STA 371G: Statistics and Modeling

Introduction to Monte Carlo Simulation

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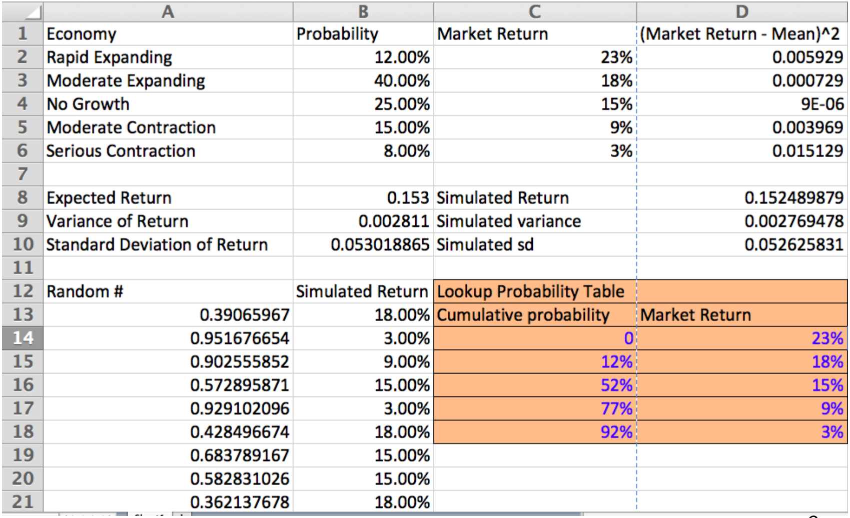
Why Simulation?

- Extremely useful to model uncertainty
- Can be incorporated into spreadsheet models
- Allow the input variables to be random
- Provide an entire distribution of results, including not only the most likely, but also the worst and the best
- Help analyze a system's sensitivity to changes in operation conditions
- Can be implemented in Excel with the RAND() function or some add-in package (such as @Risk)
- Extremely convenient to implement in R (also in Matlab, Python, ...)

Example: Simulation of Market Return

- Incorporate uncertainty into spreadsheet models



	A	B	C	D
1	Economy	Probability	Market Return	(Market Return - Mean)^2
2	Rapid Expanding	12.00%	23%	0.005929
3	Moderate Expanding	40.00%	18%	0.000729
4	No Growth	25.00%	15%	9E-06
5	Moderate Contraction	15.00%	9%	0.003969
6	Serious Contraction	8.00%	3%	0.015129
7				
8	Expected Return	0.153	Simulated Return	0.152489879
9	Variance of Return	0.002811	Simulated variance	0.002769478
10	Standard Deviation of Return	0.053018865	Simulated sd	0.052625831
11				
12	Random #	Simulated Return	Lookup Probability Table	
13	0.39065967	18.00%	Cumulative probability	Market Return
14	0.951676654	3.00%	0	23%
15	0.902555852	9.00%	12%	18%
16	0.572895871	15.00%	52%	15%
17	0.929102096	3.00%	77%	9%
18	0.428496674	18.00%	92%	3%
19	0.683789167	15.00%		
20	0.582831026	15.00%		
21	0.362137678	18.00%		

Uniform Random Variable

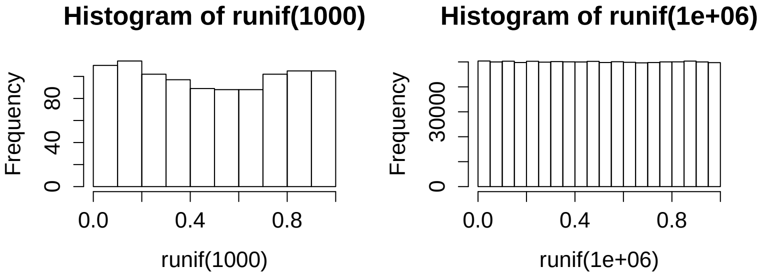
- A random number uniformly distributed between 0 and 1
- Can be used to generate any other random numbers
- In Excel: =RAND()
  - Flip a coin: =IF(RAND()>0.5,"Head","Tail")
  - Roll a die:
    - =RANDBETWEEN(1,6)
    - or use =RAND() and a lookup table
- In R: runif
  - Flip a coin: runif(1)>0.5 or sample(0:1,1)
  - Roll a die: =ceiling(runif(1)*6) or sample(1:6,1)

Uniform Random Variable

Histogram of uniformly distributed random numbers

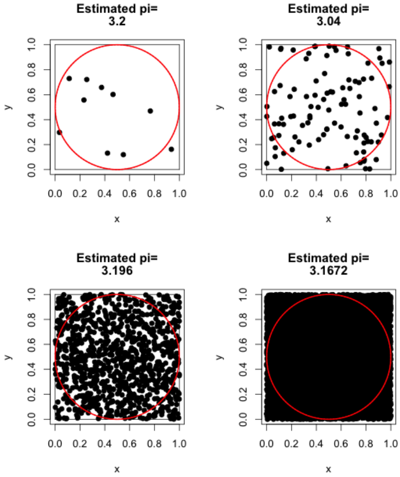
In Excel: copy RAND() to n cells , where n is the number of experiments. What if $n = 10^9$?

In R: hist(runif(n))



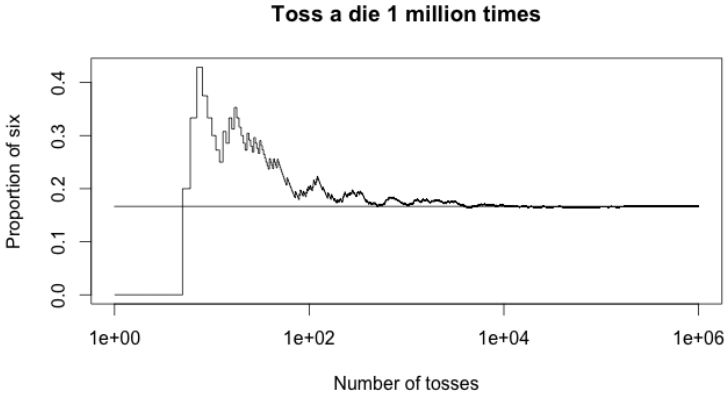
Example: Estimate π with Monte Carlo Simulation

- The area of a circle with the radius of $1/2$ is $\pi/4$
- The area of a unit square is 1
- A uniform random point in the unit square falls within the circle with probability $\pi/4$

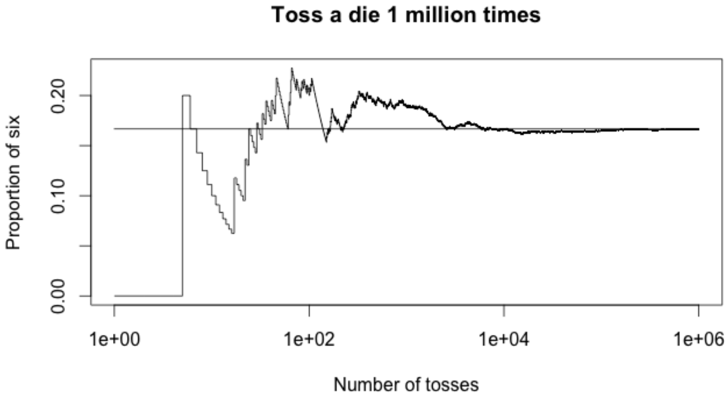


Law of Large Numbers

- Toss a die one million times
- Record the proportion of die tosses that are 6 after each toss
- The proportion tends to get closer to the probability 1/6 as the number of tosses increases



Repeat the same experiment: toss a die one million times



Law of Large Numbers

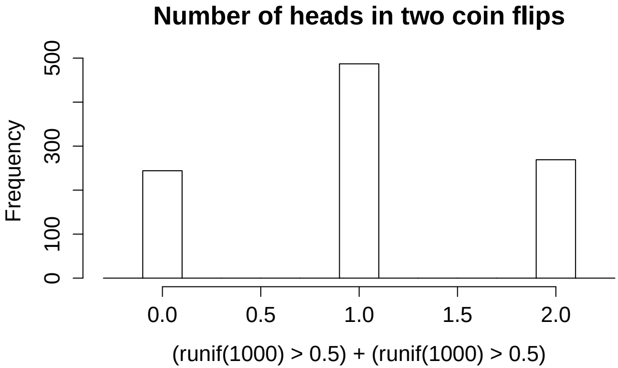
- Probability: the probability of a random outcome is the proportion of the times that it would occur if the experiment is repeated an infinite number of times.
- Law of Large Numbers: As the number of experiments n increases towards ∞, the proportion of a particular random outcome $\hat{p}_n$ converges to the true probability p of that outcome.

Example: Flip Two Coins

In Excel: copy =(RAND()>0.5)+(RAND()>0.5) to *n* cells, use scatter plot to visualize the results

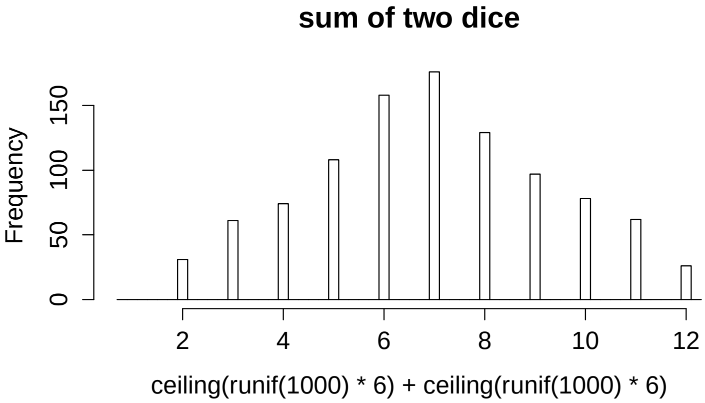
In R: x = (runif(n)>0.5) + (runif(n)>0.5)

use hist(x) and plot(x) to visualize the results



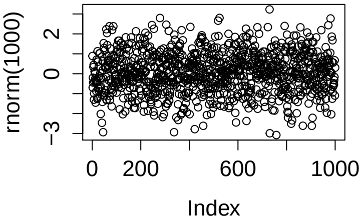
Example: Toss Two Dice

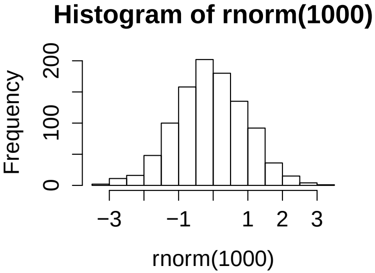
In R: hist(ceiling(runif(n)*6) + ceiling(runif(n)*6)) or
hist(sample(1:6,n,replace=TRUE) + sample(1:6,n,replace=TRUE))



Normal Random Variable

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- In Excel: $=$ NORMINV(RAND(), μ, σ)
- In R: rnorm(n,μ,σ) , where *n* is the number of normal random numbers that we ask R to generate
- Let's simulate 1000 standard normal random numbers and visualize the results:





Normal Random Variable (P(X<x))

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- In Excel: $=$ NORMINV(RAND(), μ, σ)
- In R: rnorm(n,μ,σ) , where *n* is the number of normal random numbers that we ask R to generate
- Let's simulate 1000 standard normal random numbers and find the number of $X < x$