# STA 371G: Statistics and Modeling

## Decision Making Under Uncertainty: Decision Criteria and Utility Functions

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# Payoffs and Losses

The consequence of a decision can be expressed in terms of either payoffs or losses:

- A *payoff*, or *reward*, represents the net change in your total wealth as a result of your decision and the actual state of the world. This can be either positive or negative.
- A *loss* is the nonnegative difference between the given payoff and the *highest possible payoff under that state of the world*. It is intepretated in terms of "opportunity loss."
- $L(\text{action } i, \text{state } j) = \max_k R(\text{action } k, \text{state } j) - R(\text{action } i, \text{state } j)$

# Payoffs and Losses

- State of the world: *Rain* or *No Rain*
- Action: *Carry umbrella* or *Do not carry umbrella*

- Payoff Table:

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | $-\$3$ | $-\$3$ |
| *Do not carry umbrella* | $-\$90$ | $\$0$ |

- Loss Table

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | | |
| *Do not carry umbrella* | | |

# Payoffs and Losses

- State of the world: *Rain* or *No Rain*
- Action: *Carry umbrella* or *Do not carry umbrella*

- Payoff Table:

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | −\$3 | −\$3 |
| *Do not carry umbrella* | −\$90 | \$0 |

- Loss Table

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | \$0 | |
| *Do not carry umbrella* | \$87 | |

# Payoffs and Losses

- State of the world: *Rain* or *No Rain*
- Action: *Carry umbrella* or *Do not carry umbrella*

- Payoff Table:

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | −\$3 | −\$3 |
| *Do not carry umbrella* | −\$90 | \$0 |

- Loss Table

| | *Rain* | *No Rain* |
|---|---|---|
| *Carry umbrella* | \$0 | \$3 |
| *Do not carry umbrella* | \$87 | \$0 |

# Payoffs and Losses

- State of the world: *Oil* or *No Oil*
- Action: *Drill* or *Do not drill*

- Payoff Table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | \$1,000,000 | −\$300,000 |
| *Do not drill* | \$200,000 | \$100,000 |

- Loss Table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | | |
| *Do not drill* | | |

# Payoffs and Losses

- State of the world: *Oil* or *No Oil*
- Action: *Drill* or *Do not drill*

- Payoff Table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | \$1,000,000 | −\$300,000 |
| *Do not drill* | \$200,000 | \$100,000 |

- Loss Table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | \$0 | \$400,000 |
| *Do not drill* | \$800,000 | \$0 |

# Payoffs and Losses

- Payoff Table:

| | *Competitor introduces new product* | *Competitor introduces no new product* |
|---|---|---|
| *No advertising* | \$100,000 | \$700,000 |
| *Minor ad campaign* | \$300,000 | \$600,000 |
| *Major ad campaign* | \$400,000 | \$500,000 |

- Loss Table:

| | *Competitor introduces new product* | *Competitor introduces no new product* |
|---|---|---|
| *No advertising* | | |
| *Minor ad campaign* | | |
| *Major ad campaign* | | |

# Payoffs and Losses

- Payoff Table:

| | *Competitor introduces new product* | *Competitor introduces no new product* |
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| *No advertising* | \$100,000 | \$700,000 |
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| *Major ad campaign* | \$400,000 | \$500,000 |

- Loss Table:

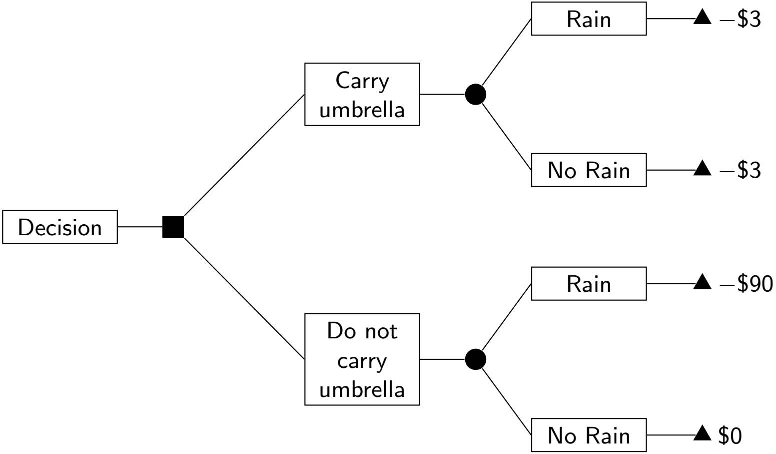
| | *Competitor introduces new product* | *Competitor introduces no new product* |
|---|---|---|
| *No advertising* | \$300,000 | \$0 |
| *Minor ad campaign* | \$100,000 | \$100,000 |
| *Major ad campaign* | \$0 | \$200,000 |

# Converting Payoffs to Losses

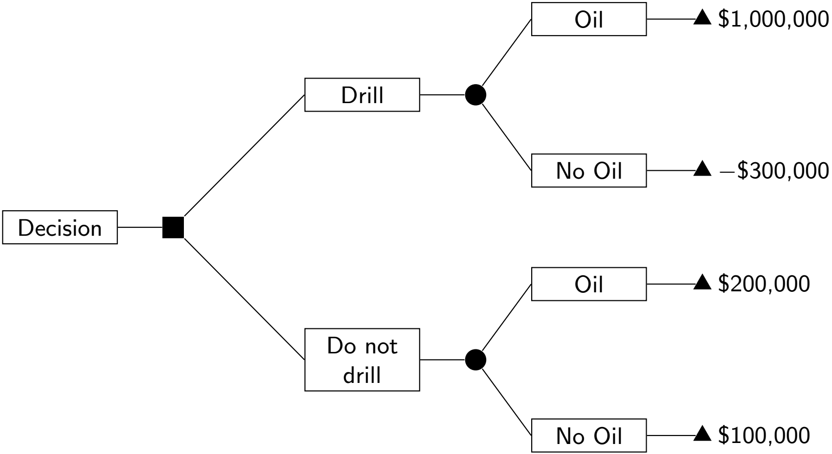
$$L(\text{action } i, \text{ state } j) = \max_k R(\text{action } k, \text{ state } j) - R(\text{action } i, \text{ state } j)$$

# Tree Diagram

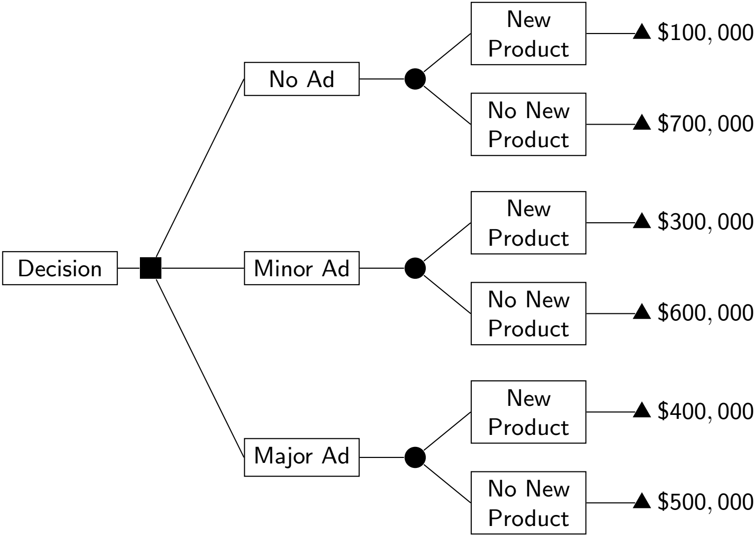
A tree diagram is a common way to present the payoffs or losses in a decision-making problem.



# Tree Diagram



# Tree Diagram



# Admissible and Inadmissible Actions

Admissible and inadmissible actions (decisions):

- An action is said to *dominate* a second action if for *each possible state of the world*, the first action leads to at least as high a payoff (or at least as small a loss) as the second action, and if for at least one state of the world, the first action leads to a higher payoff than the second action.
- If an action dominates the second option, then the second action is said to be *inadmissible*.
- A decision maker need to consider only admissible actions.
- Which action is inadmissible?

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 |
| Buy Stock C | −\$2000 | −\$4,000 | −\$8000 | \$2,000 |

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

*maximin* rule: for each action, find the smallest possible payoff, and then choose the action for which the smallest payoff is largest.

- Maximize the minimum payoff
- Assume the worst would happen
- Conservative
- A potential problem of the *maximin* rule:

| | State I | State II |
|---|---|---|
| Action 1 | \$100,000 | $-$\$1 |
| Action 2 | \$0 | \$0 |

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

*maximax* rule: for each action, find the largest possible payoff, and then choose the action for which the largest payoff is largest.

- Maximize the maximum payoff
- Assume the best would happen
- Risky
- A potential problem of the *maximax* rule:

| | State I | State II |
|---|---|---|
| Action 1 | \$100,000 | \$99,999 |
| Action 2 | \$100,001 | \$0 |

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

The *minimax loss*, or *minimax regret*, criterion: for each action, find the largest possible loss, and then choose the action for which the largest loss is smallest.

- Minimize the maximum possible loss
- Not as conservative as the *maximum* rule
- Not as risky as the *maximax* rule
- Loss tables (converted from the payoff tables from the previous two slides):

| | State I | State II |
|---|---|---|
| Action 1 | \$0 | 1 |
| Action 2 | \$100,000 | \$0 |

| | State I | State II |
|---|---|---|
| Action 1 | \$1 | \$0 |
| Action 2 | \$0 | \$99,000 |

- Which action to choose based on the *minimax loss* criterion?

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*:
- *maximax*:
- *minimax loss*:
- Payoff table

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 |

- Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*:
- *minimax loss*:
- Payoff table

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 |

- Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*: Buy Stock A
- *minimax loss*:
- Payoff table

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 |

- Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy Stock B
- *maximax*: Buy Stock A
- *minimax loss*: Buy Stock B
- Payoff table

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 |

- Loss table

| | State I | State II | State III | State IV |
|---|---|---|---|---|
| Buy Stock A | \$0 | \$0 | \$6,000 | \$0 |
| Buy Stock B | \$2000 | \$4,000 | \$2,000 | \$0 |
| Buy Stock C | \$4000 | \$10,000 | \$0 | \$0 |

# Probabilistic Criteria for Decision Making Under Uncertainty

Probabilistic Criteria:

- Expected Payoff ($ER$) criterion: choose the act with the highest expected payoff.
- Expected Loss ($EL$) criterion: choose the act with the smallest expected loss.
- $ER(action\ i) = \sum_j R(action\ i, state\ j)P(state\ j)$
- $EL(action\ i) = \sum_j L(action\ i, state\ j)P(state\ j)$
- The $ER$ and $EL$ criteria yield identical decisions:

$$
\begin{aligned}
EL(action\ i) &= \sum_j L(action\ i, state\ j)P(state\ j) \\
&= \sum_j (\max_k R(action\ k, state\ j) - R(action\ i, state\ j))P(state\ j) \\
&= \Big(\sum_j \max_k R(action\ k, state\ j)P(state\ j)\Big) - ER(action\ i)
\end{aligned}
$$

# Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- *ER* or *EL* criteria:
- $ER + EL =$
- Payoff table

| Probability: | 0.1 | 0.4 | 0.3 | 0.2 | |
|---|---|---|---|---|---|
| State: | State I | State II | State III | State IV | ER |
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 | |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 | |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 | |

- Loss table

| Probability: | 0.1 | 0.4 | 0.3 | 0.2 | |
|---|---|---|---|---|---|
| State: | State I | State II | State III | State IV | EL |
| Buy Stock A | \$0 | \$0 | \$6,000 | \$0 | |
| Buy Stock B | \$2000 | \$4,000 | \$2,000 | \$0 | |
| Buy Stock C | \$4000 | \$10,000 | \$0 | \$0 | |

# Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- *ER* or *EL* criteria: Buy Stock A
- $ER + EL = \$3000$
- Payoff table

| Probability:<br>State: | 0.1<br>State I | 0.4<br>State II | 0.3<br>State III | 0.2<br>State IV | ER |
|---|---|---|---|---|---|
| Buy Stock A | \$2,000 | \$6,000 | −\$6,000 | \$2,000 | \$1,200 |
| Buy Stock B | \$0 | \$2,000 | −\$2,000 | \$2,000 | \$600 |
| Buy Stock C | −\$2000 | −\$4,000 | \$0 | \$2,000 | −\$1,400 |

- Loss table

| Probability:<br>State: | 0.1<br>State I | 0.4<br>State II | 0.3<br>State III | 0.2<br>State IV | EL |
|---|---|---|---|---|---|
| Buy Stock A | \$0 | \$0 | \$6,000 | \$0 | \$1,800 |
| Buy Stock B | \$2000 | \$4,000 | \$2,000 | \$0 | \$2,400 |
| Buy Stock C | \$4000 | \$10,000 | \$0 | \$0 | \$4,400 |

# Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

# Utility

Will you take the bet?

- Toss a coin once, you win \$1 if the coin comes up heads, and you loose \$0.75 if the coin comes up tails.
- Toss a coin once, you win \$1000 if the coin comes up heads, and you loose \$750 if the coin comes up tails.
- Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you loose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

- You win \$10 million if a coin comes up heads and win \$10 million if the conn comes up tails.
- You win \$100 million if a coin comes up heads and win 0 if the conn comes up tails.

# St. Petersburg Paradox

Will you take the bet?

- You win $2^0 = 1$ dollar if heads appear on the first toss.
- You win $2^1 = 2$ dollar if the first heads appear on the second toss.
- ...
- You win $2^k$ dollar if the first heads appear on the $k$th toss.
- ...

$$ER(gamble) = \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

Again, based on the *ER* criterion, you should take the bet, but...

# Axioms of Utility

A utility function can be interpreted in terms of a preference relationship. The two basic axioms of utility are:

- If payoff $R_1$ is preferred to payoff $R_2$, then $U(R_1) > U(R_2)$; if $R_2$ is preferred to $R_1$, then $U(R_2) > U(R_1)$; and if neither is preferred to the other, then $U(R_1) = U(R_2)$.
- If you are indifferent between (a) receiving payoff $R_1$ for certain and (b) taking a bet or lottery in which you receive payoff $R_2$ with probability $p$ and payoff $R_3$ with probability $1 - p$, then

$$U(R_1) = pU(R_2) + (1 - p)U(R_3).$$

# Assessment of Utility Functions

Denote the most preferable payoff as $R^*$ and the least preferable payoff as $R_*$. Suppose that you let $U(R_*) = 0$ and $U(R^*) = 1$, then $0 \leq U(R) \leq 1$ for for any payoff $R$.

Consider the following choice of lotteries:

- Lottery I: Receive $R$ for certain
- Lottery II: Receive $R^*$ with probability $p$ and $R_*$ with probability $1 - p$.

If $U(R) > p$, then choose Lottery I; if $U(R) < p$, then choose Lottery II; and if $U(R) = p$, then you are indifferent between the two lotteries.

With $U(R_*) = 0$ and $U(R^*) = 1$, your utility for a payoff can be interpreted in terms of an indifference probability.

# Example: Oil Drilling

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | \$1,000,000 | $-$\$300,000 |
| *Do not drill* | \$200,000 | \$100,000 |

- Utility:
  - The utility of \$200,000 is $p_1$ if you are indifferent between:
    - Lottery I: Receive $\$200,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability $p_1$ and $-\$300,000$ with probability $1 - p_1$.
  - The utility of \$100,000 is $p_2$ if you are indifferent between:
    - Lottery I: Receive $\$100,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability $p_2$ and $-\$300,000$ with probability $1 - p_2$.
  - Utility table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | | |
| *Do not drill* | | |

# Example: Oil Drilling

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | \$1,000,000 | −\$300,000 |
| *Do not drill* | \$200,000 | \$100,000 |

- Utility:
  - The utility of \$200,000 is $p_1$ if you are indifferent between:
    - Lottery I: Receive $\$200,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability $p_1$ and $-\$300,000$ with probability $1 - p_1$.
  - The utility of \$100,000 is $p_2$ if you are indifferent between:
    - Lottery I: Receive $\$100,000$ for certain
    - Lottery II: Receive $\$1,000,000$ with probability $p_2$ and $-\$300,000$ with probability $1 - p_2$.
  - Utility table:

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | 1 | 0 |
| *Do not drill* | $p_1$ | $p_2$ |

# Example: Oil Drilling

Suppose $P(Oil) = 0.3$ and the utility table is

| | *Oil* | *No Oil* |
|---|---|---|
| *Drill* | 1 | 0 |
| *Do not drill* | 0.45 | 0.35 |

then

$$EU(Drill) = 0.3 * 1 + 0.7 * 0 = 0.30,$$

$$EU(Do\ Not\ Drill) = 0.3 * 0.45 + 0.7 * 0.35 = 0.38.$$

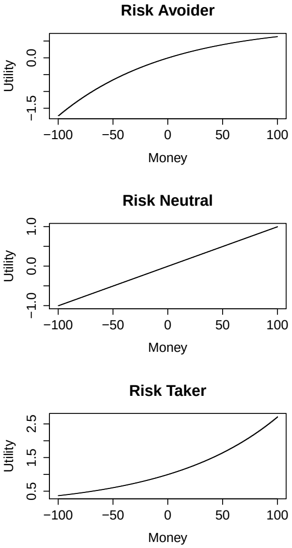
The decision maker should not drill according to the *EU* criterion.

# Utility and Money

- A Risk Avoider would be willing to pay a risk premium to avoid a fair bet.
- Risk Neutral: indifferent about a fair bet.
- A Risk Taker would be willing to pay money to take a fair bet.

When you are shopping for car insurance, are you a Risk Avoider or Taker?

When you are buying a lottery ticket, are you a Risk Avoider or Taker?



# Exponential Utility

$U(x) = 1 - e^{-x/R}$

- $x$ is the payoff
- $R > 0$ is the risk tolerance that specifies an individual's aversion to risk
- The risk tolerance $R$ can be assessed as the amount that makes the decision maker indifferent between the following two lotteries:
  - Lottery I: obtain zero payoff;
  - Lottery II: win $R$ dollars with probability $1/2$ or loose $R/2$ with probability $1/2$.
- Approximate values of $R$ for a company: 6.4% of net sales, 124% of net income, and 15.7% of equality

# Example: Exponential Utility

Venture Limited, with net sales of \$30,000,000, need to decide:

- Invest in a sure thing with a return of \$125,000;
- Invest in a low risky venture that returns $\$-500{,}000$, \$100,000, or \$1,000,000 with probabilities 0.25, 0.50, and 0.25, respectively;
- Invest in a high risky venture that returns $\$-1{,}000{,}000$, \$1,000,000, or \$3,000,000 with probabilities 0.35, 0.60, and 0.05, respectively;

Your job is to:

- find the best decision in terms of expected payoff
- find the best decision in terms of expected utility