STA 371G: Statistics and Modeling

Time Series: Autoregressive Models

Mingyuan Zhou
McCombs School of Business
The University of Texas at Austin

http://mingyuanzhou.github.io/STA371G

Time Series Regression... Hotel Occupancy Case

In a recent legal case, a Chicago downtown hotel claimed that it had suffered a loss of business due to what was considered an illegal action by a group of hotels that decided to leave the plaintiff out of a hotel directory.

In order to estimate the loss business, the hotel had to predict what its level of business (in terms of occupancy rate) would have been in the absence of the alleged illegal action.

In order to do this, experts testifying on behalf of the hotel use data collected before the period in question and fit a relationship between the hotels occupancy rate and overall occupancy rate in the city of Chicago. This relationship would then be used to predict occupancy rate during the period in question.

Example: Hotel Occupancy Case

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.7111011
R Square	0.5056648
Adjusted R Squa	0.48801
Standard Error	7.5055176
Observations	30

$$Hotel_t = \beta_0 + \beta_1 Chicago_t + \epsilon_t$$

ANOVA

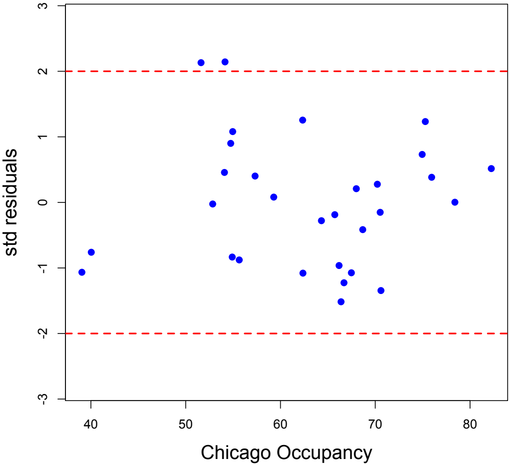
	df	SS	MS	F	Significance F
Regression	1	1613.468442	1613.4684	28.64172598	1.06082E-05
Residual	28	1577.318225	56.332794		
Total	29	3190.786667			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	16.135666	8.518889357	1.8941044	0.068584205	-1.314487337	33.5858198
ChicagoInd	0.7161318	0.133811486	5.3517965	1.06082E-05	0.442031445	0.990232246

- In the month after the omission from the directory the Chicago occupancy rate was 66. The plaintiff claims that its occupancy rate should have been 16 + 0.71*66 = 62.
- It was actually 55!! The difference added up to a big loss!!
- Under this model, what's the probability for $Hotel_{31} \leq 55$?

Example: Hotel Occupancy Case

A statistician was hired by the directory to access the regression methodology used to justify the claim. As we should know by now, the first thing he looked at was the residual plot...



Looks fine. However...

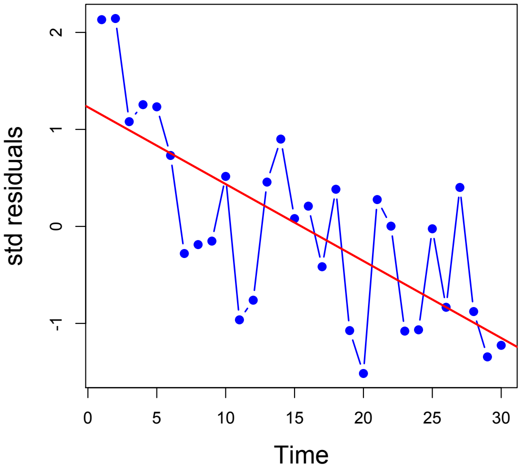
Example: Hotel Occupancy Case

... this is a *time series regression*, as we are regressing one time series on another.

In this case, we should also check whether or not the residuals show some temporal pattern.

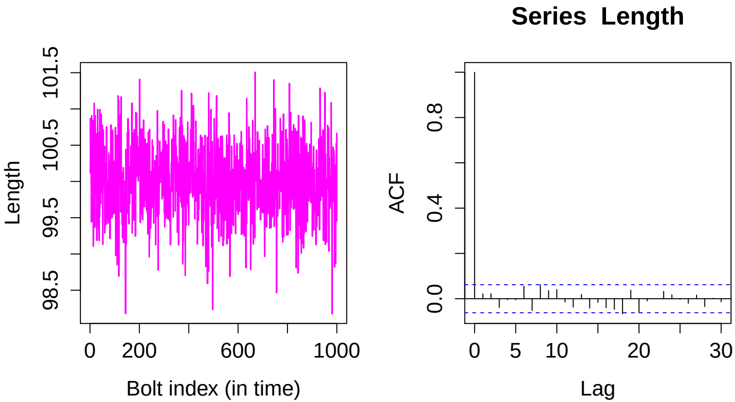
If our model is correct the residuals should look iid normal over time.

Example: Hotel Occupancy Case

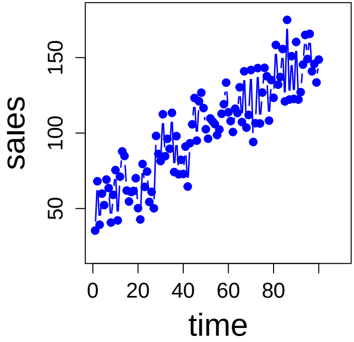


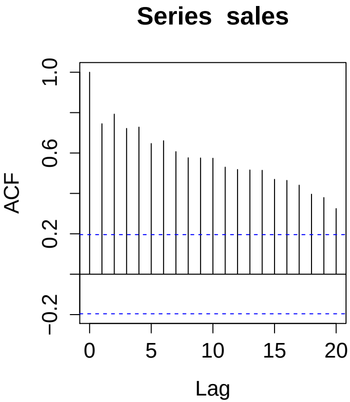
Does this look iid to you? Can you guess what does the red line represent?

Autocorrelation

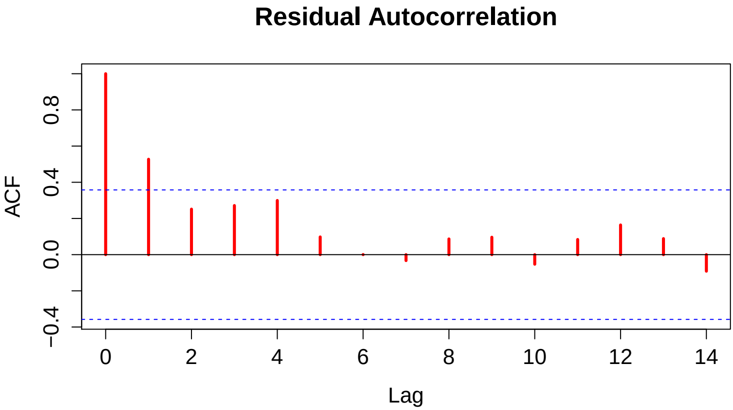


Autocorrelation





Example: Hotel Occupancy Case



Example: Hotel Occupancy Case

It looks like part of hotel occupancy (y) not explained by the Chicago downtown occupancy (x) is moving down over time. We can try to control for that by adding a trend factor to our model...

$$Hotel_t = \beta_0 + \beta_1 Chicago_t + \beta_2 t + \epsilon_t$$

SUMMARY OUTPUT

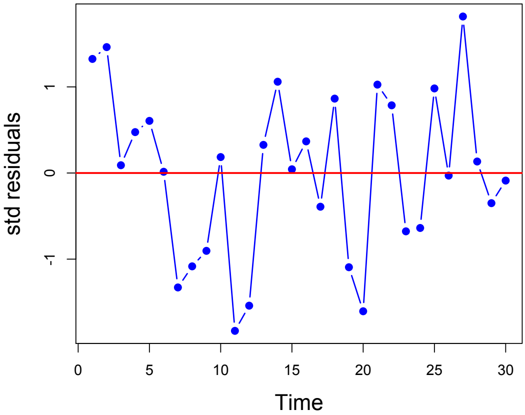
Regression Statistics	
Multiple R	0.869389917
R Square	0.755838827
Adjusted R Squ	0.737752815
Standard Error	5.37162026
Observations	30

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	2411.720453	1205.86	41.79134652	5.41544E-09
Residual	27	779.0662139	28.8543		
Total	29	3190.786667			

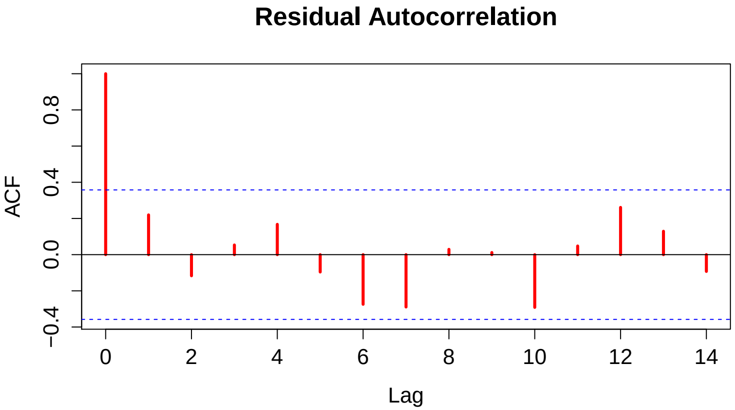
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	26.69391108	6.418837165	4.158683	0.000290493	13.52354525	39.8642769
ChicagoInd	0.69523791	0.095849831	7.253408	8.41391E-08	0.498570304	0.89190552
t	-0.596476666	0.113404099	-5.259745	1.51653E-05	-0.82916265	-0.3637907

Example: Hotel Occupancy Case



Much better!! What is the slope of the red line?

Example: Hotel Occupancy Case



Example: Hotel Occupancy Case

Okay, what happened?!

Well, once we account for the downward trend in the occupancy of the plaintiff, the prediction for the occupancy rate is

$$26 + 0.69 * 66 - 0.59 * 31 = 53.25$$

What do we conclude?

Example: Hotel Occupancy Case

Take away lessons...

- When regressing a time series on another, always check the residuals as a time series
- What does that mean... plot the residuals over time. If all is well, you should see no patterns, i.e., they should behave like iid normal samples.

Example: Hotel Occupancy Case

Question

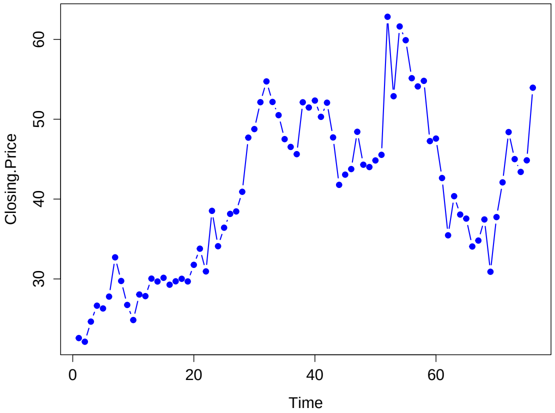
- What if we were interested in predicting the hotel occupancy ten years from now?? We should compute

$$26 + 0.69 * 66 - 0.59 * 150 = -16.96$$

- Would you trust this prediction? Could you defend it in court?
- Remember: always be careful with extrapolating relationships!

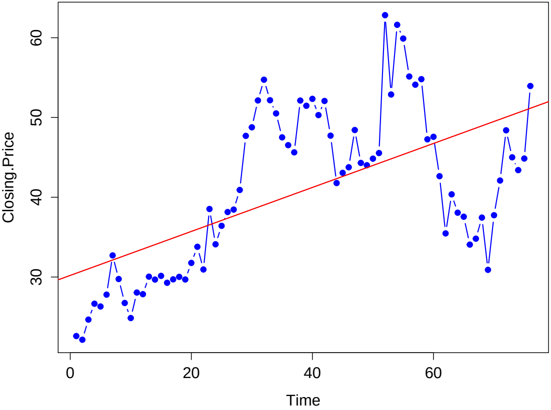
Example: Monthly Stock Closing Prices

How to model this time series?



Example: Monthly Stock Closing Prices

Let's first model the trend using $Y_t = \beta_0 + \beta_1 t + \epsilon_t$



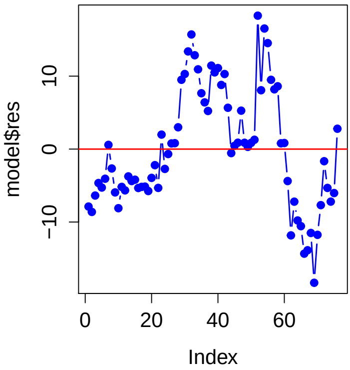
Example: Monthly Stock Closing Prices

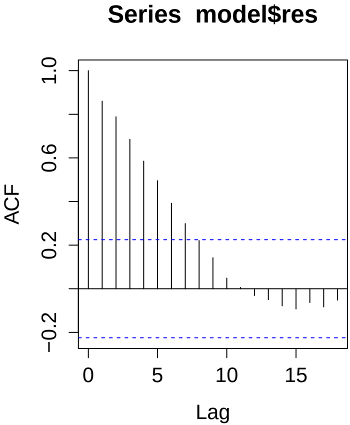
The following summary looks fine

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 30.19762    1.90827  15.825  < 2e-16 ***
Time         0.27577    0.04307   6.404 1.24e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 8.236 on 74 degrees of freedom
Multiple R-squared:  0.3565,    Adjusted R-squared:  0.3479
F-statistic:    41 on 1 and 74 DF,  p-value: 1.245e-08
```

Example: Monthly Stock Closing Prices

But let's look at the residual time series and its autocorrelation





Random Walk Model

$Y_t = Y_{t-1} + \mu + \epsilon_t$

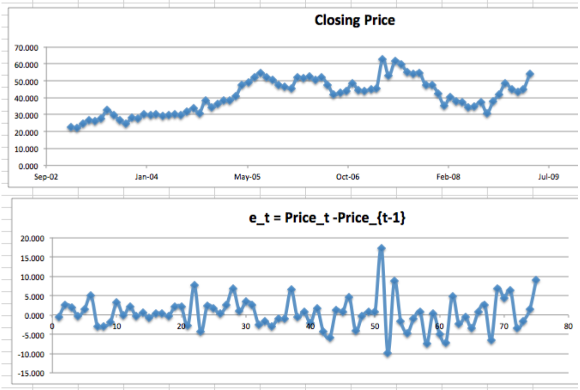
- The time series itself is not random
- The differences between consecutive times are random

$$Y_t - Y_{t-1} = \mu + \epsilon_t$$

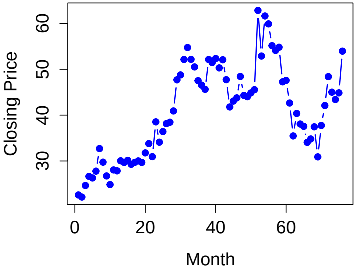
Random Walk Model

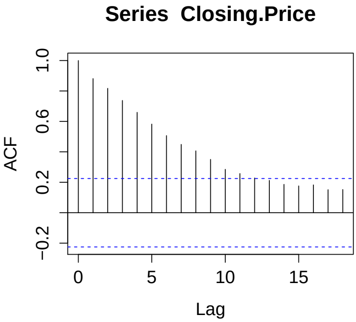
$$Y_t = Y_{t-1} + \mu + \epsilon_t$$

Analyzing monthly stock closing prices

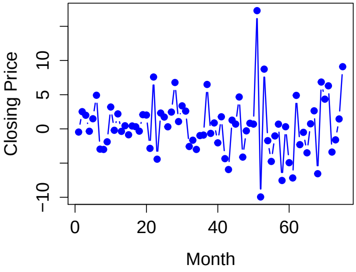


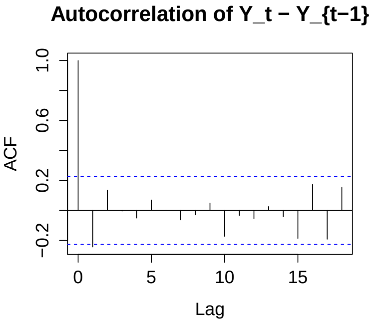
Random Walk Model



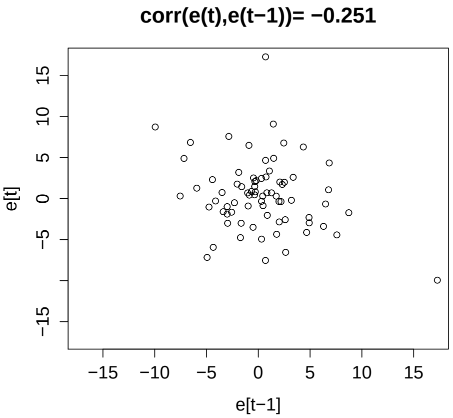


Random Walk Model





Random Walk Model



Can we do better?

The AR(1) Model

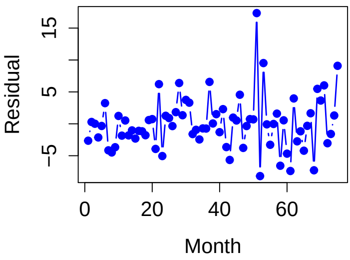
A simple way to model dependence over time in with the autoregressive model of order 1...

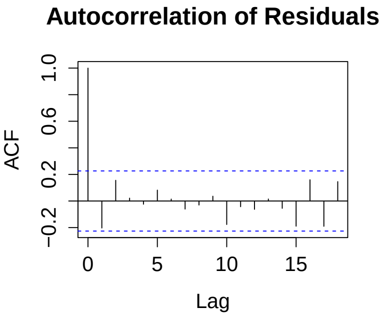
$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \epsilon_t$$

- What is the mean of Y_t for a given value of Y_{t-1}?
- If the model successfully captures the dependence structure in the data then the residuals should look iid.
- Remember: if our data is collected in time, we should always check for dependence in the residuals...

Example: Monthly Stock Closing Prices

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \epsilon_t$$





Example: Monthly Stock Closing Prices

Regression Statistics						
R	0.91171					
R Square	0.83121					
Adjusted R Square	0.8289					
Standard Error	4.15447					
Total Number Of Cases	75				Plot Area	

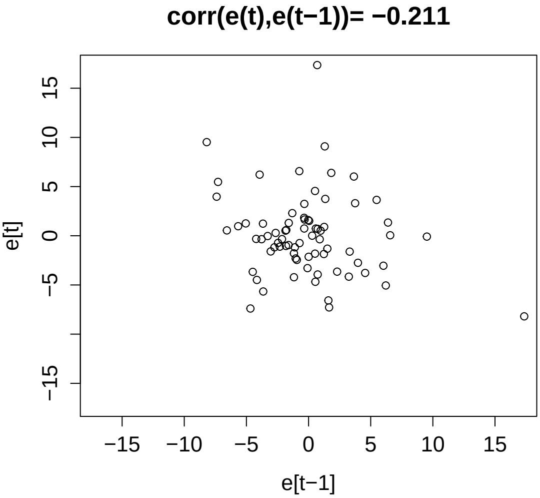
Closing Price [t] = 4.4009 + 0.9020 * Closing Price [t-1]

ANOVA					
	d.f.	*SS*	*MS*	*F*	*p-level*
Regression	1.	6,204.67064	6,204.67064	359.49056	0.E+0
Residual	73.	1,259.95229	17.25962		
Total	74.	7,464.62293			Chart Area

	Coefficients	*Standard Error*	*LCL*	*UCL*	*t Stat*	*p-level*
Intercept	4.40093	1.99197	0.43093	8.37092	2.20933	0.03029
Closing Price [t-1]	0.90199	0.04757	0.80718	0.99681	18.96024	0.E+0

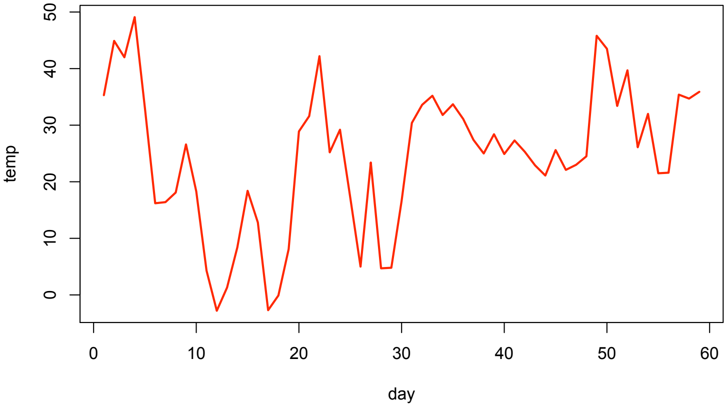
Example: Monthly Stock Closing Prices

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \epsilon_t$$



Examples: Temperatures

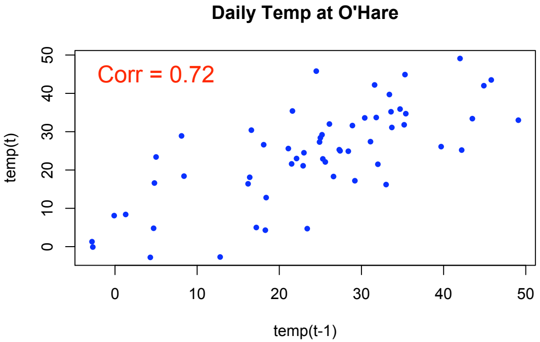
Now you need to predict tomorrow's temperature at O'Hare from (Jan-Feb).



Does this look iid? If it is iid, tomorrow's temperatures should not depend on today's... does that make sense?

Checking for Dependence

To see if Y_{t-1} would be useful for predicting Y_t, just plot them together and see if there is a relationship.



Correlation between Y_t and Y_{t-1} is called autocorrelation.

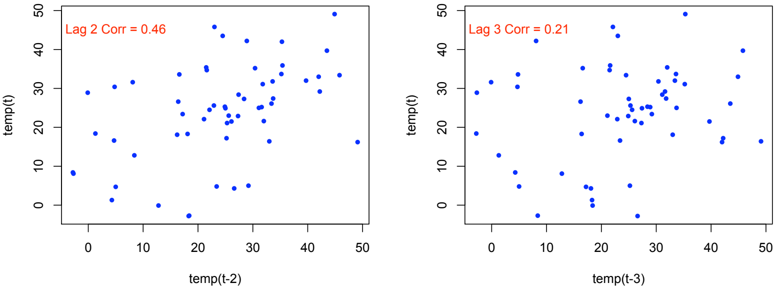
Checking for Dependence

You need to create a “lagged” variable $temp_{t-1}$... the data looks like this:

t	temp(t)	temp(t-1)
1	42	35
2	41	42
3	50	41
4	19	50
5	19	19
6	20	19
...	...	

Checking for Dependence

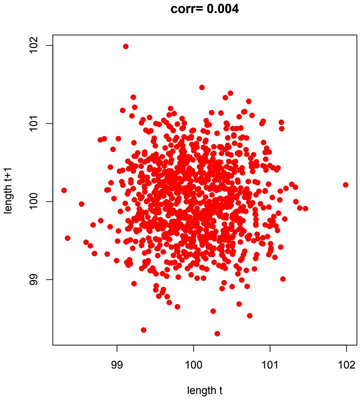
We can plot Y_t against Y_{t-L} to see L-period lagged relationships.

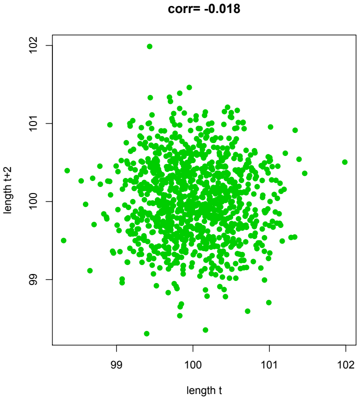


- It appears that the correlation is getting weaker with increasing L.
- How can we test for this dependence?

Checking for Dependence

Back to the "length of a bolt" example. When things are not related in time we should see...





The AR(1) Model

Again, the regression tool is our friend here... (Why?)

SUMMARY OUTPUT

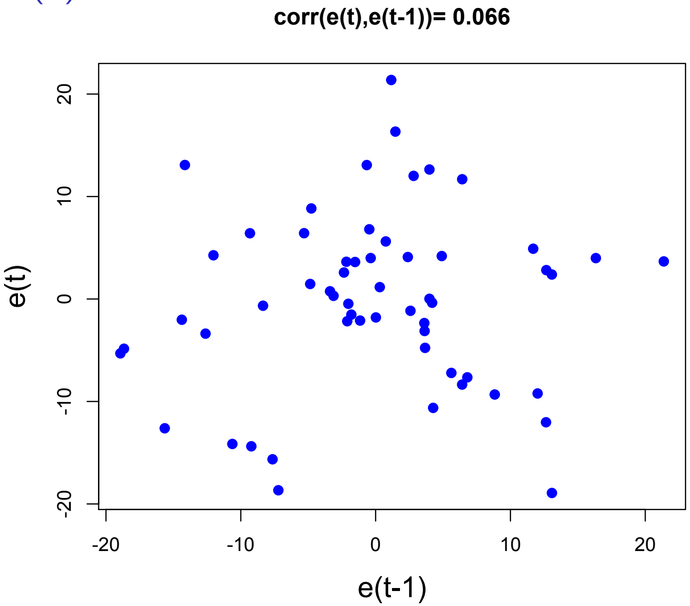
Regression Statistics	
Multiple R	0.722742583
R Square	0.522356842
Adjusted R Sq	0.5138275
Standard Erro	8.789861051
Observations	58

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	4731.684433	4731.684433	61.24233673	1.49699E-10
Residual	56	4326.652809	77.2616573		
Total	57	9058.337241			

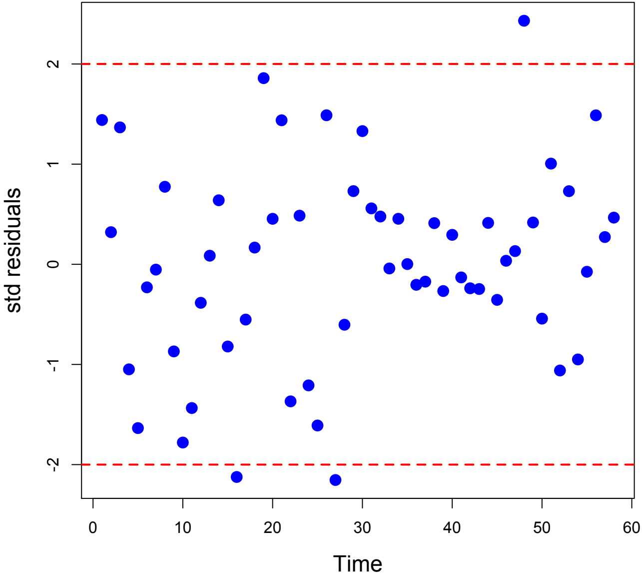
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.705800085	2.516614758	2.664611285	0.010050177	1.664414964	11.74718521
X Variable 1	0.723288866	0.092424243	7.825748317	1.49699E-10	0.53814086	0.908436873

The AR(1) Model



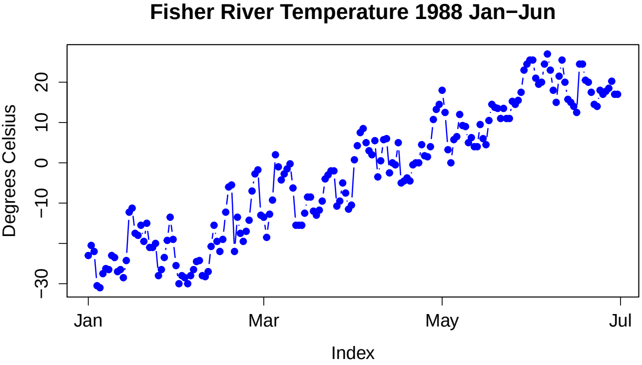
No dependence left!

The AR(1) Model



Again, looks good...

Example: Fisher River Temperature 1988 January to June



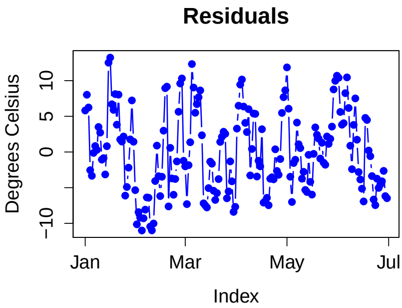
Example: Fisher River Temperature 1988 January to June

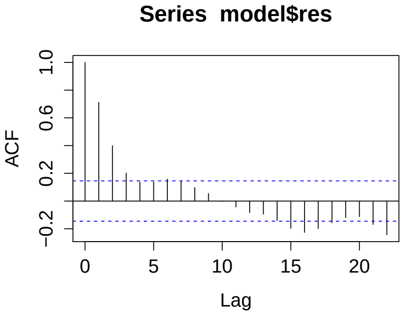
$$Y_t = \beta_0 + \beta_1 t + \epsilon_t$$

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -29.105929   0.860407  -33.83   <2e-16 ***
Time          0.288914   0.008155   35.43   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.78 on 180 degrees of freedom
Multiple R-squared:  0.8746,    Adjusted R-squared:  0.8739
F-statistic:  1255 on 1 and 180 DF,  p-value: < 2.2e-16
```

Example: Fisher River Temperature 1988 January to June

$$Y_t = \beta_0 + \beta_1 t + \epsilon_t$$



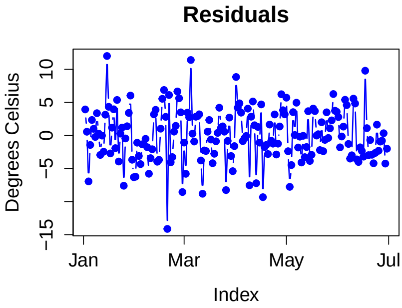


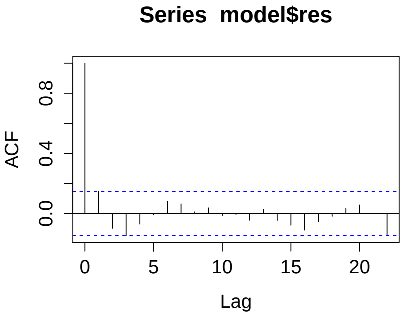
Example: Fisher River Temperature 1988 January to June

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t$$

Example: Fisher River Temperature 1988 January to June

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t$$



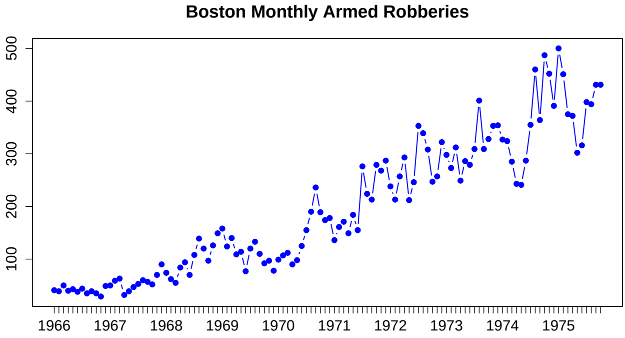


Example: Fisher River Temperature 1988 January to June

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t$$

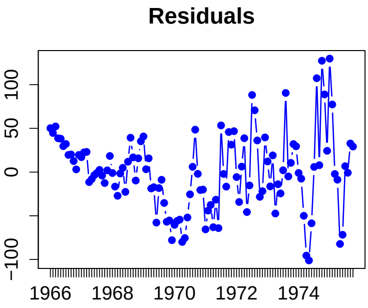
```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)     -8.03628    1.64254  -4.893 2.22e-06 ***
Fisher[1:181]    0.71663    0.05235  13.689  < 2e-16 ***
Time             0.08209    0.01624   5.054 1.07e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.045 on 178 degrees of freedom
Multiple R-squared:  0.9387,    Adjusted R-squared:  0.938
F-statistic:  1363 on 2 and 178 DF,  p-value: < 2.2e-16
```

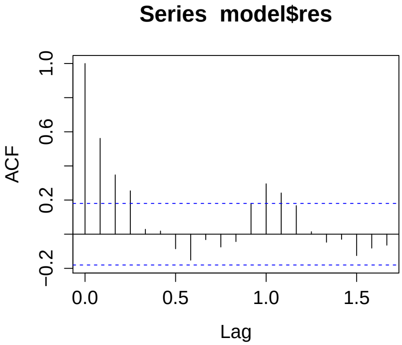
Example: Monthly Boston Armed Robberies Jan.1966-Oct.1975



Example: Boston Armed Robberies

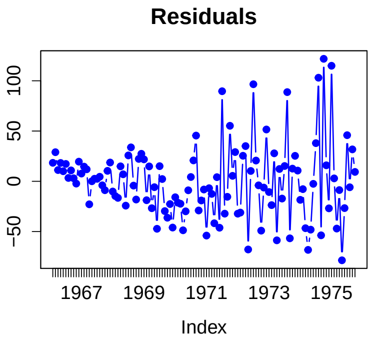
$$Y_t = \beta_0 + \beta_1 t + \epsilon_t$$

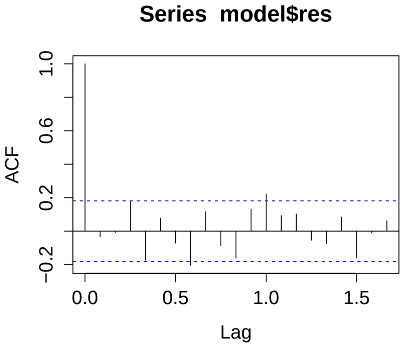




Example: Boston Armed Robberies

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t$$





Are the model assumptions violated?

Can we do better?

Example: Boston Armed Robberies

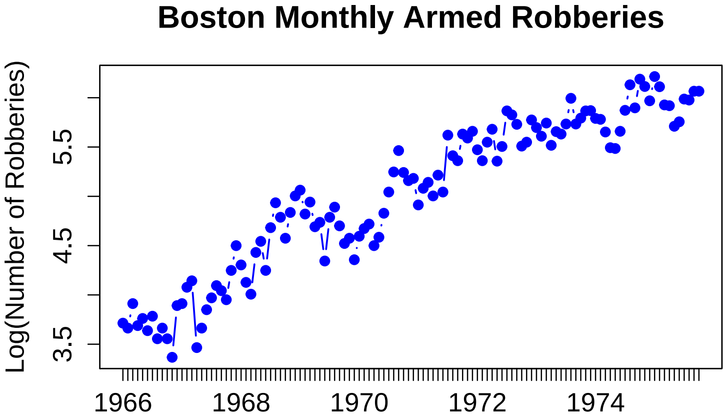
$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t$$

```
Coefficients:
                                      Estimate Std. Error t value Pr(>|t|)
(Intercept)                           -4.06758    6.89469  -0.590    0.556
BostonRobbery[1:length(BostonRobbery) - 1]  0.56397    0.07691   7.333 3.52e-11 ***
Time                                   1.56124    0.28733   5.434 3.17e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 36.7 on 114 degrees of freedom
Multiple R-squared:  0.9189,    Adjusted R-squared:  0.9175
F-statistic: 645.9 on 2 and 114 DF,  p-value: < 2.2e-16
```

Interpretation?

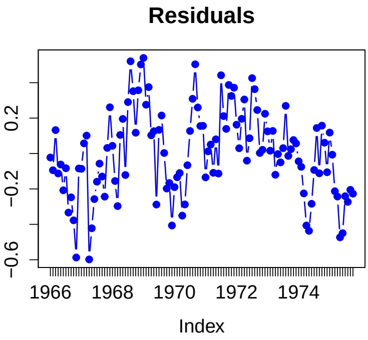
Example: Boston Armed Robberies

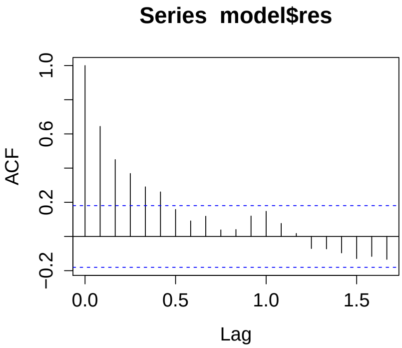
Let's take the log transformation of the number of armed robberies



Example: Boston Armed Robberies

$$Log(Y_t) = \beta_0 + \beta_1 t + \epsilon_t$$





Can we do better?

Example: Boston Armed Robberies

$$Log(Y_t) = \beta_0 + \beta_1 t + \epsilon_t$$

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 3.7142003  0.0457093   81.26   <2e-16 ***
Time        0.0218550  0.0006667   32.78   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.2467 on 116 degrees of freedom
Multiple R-squared:  0.9026,    Adjusted R-squared:  0.9017 
F-statistic:  1075 on 1 and 116 DF,  p-value: < 2.2e-16
```

Interpretation?

Example: Boston Armed Robberies

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```
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```
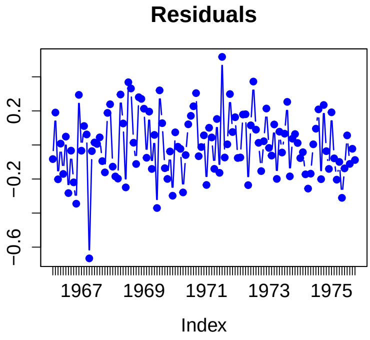
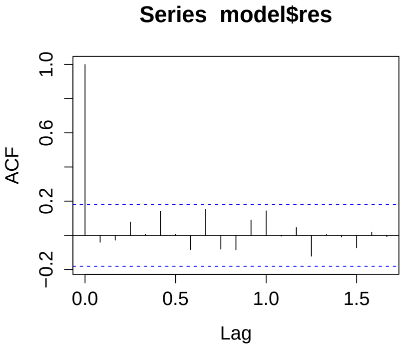

Interpretation?

The number of Armed Robberies increased by about 2.2% every month!

Example: Boston Armed Robberies

$$Log(Y_t) = \beta_0 + \beta_1 Log(Y_{t-1}) + \beta_2 t + \epsilon_t$$





Much better!

Example: Boston Armed Robberies

$$Log(Y_t) = \beta_0 + \beta_1 Log(Y_{t-1}) + \beta_2 t + \epsilon_t$$

```
Coefficients:
                                            Estimate Std. Error t value Pr(>|t|)
(Intercept)                                 1.329217   0.268444   4.952 2.57e-06 ***
log(BostonRobbery[1:length(BostonRobbery) - 1]) 0.648915   0.071723   9.047 4.57e-15 ***
Time                                        0.007598   0.001658   4.582 1.19e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1898 on 114 degrees of freedom
Multiple R-squared:  0.9419,    Adjusted R-squared:  0.9409
F-statistic: 924.5 on 2 and 114 DF,  p-value: < 2.2e-16
```