P lags
    - $|\cdot|$ is the determinant
  - Hypothesis testing based
    - General to Specific
    - Specific to General
  - Likelihood Ratio

$$(T - P_2 k^2)\left(\ln|\mathbf{\Sigma}(P_1)| - \ln|\mathbf{\Sigma}(P_2)|\right) \overset{A}{\sim} \chi^2_{(P_2 - P_1)k^2}$$

Lag Length Selection in Monetary Policy VAR

- Maximum lag: 12 (1 year)

Lag Length	AIC	HQIC	BIC	LR	P-val
0	4.014	3.762	3.605	925	0.000
1	0.279	0.079	0.000▼▲	39.6	0.000
2	0.190	0.042	0.041	40.9	0.000
3	0.096	0.000▼	0.076	29.0	0.001
4	0.050▼	0.007	0.160	7.34	0.602▼
5	0.094	0.103	0.333	29.5	0.001
6	0.047	0.108	0.415	13.2	0.155
7	0.067	0.180	0.564	32.4	0.000
8	0.007	0.172▲	0.634	19.8	0.019
9	0.000▲	0.217	0.756	7.68	0.566▲
10	0.042	0.312	0.928	13.5	0.141
11	0.061	0.382	1.076	13.5	0.141
12	0.079	0.453	1.224	–	–

Granger Causality

Granger Causality

- First fundamentally new concept
- Examines whether lags of one variable are helpful in predicting another

Definition (Granger Causality)

A scalar random variable $\{x_t\}$ is said to **not** Granger cause $\{y_t\}$ if $\mathrm{E}[y_t | x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, \ldots] = \mathrm{E}[y_t |, y_{t-1}, y_{t-2}, \ldots]$. That is, $\{x_t\}$ does not Granger cause if the forecast of y_t is the same whether conditioned on past values of x_t or not.

Granger Causality

- Translates directly into a restriction in a VAR
- Unrestricted

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Restricted so that x_t does not GC y_t

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$x_t = \phi_{01} + \phi_{11} x_{t-1} + \phi_{12} y_{t-1} + \epsilon_{1,t}$$

$$y_t = \phi_{02} + \phi_{22} y_{t-1} + \epsilon_{2,t} \Leftarrow \text{ No } x_t!$$

More Granger Causality

- In P lag model

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1\mathbf{y}_{t-1} + \mathbf{\Phi}_2\mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P\mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

  the null hypothesis is

$$H_0 : \phi_{ij,1} = \phi_{ij,2} = \ldots = \phi_{ij,P} = 0$$

- Alternative is

$$H_0 : \phi_{ij,1} \neq 0 \text{ or } \phi_{ij,2} \neq 0 \text{ or } \ldots \text{ or } \phi_{ij,P} \neq 0$$

- Likelihood Ratio test

$$\left(T - Pk^2\right)\left(\ln|\mathbf{\Sigma}_r| - \ln|\mathbf{\Sigma}_u|\right) \stackrel{A}{\sim} \chi^2_P$$

- $\mathbf{\Sigma}_u$ is the covariance of the errors from unrestricted model
- $\mathbf{\Sigma}_r$ is the covariance of the errors from restricted model
- $T - Pk^2$ is number of observations minus number of free parameters in unrestricted model
  - Why χ^2_P?

Monetary Policy VAR

- Standard tool in monetary policy analysis
  - Unemployment rate (differenced)
    - Federal Funds rate
    - Inflation rate (differenced)

$$\begin{bmatrix} \Delta\mathrm{UNEMP}_t \\ \mathrm{FF}_t \\ \Delta\mathrm{INF}_t \end{bmatrix} = \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \begin{bmatrix} \Delta\mathrm{UNEMP}_{t-1} \\ \mathrm{FF}_{t-1} \\ \Delta\mathrm{INF}_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}.$$

Granger Causality in Campbell's VAR

- Using model with lags 3 lags (HQIC)
- $H_0 : \phi_{ij,1} = \phi_{ij,2} = \phi_{ij,3} = 0$
- $H_1 : \phi_{ij,1} \neq 0$ or $\phi_{ij,2} \neq 0$ or $\phi_{ij,3} \neq 0$
- i represent series being affected by lags of series j

	Fed. Funds Rate		Inflation		Unemployment	
Exclusion	P-val	Stat	P-val	Stat	P-val	Stat
Fed. Funds Rate	–	–	0.001	13.068	0.014	8.560
Inflation	0.001	14.756	–	–	0.375	1.963
Unemployment	0.000	19.586	0.775	0.509	–	–
All	0.000	33.139	0.000	18.630	0.005	10.472

Impulse Response Functions

Impulse Response Functions

- Second fundamentally new concept
- Complicated dynamics of a VAR make direct interpretation of coefficients difficult
- Solution is to examine impulse responses
- The impulse response function of y_i with respect to a shock in ϵ_j, for any j and i, is defined as the change in y_{it+s}, $s \geq 0$ for a unit shock in ϵ_{jt}
  - Hard to decipher
- As long as $\mathbf{y}_t$ is covariance stationarity it must have a VMA representation,

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t + \boldsymbol{\Xi}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\Xi}_2 \boldsymbol{\epsilon}_{t-2} + \ldots$$

- $\boldsymbol{\Xi}_j$ are the impulse responses!
- Why?
  - Directly measure the effect in period j of any shock

AR(P) and MA(∞)

- Any stationary AR(P)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_P y_{t-P} + \epsilon_t$$

can be represented as an MA(∞)

$$y_t = \phi_0/(1 - \phi_1 - \phi_2 - \ldots - \phi_P) + \epsilon_t + \sum_{i=1}^{\infty} \theta_i \epsilon_{t-i}$$

- AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$$

becomes

$$y_t = \phi_0/(1 - \phi_1) + \epsilon_t + \sum_{i=1}^{\infty} \phi_1^i \epsilon_{t-i}$$

- Stationary VAR(P) have the same relationship to VMA(∞)

$$\mathbf{y}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_P \mathbf{y}_{t-P} + \boldsymbol{\epsilon}_t$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t + \mathbf{\Xi}_1 \boldsymbol{\epsilon}_{t-1} + \mathbf{\Xi}_2 \boldsymbol{\epsilon}_{t-2} + \ldots$$

Solving IR

- Easy in VAR(1)

$$\mathbf{y}_t = (\mathbf{I}_K - \mathbf{\Phi}_1)^{-1}\mathbf{\Phi}_0 + \boldsymbol{\epsilon}_t + \mathbf{\Phi}_1\boldsymbol{\epsilon}_{t-1} + \mathbf{\Phi}_1^2\boldsymbol{\epsilon}_{t-2} + \ldots.$$

- $\mathbf{\Xi}_j = \mathbf{\Phi}_1^j$
- In the general VAR(P),

$$\mathbf{\Xi}_j = \mathbf{\Phi}_1\mathbf{\Xi}_{j-1} + \mathbf{\Phi}_2\mathbf{\Xi}_{j-2} + \ldots + \mathbf{\Phi}_P\mathbf{\Xi}_{j-P}$$

  where $\mathbf{\Xi}_0 = \mathbf{I}_k$ and $\mathbf{\Xi}_m = \mathbf{0}$ for $m < 0$.
  - In a VAR(2),

$$\mathbf{y}_t = \mathbf{\Phi}_1\mathbf{y}_{t-1} + \mathbf{\Phi}_2\mathbf{y}_{t-2} + \boldsymbol{\epsilon}_t$$

    - $\mathbf{\Xi}_0 = \mathbf{I}_k$, $\mathbf{\Xi}_1 = \mathbf{\Phi}_1$, $\mathbf{\Xi}_2 = \mathbf{\Phi}_1^2 + \mathbf{\Phi}_2$, and $\mathbf{\Xi}_3 = \mathbf{\Phi}_1^3 + \mathbf{\Phi}_1\mathbf{\Phi}_2 + \mathbf{\Phi}_2\mathbf{\Phi}_1$.
- Confidence intervals are also somewhat painful
  - Explained in notes

Considerations for Shocks

- Simple bivariate VAR(1)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{01} \\ \phi_{02} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- How you *shock* matters
- Depends on correlation between $\epsilon_{1,t}$ and $\epsilon_{2,t}$
- 3 methods
  - Ignore correlation and just shock $\epsilon_{j,t}$ with a 1 standard deviation shock
  - Use Cholesky to factor $\mathbf{\Sigma}$ and use $\mathbf{\Sigma}^{1/2}\mathbf{e}_j$ where $\mathbf{e}_j$ is a vector of zeros with 1 in the jth position

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \quad \mathbf{\Sigma}_C^{1/2} = \begin{bmatrix} 1 & 0 \\ .5 & .866 \end{bmatrix}$$

    - Variable order matters
  - "Generalized" impulse response that uses a projection method

Example of the different shocks

- Define the error covariance

$$\mathbf{\Sigma} = \left[\begin{array}{cc} \sigma_x^2 & \sigma_x \sigma_y \rho \\ \sigma_x \sigma_y \rho & \sigma_y^2 \end{array} \right]$$

  - Standardized

$$\left[\begin{array}{c} \sigma_x \\ 0 \end{array} \right] \text{ and } \left[\begin{array}{c} 0 \\ \sigma_y \end{array} \right]$$

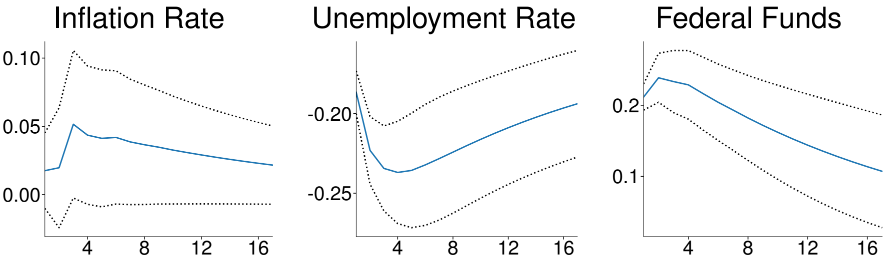
  - Cholesky

$$\mathbf{\Sigma}_C^{1/2} \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{cc} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1-\rho^2} \end{array} \right] \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{c} 0 \\ \sigma_y \sqrt{1-\rho^2} \end{array} \right]$$

$$\left[\begin{array}{cc} \sigma_x & 0 \\ \sigma_y \rho & \sigma_y \sqrt{1-\rho^2} \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = \left[\begin{array}{c} \sigma_x \\ \sigma_y \rho \end{array} \right] \text{,other is } \left[\begin{array}{c} 0 \\ \sigma_y \sqrt{1-\rho^2} \end{array} \right]$$

Impulse Responses

- Federal Funds ordered first
- Response to Federal Funds Shock
- Cholesky factorization



Cointegration

Cointegration

- Cointegration is the VAR version of unit roots
- Establishes long run relationships between two unit root variables
  - Consumption has a unit root, income has a unit root
  - Consumption - Income : ????

Definition (Integrated of Order 1)

A variable y_t is integrated of order 1 (I(1)) if y_t is non-stationary and $\Delta y_t = y_t - y_{t-1}$ is stationary.

Cointegration

Definition (Bivariate Cointegration)

If x_t and y_t are are cointegrated if both are I(1) and there exists a vector β with both elements non-zero such that

$$\beta_1 x_t - \beta_2 y_t \sim I(0)$$

- Strong link between x_t and y_t
- Both are random walks but difference is mean reverting
- Mean reversion to the trend (stochastic trend)

What does cointegration look like?

$$\mathbf{y}_t = \mathbf{\Phi}_{ij}\mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

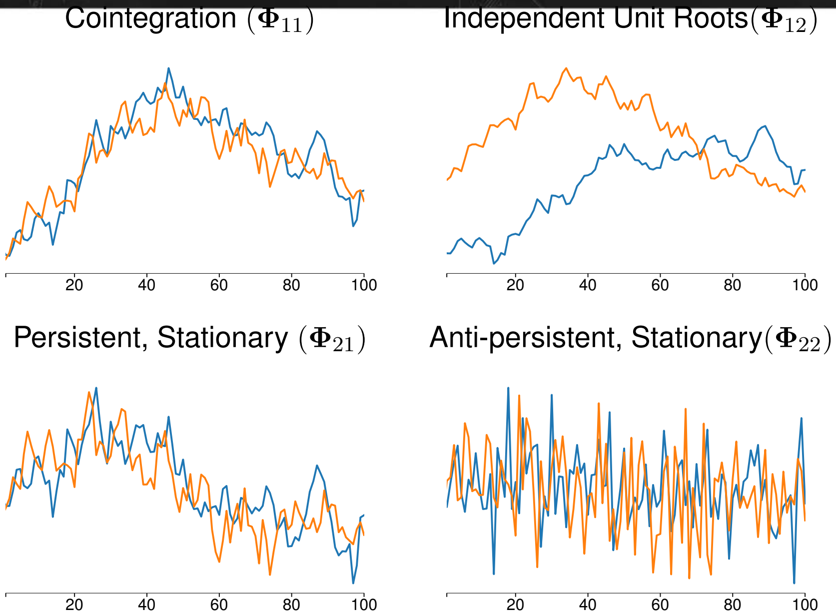
$$\mathbf{\Phi}_{11} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \qquad \mathbf{\Phi}_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_i = 1, 0.6 \qquad \lambda_i = 1, 1$$

$$\mathbf{\Phi}_{21} = \begin{bmatrix} .7 & .2 \\ .2 & .7 \end{bmatrix} \qquad \mathbf{\Phi}_{22} = \begin{bmatrix} -.3 & .3 \\ .1 & -.2 \end{bmatrix}$$

$$\lambda_i = 0.9, 0.5 \qquad \lambda_i = -0.43, -0.06$$

Persistence, Anti-persistence and Cointegration



How do we know when a VAR is cointegrated?

- Eigenvalue condition determines whether a VAR(1) is cointegrated

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Cointegrated if only 1 eigenvalue is unity.
- If all less than 1: ?
- If both 1: two independent unit roots

$$\mathbf{\Phi}_{11} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \qquad \mathbf{\Phi}_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_i = 1, 0.6 \qquad \lambda_i = 1, 1$$

$$\mathbf{\Phi}_{21} = \begin{bmatrix} .7 & .2 \\ .2 & .7 \end{bmatrix} \qquad \mathbf{\Phi}_{22} = \begin{bmatrix} -.3 & .3 \\ .1 & -.2 \end{bmatrix}$$

$$\lambda_i = 0.9, 0.5 \qquad \lambda_i = -0.43, -0.06$$

Error Correction Models

Error Correction Models

- Major point of cointegration
  - Cointegrated $\Leftrightarrow$ Error correction model
- What is an error correction model?
  - Cointegrated VAR:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

  - Error correction model:

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

  - Normalized form

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- $[1 \quad -1]$ is cointegrating vector
- $[-.2 \;\; .2]'$ measures the speed of adjustment

From VAR to VECM

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Subtracting $[y_{t-1}\ x_{t-1}]'$ from both sides

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} - \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} - \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \left(\begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

Cointegrating vectors

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

- Cointegrating relationship can always be decomposed

$$\Delta \mathbf{y}_t = \boldsymbol{\pi} \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$$

- $\boldsymbol{\alpha}$ measures the speed of convergence
- $\boldsymbol{\beta}$ contain the cointegrating vectors
- Number of cointegrating vectors is rank$(\boldsymbol{\alpha}\boldsymbol{\beta}')$

$$\boldsymbol{\alpha}\boldsymbol{\beta}' = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

- How many?

Determining the cointegrating vectors

$$\Delta \mathbf{y}_t = \boldsymbol{\pi} \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\pi} = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

- Put $\boldsymbol{\pi}$ in row echelon form

$$\text{Row Echelon Form} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Recall $\boldsymbol{\pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -.3 \end{bmatrix} \quad \boldsymbol{\alpha} = \begin{bmatrix} .3 & .2 \\ .2 & .5 \\ -.3 & -.3 \end{bmatrix}$$

Solving for the cointegrating vectors

$$\boldsymbol{\alpha}\boldsymbol{\beta}' = \begin{bmatrix} 0.3 & 0.2 & -0.36 \\ 0.2 & 0.5 & -0.35 \\ -0.3 & -0.3 & 0.39 \end{bmatrix}$$

$$\text{Row-Echelon Form} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \beta_1 & \beta_2 \end{bmatrix}$$

and $\boldsymbol{\alpha}$ has 6 unknown parameters. $\boldsymbol{\alpha}\boldsymbol{\beta}'$ can be combined to produce

$$\boldsymbol{\pi} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{11}\beta_1 + \alpha_{12}\beta_2 \\ \alpha_{21} & \alpha_{22} & \alpha_{21}\beta_1 + \alpha_{22}\beta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{31}\beta_1 + \alpha_{32}\beta_2 \end{bmatrix}$$

Testing for Cointegration

Testing for Cointegration

- Two tests for cointegration
  - Engle-Granger
  - Johansen
- We will focus on Engle-Granger
  - Simple and intuitive
  - Only applicable with 1 cointegrating relationship
- Test key property of cointegration: difference is I(0)
- Most of the work is a simple OLS

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

- Rest of work is testing $\hat{\epsilon}_t$ for a unit root
- Johansen tests eigenvalues of $\boldsymbol{\pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ directly.

Engle-Granger Procedure

Algorithm (Engle-Granger Test)

1. *Begin by analyzing x_t and y_t in isolation. Both must be unit roots to consider cointegration.*
2. *Estimate the long run relationship*

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

and test $H_0 : \gamma = 0$ against $H_0 : \gamma < 0$ in the ADF regression

$$\Delta\hat{\epsilon}_t = \gamma\hat{\epsilon}_{t-1} + \delta_1\Delta\hat{\epsilon}_{t-1} + \ldots + \delta_p\Delta\hat{\epsilon}_{t-P} + \eta_t.$$

3. *Using the estimated parameters, specify and estimate the error correction form of the relationship,*

$$\left[\begin{array}{c} \Delta x_t \\ \Delta y_t \end{array}\right] = \begin{array}{c} \pi_{01} \\ \pi_{02} \end{array} + \begin{array}{c} \alpha_1\hat{\epsilon}_t \\ \alpha_2\hat{\epsilon}_t \end{array} + \boldsymbol{\pi}_1 \left[\begin{array}{c} \Delta x_{t-1} \\ \Delta y_{t-1} \end{array}\right] + \ldots + \boldsymbol{\pi}_P \left[\begin{array}{c} \Delta x_{t-P} \\ \Delta y_{t-P} \end{array}\right] + \left[\begin{array}{c} \eta_{1,t} \\ \eta_{2,t} \end{array}\right]$$

4. *Assess the model*

Engle-Granger Considerations

- Deterministic terms
  - No deterministic terms: only in special circumstances

$$y_t = \beta x_t + \epsilon_t$$

  - Constant: standard case

$$y_t = \delta_0 + \beta x_t + \epsilon_t$$

  - Time trend and constant: allow different growth rates/time trends in variables

$$y_t = \delta_0 + \delta_1 t + \beta x_t + \epsilon_t$$

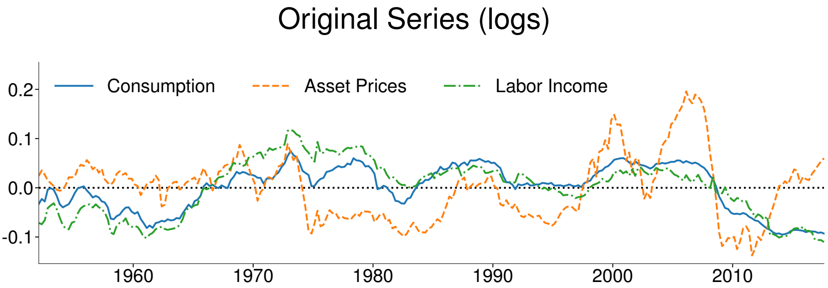
- Critical Values
  - Critical values depend on the deterministics in the CI regression
    - Models with more deterministics have lower (more negative) critical values
  - Critical values depend on number of RHS $I(1)$ variables
    - Larger models have lower critical values

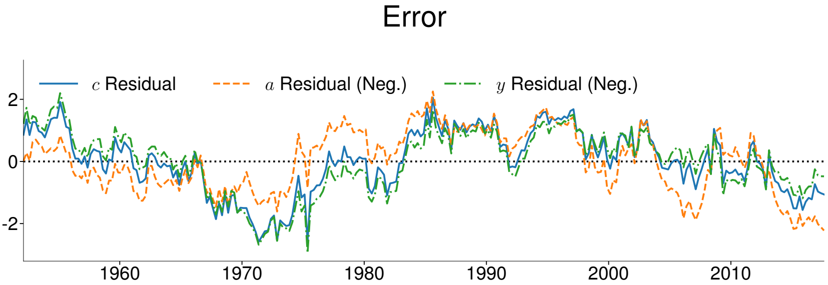
Example: cay

- Consumption-Aggregate Wealth has been an interesting cointegrated series in recent finance literature
- Has revived the CCAPM
- Three components:
  - Consumption (c)
  - Asset Wealth (a)
  - Labor Income (Human Wealth) (y)
- Deviation from long run related to expected return
- Cointegrating relationship: $c_t + .643 - 0.249a_t - 0.785y_t$

Unit Root Tests

Series	T-stat	P-val	ADF Lags
c	-1.198	0.674	5
a	-0.205	0.938	3
y	-2.302	0.171	0
$\hat{\epsilon}_t^c$	-2.706	0.383	1
$\hat{\epsilon}_t^a$	-2.573	0.455	0
$\hat{\epsilon}_t^y$	-2.679	0.398	1


Original Series (logs)
Consumption
Asset Prices
Labor Income
0.2
0.1
0.0
-0.1
1960
1970
1980
1990
2000
2010


Error
c Residual
a Residual (Neg.)
y Residual (Neg.)
2
0
-2
1960
1970
1980
1990
2000
2010

Vector Error Correction Model

- VECM estimated using the residuals from cointegrating regression

$$\begin{bmatrix} \Delta c_t \\ \Delta a_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} \underset{(0.000)}{0.003} \\ \underset{(0.014)}{0.004} \\ \underset{(0.000)}{0.003} \end{bmatrix} + \begin{bmatrix} \underset{(0.281)}{-0.000} \\ \underset{(0.037)}{0.002} \\ \underset{(0.515)}{0.000} \end{bmatrix} \hat{\epsilon}_{t-1} + \begin{bmatrix} \underset{(0.005)}{0.192} & \underset{(0.000)}{0.102} & \underset{(0.004)}{0.147} \\ \underset{(0.116)}{0.282} & \underset{(0.006)}{0.220} & \underset{(0.414)}{-0.149} \\ \underset{(0.000)}{0.369} & \underset{(0.088)}{0.061} & \underset{(0.140)}{-0.139} \end{bmatrix} \begin{bmatrix} \Delta c_{t-1} \\ \Delta a_{t-1} \\ \Delta y_{t-1} \end{bmatrix} + \boldsymbol{\eta}_t$$

- P-values in parentheses
- Estimation of cointegration relationship has no effect on standard errors
  - Converges fast (T)
  - VECM parameters converge at rate $\sqrt{T}$

Spurious Regression

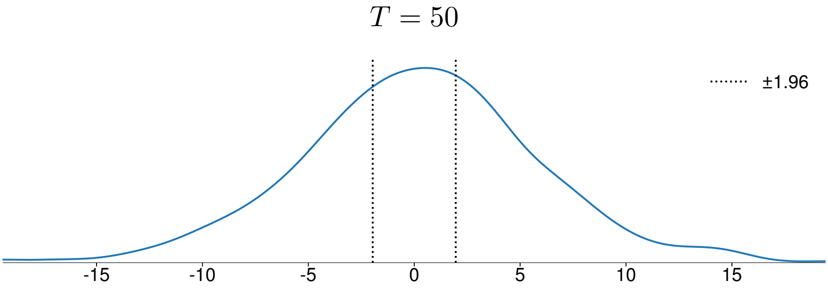
Spurious Regression and Balance

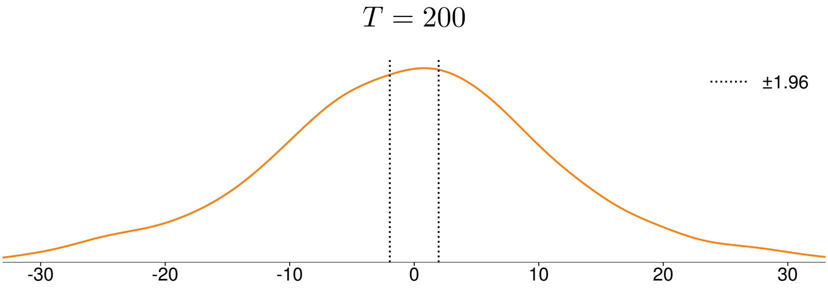
- Caution is needed when working with I(1) data
  - I(0) on I(0): The usual case. Standard asymptotic arguments apply.
  - I(1) on I(0): This regression is unbalanced.
  - I(1) on I(1): Cointegration or spurious regression.
  - I(0) on I(1): This regression is unbalanced.
- Spurious regression can lead to large t-stats when the series are independent.
  - Two unrelated I(1) processes, x_t and y_t

$$x_t = x_{t-1} + \epsilon_t$$

$$y_t = y_{t-1} + \eta_t$$

  - When $T = 50$, approx 80% of t-stats are significant
  - Always check for I(1) when using time-series data
  - If both I(1), make sure cointegrated.


$T = 50$
±1.96
-15
-10
-5
0
5
10
15


$T = 200$
±1.96
-30
-20
-10
0
10
20
30

Revisiting Cross-Secitonal Regression

Cross-section Regression with Time Series Data

- It is common to run regressions using time-series data

$$y_t = \mathbf{x}_t \boldsymbol{\beta} + \epsilon_t$$

- Using time-series data in a cross-sectional regression may require modification to inference
- Modification is needed if the scores $\{\mathbf{x}_t \epsilon_t\}$ are autocorrelated

$$\begin{aligned}
\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} &= \left(\frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \epsilon_t \\
\Rightarrow \mathrm{V}\left[\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right] &\approx \boldsymbol{\Sigma}_{\mathbf{XX}}^{-1} \mathrm{V}\left[\frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \epsilon_t \right] \boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}
\end{aligned}$$

  - Usually occurs when the errors ϵ_t are autocorrelated due to mis- or under-specification of the model

Why the difference?

- Consider the estimation of the mean when y_t has white noise errors

$$y_t = \mu + \epsilon_t$$

- Obviously
- The sample mean is unbiased

$$\begin{aligned} \mathrm{E}[\hat{\mu}] &= \mathrm{E}\left[T^{-1}\sum_{t=1}^{T} y_t\right] \\ &= T^{-1}\sum_{t=1}^{T}\mathrm{E}\left[y_t\right] \\ &= \mu \end{aligned}$$

Why the difference?

- The variance of the sample mean

$$
\begin{aligned}
\mathrm{V}[\hat{\mu}] &= \mathrm{E}\left[\left(T^{-1}\sum_{t=1}^{T} y_t - \mu\right)^2\right] \\
&= \mathrm{E}\left[T^{-2}\left(\sum_{t=1}^{T}\epsilon_t^2 + \sum_{r=1}^{T}\sum_{s=1, r\neq s}^{T}\epsilon_r\epsilon_s\right)\right] \\
&= T^{-2}\sum_{t=1}^{T}\mathrm{E}[\epsilon_t^2] + T^{-2}\sum_{r=1}^{T}\sum_{s=1, r\neq s}^{T}\mathrm{E}[\epsilon_r\epsilon_s] \\
&= T^{-2}\sum_{t=1}^{T}\sigma^2 + T^{-2}\sum_{r=1}^{T}\sum_{s=1, r\neq s}^{T} 0 \\
&= \frac{\sigma^2}{T},
\end{aligned}
$$

- Due to white noise, $\mathrm{E}[\epsilon_i\epsilon_j]$=0 whenever $i \neq j$.
- This is the usual result

The case of an MA(1) error

- Now suppose that the error follows an MA(1)

$$\eta_t = \theta \epsilon_{t-1} + \epsilon_t$$

  where $\{\epsilon_t\}$ is a white noise process
- Error is mean 0 and so sample mean is still unbiased
- Variance of sample mean is *different* since η_t is autocorrelated
  - $\mathrm{E}[\eta_t \eta_{t-1}] \neq 0$.

$$\begin{aligned}
\mathrm{V}[\hat{\mu}] &= \mathrm{E}\left[\left(T^{-1}\sum_{t=1}^{T}\eta_t\right)^2\right] \\
&= \mathrm{E}\left[T^{-2}\left(\sum_{t=1}^{T}\eta_t^2 + 2\sum_{t=1}^{T-1}\eta_t\eta_{t+1} + 2\sum_{t=1}^{T-2}\eta_t\eta_{t+2} + \ldots + \right.\right. \\
&\qquad\qquad \left.\left. 2\sum_{t=1}^{2}\eta_t\eta_{t+T-2} + 2\sum_{t=1}^{1}\eta_t\eta_{t+T-1}\right)\right]
\end{aligned}$$

The case of an MA(1) error

- In terms of autocovariances,

$$\mathrm{V}[\hat{\mu}] = T^{-2}\sum_{t=1}^{T}\mathrm{E}[\eta_t^2] + 2T^{-2}\sum_{t=1}^{T-1}\mathrm{E}[\eta_t\eta_{t+1}] + 2T^{-2}\sum_{t=1}^{T-2}\mathrm{E}[\eta_t\eta_{t+2}] + \ldots +$$
$$2T^{-2}\sum_{t=1}^{2}\mathrm{E}[\eta_t\eta_{t+T-2}] + 2T^{-2}\sum_{t=1}^{1}\mathrm{E}[\eta_t\eta_{t+T-1}]$$
$$= T^{-2}\sum_{t=1}^{T}\gamma_0 + 2T^{-2}\sum_{t=1}^{T-1}\gamma_1 + 2T^{-2}\sum_{t=1}^{T-2}\gamma_2 + \ldots + 2T^{-2}\sum_{t=1}^{1}\gamma_{T-1}$$

- $\gamma_0 = \mathrm{V}[\eta_t]$=$\left(1+\theta^2\right)\mathrm{V}\left[\epsilon_t\right]$ and $\gamma_s = \mathrm{E}[\eta_t\eta_{t-s}]$
- An MA(1) has 1 non-zero autocovariance,

$$\begin{aligned}\gamma_1 &= \mathrm{E}[\eta_t\eta_{t-1}]\\ &= \mathrm{E}[(\theta\epsilon_{t-1}+\epsilon_t)(\theta\epsilon_{t-2}+\epsilon_{t-1})]\\ &= \theta^2\mathrm{E}[\epsilon_{t-1}\epsilon_{t-2}] + \theta\mathrm{E}[\epsilon_{t-1}^2] + \theta\mathrm{E}[\epsilon_t\epsilon_{t-2}] + \mathrm{E}[\epsilon_t\epsilon_{t-1}]\\ &= \theta\sigma^2\end{aligned}$$

- Putting it all together

$$\begin{aligned}
\mathrm{V}[\hat{\mu}] &= T^{-2}\sum_{t=1}^{T}\gamma_0 + 2T^{-2}\sum_{t=1}^{T+1}\gamma_1 \\
&= T^{-2}T\gamma_0 + 2T^{-2}(T-1)\gamma_1 \\
&\approx \frac{\gamma_0 + 2\gamma_1}{T} \\
&= \frac{\sigma^2\left(1+\theta^2+2\theta\right)}{T}
\end{aligned}$$

Can be larger or smaller ($-2 < \theta < 0$)

The variance of the sum is the sum of the variance *only* when the errors are uncorrelated

Estimating the parameter covariance (from CS lecture)

- When the scores are uncorrelated (a Martingale Difference sequence (MDS)) White's covariance estimator is consistent

Theorem (Consistency of Asymptotic Covariance Estimator)

Under the large sample assumptions,

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_{\mathbf{XX}} =& T^{-1}\mathbf{X}'\mathbf{X} \xrightarrow{p} \boldsymbol{\Sigma}_{\mathbf{XX}} \\ \hat{\mathbf{S}} =& T^{-1}\sum_{t=1}^{T}\hat{\epsilon}_t^2\mathbf{x}_t'\mathbf{x}_t \xrightarrow{p} \mathbf{S}\end{aligned}$$

and

$$\hat{\boldsymbol{\Sigma}}_{\mathbf{XX}}^{-1}\hat{\mathbf{S}}\hat{\boldsymbol{\Sigma}}_{\mathbf{XX}}^{-1} \xrightarrow{p} \boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}\mathbf{S}\boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}$$

Modification to regression parameter covariance

- White's estimator is only heteroskedasticity robust – not heteroskedasticity and autocorrelation robust

$$\hat{\mathbf{S}} = T^{-1} \sum_{t=1}^{T} \hat{\epsilon}_t^2 \mathbf{x}_t' \mathbf{x}_t \xrightarrow{p} \mathbf{S}$$

- Solution is to use a Newey-West covariance for the scores ($\mathbf{x}_t \epsilon_t$)

Definition (Newey-West Covariance Estimator)

Let $\mathbf{z}_t$ be a k by 1 vector series that may be autocorrelated and define $\mathbf{z}_t^* = \mathbf{z}_t - \bar{\mathbf{z}}$ where $\bar{\mathbf{z}} = T^{-1} \sum_{t=1}^{T} \mathbf{z}_t$. The L-lag Newey-West covariance estimator for the variance of $\bar{\mathbf{z}}$ is

$$\hat{\mathbf{\Sigma}}_{NW} = \hat{\mathbf{\Gamma}}_0 + \sum_{l=1}^{L} w_l \left(\hat{\mathbf{\Gamma}}_l + \hat{\mathbf{\Gamma}}_l' \right)$$

where $\hat{\mathbf{\Gamma}}_l = T^{-1} \sum_{t=l+1}^{T} \mathbf{z}_t^* \mathbf{z}_{t-l}^{*\prime}$ and $w_l = 1 - \frac{l}{L+1}$.

Modification to regression parameter covariance

- Applied to a cross-sectional regression with time-series data

$$\hat{\mathbf{S}}_{NW} = T^{-1}\left(\sum_{t=1}^{T} e_t^2 \mathbf{x}_t'\mathbf{x}_t + \sum_{l=1}^{L} w_l \left(\sum_{s=l+1}^{T} e_s e_{s-l}\mathbf{x}_s'\mathbf{x}_{s-l} + \sum_{q=l+1}^{T} e_{q-l}e_q \mathbf{x}_{q-l}'\mathbf{x}_q\right)\right)$$

$$= \hat{\mathbf{\Gamma}}_0 + \sum_{l=1}^{L} w_l \left(\hat{\mathbf{\Gamma}}_l + \hat{\mathbf{\Gamma}}_l'\right)$$

- The HAC robust covariance of $\hat{\boldsymbol{\beta}}$ is

$$\hat{\mathbf{\Sigma}}_{\mathbf{XX}}^{-1}\hat{\mathbf{S}}_{NW}\hat{\mathbf{\Sigma}}_{\mathbf{XX}}^{-1}$$

Considerations when using Newey-West an estimator

- Is a Newey-West estimator needed? Complex estimators have worse finite sample performance
- It **must** be the case that $L \to \infty$ as $T \to \infty$
- Even if the scores follow a MA(1)!
- Optimal rate is $O(T^{\frac{1}{3}})$ so $L \propto T^{\frac{1}{3}}$ or $L = cT^{\frac{1}{3}}$ for some (unknown) c
- Other HAC estimators available and may work well if the scores very persistent
  - Den Haan-Levin
- Alternative is to include lagged regressand(s) in the regression

$$y_t = \mathbf{x}_t \boldsymbol{\beta} + \sum_{p=1}^{P} \phi_p y_{t-p} + \epsilon_t$$

  - Not popular when focus is on cross-section component of model