Lecture 5
Calculus (III)

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黎曼积分
Riemann Integration

- 我们前面所讲的积分，实际上是黎曼积分。现在我们要交代一个事实，即黎曼积分并不是总是存在。不存在的情况一是由函数$f(x)$的性质引起，二是由积分区间引起。

  The integrals in previous lecture are actually the traditional Riemann integration. Notably that the Riemann integration not always exists. One cause could be due to the property of the function $f(x)$, and another cause could be due to the interval to be integrated.

- 为了解决第一个问题，我们需要引入勒贝格积分。作为“实变函数”这门课程的核心，虽然勒贝格积分更加抽象，但可以很容易地推广到其它抽象的数学结构上。

  To solve the first issue, we introduce the Lebesgue integral. As the core for the subject Real Analysis, Lebesgue integrals are more abstract, however, it can be easily generalized to other abstract mathematical structures.

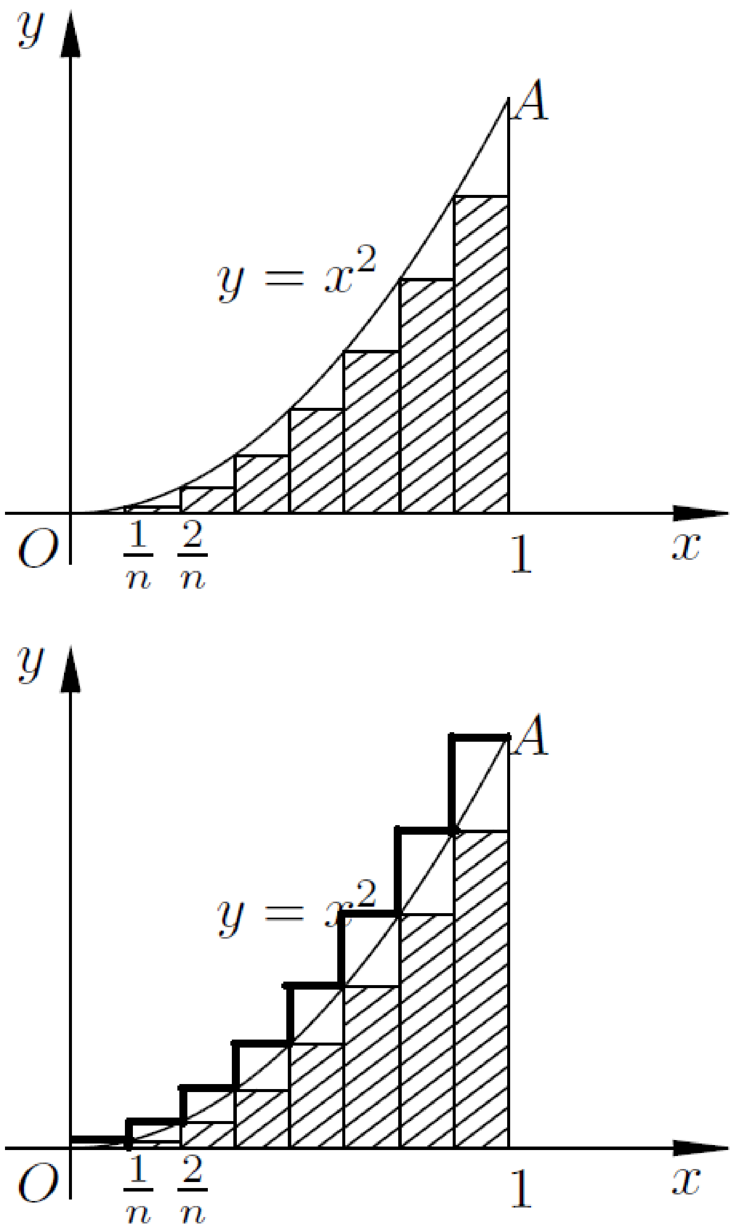
- 为了解决第二个问题，我们需要引入反常积分的概念，这个概念与我们后面要讲的傅里叶变换与Z变换有一定程度的联系。

  To solve the second problem, we introduce the concept of improper integrals, which are related to the Fourier transform and Z transform that we will talk about later.

黎曼积分
Riemann Integration

- 回忆在介绍定积分时（同时也是黎曼积分），当我们按方式T对积分区间$[a,b]$进行剖分时，考虑$[x_{i-1},x_i]$，令 $\xi_i^l=\min\{f(x)|x\in[x_{i-1},x_i]\}$ ， $\xi_i^h=\max\{f(x)|x\in[x_{i-1},x_i]\}$，令$S^l=\sum_{i=0}^n\xi_i^l(x_i-x_{i-1})$，$S^h=\sum_{i=0}^n\xi_i^h(x_i-x_{i-1})$ ， 则 当 $\lambda(T)=\max_i|x_i-x_{i-1}|\to 0$时，有$S^l=S^h\triangleq\int_a^b f(x)dx$。

  Recall that when we introduce the definite integral (also Riemann Integral), suppose we segment the interval $[a,b]$ according to T. Let $\xi_i^l=\min\{f(x)|x\in[x_{i-1},x_i]\}$, $\xi_i^h=\max\{f(x)|x\in[x_{i-1},x_i]\}$ where $[x_{i-1},x_i]$ is one segment contained in $[a,b]$. Let $\xi_i^l=\min\{f(x)|x\in[x_{i-1},x_i]\}$, $\xi_i^h=\max\{f(x)|x\in[x_{i-1},x_i]\}$, $S^l=\sum_{i=0}^n\xi_i^l(x_i-x_{i-1})$ and $S^h=\sum_{i=0}^n\xi_i^h(x_i-x_{i-1})$. When $\lambda(T)=\max_i|x_i-x_{i-1}|\to 0$, we have $S^l=S^h\triangleq\int_a^b f(x)dx$.



黎曼积分
Riemann Integration

- 考虑如下定义在区间$[0,1]$上的函数$f(x)$，实际上无论T如何选取，当$\lambda(T)=\max_i|x_i-x_{i-1}|\to 0$时，始终有$S^l\neq S^h$。则按照定积分的定义，则$\int_0^1 f(x)dx$不存在。

  Consider the function $f(x)$ defined below on interval $[0,1]$, actually, no matter how we divide the interval, when $\lambda(T)=\max_i|x_i-x_{i-1}|\to 0$, we always have $S^l\neq S^h$. Therefore, $\int_0^1 f(x)dx$ doesn't exist according to the definition.

$$f(x)=\begin{cases}0 & x\ is\ a\ rational\ number\\ 1 & x\ is\ an\ irrational\ number\end{cases}$$

- 只所以有$S^l\neq S^h$，是因为有理数在是数轴上是处处稠密的，即无论两点x_{i-1}、x_i之间距离$|x_i-x_{i-1}|$如何小，总存在着一个有理数，其导致$S^l=0$。同理，由于无理数在数轴上也是处处稠密的，则$S^h=1$，这导致了$S^l\neq S^h$。

  The reason for $S^l\neq S^h$ is because rational numbers are dense everywhere on the number line, that is, no matter how adjacent two points x_{i-1}, x_i away from each other, there is always a rational number in-between. This causes $S^l=0$. Similarly, since irrational numbers are also dense everywhere on the number line, so $S^h=1$. Put it together, it leads to $S^l\neq S^h$.

黎曼积分
Riemann Integration

- 我们重新考虑划分T：$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$。如果不考虑端点的情况，有$[x_{i-1}, x_i] \cap [x_{j-1}, x_j] = \emptyset$，$\bigcup_i [x_{i-1}, x_i] = [a, b]$。我们觉得这种不重不漏的划分很自然，特别地，对于传统的划分，对$\forall \xi_i \in (x_{i-1}, x_i)$，$\exists \delta > 0$，当$|x - \xi_i| < \delta$时，$x$仅属于$[x_{i-1}, x_i]$。

  We reconsider the division T: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Regardless of the endpoints, we have $[x_{i-1}, x_i] \cap [x_{j-1}, x_j] = \emptyset$, $\bigcup_i [x_{i-1}, x_i] = [a, b]$. We think this non-duplicate and non-overlapping division is natural, especially, for $\forall \xi_i \in (x_{i-1}, x_i)$, $\exists \delta > 0$, when $|x - \xi_i| < \delta$, x is uniquely belongs to $[x_{i-1}, x_i]$.

- 实际上，最后一个数学表达式只是使看起来直观，并非划分本身的要求。如果去掉最后一个限制，将前面的两个表述推广，实际上我们想找的是集合$[a, b]$的一个划分方式，其能划为子集$S_1, S_2, \cdots, S_n$，满足$S_i \cap S_j = \emptyset$，$\bigcup_i S_i = [a, b]$。

  In fact, the last expression is only to make the operation look intuitive, not a requirement for the division itself. If we remove the last restriction and generalize the first two statements, what we are actually looking for is a division of set $[a, b]$, which can be divided into subsets $S_1, S_2, \cdots, S_n$, satisfying $S_i \cap S_j = \emptyset$, $\bigcup_i S_i = [a, b]$.

勒贝格积分
Lebesgue Integration

- 针对上例，我们考虑将区间$[0,1]$分为有理数集合Q与无理数集合I。则有$Q \cap I = \emptyset$，$Q \cup I = [0,1]$。对于传统划分区间$[x_{i-1}, x_i]$，我们知道其长度为$|x_i - x_{i-1}|$。因此，现在我们需要一个手段，来度量集合Q与I的大小，或者说长度。

  For the above example, consider dividing the interval$[0,1]$into the set of rational numbers, denoted by Q, and the set of irrational numbers, denoted by I. Then we have $Q \cap I = \emptyset$, $Q \cup I = [0,1]$. For the traditional division segment $[x_{i-1}, x_i]$, we know that its length is $|x_i - x_{i-1}|$. Therefore, we now need a means to measure the size, or length, of the sets Q and I.

- 我们先考虑有界区间I，端点a和b（$a < b$）。这个有界区间I的长度为$\ell(I) = b - a$；对于无界区间的长度，如(a, ∞)、$(-\infty,\ b)$或$(-\infty, \infty)$，被定义为无限。显然，线段的长度很容易量化。但是，如果我们想测量$\mathbb{R}$的任意子集，我们应该怎么做呢？

  Let us consider a bounded interval I with endpoints a and b ($a < b$). The length of this bounded interval I is defined by $\ell(I) = b - a$. In contrast, the length of an unbounded interval, such as (a, ∞), $(-\infty,\ b)$ or $(-\infty, \infty)$, is defined to be infinite. Obviously, the length of a line segment is easy to quantify. However, what should we do if we want to measure an arbitrary subset of $\mathbb{R}$?

勒贝格积分
Lebesgue Integration

- 以亨利·勒贝格命名的勒贝格测度是帮助我们解决上述问题的方法之一。给定一个实数的集合E，我们用$\mu(E)$表示集合E的勒贝格测度。为了与线段的长度相对应，一个集合A的测度应保持以下属性：

  The Lebesgue measure, named after Henri Lebesgue, is one of the approaches that helps us to investigate the problem above. Given a set E of real numbers, we denote the Lebesgue measure of set E by $\mu(E)$: To correspond with the length of a line segment, the measure of a set A should keep the following properties:

  (1) If A is an interval, then $\mu(E) = \ell(I)$.

  (2) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

  (3) Given $A \subseteq \mathbb{R}$ and$x_0 \in \mathbb{R}$, define $A + x_0 = \{x + x_0 : x \in A\}$, then $\mu(A) = \mu(A + x_0)$.

  (4) If A and B are disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}$ is a sequence of disjoint sets, then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.

勒贝格积分
Lebesgue Integration

- 我们接下来定义勒贝格外测度的概念，以此来推导勒贝格测度的一些性质：设E是$\mathbb{R}$的子集，$\{I_0\}$是开区间序列。则E的勒贝格外测度由下式定义，

  Now define the concept of Lebesgue outer measure to help deduce some properties of the Lebesgue measure. Let E be a subset of $\mathbb{R}$. Let $\{I_k\}$ be a sequence of open intervals. The Lebesgue outer measure of E is defined by:

$$\mu^*(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$$

- 我们不再证明勒贝格外测度具有如下性质：

  We will enumerate the properties of Lebesgue outer measure with proof:

  a) If $E_1 \subseteq E_1$, then $\mu^*(E_1) \leq \mu^*(E_2)$

  b) The Lebesgue outer measure of any countable set is zero.

  c) The Lebesgue outer measure of the empty set is zero.

  d) Lebesgue outer measure is invariant under translation, that is, $\mu^*(E + x_0) = \mu^*(E)$:

勒贝格积分
Lebesgue Integration

e) Lebesgue outer measure is countably sub-additive, that is, $\mu^*(\bigcup_{i=1}^{\infty} E_k) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$

f) For any interval I, $\mu^*(I) = \ell(I)$

- 注意，一个集合是可数集，是指存在集合元素到自然数集或自然数的一个子集的一一映射。可数集往往是具有无限多个元素，但具有无限多个元素的不一定是自然数集。例如特定区间的无理数集是一个无限集合，但由于不存在到自然数集的一一映射，因此是不可数集。

  Note that a set is a countable if there exists a one-to-one mapping between its elements and natural numbers, or a subset of natural numbers. Countable sets tend to have an infinite number of elements, but sets with an infinite number of elements may not be countable. For example, the set of irrational numbers in an interval is an infinite set, but it is uncountable because there is no one-to-one mapping to the set of natural numbers.

- 现在我们说明$[0,1]$之间的无理数集合的勒贝格外测度为0。

  We now show that he outer measure of the set of irrational numbers in the interval $[0,1]$ is 0.

  Let A be the set of irrational numbers in $[0,1]$. Since $A \subseteq [0,1]$, then $\mu^*(A) \leq 0$.

  Let Q be the set of rational numbers in $[0,1]$. Note that $[0,1] = A \cup Q$, by the above property (e) and (f), we can conclude that $1 \leq \mu^*(A) + \mu^*(Q)$.

  Since Q is countable, then by the property (b), $\mu^*(Q) = 0$, Therefore, $\mu^*(A) = 1$.

勒贝格积分
Lebesgue Integration

- 现在我们来定义测度的概念。集合$E\subseteq\mathbb{R}$是勒贝格可测的，当对$\forall A\subseteq\mathbb{R}$，有$\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$。如果$E$是勒贝格可测集，则$E$的勒贝格测度为其勒贝格外测度，记为$\mu(E)$。由于勒贝格外测度满足次可加性，即总有$\mu^*(A)\leq\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$，因此，勒贝格测度额外要求逆不等式成立，即$\mu^*(A)\geq\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$。

  A set $E\subseteq\mathbb{R}$ is Lebesgue measurable if for each set $A\subseteq\mathbb{R}$, the equality $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$ is satisfied. If E is a Lebesgue measurable set, then the Lebesgue measure of E is its Lebesgue outer measure and will be written as $\mu(E)$. Since the Lebesgue outer measure satisfies the property of subadditivity, then we always have $\mu^*(A)\leq\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$. Hence, the Lebesgue measure requires that the reverse inequality holds, namely, $\mu^*(A)\geq\mu^*(A\cap E)+\mu^*(A\cap\bar{E})$.

- 粗略地说，一把尺子是可度量的，则一定能去度量别的东西。勒贝格测度的定义中，当用E作掩码去剪切集合A时，E一定不能有奇怪的性质，导致出现集合度量的差异。同时，这意味如果A有不好的性质，经过E的剪切，至少在E的部分，有着良好的性质。

  Roughly speaking, if a ruler is measurable, it can be used to measure other things. When E is Lebesgue measurable and used as a "mask" to "clip" a set A, E itself must not have some curious properties which causes a discrepancy in the measure. Meantime, when A has some improper properties, clipped by E will introduce the well-defined property to the masked set.

勒贝格积分
Lebesgue Integration

- 我们将不加证明地列举Lebesgue测度的如下性质。定义在ℝ上的勒贝格可测集的集合具有以下性质：

  We will list the properties of Lebesgue measure without proof. The collection of measurable sets defined on ℝ has the following properties:

  a) Both ∅ and ℝ are measurable.

  b) If E is measurable, then $\bar{E}$ is measurable.

  c) If $\mu^*(E) = 0$, then E is measurable.

  d) If E_1 and E_2 are measurable, then $E_1 \cup E_2$and $E_1 \cap E_2$ are measurable.

  e) If E is measurable, then $E + x_0$ is measurable.

- 同理，我们可以讨论内测度的概念。设E是ℝ的子集，则E的内测度定义为：

  We can also discuss the concept of inner measure. Let E be a subset of ℝ, the inner measure of E is defined by:

$$\mu_*(E) = \sup\left\{\sum_{k=1}^{\infty} \ell(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$$

勒贝格积分

Lebesgue Integration

- 定理：设A和E是$\mathbb{R}$的子集：
  1. 假设$\mu^*(E) < \infty$，那么当且仅当$\mu_*(E) = \mu^*(E)$时，E是可测的：
  2. 如果E是可测的且$A \subseteq E$；那么$\mu(E) = \mu_*(A) + \mu^*(E - A)$.

  Let A and E be subsets of $\mathbb{R}$:
  1. Suppose that $\mu^*(E) < \infty$. Then E is measurable if and only if $\mu_*(E) = \mu^*(E)$:
  2. If E is measurable and $A \subseteq E$; then $\mu(E) = \mu_*(A) + \mu^*(E - A)$.
- 下面，我们证明区间$[0,1]$内无理数集合I的测度为1。注意，我们已经证明有理数集合Q，$\mu^*(Q) = 0$，$\mu^*(I) = 1$。根据上面定理：$\mu([0,1]) = \mu_*(Q) + \mu^*([0,1] - Q)$。注意，$\mu([0,1]) = 1$，$\mu^*([0,1] - Q) = \mu^*(I) = 1$，则$\mu_*(Q) = 0$。即$\mu_*(Q) = \mu^*(Q) = 0$，所以$\mu(Q) = 0$。

  Now we prove that the set of irrational numbers in the interval $[0,1]$ is with the Lebesgue measure 0. Note we have proved that the Lebesgue outer measure that $\mu^*(Q) = 0$, $\mu^*(I) = 1$. According to the theorem above, we have $\mu([0,1]) = \mu_*(Q) + \mu^*([0,1] - Q)$. Note$\mu([0,1]) = 1$, $\mu^*([0,1] - Q) = \mu^*(I) = 1$, then $\mu_*(Q) = 0$. That is $\mu_*(Q) = \mu^*(Q) = 0$, which means$\mu(Q) = 0$.

勒贝格积分
Lebesgue Integration

- 我们先回忆一下黎曼积分的构造步骤：
  - 将函数的域（通常是封闭的有界区间）细分为无限多的子区间（分割）
  - 构造一个阶梯函数，该函数在分区的每个子区间（上限和下限总和）上取常量值
  - 当分割的点越来越密集时时，取这些阶梯函数的和的极限

- We first recall the construction of the Riemann integral in the following steps:
  - Subdivide the domain of the function (usually a closed, bounded interval) into infinitely many subintervals (the partition).
  - Construct a step function that has a constant value on each of the subintervals of the partition (the Upper and Lower sums).
  - Take the limit of the sum of these step functions as you add more and more points to the partition.

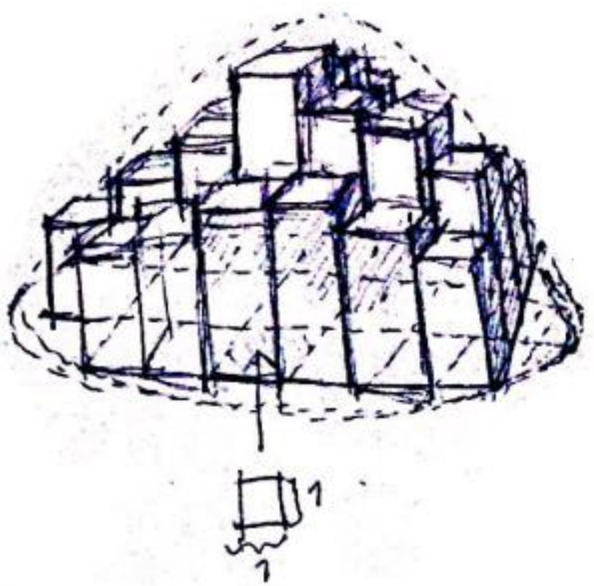
勒贝格积分
Lebesgue Integration

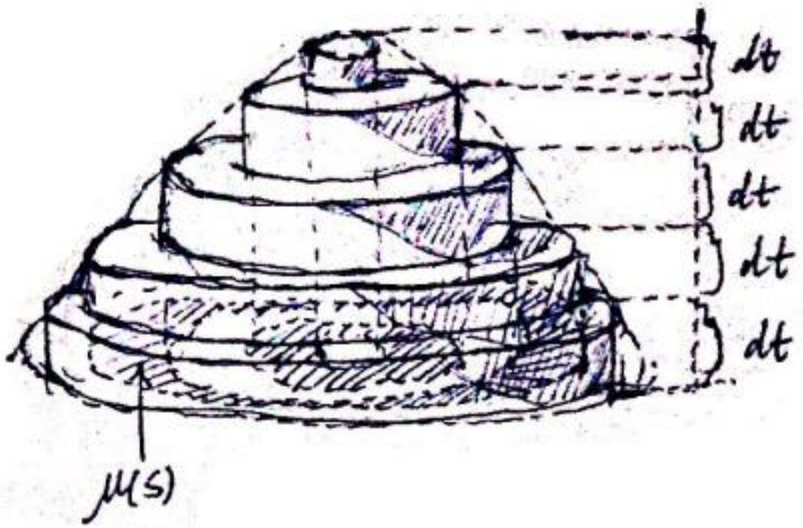
- 与之相对的方法，称为勒贝格积分，即按以下步骤操作：
  - 将函数的取值范围细分为无限多个区间$[f_{i-1}, f_i]$
  - 构造定义域的子集合D_i，其元素的对应的函数值是特定的划分区间$[f_{i-1}, f_i]$内的点
  - 当对函数的范围内划分越来越细的时候，取这些简单函数$\xi_i \cdot m(D_i)$的和的极限，其中$\xi_i \in [f_{i-1}, f_i]$。

- The "opposite" approach, named the Lebesgue integral, will be operated in the following steps:
  - Subdivide the range of the function into infinitely many intervals $[f_{i-1}, f_i]$
  - Construct subset D_is with elements from the domain of the function. The elements in D_i are with the corresponding values in $[f_{i-1}, f_i]$.
  - Take the limit of sum of these simple functions $\xi_i \cdot m(D_i)$, where $\xi_i \in [f_{i-1}, f_i]$, as you add more and more points into the division of the range of the original function.

勒贝格积分
Lebesgue Integration

- 勒贝格积分可形式上记为$\int_{\mathbb{R}} \phi dm = \sum_{i=1}^{n} a_i m(S_i)$，其直观意义如下图所示：
- The Lebesgue integral can be formally denoted as $\int_{\mathbb{R}} \phi dm = \sum_{i=1}^{n} a_i m(S_i)$, an intuitive interpretation to it is as follows:





反常积分
Improper Integration

- 现在，我们来讨论黎曼积分的第二个问题，即积分区间含有无穷上限或下限的情况，这个时候，我们称积分为反常积分，有时又称为无穷限广义积分。当然，反常积分还包括被积函数含有瑕点的情况，此时又称瑕积分或无界函数的反常积分。

  Now, let's discuss the second problem of the Riemann integral, that is, the case where the integral interval contains an infinite positive or negative limit or both. At this time, we call the integral an improper integral, and sometimes it is also called improper integral of unbounded intervals. Another type of improper integral refers to the case that the interval contains point that the integrand is not well-defined. Now it is also called the improper integral of unbounded function.

- 注意，这里说的瑕点与勒贝格积分的情况不同，这里的瑕点往往是有限个瑕点，并且通常表现为被积函数在这些瑕点上无界。

  Note that the points where the function is not well-defined are different from the Lebesgue integral case. These points are often of a finite number, and it usually appears that the integrand is unbounded on these points.

反常积分
Improper Integration

- 对于积分区间含有无穷上限或下限的情况，如$\int_0^{+\infty} f(x)dx$，定义$a_n = \int_0^n f(x)dx$。考察序列$\{a_n\}$，若$n \to \infty$时，a_n趋近于定值a，则显然$\int_0^{+\infty} f(x)dx$可积，且$\int_0^{+\infty} f(x)dx = a$。实际上，具体计算时，我们也是求出$\int_0^n f(x)dx$的表达式，再令$n \to \infty$，作为$\int_0^{+\infty} f(x)dx$的结果。

  For the interval to be integrated contains infinite positive or negative limit, for example, $\int_0^{+\infty} f(x)dx$, define $a_n = \int_0^n f(x)dx$ and consider the sequence $\{a_n\}$. If $a_n \to a$ when $n \to \infty$, then $\int_0^{+\infty} f(x)dx$ is integrable and $\int_0^{+\infty} f(x)dx = a$. Actually, to compute $\int_0^{+\infty} f(x)dx$, we first calculate $\int_0^n f(x)dx$, then let $n \to \infty$, the final result will be treated as $\int_0^{+\infty} f(x)dx$.

- 另外，记$I_n = \int_n^{n+1} f(x)dx$，考察序列$\{I_n\}$，令$S_n = \sum_{k=0}^n I_k$表示序列的部分和。当$f(x)$满足特定条件时，若当$n \to \infty$时，$S_n$存在且有限，则我们认为$\int_0^{+\infty} f(x)dx$存在且有限，且$\int_0^{+\infty} f(x)dx = \lim_{n\to\infty} S_n$。

  In addition, let $I_n = \int_n^{n+1} f(x)dx$ and consider the sequence $\{I_n\}$. Let $S_n = \sum_{k=0}^n I_k$ denote the partial sum of the $\{I_n\}$. When $f(x)$ satisfies certain conditions and S_n exists and is bounded when $n \to \infty$, we assert that $\int_0^{+\infty} f(x)dx$ exists and $\int_0^{+\infty} f(x)dx = \lim_{n\to\infty} S_n$.

函数逼近
Function Approximation

- 上述两种方式对反常积分而言，都能得到同样的结果，但第二种方式其实包含着一种更一般的思想。实际上，对$\forall\varepsilon > 0$，$\exists N \in \mathbb{N}$，当$n > N$时，有$\left|\int_0^{+\infty} f(x)dx - S_n\right| < \varepsilon$。即部分和$S_n$可以以任意精度逼近$\int_0^{+\infty} f(x)dx$。由于定积分并不总是能精确求解，在某些情况下我们可能采取构造一序列，用其部分和作为积分的数值逼近。

  Although the above two approaches lead to the same result, however, the way indicates a more general idea. Actually, for $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$, when $n > N$, we have $\left|\int_0^{+\infty} f(x)dx - S_n\right| < \varepsilon$. That is the partial sum can approximate the $\int_0^{+\infty} f(x)dx$ in arbitrary precision. Since it is not always possible to solve definite integral in an analytic way. In some situation we have to construct a sequence with the partial sum approximating the integral in arbitrary precision as the numerical approximation.

- 推而广之，若对任一定义在区间I上的函数$f(x)$，考查序列$\{e_n(x)\}$。记$S_n(x) = \sum_{i=0}^{n} e_n(x)$，若对$\forall\varepsilon > 0$，$\exists N \in \mathbb{N}$，当$n > N$时，对$\forall x \in I$，均有$\left|\int_0^{+\infty} f(x)dx - S_n(x)\right| < \varepsilon$，则我们称$S_n(x)$为$f(x)$的逼近。

  Generalize the above approach, for any We know that it is not always possible to solve definite integral in an analytic way. In some situation we have to turn to numerical approximation. Suppose we can construct a sequence with the partial sum approximating the integral in arbitrary precision, then we can take the partial sum as the substitution of the integral.

函数逼近

Function Approximation

- 实际上，函数逼近的情况，在一定程度上可视为向量空间的推广。一般情况上，向量空间$\mathcal{L}$的基为有限个数的，即$\dim(\mathcal{L}) = n < \infty$，基表示为$\{e_i\}_{i=1}^{n}$。通常，这$n$个基地位均等，共同撑起了向量空间$\mathcal{L}$。当把基推广到无限维空间，即基的个数是可数个时，基的地位将不再均等。即考虑连续函数空间$\mathcal{C}[a,b]$，对$\forall f \in \mathcal{C}[a,b]$与基函数集合$\{e_n(x)\}_{n=1}^{\infty}$，理论上有$f = \sum_{n=1}^{\infty} a_n e_n(x)$，但实际上，对于高度非线性的函数$f$，当用基函数逼近时，不可能用到无穷多个。因此，我们会保留基函数中，对逼近影响精度大的。因此，基于这种想法与实践，我们引入一个重要的内容，即傅里叶分析。

  Actually, the function approximated can be regarded as the generalization of vector space. Generally, a vector space $\mathcal{L}$ is of finite number of bases $\{e_i\}_{i=1}^{n}$, indicating $\dim(\mathcal{L}) = n < \infty$. In addition, these bases are usually of equal importance, all of which span the vector space. However, when generalize these concept into infinite dimensional space, things might be different. Consider the space $\mathcal{C}[a,b]$ constituted by continuous functions defined on $[a,b]$, for $\forall f \in \mathcal{C}[a,b]$ and bases function collection $\{e_n(x)\}_{n=1}^{\infty}$, in theory we have $f = \sum_{n=1}^{\infty} a_n e_n(x)$. But for high-linear function f, if we have no choice but have to resort to approximation, usually we can not do it by using infinite number of base functions. In this case, we will retain the base functions which are critical to the precision of the approximation and drop the insignificant ones. Such ideas and practice lead to another important content in mathematics, namely, Fourier analysis.

函数逼近
Function Approximation

- 至于在有限维度的时候，基的地位是否均等，至少在线性代数里是这样的。但在物理的弦论中，认为现实的空间由11维构成，有的维度蜷曲起来，不被我们觉察。因此从这个角度，即使对有限空间，基的地位也可能不是均等的，当然，对这个的深度讨论不在我们这门课程之列。

  As for the case in finite dimensions, the basis rendering equal important is at least true in linear algebra. However, in the physics of string theory, it is believed that reality is composed of 11 dimensions, some of which are curled up and not perceived by us. Therefore, from this perspective, even in finite space, the important of the basis may not be equal. Of course, a deep discussion of this is not within the scope of our course.

- 傅里叶分析粗略地讲，就是属于特定函数空间，如$C[a,b]$的任何函数（包括常函数）可以表示为正弦函数与余弦函数的加权和。这也就是说，$\left\{\sin\frac{2\pi}{b-a}nx\right\}_{n=1}^{\infty}\cup\left\{\cos\frac{2\pi}{b-a}nx\right\}_{n=1}^{\infty}$构成了函数空间的一组基。至于为什么是这样的情况，也许更深层次的原因是波粒二象性。当然，对这个的深度讨论不在我们这门课程之列。

  Roughly speaking, Fourier analysis means that any function (including a constant function) can be represented as a weighted sum of sine and cosine functions. This means that $\left\{\sin\frac{2\pi}{b-a}nx\right\}_{n=1}^{\infty}\cup\left\{\cos\frac{2\pi}{b-a}nx\right\}_{n=1}^{\infty}$ form a basis for the function space. As for why this is the case, perhaps the deeper reason lies in the wave-particle duality. Of course, a deep discussion of this is not within the scope of our course.

傅里叶分析
Fourier Analysis

- 从概念上讲，傅立叶分析是研究如何将一般函数分解为具有确定频率的三角函数或指数函数。根据函数的特点，有两种类型的傅立叶展开：
  - 傅立叶级数：如果一个（表现良好的）函数是周期性的，那么它可以写成具有特定频率的三角函数或指数函数的离散和。
  - 傅里叶变换：不一定是周期性的一般函数（但仍然表现得相当好）可以写成三角函数或指数函数的连续积分，具有连续的可能频率。
- From the mathematical concept, Fourier analysis is the study of how general functions can be decomposed into trigonometric or exponential functions with definite frequencies. According to the characteristics of the underlying function, there are two types of Fourier expansions:
  - Fourier series: If a (reasonably well-behaved) function is periodic, then it can be written as a discrete sum of trigonometric or exponential functions with specific frequencies.
  - Fourier transform: A general function that isn’t necessarily periodic (but that is still reasonably well-behaved) can be written as a continuous integral of trigonometric or exponential functions with a continuum of possible frequencies.

傅里叶级数
Fourier Series

- 对于定义良好的以周期为T的函数$f(x)$而言，其可以表示为正弦和余弦的级数展开，即：

  For an well-defined function $f(x)$ with T, namely, $f(x) = f(x + nT)$, $n \in \mathbb{Z}$, it may be represented as a series expansion of sines and cosines, that is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi}{T}nx + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi}{T}nx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2\pi}{T}nx + b_n \sin\frac{2\pi}{T}nx\right)$$

- 在上面的傅里叶展式，相关系数的求法如下所示：

  In the Fourier expansion above, the way to find the coefficients are as follows:

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\frac{2\pi}{T}nx\,dx$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\frac{2\pi}{T}nx\,dx$$

傅里叶级数
Fourier Series

- 我们之所以说傅里叶分析，特别是傅里叶级数在一定程度上是向量空间的推广，是因为这些基函数满足正交性。令$\omega=\frac{2\pi}{T}$，有：

  The reason why we say Fourier analysis, especially Fourier series, is a generalization of vector space to some extent, is because these basis functions satisfy orthogonality. Let$\omega=\frac{2\pi}{T}$, then we have:

$$\frac{1}{T}\int_{-T/2}^{T/2}\cos m\omega x\cos n\omega x\,dx=\frac{1}{T}\int_{-T/2}^{+T/2}\cos(m+n)\omega x\,dx+\frac{1}{T}\int_{-T/2}^{+T/2}\cos(m-n)\omega x\,dx=$$

$$=\frac{1}{T}\frac{1}{m+n}\cos(m+n)\omega x\Big|_{-\frac{T}{2}}^{+\frac{T}{2}}+\frac{1}{T}\frac{1}{m-n}\cos(m-n)\omega x\Big|_{-\frac{T}{2}}^{+\frac{T}{2}}=\begin{cases}0 & m\neq n\\ 1/2 & m=n>0\\ 1 & m=n=0\end{cases}$$

$$\frac{1}{T}\int_{-T/2}^{T/2}\sin m\omega x\sin n\omega x\,dx=\begin{cases}0 & m\neq n\\ 1/2 & m=n>0\\ 1 & m=n=0\end{cases}$$

$$\frac{1}{T}\int_{-T/2}^{T/2}\cos m\omega x\sin n\omega x\,dx=0$$

傅里叶级数
Fourier Series

- 例：求周期为T的方波的傅里叶级数：

  Ex: find the Fourier series for the square function of period T:

$$f(t)=\begin{cases}-1 & -\frac{T}{2}\le t<0\\ 1 & 0\le t<\frac{T}{2}\end{cases}$$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos n\omega t\,dt=\frac{2}{T}\left[\int_{-T/2}^{0}-1\cdot\cos n\omega t\,dt+\int_{0}^{T/2}\cos n\omega t\,dt\right]=0$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin n\omega t\,dt=\frac{2}{T}\left[\int_{-T/2}^{0}-1\cdot\sin n\omega t\,dt+\int_{0}^{T/2}\sin n\omega t\,dt\right]=\frac{4}{T}\int_{0}^{T/2}\sin n\omega t\,dt$$

$$=\frac{4}{T}\frac{1}{n\omega}(-\cos n\omega t)\Big|_0^{T/2}=\frac{2}{n\pi}[-\cos n\pi+1]=\frac{2}{n\pi}[(-1)^{n+1}+1]$$

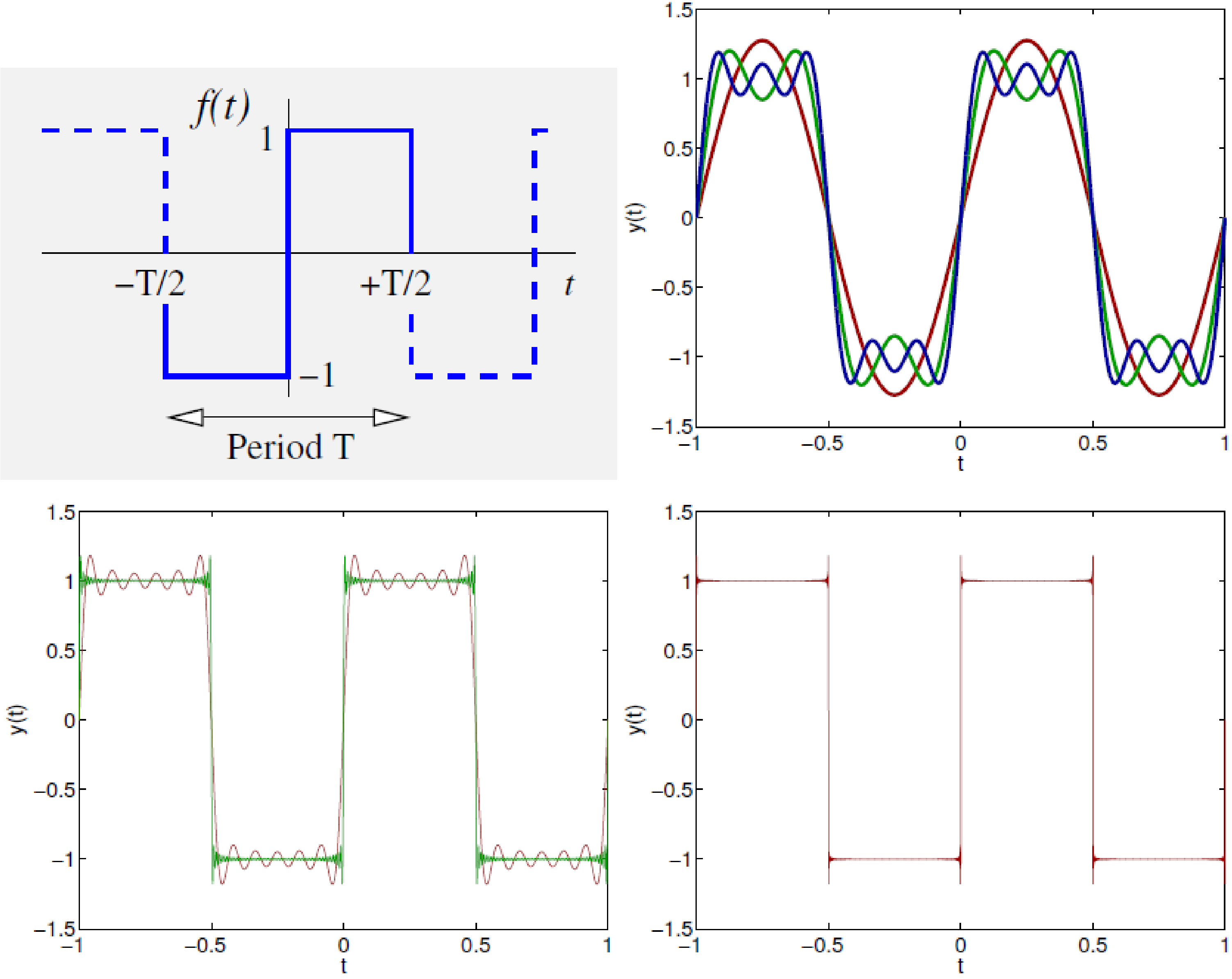
因此，我们有：

$$f(t)=\frac{4}{\pi}\left[\sin\omega t+\frac{1}{3}\sin 3\omega t+\frac{1}{5}\sin 5\omega t+\frac{1}{7}\sin 7\omega t\cdots\right]$$

傅里叶级数
Fourier Series

- 右边为逐渐增加级数项数之后的逼近结果。注意到在不连续点会有一个尖峰，这叫做吉布斯效应，但这里不作深入讲述。

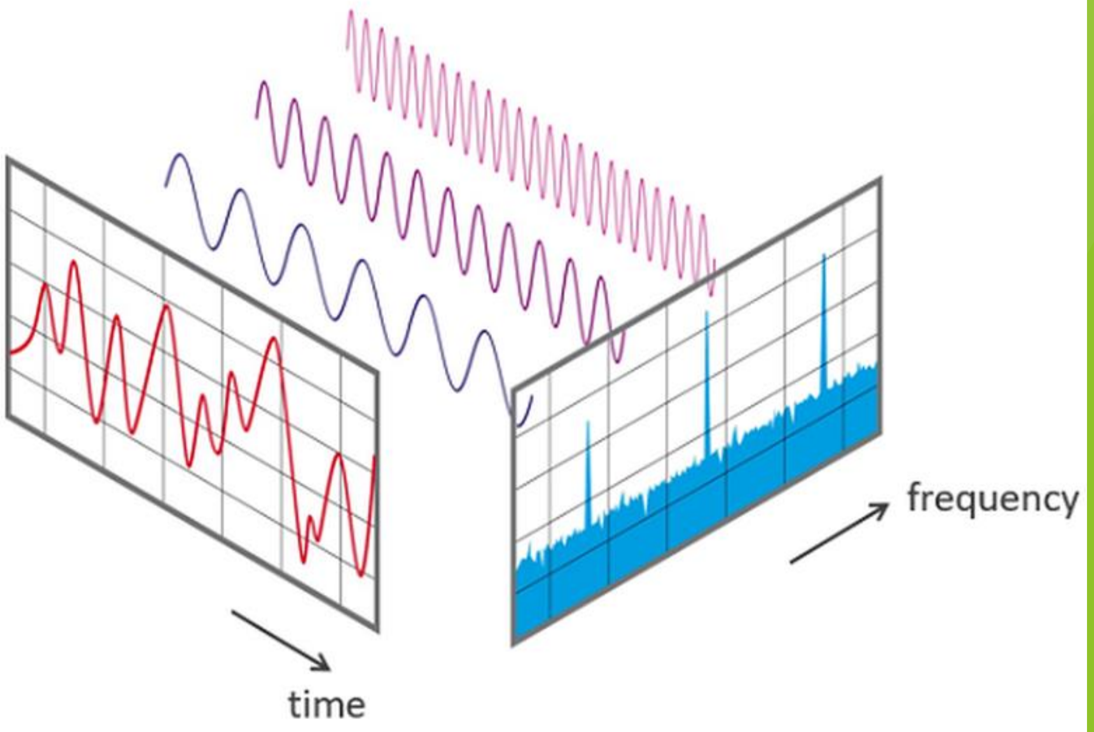
  The right figure shows the approximation of Fourier series with different number of items. It is obvious the more the better. The last sub-figure demonstrates the slight “chip” at the discontinuity is a result of the Gibbs’ phenomenon, which is out of discussion in this lecture.



傅里叶分析
Fourier Analysis

- 实际上，在实际中许多对物理过程的描述是在时域上的，例如运动、震动，如果我们$f(t)$来表示并称为信号的话，我们希望知道$f(t)$的性质。我们当然可以根据函数的概念，来谈$f(t)$的性质。但傅里叶变换实际上提供了另外一种工具，通过傅里叶级数或傅里叶变换，我们可以从另外一个空间，即频域空间，来分析函数的性质。

  In fact, many descriptions of physical processes in practice are in the time domain, such as motion and vibration. If we use $f(t)$ to represent and call it a signal, we want to know the properties of $f(t)$. Of course, we can talk about the properties of $f(t)$ based on the concept of functions. However, Fourier transform actually provides another tool. Via Fourier series or Fourier transform, we can analyze the properties of functions from another space, namely, the frequency domain.



傅里叶分析
Fourier Analysis

- 例如，我们常说的低频信号与高频信号，口头表述上可能仅仅能说是变化的快慢，当在进行完傅里叶变换之后，其频率轴上幅值的分布，可以清楚的刻画信号的特征，尽管这个特征往往是全局特征而不是局部特征。

  For example, when we talk about low-frequency signals and high-frequency signals, we may only be able to describe them verbally as the speed of change. However, after performing Fourier transform, the distribution of amplitude on the frequency axis can clearly unveil the characteristics of the signal, although this characteristic is often a global feature rather than a local feature.