Lecture 4
Calculus (II)

明玉瑞 Yurui Ming

yrming@gmail.com

声明
Disclaimer

- 本讲义在准备过程中由于时间所限，所用材料来源并未规范标示引用来源。所引材料仅用于教学所用，作者无意侵犯原著者之知识版权，所引材料之知识版权均归原著者所有；若原著者介意之，请联系作者更正及删除。

  The time limit during the preparation of these slides incurs the situation that not all the sources of the used materials (texts or images) are properly referenced or clearly manifested. However, all materials in these slides are solely for teaching and the author is with no intention to infringe the copyright bestowed on the original authors or manufacturers. All credits go to corresponding IP holders. Please address the author for any concern for remedy including deletion.

积分
Integral

- 在上一讲中我们讲了导数的相关概念，这一章我们来讲与之相对的另一个概念，称为积分。

  We lectured the concept of derivatives in the previous lecture. In this chapter we will talk about another concept which can be treated as the inversion of derivatives, namely, integral.

- 我们从求解不规则图形的面积开始。求解形状更复杂的区域的面积的一个直观的想法，是使用小的规则形状的区域之和来逼近形状不规则的区域。

  We begin with the calculation of areas of irregular shapes. One intuitive idea to find the areas of complicated shapes is to use the sum of areas of small regular shapes to approach the areas of irregular shapes.

- 我们通过一个例子来介绍这个想法。考虑下图中的抛物线$y = x^2$。设C是x轴上的一个点，使得线段OC的长度等于1。从C绘制一条垂直线，该线在点A处与抛物线相交。我们的目的是计算一条弯曲边的三角形OAC的面积。

  We introduce this idea by an example. Consider the parabola $y = x^2$ in the following figure. Let C be a point on the x-axis such that the length of the line segment OC equals to 1. Draw a vertical line from C that intersects the parabola at point A. We are going to calculate the area of the triangle OAC with a curved side.

定积分
Definite Integral

- 我们首先考虑曲线下的长方形。当$n=k$，表示将OC分为k份，求所有这些以每一份为底，高度不超过曲线的长方形的面积和，下面展示了从$k=3$开始的情况：

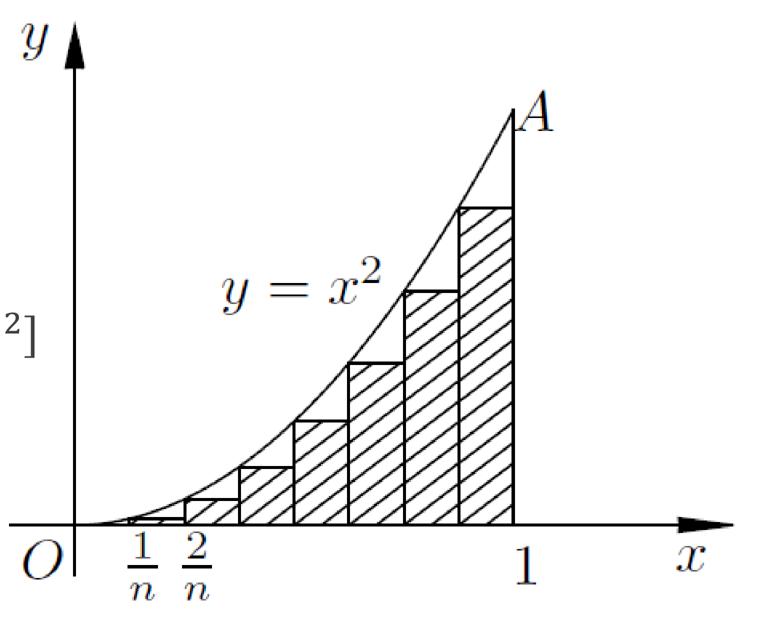
  We first consider the rectangles under the curve. $n=k$ means to segment OC into k equal sections, and sum the areas of all these rectangles which take the segments as widths and are with heights no protruding over the curve. The following demonstrates the cases begin from $k=3$.

$$S_3^l=\frac{1}{3}\left[\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]=\frac{1}{3^3}[1^2+2^2]=\frac{5}{27}$$

$$S_4^l=\frac{1}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{2}{4}\right)^2+\left(\frac{3}{4}\right)^2\right]=\frac{1}{4^3}[1^2+2^2+3^2]=\frac{7}{32}$$

$$S_n^l=\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]=\frac{1}{n^3}[1^2+2^2+\cdots+(n-1)^2]$$

$$=\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}=\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$



定积分
Definite Integral

- 我们再求所有这些以每一份为底，高度不低于曲线的长方形的面积和，下面展示了从$k=3$开始的情况：

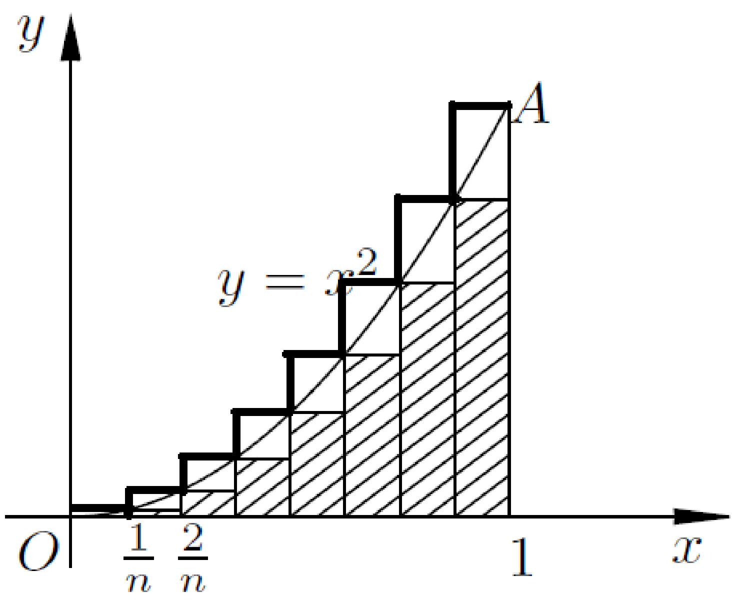
  We now sum the areas of all these rectangles which take the segments as widths and are with heights no lowering beneath the curve. The following demonstrates the cases begin from $k=3$.

$$S_3^h=\frac{1}{3}\left[\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{3}{3}\right)^2\right]=\frac{1}{3^3}[1^2+2^2+3^2]=\frac{14}{27}$$

$$S_4^h=\frac{1}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{2}{4}\right)^2+\left(\frac{3}{4}\right)^2\right]=\frac{1}{4^3}[1^2+2^2+3^2+4^2]=\frac{30}{32}$$

$$S_n^h=\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right]=\frac{1}{n^3}[1^2+2^2+\cdots+n^2]$$

$$=\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}=\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$



定积分
Definite Integral

- 当$n \to +\infty$时，$\lim_{n\to\infty} S_n^l = \lim_{n\to\infty} S_n^h = \frac{1}{3}$，根据夹逼定理，我们说积分的面积为$\frac{1}{3}$。

  When $n \to +\infty$, $\lim_{n\to\infty} S_n^l = \lim_{n\to\infty} S_n^h = \frac{1}{3}$. According to the sandwich theorem, we can assert the area under the curve, or the integral of the curve is $\frac{1}{3}$.

- 实际上，按照积分的定义，上面的划分OC的方式，需要以任意方式划分，S_n^l与S_n^h的极限存在且相等，均等于同一值时，积分才可以说存在，我们称其为函数在给定区间上的定积分。

  Actually, according to the definition of integration, the way for segmenting OC should be in arbitrary ways, all lead to the same value, then we can say the integral exists and equals to $\frac{1}{3}$. And we call it the definite integral of the function on given intervals.

- 我们考虑一般情况，假设$f(x)$在$[a,b]$上连续且大于0。将$[a,b]$按方式T划分为n个子区间（每个子区间的长度不一定相等），并将分界点集表示为：

  Without loss of generosity, assume that $f(x)$ is continuous and greater than zero on $[a,b]$. Partitioning $[a,b]$ in a specific way T into n subintervals (the length of each subinterval is not necessarily equal), and denote the set of dividing points as:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

定积分
Definite Integral

- 令ξ_i为子区间$[x_{i-1}, x_i]$中任意一点，考虑以子区间$[x_{i-1}, x_i]$为底，$f(\xi_i)$为高的矩形，其面积为$f(\xi_i)(x_i - x_{i-1})$。所有这些矩形的面积之和为：

  Let ξ_i an arbitrary point in the subinterval $[x_{i-1}, x_i]$. Consider the rectangle with the subinterval $[x_{i-1}, x_i]$ as the base and $f(\xi_i)$ as the height. The area of this rectangle is $f(\xi_i)(x_i - x_{i-1})$. The sum of the areas of all such rectangles is:

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

- 上式即区域S的近似值。当区间$[a, b]$的划分越细，此和的值与S之间的差值越小。设$\lambda = \max\limits_{1 \le i \le n}(x_i - x_{i-1})$是$(x_1 - x_0)$，…，$(x_n - x_{n-1})$中最长区间的长度，则曲线$f(x)$下区域的面积可以定义为：

  The above expression is an approximation of the area S that we want to find. The finer the partition of the interval $[a, b]$, the smaller the difference between the value of this sum and S. Let $\lambda = \max\limits_{1 \le i \le n}(x_i - x_{i-1})$ be the length of the longest interval among $(x_1 - x_0)$， …， $(x_n - x_{n-1})$, then the area of the region under the curve f(x) can be defined by:

$$S = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

定积分
Definite Integral

- 实际上，上式的S又称函数$f(x)$在区间$[a,b]$上的部分和，记为σ；同时，注意到λ的值依赖于T的选取。如果不管划分T的差异和ξ_i的选择如何，当$\lambda(T)\to 0$时，均有部分和σ接近同一值I，则我们记为$I=\int_a^b f(x)dx$，称I为$f(x)$在区间$[a,b]$上的定积分。其中$f(x)$称为被积函数，x为积分变量，a、b分别称为下限和上限。符号$\int$是积分符号，可以想象成一个拉长的表示求和的字母S。

  Actually, S is also called the partial sum of the function $f(x)$ on the interval $[a,b]$, denoted as σ. Meanwhile, note that the value of λ depends on the partition T. If the partial sum σ approaches a value I as$\lambda(T)$ approaches zero regardless of the difference in partition and the choices of ξ_is, we denote it as $I=\int_a^b f(x)dx$ and call it the definite integral of $f(x)$ on the interval $[a,b]$. The function $f(x)$ is called the integrand, x is the integral variable and a, b are called the lower limit and the upper limit respectively. The symbol $\int$ is the integral sign which can be thought of as an elongated letter S standing for sum.

- 函数$f(x)$在区间$[a,b]$上的定积分是函数和区间的全局属性，它的值取决于函数在区间的每个部分的值，与积分变量的名称无关，即$\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(\xi)d\xi=\cdots$。

  The definite integral of a function $f(x)$ on an interval $[a,b]$ is a global property of the function and the interval. Its value depends on the function’s values on every part of the interval and has nothing to do with the name of the integration variable, i.e., $\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(\xi)d\xi=\cdots$

定积分
Definite Integral

- 因此，要找到以曲线$y=f(x)$和直线$x=a$、$x=b$、$y=0$为界的区域的面积，就是计算定积分$\int_a^b f(x)dx$。

  Thus, to find the area of a region bounded by the curve $y=f(x)$ and straight lines $x=a$, $x=b$, $y=0$, is to calculate the definite integral $\int_a^b f(x)dx$.

- 要找到从时间t_0到时间T沿直线移动的物体的距离（位移），需要计算定积分$\int_{t_0}^T v(t)dt$，其中$v(t)$是时间t的速度函数。

  To find the distance (displacement) of an object moving along a straight line from time t_0 to time T is to evaluate the definite integral $\int_{t_0}^T v(t)dt$ where $v(t)$ is the velocity function of time t.

- 求一个力在从点s_0到点s的直线上移动的物体上做的功是计算定积分$\int_{s_0}^s F(s)ds$，其中$F(s)$是距离s的力的函数。

  To find the work of a force on an object moving along a straight line from point s_0 to point s is to evaluate the definite integral $\int_{s_0}^s F(s)ds$, where $F(s)$ is the force function of distance s.

定积分
Definite Integral

- 在定积分$\int_a^b f(x)dx$的定义中，我们假设$a < b$。如果$a > b$，我们定义$\int_a^b f(x)dx = -\int_b^a f(x)dx$。特别地，当$a = b$时，$\int_a^b f(x)dx = 0$。

  In the definition of the definite integral $\int_a^b f(x)dx$, we assumed that $a < b$. However if $a > b$, we define that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. When $a = b$, $\int_a^b f(x)dx = 0$.
- 以下性质可以直接从定积分的定义中推导出来。

  The following properties can be derived directly from the definition of definite integrals.

$$\int_a^b \big(f(x) + g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx, \qquad c \in \mathbb{R}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad a, b, c \in \mathbb{R}$$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx, \qquad f(x) \leq g(x) \wedge x \in [a, b]$$

定积分
Definite Integral

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

- 上面式子推导如下（The above inequality can be deduced below）：

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\Rightarrow -\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

- 对于在区间$[a,b]$上定义的任何连续函数$f(x)$，存在一个点$\xi \in [a,b]$，使得如下式子成立：

  For any continuous function $f(x)$ defined on an interval $[a,b]$, there exists a point $\xi \in [a,b]$, such that:

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

定积分
Definite Integral

- 事实上，如果令m与M是上$f(x)$在$[a,b]$上的最小值和最大值，则下式成立：

  In fact, if m and M are the minimum and the maximum values of $f(x)$ on $[a,b]$, then we have:

  $$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

  or

  $$m \leq \frac{\int_a^b f(x)dx}{(b-a)} \leq M$$

  Let $\mu = \frac{\int_a^b f(x)dx}{(b-a)}$, then $m \leq \mu \leq M$. By the continuity of the function, there exists a point ξ in $[a,b]$, such that $f(\xi) = \mu$. Hence

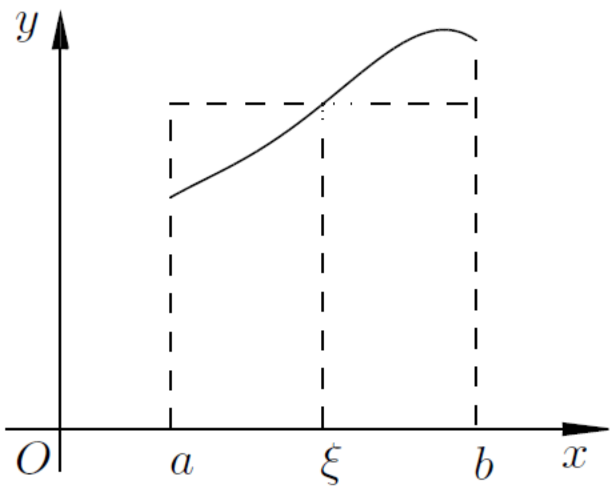
  $$\frac{\int_a^b f(x)dx}{(b-a)} = f(\xi)$$

  or

  $$\int_a^b f(x)dx = f(\xi)(b-a)$$

- 上式称为积分的均值定理。这个定理的几何意义如下图所示：

  The above formulas is called the Mean-Value Theorem for Integrals. The geometric meaning of this theorem is as follows:



微积分基本定理
The Fundamental Theorem of Calculus

- 为了进行定积分的计算，我们需要介绍微积分的两个基本定理。

  To calculate the integrals, we introduce the fundamental theorem of calculus.

- 定理1（微积分基本定理--第一部分）如果$f(x)$在$[a,b]$上连续，则如下定义的函数$F(x)$：$F(x)=\int_a^x f(t)dt, a\le x\le b$在$[a,b]$上连续，在$(a,b)$上可微分，且$F'(x)=f(x)$。$F(x)$称为$\int_a^x f(t)dt$的一个反导数。

  Theorem 1 (Fundamental Theorem of Calculus - Part I). If $f(x)$ is continuous on $[a,b]$, then the function $F(x)$ defined by: $F(x)=\int_a^x f(t)dt, a\le x\le b$ is continuous on $[a,b]$, differentiable on (a,b) and $F'(x)=f(x)$. $F(x)$ is called a anti-derivative of $\int_a^b f(x)dx$.

- 定理2（微积分基本定理--第二部分）如果$f(x)$在$[a,b]$上连续，则：$\int_a^b f(x)dx=F(b)-F(a)$。其中$F(x)$是$f(x)$的任一反导数。

  Theorem 2 (Fundamental Theorem of Calculus - Part II). If $f(x)$ is continuous on $[a,b]$, then: $\int_a^b f(x)dx=F(b)-F(a)$ where $F(x)$ is any antiderivative of $f(x)$.

微积分基本定理
The Fundamental Theorem of Calculus

- PROOF OF FTC - PART I

Let $x \in [a, b]$, let $\epsilon > 0$. Because $f(x)$ is continuous at x, there exists $\delta > 0$ satisfying $|f(t) - f(x)| < \epsilon$ when $|t - x| < \delta$. Let Δx be such that $x + \Delta x < b$ and $0 < \Delta x < \delta$. Then:

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} = \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}$$

In particular, if $t \in [x, x + \Delta x]$, we have $x \leq t \leq x + \Delta x$, so $0 \leq t - x \leq \Delta x < \delta$, and so in particular $|t - x| < \delta$, and so we get $|f(t) - f(x)| < \epsilon$.

This implies that $-\epsilon < f(t) - f(x) < \epsilon$, so $f(x) - \epsilon < f(t) < f(x) + \epsilon$. Integrating this over $[x, x + \Delta x]$, and using our comparison inequalities, we get:

$$f(x) - \epsilon < f(t) < f(x) + \epsilon$$

$$\int_x^{x+\Delta x} (f(x) - \epsilon)dt < \int_x^{x+\Delta x} f(t)dt < \int_x^{x+\Delta x} (f(x) + \epsilon)dt$$

Because $f(x) - \epsilon$ and $f(x) + \epsilon$ are constants with respect to t, we have:

$$(f(x) - \epsilon)\int_x^{x+\Delta x} dt < \int_x^{x+\Delta x} f(t)dt < (f(x) + \epsilon)\int_x^{x+\Delta x} dt$$

微积分基本定理
The Fundamental Theorem of Calculus

$$(f(x)-\epsilon)(x+\Delta x-x)<\int_x^{x+\Delta x} f(t)dt<(f(x)+\epsilon)(x+\Delta x-x)$$

$$(f(x)-\epsilon)\Delta x<\int_x^{x+\Delta x} f(t)dt<(f(x)+\epsilon)\Delta x$$

$$f(x)-\epsilon<\frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}<f(x)+\epsilon$$

$$f(x)-\epsilon<\frac{F(x+\Delta x)-F(x)}{\Delta x}<f(x)+\epsilon$$

$$\left|\frac{F(x+\Delta x)-F(x)}{\Delta x}-f(x)\right|<\epsilon$$

And so we've shown that:

$$\lim_{\Delta x\to 0^+}\frac{F(x+\Delta x)-F(x)}{\Delta x}=f(x)$$

Similarly, one can show that:

$$\lim_{\Delta x\to 0^-}\frac{F(x+\Delta x)-F(x)}{\Delta x}=f(x)$$

微积分基本定理
The Fundamental Theorem of Calculus

And hence, we get:

$$\lim_{\Delta x \to 0^-} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$$

By definition of a derivative, we have:

$$\lim_{\Delta x \to 0^-} \frac{F(x+\Delta x) - F(x)}{\Delta x} = F'(x)$$

So we finally have $F'(x) = f(x)$.

- PROOF OF FTC - PART II

Let F be an antiderivative of f, as in the statement of the theorem. Now define a new function g as follows:

$$G(x) = \int_a^x f(t)dt$$

By FTC Part I, G is continuous on $[a, b]$ and differentiable on (a, b) and $G'(x) = f(x)$ for every x in (a, b).

Now define another new function h as follows:

$$h(x) = G(x) - F(x)$$

微积分基本定理
The Fundamental Theorem of Calculus

Then $h(x)$ is continuous on $[a,b]$ and differentiable on (a,b) as a difference of two functions with those two properties. Moreover, if $x \in (a,b)$, $h'(x) = G'(x) - F'(x)$. But $G'(x) = f(x)$ by FTC Part I, and $F'(x) = f(x)$ by definition of antiderivative. So $h'(x) = 0$ for every $x \in (a,b)$.

Furthermore, because h is continuous at a and b, h is constant on $[a,b]$, hence $h(a) = h(b)$.

By definition of h,

$$h(a) = G(a) - F(a) \wedge h(b) = G(b) - F(b) \wedge h(a) = h(b)$$
$$\Rightarrow G(b) - F(b) = G(a) - F(a)$$
$$\Rightarrow G(b) = G(a) + \big(F(b) - F(a)\big)$$
$$\Rightarrow \int_a^b f(t)dt = \int_a^a f(t)dt + \big(F(b) - F(a)\big)$$
$$\Rightarrow \int_a^b f(t)dt = 0 + \big(F(b) - F(a)\big)$$
$$\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$$

Therefore we prove the FTC.

不定积分
Indefinite Integral

- 从前面所讲中，我们可以看出定积分计算有个问题，即实际计算时，我们往往需要知道反导数。为解决这个问题，我们引入不定积分的概念。

  We previously introduce the concepts about derivatives and definite integral. However for definite integral there is one problem, which is the existence of antiderivatives. To address this challenge we introduce indefinite integral now.

- 定义：给定一个函数$f(x)$，$f(x)$的反导数或原函数，是满足$F'(x) = f(x)$的任一函数$F(x)$。如果$F(x)$是$f(x)$的任何反导数，则$f(x)$的形式化反导数称为不定积分并记为$\int f(x)dx = F(x) + c$，c为任意常数。在此定义中，$\int$称为积分符号，$f(x)$称为积分，x称为积分变量，c称为积分常数。

  Given a function, $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. If $F(x)$ is any anti-derivative of $f(x)$, then the most general anti-derivative of $f(x)$ is called an indefinite integral and denoted as $\int f(x)dx = F(x) + c$, c is an arbitrary constant. In this definition the $\int$ is called the integral symbol, $f(x)$ is called the integrand, x is called the integration variable and the "c" is called the constant of integration.

不定积分
Indefinite Integral

- 注意，我们通常只说积分而不是不定积分（或者当我们需要明确时，说定积分）。一般地从问题的上下文中我们可以清楚地知道正在谈论是不定积分还是定积分。

  Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).

- 找到不定积分的过程称为积分或对$f(x)$进行积分。如果我们需要具体说明积分变量，我们会说我们正在对$f(x)$依x进行积分。

  The process of finding the indefinite integral is called integration or integrating $f(x)$. If we need to be specific about the integration variable we will say that we are integrating $f(x)$ with respect to x.

- 定义中的dx称为x的微分，即我们在上一讲中讲过的微分的概念。

  We called the dx the differential of x in the definition, which is exactly what we talked about in the last lecture.

不定积分
Indefinite Integral

- 按照定义，不定积分与导数形式上类似逆运算关系，则我们可以如下方式求解。

  By definition, the indefinite integral and the derivative are formally reciprocal to each other, then we can solve as follows.

- 例：求下面式子的不定积分（Find the indefinite integral of the following expressions)：

（1）$\int x^2 dx$：

Since $\frac{d}{dx}\left(\frac{x^3}{3}+c\right)=x^2$， we have $\int x^2 dx=\frac{x^3}{3}+c$

（2）$\int \frac{1}{x^2} dx$：

Since $\frac{d}{dx}\left(-\frac{1}{x}+c\right)=\frac{1}{x^2}$， we have $\int \frac{1}{x^2} dx=-\frac{1}{x}+c$

（3）$\int e^x dx$：

Since $\frac{d}{dx}(e^x+c)=e^x$， we have $\int e^x dx=e^x+c$

不定积分
Indefinite Integral

- 通常情况下，求不定积分或反导数是通过查表，是向特定反导数添加一个常数来获得的。

  The most general antiderivative for an integral is obtained by adding a constant to a particular antiderivative, which is searched in a table.

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

不定积分的性质
Properties of the Indefinite Integral

- 对任意$k \in \mathbb{R}$，我们可以从不定积分中分离出乘法常数。

  For arbitrary $k \in \mathbb{R}$, we can factor multiplicative constants out of indefinite integrals.

$$\int kf(x)dx = k\int f(x)dx$$

- 令$k = -1$，特别地，我们有：

  Set $k = -1$, then we have:

$$\int -f(x)dx = -\int f(x)dx$$

- 函数的和或差的积分是各个积分的和或差，此规则可以扩展到任意数量的函数。

  In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$$

积分的性质
Properties of the Integral

- 由于微积分基本定理（FTC），如果我们知道反导数，即不定积分，我们才可以准确积分一个函数。

  Due to the Fundamental Theorem of Calculus (FTC), we can integrate a function in a precise way if we know an antiderivative, that is, an indefinite integral.

- 积分并不像微分那么简单，没有规则可以绝对保证求出函数的不定积分。

  Integration is not as straightforward as differentiation. There are no rules that absolutely guarantee obtaining an indefinite integral of a function.

- 当没有办法以解析的方式求不定积分时，则对应的定积分只能按照定积分的定义，以数值逼近的方式进行。

  When it is not possible to find the indefinite integral analytically, the corresponding definite integral can only be carried out by numerical approximation according to the definition of the definite integral.

定积分的数值计算
Numerical Computation of Definite Integral

- 下面代码展示了JAX对定积分数值计算的支持：

  The following code demonstrates the support of JAX for numerical computation of definite integrals:

  f = jax.jit(lambda x: x * x) # f(x) = x ^ 2

  x = jax.numpy.arange(1000000) / 1000000 # divide [0, 1] into 1000000 segments

  s = jax.numpy.trapz(f(x), dx=1/1000000) # integrates using the composite trapezoidal rule

  # DeviceArray(0.33333233, dtype=float32)

  # analytic calculation of the integration of x ^ 2 from 0 to 1 yields to 0.3333333…

  f = jax.jit(lambda x: jax.numpy.sin(x)) # f(x) = sin(x)

  x = jax.numpy.arange(1000000) / 1000000 * jax.numpy.pi / 2 # divide [0, π/2] into 1000000 segments

  s = jax.numpy.trapz(f(x), dx=1/1000000 * jax.numpy.pi / 2) # integrates using the composite trapezoidal rule

  # DeviceArray(0.9999985, dtype=float32)

  # analytic calculation of the integration of sin(x) from 0 to π/2 yields to 1

换元法
Substitution Rule

- 由于积分并不像微分那么简单，因此具体计算时，可能需要用到一些技巧。

  Since integration is not as straightforward as differentiation, there are some tricks which can be utilized when do the integration.

- 实际上，每个微分规则都有相应的积分规则：

  Every differentiation rule has a corresponding integration rule:

- 积分的换元法对应于微分的链式法则。

  The Substitution Rule for integration corresponds to the Chain Rule for differentiation.

- 积分的乘积法则对应于微分的分部积分法则。

  The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

分部积分法
Rule for Integration by Parts

- 例如，计算$\int x\sin x\,dx$

  实际上，计算$f(x)=x\cos x$的微分，利用乘法法则，我们有：

  Actually, to compute the derivative of $f(x)=x\cos x$, by the product rule, we have:

$$\frac{df(x)}{dx}=\cos x - x\sin x=\frac{d\sin x}{dx}-x\sin x$$

  因此，利用分部积分法，我们可以计算$\int x\sin x\,dx$如下：

  Now we can calculate$\int x\sin x\,dx$ as below by using the rule for integration by parts:

$$\int x\sin x\,dx=\int xd(-\cos x)=x(-\cos x)-\int(-\cos x)dx$$

$$=-x\cos x+\int \cos x\,dx=-x\cos x+\sin x+C$$

分部积分法
Rule for Integration by Parts

- 实际上，根据乘积法则，若$f(x)$与$g(x)$可微，则我们有：

  Actually, The Product Rule states that, if f and g are differentiable functions, then

$$\frac{d[f(x)g(x)]}{dx} = f'(x)g(x) + f(x)g'(x)$$

  利用不定积分，上式改写为：

  In the notation for indefinite integrals, this equation becomes:

$$\int [f'(x)g(x) + f(x)g'(x)]dx = f(x)g(x)$$

  等价地，

$$\int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x)$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

  此时，若$f'(x)g(x)$可积，则可以利用分部积分法求出原$f(x)g'(x)$的积分。

  Now we can integrate the original $f(x)g'(x)$ if the integration $f'(x)g(x)$ exists.