STA 371G: Statistics and Modeling

Decision Making Under Uncertainty: Decision Criteria and Utility Functions

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Payoffs and Losses

The consequence of a decision can be expressed in terms of either payoffs or losses:

- A *payoff*, or *reward*, represents the net change in your total wealth as a result of your decision and the actual state of the world. This can be either positive or negative.
- A *loss* is the nonnegative difference between the given payoff and the *highest possible payoff under that state of the world*. It is intepretated in terms of "opportunity loss."
- $L(\text{action } i, \text{state } j) = \max_k R(\text{action } k, \text{state } j) - R(\text{action } i, \text{state } j)$

Payoffs and Losses

- State of the world: *Rain* or *No Rain*
- Action: *Carry umbrella* or *Do not carry umbrella*

- Payoff Table:

	Rain	*No Rain*
Carry umbrella	$-\$3$	$-\$3$
Do not carry umbrella	$-\$90$	$\$0$

- Loss Table

	Rain	*No Rain*
Carry umbrella		
Do not carry umbrella		

Payoffs and Losses

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Carry umbrella	$-\$3$	$-\$3$
Do not carry umbrella	$-\$90$	$\$0$

- Loss Table

	Rain	*No Rain*
Carry umbrella	$\$0$	
Do not carry umbrella	$\$87$	

Payoffs and Losses

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- Payoff Table:

	Rain	*No Rain*
Carry umbrella	$-\$3$	$-\$3$
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- Loss Table

	Rain	*No Rain*
Carry umbrella	$\$0$	$\$3$
Do not carry umbrella	$\$87$	$\$0$

Payoffs and Losses

- State of the world: *Oil* or *No Oil*
- Action: *Drill* or *Do not drill*

- Payoff Table:

	Oil	*No Oil*
Drill	\$1,000,000	−\$300,000
Do not drill	\$200,000	\$100,000

- Loss Table:

	Oil	*No Oil*
Drill		
Do not drill		

Payoffs and Losses

- State of the world: *Oil* or *No Oil*
- Action: *Drill* or *Do not drill*

- Payoff Table:

	Oil	*No Oil*
Drill	\$1,000,000	−\$300,000
Do not drill	\$200,000	\$100,000

- Loss Table:

	Oil	*No Oil*
Drill	\$0	\$400,000
Do not drill	\$800,000	\$0

Payoffs and Losses

- Payoff Table:

	Competitor introduces new product	*Competitor introduces no new product*
No advertising	\$100,000	\$700,000
Minor ad campaign	\$300,000	\$600,000
Major ad campaign	\$400,000	\$500,000

- Loss Table:

	Competitor introduces new product	*Competitor introduces no new product*
No advertising		
Minor ad campaign		
Major ad campaign		

Payoffs and Losses

- Payoff Table:

	Competitor introduces new product	*Competitor introduces no new product*
No advertising	\$100,000	\$700,000
Minor ad campaign	\$300,000	\$600,000
Major ad campaign	\$400,000	\$500,000

- Loss Table:

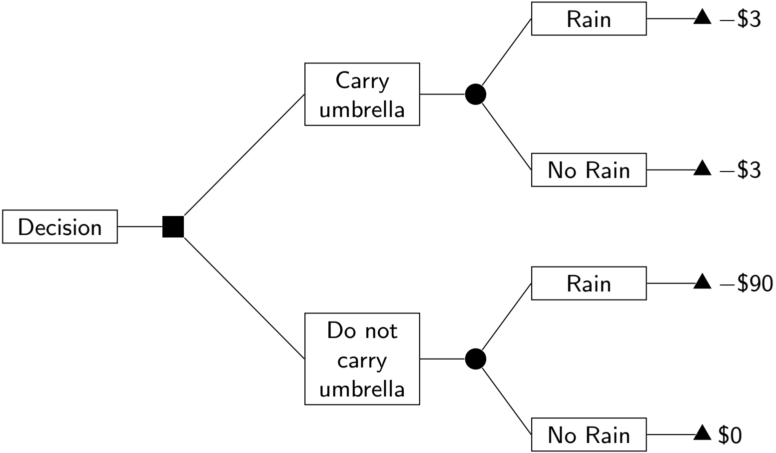
	Competitor introduces new product	*Competitor introduces no new product*
No advertising	\$300,000	\$0
Minor ad campaign	\$100,000	\$100,000
Major ad campaign	\$0	\$200,000

Converting Payoffs to Losses

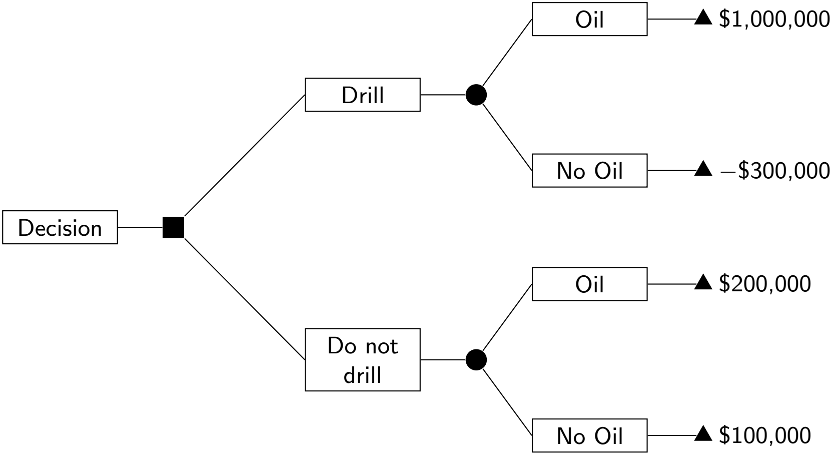
$$L(\text{action } i, \text{ state } j) = \max_k R(\text{action } k, \text{ state } j) - R(\text{action } i, \text{ state } j)$$

Tree Diagram

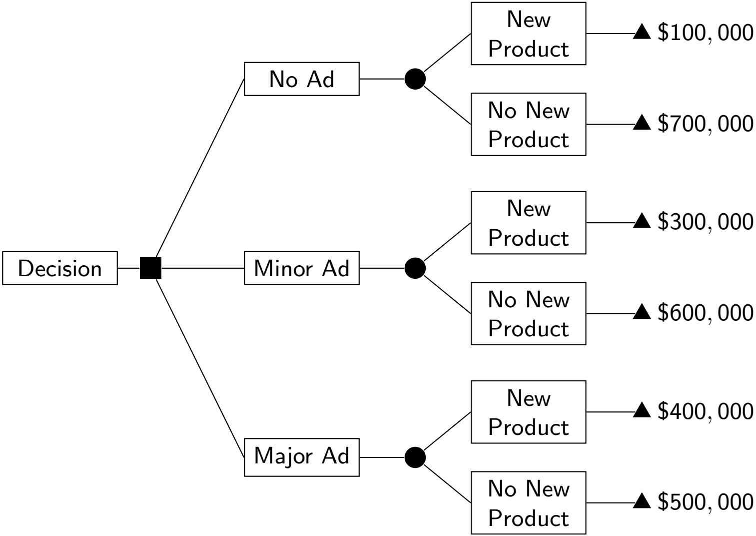
A tree diagram is a common way to present the payoffs or losses in a decision-making problem.



Tree Diagram



Tree Diagram



Admissible and Inadmissible Actions

Admissible and inadmissible actions (decisions):

- An action is said to *dominate* a second action if for *each possible state of the world*, the first action leads to at least as high a payoff (or at least as small a loss) as the second action, and if for at least one state of the world, the first action leads to a higher payoff than the second action.
- If an action dominates the second option, then the second action is said to be *inadmissible*.
- A decision maker need to consider only admissible actions.
- Which action is inadmissible?

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000
Buy Stock C	−\$2000	−\$4,000	−\$8000	\$2,000

Nonprobabilistic Criteria for Decision Making Under Uncertainty

maximin rule: for each action, find the smallest possible payoff, and then choose the action for which the smallest payoff is largest.

- Maximize the minimum payoff
- Assume the worst would happen
- Conservative
- A potential problem of the *maximin* rule:

	State I	State II
Action 1	\$100,000	−\$1
Action 2	\$0	\$0

Nonprobabilistic Criteria for Decision Making Under Uncertainty

maximax rule: for each action, find the largest possible payoff, and then choose the action for which the largest payoff is largest.

- Maximize the maximum payoff
- Assume the best would happen
- Risky
- A potential problem of the *maximax* rule:

	State I	State II
Action 1	\$100,000	\$99,999
Action 2	\$100,001	\$0

Nonprobabilistic Criteria for Decision Making Under Uncertainty

The *minimax loss*, or *minimax regret*, criterion: for each action, find the largest possible loss, and then choose the action for which the largest loss is smallest.

- Minimize the maximum possible loss
- Not as conservative as the *maximum* rule
- Not as risky as the *maximax* rule
- Loss tables (converted from the payoff tables from the previous two slides):

	State I	State II
Action 1	\$0	1
Action 2	\$100,000	\$0

	State I	State II
Action 1	\$1	\$0
Action 2	\$0	\$99,000

- Which action to choose based on the *minimax loss* criterion?

Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*:
- *maximax*:
- *minimax loss*:
- Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000

- Loss table

Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*:
- *minimax loss*:
- Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000

- Loss table

Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*: Buy Stock A
- *minimax loss*:
- Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000

- Loss table

Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*: Buy Stock A
- *minimax loss*: Buy Stock B
- Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000

- Loss table

	State I	State II	State III	State IV
Buy Stock A	\$0	\$0	\$6,000	\$0
Buy Stock B	\$2000	\$4,000	\$2,000	\$0
Buy Stock C	\$4000	−\$10,000	\$0	\$0

Probabilistic Criteria for Decision Making Under Uncertainty

Probabilistic Criteria:

- Expected Payoff (ER) criterion: choose the act with the highest expected payoff.
- Expected Loss (EL) criterion: choose the act with the smallest expected loss.
- $ER(action\ i) = \sum_j R(action\ i, state\ j)P(state\ j)$
- $EL(action\ i) = \sum_j L(action\ i, state\ j)P(state\ j)$
- The ER and EL criteria yield identical decisions:

$$\begin{aligned}
EL(action\ i) &= \sum_j L(action\ i, state\ j)P(state\ j) \\
&= \sum_j (\max_k R(action\ k, state\ j) - R(action\ i, state\ j))P(state\ j) \\
&= \Big(\sum_j \max_k R(action\ k, state\ j)P(state\ j)\Big) - ER(action\ i)
\end{aligned}$$

Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- *ER* or *EL* criteria:
- $ER + EL =$
- Payoff table

Probability: State:	0.1 State I	0.2 State II	0.3 State III	0.2 State IV	ER
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000	
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000	
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000	

- Loss table

Probability: State:	0.1 State I	0.2 State II	0.3 State III	0.2 State IV	EL
Buy Stock A	\$0	\$0	\$6,000	\$0	
Buy Stock B	\$2000	\$4,000	\$2,000	\$0	
Buy Stock C	\$4000	−\$10,000	\$0	\$0	

Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- *ER* or *EL* criteria: Buy Stock A
- $ER + EL = \$3000$
- Payoff table

Probability:	0.1	0.2	0.3	0.2	
State:	State I	State II	State III	State IV	ER
Buy Stock A	\$2,000	\$6,000	−\$6,000	\$2,000	\$1,200
Buy Stock B	\$0	\$2,000	−\$2,000	\$2,000	\$600
Buy Stock C	−\$2000	−\$4,000	\$0	\$2,000	−\$1,400

- Loss table

Probability:	0.1	0.2	0.3	0.2	
State:	State I	State II	State III	State IV	EL
Buy Stock A	\$0	\$0	\$6,000	\$0	\$1,800
Buy Stock B	\$2000	\$4,000	\$2,000	\$0	\$2,400
Buy Stock C	\$4000	−\$10,000	\$0	\$0	\$4,400

Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.

Which bet will you take?

Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.

Which bet will you take?

Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Which bet will you take?

Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

- You win \$10 million if a coin comes up heads and win \$10 million if the conn comes up tails.
- You win \$100 million if a coin comes up heads and win 0 if the conn comes up tails.

St. Petersburg Paradox

Will you take the bet?

- You will $2^0 = 1$ dollar if heads appear on the first toss.
- You will $2^1 = 2$ dollar if the first heads appear on the second toss.
- ...
- You will 2^k dollar if the first heads appear on the kth toss.
- ...

$$ER(gamble) = \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

Again, based on the *ER* criterion, you should take the bet, but...

Axioms of Utility

A utility function can be interpreted in terms of a preference relationship. The two basic axioms of utility are:

- If payoff R_1 is preferred to payoff R_2, then $U(R_1) > U(R_2)$; if R_2 is preferred to R_1, then $U(R_2) > U(R_1)$; and if neither is preferred to the other, then $U(R_1) = U(R_2)$.
- If you are indifferent between (a) receiving payoff R_1 for certain and (b) taking a bet or lottery in which you receive payoff R_2 with probability p and payoff R_3 with probability $1 - p$, then

$$U(R_1) = pU(R_2) + (1 - p)U(R_3).$$

Assessment of Utility Functions

Denote the most preferable payoff as R^* and the least preferable payoff as R_*. Suppose that you let $U(R_*) = 0$ and $U(R^*) = 1$, then $0 \leq U(R) \leq 1$ for for any payoff R.

Consider the following choice of lotteries:

- Lottery I: Receive R for certain
- Lottery II: Receive R^* with probability p and R_* with probability $1 - p$.

If $U(R) > p$, then choose Lottery I; if $U(R) < p$, then choose Lottery II; and if $U(R) = p$, then you are indifferent between the two lotteries.

With $U(R_*) = 0$ and $U(R^*) = 1$, your utility for a payoff can be interpreted in terms of an indifference probability.

Example: Oil Drilling

	Oil	*No Oil*
Drill	\$1,000,000	−\$300,000
Do not drill	\$200,000	\$100,000

- Utility:
  - The utility of \$200,000 is p_1 if you are indifferent between:
    - Lottery I: Receive $\$200,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability p_1 and $-\$300,000$ with probability $1 - p_1$.
  - The utility of \$100,000 is p_2 if you are indifferent between:
    - Lottery I: Receive $\$100,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability p_2 and $-\$300,000$ with probability $1 - p_2$.
  - Utility table:

	Oil	*No Oil*
Drill		
Do not drill		

Example: Oil Drilling

	Oil	*No Oil*
Drill	\$1,000,000	−\$300,000
Do not drill	\$200,000	\$100,000

- Utility:
  - The utility of \$200,000 is p_1 if you are indifferent between:
    - Lottery I: Receive $\$200,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability p_1 and $-\$300,000$ with probability $1 - p_1$.
  - The utility of \$100,000 is p_2 if you are indifferent between:
    - Lottery I: Receive $\$100,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability p_2 and $-\$300,000$ with probability $1 - p_2$.
  - Utility table:

	Oil	*No Oil*
Drill	1	0
Do not drill	p_1	p_2

Example: Oil Drilling

Suppose $P(Oil) = 0.3$ and the utility table is

	Oil	*No Oil*
Drill	1	0
Do not drill	0.45	0.35

then

$$EU(Drill) = 0.3 * 1 + 0.7 * 0 = 0.30,$$

$$EU(Do\ Not\ Drill) = 0.3 * 0.45 + 0.7 * 0.35 = 0.38.$$

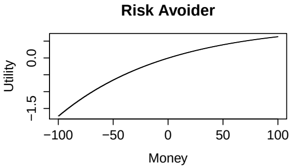
The decision maker should not drill according to the *EU* criterion.

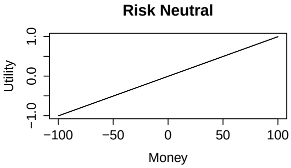
Utility and Money

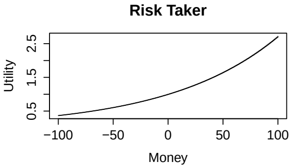
- A Risk Avoider would be willing to pay a risk premium to avoid a fair bet.
- Risk Neutral: indifferent about a fair bet.
- A Risk Taker would be willing to pay money to take a fair bet.

When you are shopping for car insurance, are you a Risk Avoider or Taker?

When you are buying a lottery ticket, are you a Risk Avoider or Taker?







Exponential Utility

$U(x) = 1 - e^{-x/R}$

- x is the payoff
- $R > 0$ is the risk tolerance that specifies an individual's aversion to risk
- The risk tolerance R can be assessed as the amount that makes the decision maker indifferent between the following two lotteries:
  - Lottery I: obtain zero payoff;
  - Lottery II: win R dollars with probability $1/2$ or loose $R/2$ with probability $1/2$.
- Approximate values of R for a company: 6.4% of net sales, 124% of net income, and 15.7% of equality

Example: Exponential Utility

Venture Limited, with net sales of \$30,000,000, need to decide:

- Invest in a sure thing with a return of \$125,000;
- Invest in a low risky venture that returns $\$-500,000$, \$100,000, or \$1,000,000 with probabilities 0.25, 0.50, and 0.25, respectively;
- Invest in a high risky venture that returns $\$-1,000,000$, \$1,000,000, or \$3,000,000 with probabilities 0.35, 0.60, and 0.05, respectively;

Your job is to:

- find the best decision in terms of expected payoff
- find the best decision in terms of expected utility