STA371G Homework Assignment 4: Solutions

Problem 1

A company sets different prices for a particular stereo system in eight different regions of the country. The table below shows the numbers of units sold (in 1000s of units) and the corresponding prices (in hundreds of dollars).

Sales	420	380	350	400	440	380	450	420
Price	5.5	6.0	6.5	6.0	5.0	6.5	4.5	5.0

(a) Let X denote Price and Y denote Sales. Using Excel or R: calculate the sample means of X and Y, sample standard deviations of X and Y, sample covariance between X and Y, and sample correlation between X and Y.

$\bar{X} = 5.625$, $\bar{Y} = 405$

$s_x = 0.744$, $s_y = 33.806$

$Cov(X, Y) = -23.571$

$r_{xy} = -0.937$

(b) In Excel or R, regress sales on price and obtain the estimates of the intercept b_0, slope b_1 and coefficient of determination R^2.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.937137027
R Square	0.878225806
Adjusted R Square	0.857930108
Standard Error	12.74227575
Observations	8

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	7025.806452	7025.806452	43.27152318	0.000592135
Residual	6	974.1935484	162.3655914		
Total	7	8000			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	644.516129	36.68873299	17.56714055	2.18343E-06	554.7420336	734.2902244
X Variable 1	-42.58064516	6.473082556	-6.578109392	0.000592135	-58.41970755	-26.74158277

(c) Using the results obtained in (a) to calculate the intercept b_0, slope b_1 and the coefficient of determination R^2. Are the results the same as those obtained in (b)?

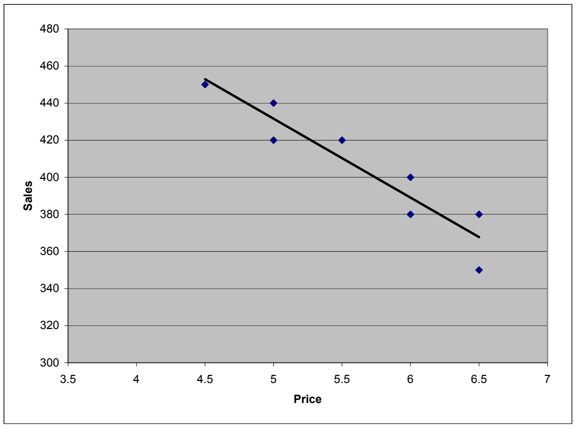
$b_1 = r_{xy}\frac{s_y}{s_x} = -0.937 \times \frac{33.806}{0.744} = -42.58$

$b_0 = \bar{Y} - b_1\bar{X} = 644.51$

$R^2 = r_{xy}^2 = (-0.937)^2 = 0.878$

The results are the same as those in (b)

(d) Present a plot with the data and the regression line.



(e) Based on this analysis, briefly describe your understanding of the relationship between sales and prices.

The sales tend to decrease as the price increases: the sales decreases by about 42.6×1000 units as the price per unit increases by 100 dollars.

Problem 2

Suppose we are modeling house price as depending on house size. Price is measured in thousands of dollars and size is measured in thousands of square feet.
Suppose our model is:

$$P = 20 + 50\,s + \epsilon, \quad \epsilon \sim N(0, 15^2).$$

(a) Given you know that a house has size $s = 1.6$, give a 95% predictive interval for the price of the house.

The point prediction is $\hat{P}_f = 20 + 50 \times 1.6 = 100$

The prediction interval is $[100 \pm 2 \times 15] = [70; 130]$

(b) Given you know that a house has size $s = 2.2$, give a 95% predictive interval for the price.

The point prediction is $\hat{P}_f = 20 + 50 \times 2.2 = 130$

The prediction interval is $[130 \pm 2 \times 15] = [100; 160]$

(c) In our model the slope is 50. What are the units of this number?

1,000$ / 1,000 Sq. Feet = $/Sq. Feet

(d) What are the units of the intercept 20?

1,000$ (same as P)

(e) What are the units of the the error standard deviation 15?

1,000$ (same as P)

(f) Suppose we change the units of price to dollars and size to square feet
What would the values and units of the intercept, slope, and error standard deviation?

Intercept: 20,000 $
Slope: 50 $/Sq. Feet
error standard deviation: 15,000 $

(g) If we plug $s = 1.6$ into our model equation, P is a constant plus the normal random variables ϵ. Given $s = 1.6$, what is the distribution of P?

When $s = 1.6$ the mean of house prices is $20 + 50 \times 1.6 = 100$. The error standard deviation is the same, 15. Therefore

$$P|s = 1.6 \sim N(100, 15^2)$$

Problem 3

Read the case "Waite First Securities" in the course packet. The data file is available on the course website. Consider the regression model

$$TI_t = \alpha + \beta SP500_t + \epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

where TI_t represents the return on Texas Instruments in month t and $SP500_t$ represents the return on the S&P 500 in month t.

(a) What is the interpretation of β in terms of a measure of risk of the stock?

β is a measure of the risk of the stock relative to the market. If the return on the market goes up (or down) by 1% then we expect the return on the stock to go up (or down), on average, by β%.

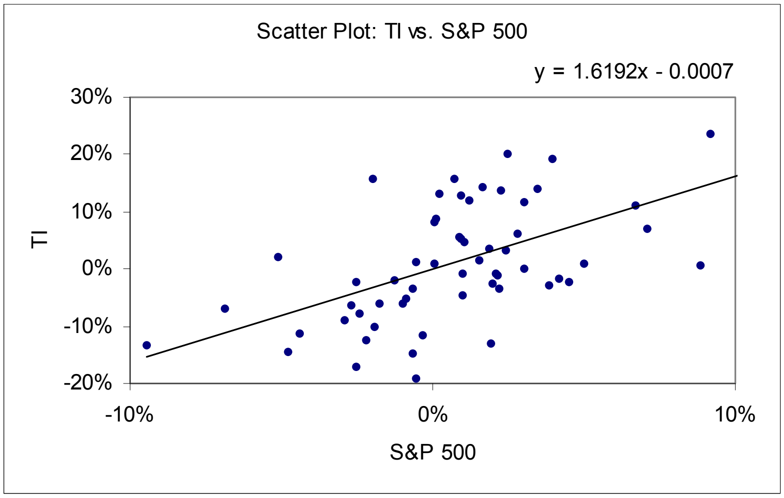
(b) Plot TI against $SP500$. What graphical evidence is there of a relationship between TI and $SP500$? Does the relationship appear to be linear? Why or why not?

Yes, by looking at the plot it appears that TI and $SP500$ are linearly related...

(c) Estimate β. What is the interpretation of this estimate in terms of the risk of the stock? Why is Mr. Gagnon interested in this estimate?

The estimate of β is 1.619. This implies that TI is riskier than the market as a 1% change in the market returns would result in a 1.619% change, on average, in TI's return.

Mr. Gagnon is interested in the risk of the stock and hence he is interest in β...



SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.555135037
R Square	0.308174909
Adjusted R Square	0.29624689
Standard Error	0.090755054
Observations	60

ANOVA

	df	*SS*	*MS*
Regression	1	0.212799498	0.212799498
Residual	58	0.477715829	0.00823648
Total	59	0.690515326	

	Coefficients	*Standard Error*	*t Stat*
Intercept	-0.000682785	0.012089016	-0.056479795
S&P 500	1.61919734	0.318555642	5.082934116

(d) Is the estimate of β obtained in part (c) the actual value of β? Why or why not?

No, it is an estimate and indeed our best guess about the "true" β.

(e) Now consider the regression models

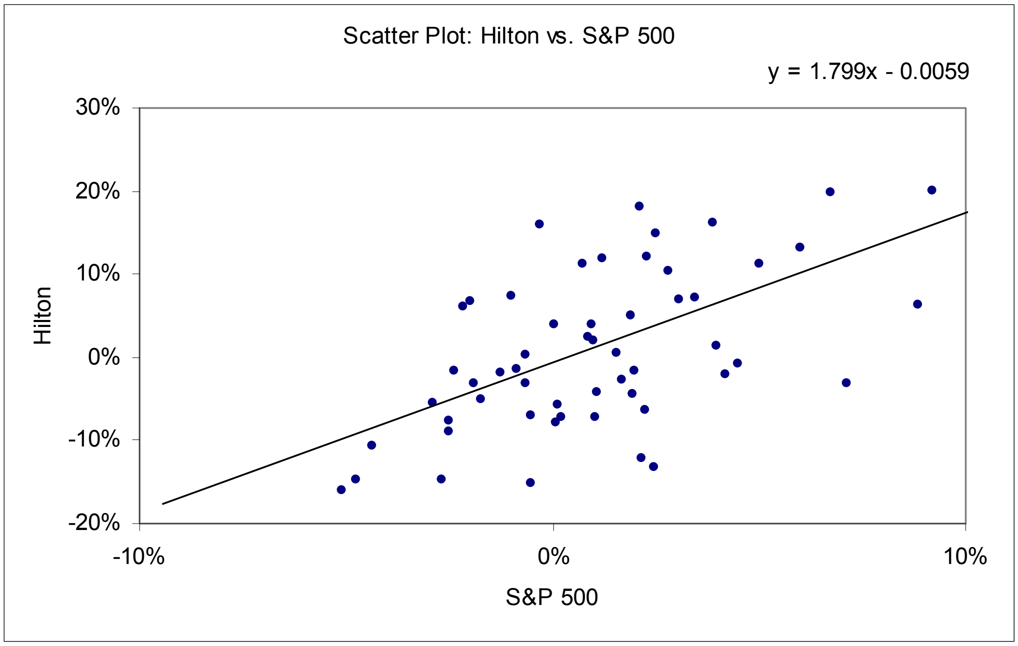
$$Hilton_t = \alpha + \beta SP500_t + \epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

where $Hilton_t$ represents the return on Hilton in month t, and

$$Giant_t = \alpha + \beta SP500_t + \epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

where $Giant_t$ represents the return on Giant in month t. How does the beta risk of the three companies compare? If Mr. Gagnon wants to lower the overall market risk of his portfolio should he buy Giant, Hilton or Texas Instruments?

The market risk of Giant is estimated to be the smallest, with TI as the second riskier and Hilton as the stock with the highest market risk. Mr. Gagnon should buy Giant since he is interested in lowering the overall market risk of his portfolio. (see results below)



SUMMARY OUTPUT

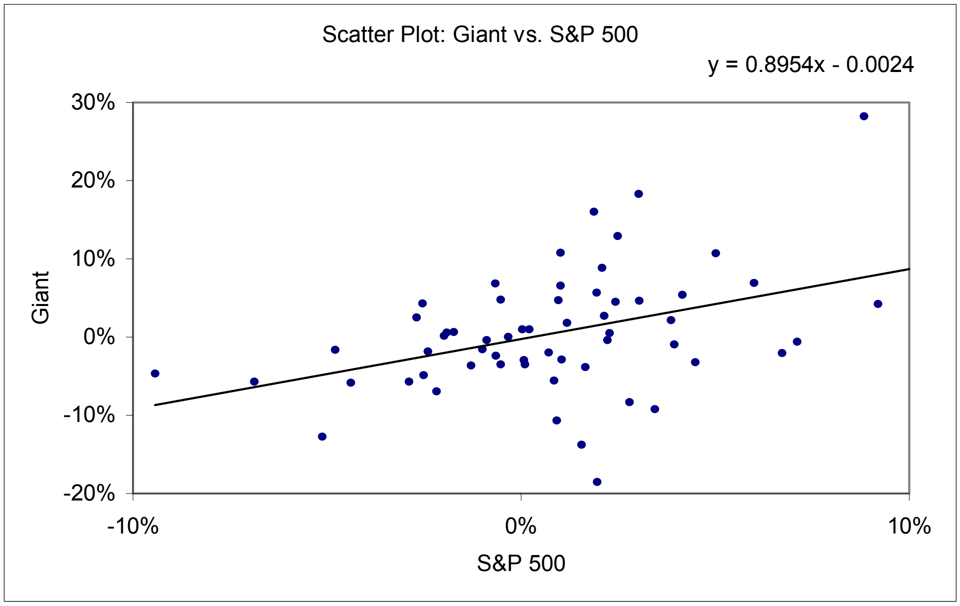
Regression Statistics	
Multiple R	**0.563390647**
R Square	**0.317409021**
Adjusted R Square	**0.305640211**
Standard Error	**0.09869166**
Observations	**60**

ANOVA

	df	*SS*	*MS*
Regression	**1**	**0.262692469**	**0.262692469**
Residual	**58**	**0.564922536**	**0.009740044**
Total	**59**	**0.827615005**	

	Coefficients	*Standard Error*	*t Stat*
Intercept	**-0.00592944**	**0.013146211**	**-0.451037971**
S&P 500	**1.799029535**	**0.346413601**	**5.193299362**

GIANT



SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.42681471
R Square	0.182170797
Adjusted R Square	0.168070293
Standard Error	0.070974288
Observations	60

ANOVA

	df	*SS*	*MS*
Regression	1	0.065079802	0.065079802
Residual	58	0.292166272	0.00503735
Total	59	0.357246074	

	Coefficients	*Standard Error*	*t Stat*
Intercept	-0.002417008	0.009454122	-0.255656515
S&P 500	0.895442274	0.249123975	3.594364104

Problem 4

Read the "Milk and Money" case in the course notebook. The data file is available on the course website.

Important information:

1. The Federal government, through the Agricultural Marketing Service (AMS), sets the price that dairy farmers receive for different "classes" of milk (these classes are called Class I, Class II, etc.). The prices set by the AMS depend on, among other things, the wholesale price of milk and can vary significantly over a two or three year period. In this problem, we will be concerned only with Class III milk prices.

2. A farmer can purchase a put option that gives him the right but not the obligation to sell a futures contract on Class III milk at the "strike" price on or before the expiration date of the option. This puts a "floor" under the price that the farmer will receive for his Class III milk. He removes the downside risk but still has the upside potential.

   For example, suppose the strike price on a December 15 Class III milk put option is \$12/cwt (cwt is a unit of measurement that is roughly 100 pounds of milk). If the AMS price on December 15 is below \$12/cwt, the put option allows the farmer to sell his milk for \$12/cwt. If the AMS price is greater than \$12/cwt then he will sell his milk at the AMS price.

   The cost of the put option is the price a farmer must pay someone to take on the downside risk. For example, the cost of a \$12/cwt December 15 put option purchased in June might be \$0.45/cwt.

   The farmer must also pay trading costs for purchasing the option (e.g. brokers commission, etc.). For example, the trading cost on a \$12/cwt December 15 option might be \$0.05/cwt.

   Strike prices on put options for Class III milk are available every \$0.25. For example, \$11.50/cwt, \$11.75/cwt, \$12/cwt, \$12.25/cwt, etc.

3. For historical and legal reasons, California dairy farmers participate in a California pricing system rather than the federal AMS pricing system. The price a California dairy farmer receives for his milk, called the "mailbox" price, is determined by a complex formula that depends on the value of various dairy products on the wholesale market. The California mailbox price varies a great deal over time just as the federal AMS price does. For example, between 2005 and 2007 the mailbox price varied between \$10.16/cwt and \$19.98/cwt with an average price in 2006 of \$11.28/cwt. The dairy farmer in the case, Gerard, estimates his costs are \$12/cwt so a price of \$11.28/cwt creates a significant financial problem for him.

4. Gerard is interested in hedging his revenue six months in advance and guaranteeing a price of at least \$12/cwt for his milk. For example, in June he wants to hedge his December 15 revenue.

5. Put options on the California mailbox price are not available. The federal Class III milk price is closely related, although not the same as, the California mailbox price

that Gerard will receive. For this reason, Gerard will use put options on the federal Class III milk price to hedge his revenue.

6. Gerard wants the probability to be at least 95% that his revenue will be \$12/cwt or more no matter what the California mailbox price is.

Parts (a) and (b) below provide an example of how to determine the value of a put option on Class III milk on its expiration date. The same idea is used in a slightly more complicated context in parts (c) – (j).
The discussion of put options on pages 6-8 of the case, and in particular the example at the bottom of page 7 and top of page 8 will be helpful in answering parts (a) and (b).

(a) Suppose Gerard buys a December 15 put option on Class III milk in June with a strike price of \$12/cwt. If the Class III milk price on December 15 is \$11.50/cwt, how much is the put option worth when it expires on this day?

    The put option is worth \$0.50. The reason is that the option allows the holder to sell Class III milk for \$12/cwt on December 15 while the price of Class III milk through the AMS government program is only \$11.50/cwt. This means the option provides the holder with an additional \$0.50/cwt in revenue that he would not receive if he sold his milk through the government program. Therefore, someone would be willing to pay up to \$0.50 to purchase the option on the day it expires.

(b) Suppose the price for the put option in part (a) is \$0.30/cwt and that the trading costs for purchasing the option are \$0.05/cwt. Combining the value of the option obtained in part (a) with the cost information, what is Gerard's net gain on the option (i.e. what is the value of the put option minus the option cost and trading cost)?

    Gerard's net gain is \$0.15. This includes the \$0.50 he makes on the put option minus the \$0.30 premium paid for the option minus the \$0.05 paid in trading costs. Therefore, his net gain is \$0.50 - (\$0.30 + \$0.05) = \$0.15.

For parts (c)–(j), suppose Gerard in June decides to hedge his December 15 revenue by purchasing a put option on Class III milk with a strike price of \$14.25/cwt and an expiration date of December 15.
You should do the calculations for parts (c)–(h) assuming the costs of the option are zero. A way to incorporate the additional costs into the hedging process is discussed in parts (i) and (j).

(c) Plot the Class III milk price against the California mailbox price and add the estimated regression line to the plot. To add the estimated regression line right click on any data point, click on Add Trendline, click on the box above Linear, click on the Options tab, check the box next to Display equation on chart and click OK. What is the equation of the estimated regression line? How much do you expect the Class III milk price to change on average for a \$1/cwt change in the California mailbox price?

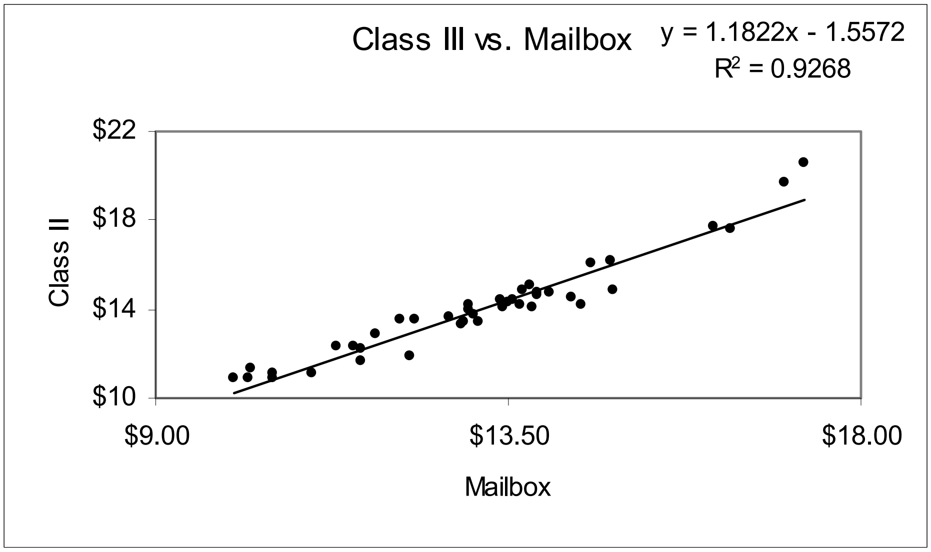


Figure 1: The Class III milk price is expected to change \$1.18/cwt for every \$1/cwt change in the California mailbox price.

For the remainder of the problem use 0.60 as the value of σ (the standard deviation of the error term) and the estimated regression line as if it were the true regression line to answer the following questions.

(d) What is the probability that Gerard's December 15 put option on Class III milk with a strike price of \$14.25 is in the money (i.e. worth something) on December 15 if the California mailbox price on December 15 is \$12.50/cwt?

Note that for a December 15 put option on Class III milk with a strike price of \$14.25/cwt to be in the money on December 15, the Class III milk price must be less than \$14.25/cwt on December 15 (because the value of the put option on its expiration date is the difference between the strike price of \$14.25 and the actual Class III milk price if the strike price is greater than the Class III milk price).

This means the probability a put option with a strike price of \$14.25/cwt is in the money on December 15 is the probability that the Class III milk price on December 15 is less than \$14.25.

Using the regression line and the distribution for the Class III milk price associated with a mailbox price of \$12.50 (see the graph at the end of the answer to this problem), the distribution for the Class III milk price given that the mailbox price is \$12.50/cwt is

$$N(13.22, 0.60^2).$$

Note that the mean is the point on the regression line when the mailbox price is 12.50, i.e. $-1.5572 + 1.1822 \times (12.50) = 13.22$. Therefore,

$$\begin{aligned} Pr(\text{Option is in the money}) &= Pr(\text{Class III Price} < 14.25) \\ &= Pr(\frac{\text{Class III Price} - 13.22}{0.60} < \frac{14.25 - 13.22}{0.60}) \\ &= Pr(Z < 1.72) \\ &= 95.7\% \end{aligned}$$

(e) Suppose the California mailbox price on December 15 is \$12.00/cwt. What is the probability that the value of the put option will be greater than \$0.50?

The value of the put option will be greater than \$0.50 if the Class III milk price is less than \$13.75/cwt (because the value of the put option on its expiration date is the difference between the strike price of \$14.25 and the actual Class III milk price if the strike price is greater than the Class III milk price).

Therefore, the probability that the value of the put option will be greater than \$0.50 is the probability that the Class III milk price is less than \$13.75/cwt.

To compute this probability, note that the distribution for the Class III milk price if the mailbox price is \$12.00/cwt is

$$N(12.63, 0.60^2)$$

see the graph at the end of the answer to this problem. Note that the mean is the point on the regression line when the mailbox price is 12.00, i.e. $-1.5572 + 1.1822 \times (12.00) = 12.63$. Therefore,

$$\begin{aligned} Pr(\text{Option value is greater than } 0.50) &= Pr(\text{Class III Price} < 13.75) \\ &= Pr(\frac{\text{Class III Price} - 12.63}{0.60} < \frac{13.75 - 12.63}{0.60}) \\ &= Pr(Z < 1.87) \\ &= 96.9\% \end{aligned}$$

(f) Using your answer to part (e), what is the probability that Gerard's net revenue (mailbox price plus payoff from the option) will exceed \$12.50/cwt if the California mailbox price is \$12.00/cwt?

If the mailbox price is \$12.00/cwt then the put option must be worth at least \$0.50 for Gerard's net revenue to be at least \$12.50/cwt.

The result in part (e) means there is a 96.9% chance Gerard's net revenue will be at least \$12.50/cwt if the mailbox price is \$12.00/cwt. The reason is that he will receive the mailbox price of \$12.00/cwt plus the revenue from selling the put option, and there is a 96.9% chance that the value of the put option will be at least \$0.50.

(g) Now suppose the mailbox price on December 15 is \$11.50/cwt. What is the probability that the value of the put option will be greater than \$1? Using your answer to this question, what is the probability that Gerard's net revenue (mailbox price plus payoff from the option) will exceed \$12.50/cwt?

If the mailbox price is \$11.50/cwt then the put option must be worth at least \$1 for Gerard's net revenue to be at least \$12.50/cwt.

The value of the put option will be greater than \$1 if the Class III milk price is less than \$13.25/cwt (because the value of the put option on its expiration date is the difference between the strike price of \$14.25 and the actual Class III milk price if the strike price is greater than the Class III milk price).

Therefore, the probability that the value of the put option will be greater than \$1 is the probability that the Class III milk price is less than \$13.25/cwt.

To compute this probability, note that the distribution for the Class III price if the mailbox price is \$11.50/cwt is

$$N(12.04, 0.60^2)$$

see the graph at the end of the answer to this problem. Note that the mean is the point on the regression line when the mailbox price is 11.50, i.e. $-1.5572+1.1822\times(11.50) = 12.04$. Therefore,

$$\begin{aligned}
Pr(\text{Option value is greater than 1.00}) &= Pr(\text{Class III Price} < 13.25) \\
&= Pr(\frac{\text{Class III Price} - 12.04}{0.60} < \frac{13.25 - 12.04}{0.60}) \\
&= Pr(Z < 2.02) \\
&= 97.8\%
\end{aligned}$$

This means there is a 97.8% chance Gerard's net revenue will be at least \$12.50/cwt if the mailbox price is \$11.50/cwt. The reason is that he will receive the mailbox price of \$11.50/cwt plus the revenue from selling the put option, and there is a 97.8% chance that the value of the put option will be at least \$1.00.

(h) Is the probability at least 95% that his net revenue (mailbox price plus payoff from the option) will equal or exceed \$12.50/cwt for any mailbox price below \$12.50/cwt? Why or why not?

The probability will be at least 95% that his net revenue will equal or exceed \$12.50/cwt for any mailbox price below \$12.50/cwt.

One way to solve the problem:

Parts (e) and (f) showed that the probability is 96.9% that his net revenue will equal or exceed \$12.50/cwt if the mailbox price is \$12.00/cwt. Part (g) showed that the probability is 97.8% that his net revenue will equal or exceed \$12.50/cwt if the mailbox price is \$11.50/cwt.

A similar calculation can be done for any mailbox price less than \$12.50/cwt (e.g. \$12.49/cwt, \$12.48/cwt, etc.) to show the probability is at least 95% that his net revenue will equal or exceed \$12.50/cwt for any mailbox price below \$12.50/cwt. This is a labor intensive way to do it but it is one way to verify the statement.

A graphical explanation for why this is happening is that the regression line (which gives the mean of the Class III price distribution for a given mailbox price) is decreasing at the rate of \$1.18 for every \$1.00 decrease in the mailbox price while the line that connects the Class III prices that guarantee Gerard at least \$12.50/cwt in net revenue is decreasing at the rate of \$1.00 for every \$1.00 decrease in the mailbox price. To see that the latter statement is true, draw in a line on the graph at the end of the answer to this question that connects the points: (12.50, 14.25), (12.00, 13.75) and (11.50, 13.25).

Carefully viewing this line will show that the Class III prices that guarantee Gerard at least \$12.50/cwt in net revenue are moving farther out into the tails of the Class III price distributions as the mailbox price decreases from \$12.50/cwt. This means the probabilities that the Class III price is below the price that guarantees Gerard net revenue of at least \$12.50/cwt will increase from the value of 0.957 computed for the mailbox price of \$12.50/cwt. Therefore, the probability will be at least 95% that his net revenue will equal or exceed \$12.50/cwt for any mailbox price below \$12.50/cwt.

Another way to solve the problem:

Denote X as the mailbox price and Y as the Class III price. Since $Y \sim N(1.1822 \times X - 1.5572, 0.6^2)$, supposing that $X < 12.50$, then the net revenue would be

$$\max\{X, X + (14.25 - Y)\}.$$

Denote $R = X + (14.25 - Y)$, then R is a normal random variable distributed as

$$R \sim N(14.25 + 1.5572 + X - 1.1822X, 0.6^2) = N(15.8072 - 0.1822X, 0.6^2)$$

The probability for the net revenue R to be equal or greater than \$12.50/cwt would be

$$\begin{aligned} P(R > 12.5) &= P\left(\frac{R - (15.8072 - 0.1822X)}{0.6} > \frac{12.5 - (15.8072 - 0.1822X)}{0.6}\right) \\ &= P\left(Z > \frac{12.5 - (15.8072 - 0.1822X)}{0.6}\right) \\ &= P\left(Z > \frac{0.1822X - 3.3072}{0.6}\right) \\ &= P\left(Z > 0.3036X - 5.512\right) \end{aligned}$$

Thus when $X < 12.5$,

$$\{P(R > 12.5) = P\left(Z > 0.3036X - 5.512\right)\} > \{P\left(Z > 0.3036 \times 12.5 - 5.512\right) = 95.7\%\} \tag{1}$$

Therefore, the probability will be at least 95% (in fact, at least 95.7%) that his net revenue will equal or exceed \$12.50/cwt for any mailbox price below \$12.50/cwt.

For parts (i) and (j), suppose the price in June for a December 15 put option on Class III milk with a strike price of \$14.25/cwt is \$0.45/cwt and the trading cost is \$0.05/cwt.

(i) Is the probability at least 95% that Gerard's net revenue (mailbox price plus payoff from the option minus option cost and trading cost) will equal or exceed his production costs of \$12.00/cwt no matter what the California mailbox price is?

From part (h), the probability is at least 95% that his revenue before the costs of the option are included will be \$12.50/cwt or more no matter what the California mailbox price is. This means that after the premium and trading costs of \$0.50/cwt are accounted for the probability is at least 95% that his net revenue will be \$12.00/cwt or more no matter what the California mailbox price is.

(j) Has Gerard effectively hedged his net revenue (mailbox price plus payoff from the option minus option cost and trading cost) if \$12/cwt is the amount he needs to receive for his milk to cover his production costs?

Gerard has effectively hedged his net revenue because there is at least a 95% chance his net revenue will be \$12.00/cwt or more. There is a small chance (5% or less) that his net revenue will be less than \$12/cwt but this is a risk he is willing to accept.