

# STA 371G: Statistics and Modeling

## Decision Making Under Uncertainty: Decision Criteria and Utility Functions

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# Payoffs and Losses

The consequence of a decision can be expressed in terms of either payoffs or losses:

- ▶ A *payoff*, or *reward*, represents the net change in your total wealth as a result of your decision and the actual state of the world. This can be either positive or negative.
- ▶ A *loss* is the nonnegative difference between the given payoff and the *highest possible payoff under that state of the world*. It is interpreted in terms of “opportunity loss.”
- ▶  $L(\text{action } i, \text{state } j) = \max_k R(\text{action } k, \text{state } j) - R(\text{action } i, \text{state } j)$

# Payoffs and Losses

- ▶ State of the world: *Rain* or *No Rain*
- ▶ Action: *Carry umbrella* or *Do not carry umbrella*
- ▶ Payoff Table:

	<i>Rain</i>	<i>No Rain</i>
<i>Carry umbrella</i>	−\$3	−\$3
<i>Do not carry umbrella</i>	−\$90	\$0

- ▶ Loss Table

	<i>Rain</i>	<i>No Rain</i>
<i>Carry umbrella</i>		
<i>Do not carry umbrella</i>		

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	<i>Rain</i>	<i>No Rain</i>
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<i>Do not carry umbrella</i>	\$87	\$0

# Payoffs and Losses

- ▶ State of the world: *Oil* or *No Oil*
- ▶ Action: *Drill* or *Do not drill*

- ▶ Payoff Table:

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	\$1,000,000	−\$300,000
<i>Do not drill</i>	\$200,000	\$100,000

- ▶ Loss Table:

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>		
<i>Do not drill</i>		

# Payoffs and Losses

- ▶ State of the world: *Oil* or *No Oil*
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	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	\$1,000,000	−\$300,000
<i>Do not drill</i>	\$200,000	\$100,000

- ▶ Loss Table:

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	\$0	\$400,000
<i>Do not drill</i>	\$800,000	\$0

# Payoffs and Losses

► Payoff Table:

	<i>Competitor introduces new product</i>	<i>Competitor introduces no new product</i>
<i>No advertising</i>	\$100,000	\$700,000
<i>Minor ad campaign</i>	\$300,000	\$600,000
<i>Major ad campaign</i>	\$400,000	\$500,000

► Loss Table:

	<i>Competitor introduces new product</i>	<i>Competitor introduces no new product</i>
<i>No advertising</i>		
<i>Minor ad campaign</i>		
<i>Major ad campaign</i>		



# Payoffs and Losses

► Payoff Table:

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► Loss Table:

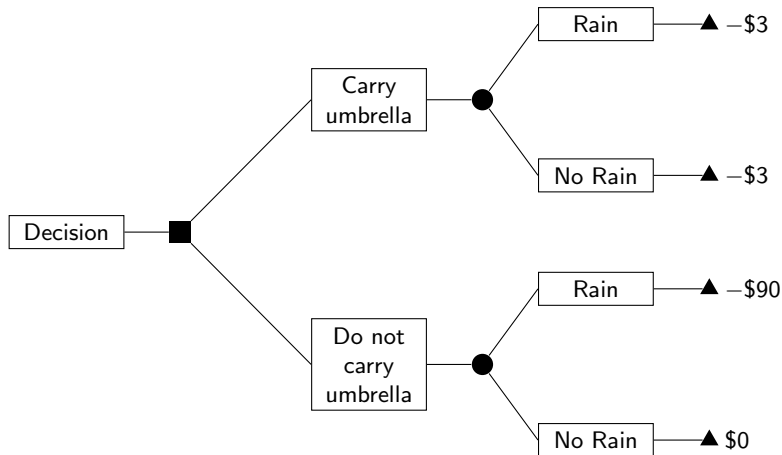
	<i>Competitor introduces new product</i>	<i>Competitor introduces no new product</i>
<i>No advertising</i>	\$300,000	\$0
<i>Minor ad campaign</i>	\$100,000	\$100,000
<i>Major ad campaign</i>	\$0	\$200,000

## Converting Payoffs to Losses

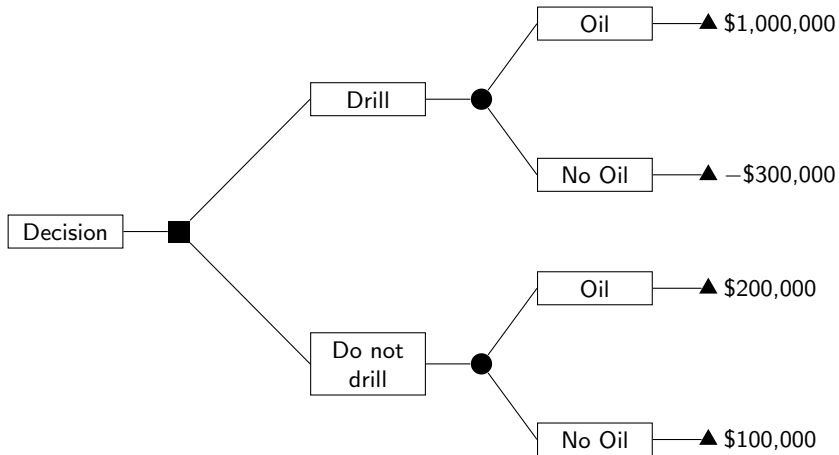
$$L(\text{action } i, \text{state } j) = \max_k R(\text{action } k, \text{state } j) - R(\text{action } i, \text{state } j)$$

# Tree Diagram

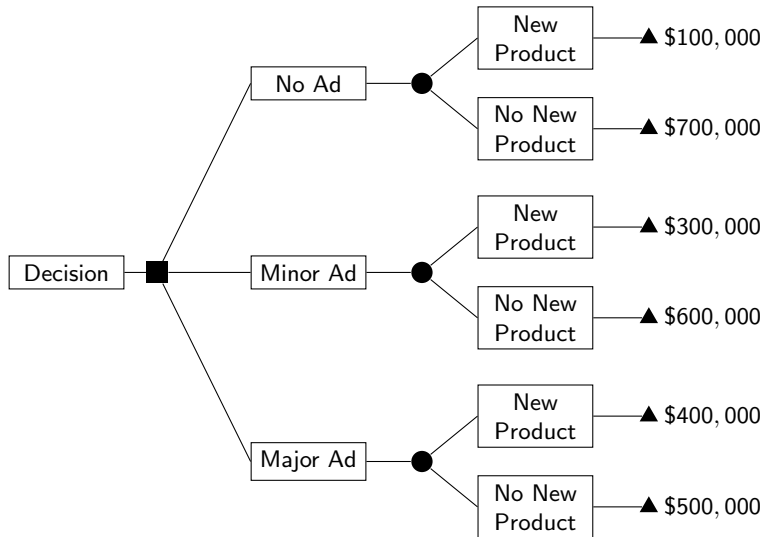
A tree diagram is a common way to present the payoffs or losses in a decision-making problem.



# Tree Diagram



# Tree Diagram



# Admissible and Inadmissible Actions

Admissible and inadmissible actions (decisions):

- ▶ An action is said to *dominate* a second action if for *each possible state of the world*, the first action leads to at least as high a payoff (or at least as small a loss) as the second action, and if for at least one state of the world, the first action leads to a higher payoff than the second action.
- ▶ If an action dominates the second option, then the second action is said to be *inadmissible*.
- ▶ A decision maker need to consider only admissible actions.
- ▶ Which action is inadmissible?

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000
Buy Stock C	-\$2000	-\$4,000	-\$8000	\$2,000

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

*maximin* rule: for each action, find the smallest possible payoff, and then choose the action for which the smallest payoff is largest.

- ▶ Maximize the minimum payoff
- ▶ Assume the worst would happen
- ▶ Conservative
- ▶ A potential problem of the *maximin* rule:

	State I	State II
Action 1	\$100,000	-\$1
Action 2	\$0	\$0

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

*maximax* rule: for each action, find the largest possible payoff, and then choose the action for which the largest payoff is largest.

- ▶ Maximize the maximum payoff
- ▶ Assume the best would happen
- ▶ Risky
- ▶ A potential problem of the *maximax* rule:

	State I	State II
Action 1	\$100,000	\$99,999
Action 2	\$100,001	\$0



# Nonprobabilistic Criteria for Decision Making Under Uncertainty

The *minimax loss*, or *minimax regret*, criterion: for each action, find the largest possible loss, and then choose the action for which the largest loss is smallest.

- ▶ Minimize the maximum possible loss
- ▶ Not as conservative as the *maximum* rule
- ▶ Not as risky as the *maximax* rule
- ▶ Loss tables (converted from the payoff tables from the previous two slides):

	State I	State II
Action 1	\$0	1
Action 2	\$100,000	\$0

	State I	State II
Action 1	\$1	\$0
Action 2	\$0	\$99,000

- ▶ Which action to choose based on the *minimax loss* criterion?

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*:
- ▶ *maximax*:
- ▶ *minimax loss*:
- ▶ Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000
Buy Stock C	-\$2000	-\$4,000	\$0	\$2,000

- ▶ Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*: Buy Stock B
- ▶ *maximax*:
- ▶ *minimax loss*:
- ▶ Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000
Buy Stock C	-\$2000	-\$4,000	\$0	\$2,000

- ▶ Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*: Buy Stock B
- ▶ *maximax*: Buy Stock A
- ▶ *minimax loss*:
- ▶ Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000
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- ▶ Loss table

# Nonprobabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*: Buy Stock B
- ▶ *maximax*: Buy Stock A
- ▶ *minimax loss*: Buy Stock B
- ▶ Payoff table

	State I	State II	State III	State IV
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000
Buy Stock C	-\$2000	-\$4,000	\$0	\$2,000

- ▶ Loss table

	State I	State II	State III	State IV
Buy Stock A	\$0	\$0	\$6,000	\$0
Buy Stock B	\$2000	\$4,000	\$2,000	\$0
Buy Stock C	\$4000	\$10,000	\$0	\$0

# Probabilistic Criteria for Decision Making Under Uncertainty

Probabilistic Criteria:

- ▶ Expected Payoff ( $ER$ ) criterion: choose the act with the highest expected payoff.
- ▶ Expected Loss ( $EL$ ) criterion: choose the act with the smallest expected loss.
- ▶  $ER(\text{action } i) = \sum_j R(\text{action } i, \text{state } j)P(\text{state } j)$
- ▶  $EL(\text{action } i) = \sum_j L(\text{action } i, \text{state } j)P(\text{state } j)$
- ▶ The  $ER$  and  $EL$  criteria yield identical decisions:

$$\begin{aligned} EL(\text{action } i) &= \sum_j L(\text{action } i, \text{state } j)P(\text{state } j) \\ &= \sum_j (\max_k R(\text{action } k, \text{state } j) - R(\text{action } i, \text{state } j))P(\text{state } j) \\ &= \left( \sum_j \max_k R(\text{action } k, \text{state } j)P(\text{state } j) \right) - ER(\text{action } i) \end{aligned}$$

# Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- ▶ *ER* or *EL* criteria:
- ▶  $ER + EL =$
- ▶ Payoff table

Probability: State:	0.1 State I	0.4 State II	0.3 State III	0.2 State IV	ER
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000	
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000	
Buy Stock C	-\$2000	-\$4,000	\$0	\$2,000	

- ▶ Loss table

Probability: State:	0.1 State I	0.4 State II	0.3 State III	0.2 State IV	EL
Buy Stock A	\$0	\$0	\$6,000	\$0	
Buy Stock B	\$2000	\$4,000	\$2,000	\$0	
Buy Stock C	\$4000	\$10,000	\$0	\$0	

# Probabilistic Criteria for Decision Making Under Uncertainty

Which action to take?

- ▶ *maximin*: Buy B, *maximax*: Buy A, *minimax loss*: Buy B
- ▶ *ER* or *EL* criteria: Buy Stock A
- ▶  $ER + EL = \$3000$
- ▶ Payoff table

Probability: State:	0.1 State I	0.4 State II	0.3 State III	0.2 State IV	ER
Buy Stock A	\$2,000	\$6,000	-\$6,000	\$2,000	\$1,200
Buy Stock B	\$0	\$2,000	-\$2,000	\$2,000	\$600
Buy Stock C	-\$2000	-\$4,000	\$0	\$2,000	-\$1,400

- ▶ Loss table

Probability: State:	0.1 State I	0.4 State II	0.3 State III	0.2 State IV	EL
Buy Stock A	\$0	\$0	\$6,000	\$0	\$1,800
Buy Stock B	\$2000	\$4,000	\$2,000	\$0	\$2,400
Buy Stock C	\$4000	\$10,000	\$0	\$0	\$4,400



# Utility

Will you take the bet?

- ▶ Toss a coin once, you win \$1 if the coin comes up heads, and you lose \$0.75 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- ▶ Toss a coin once, you win \$1 if the coin comes up heads, and you lose \$0.75 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1000 if the coin comes up heads, and you lose \$750 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- ▶ Toss a coin once, you win \$1 if the coin comes up heads, and you lose \$0.75 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1000 if the coin comes up heads, and you lose \$750 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you lose \$750,000 if the coin comes up tails.

Which bet will you take?

# Utility

Will you take the bet?

- ▶ Toss a coin once, you win \$1 if the coin comes up heads, and you lose \$0.75 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1000 if the coin comes up heads, and you lose \$750 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you lose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

# Utility

Will you take the bet?

- ▶ Toss a coin once, you win \$1 if the coin comes up heads, and you lose \$0.75 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1000 if the coin comes up heads, and you lose \$750 if the coin comes up tails.
- ▶ Toss a coin once, you win \$1,000,000 if the coin comes up heads, and you lose \$750,000 if the coin comes up tails.

Based on the *ER* criterion, you should take the bet, but...

Which bet will you take?

- ▶ You win \$10 million if a coin comes up heads and win \$10 million if the coin comes up tails.
- ▶ You win \$100 million if a coin comes up heads and win 0 if the coin comes up tails.

# St. Petersburg Paradox

Will you take the bet?

- ▶ You win  $2^0 = 1$  dollar if heads appear on the first toss.
- ▶ You win  $2^1 = 2$  dollar if the first heads appear on the second toss.
- ▶ ...
- ▶ You win  $2^k$  dollar if the first heads appear on the  $k$ th toss.
- ▶ ...

$$ER(\text{gamble}) = \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

Again, based on the  $ER$  criterion, you should take the bet, but...

# Axioms of Utility

A utility function can be interpreted in terms of a preference relationship. The two basic axioms of utility are:

- ▶ If payoff  $R_1$  is preferred to payoff  $R_2$ , then  $U(R_1) > U(R_2)$ ; if  $R_2$  is preferred to  $R_1$ , then  $U(R_2) > U(R_1)$ ; and if neither is preferred to the other, then  $U(R_1) = U(R_2)$ .
- ▶ If you are indifferent between (a) receiving payoff  $R_1$  for certain and (b) taking a bet or lottery in which you receive payoff  $R_2$  with probability  $p$  and payoff  $R_3$  with probability  $1 - p$ , then

$$U(R_1) = pU(R_2) + (1 - p)U(R_3).$$

# Assessment of Utility Functions

Denote the most preferable payoff as  $R^*$  and the least preferable payoff as  $R_*$ . Suppose that you let  $U(R_*) = 0$  and  $U(R^*) = 1$ , then  $0 \leq U(R) \leq 1$  for any payoff  $R$ .

Consider the following choice of lotteries:

- ▶ Lottery I: Receive  $R$  for certain
- ▶ Lottery II: Receive  $R^*$  with probability  $p$  and  $R_*$  with probability  $1 - p$ .

If  $U(R) > p$ , then choose Lottery I; if  $U(R) < p$ , then choose Lottery II; and if  $U(R) = p$ , then you are indifferent between the two lotteries.

With  $U(R_*) = 0$  and  $U(R^*) = 1$ , your utility for a payoff can be interpreted in terms of an indifference probability.



## Example: Oil Drilling

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	\$1,000,000	−\$300,000
<i>Do not drill</i>	\$200,000	\$100,000

► Utility:

- The utility of \$200,000 is  $p_1$  if you are indifferent between:
  - Lottery I: Receive \$200,000 for certain
  - Lottery II: Receive \$1,000,000 with probability  $p_1$  and −\$300,000 with probability  $1 - p_1$ .
- The utility of \$100,000 is  $p_2$  if you are indifferent between:
  - Lottery I: Receive \$100,000 for certain
  - Lottery II: Receive \$1,000,000 with probability  $p_2$  and −\$300,000 with probability  $1 - p_2$ .
- Utility table:

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>		
<i>Do not drill</i>		

## Example: Oil Drilling

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	\$1,000,000	−\$300,000
<i>Do not drill</i>	\$200,000	\$100,000

► Utility:

- The utility of \$200,000 is  $p_1$  if you are indifferent between:
  - Lottery I: Receive \$200,000 for certain
  - Lottery II: Receive \$1,000,000 with probability  $p_1$  and −\$300,000 with probability  $1 - p_1$ .
- The utility of \$100,000 is  $p_2$  if you are indifferent between:
  - Lottery I: Receive \$100,000 for certain
  - Lottery II: Receive \$1,000,000 with probability  $p_2$  and −\$300,000 with probability  $1 - p_2$ .
- Utility table:

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	1	0
<i>Do not drill</i>	$p_1$	$p_2$

## Example: Oil Drilling

Suppose  $P(Oil) = 0.3$  and the utility table is

	<i>Oil</i>	<i>No Oil</i>
<i>Drill</i>	1	0
<i>Do not drill</i>	0.45	0.35

then

$$EU(Drill) = 0.3 * 1 + 0.7 * 0 = 0.30,$$

$$EU(Do\ Not\ Drill) = 0.3 * 0.45 + 0.7 * 0.35 = 0.38.$$

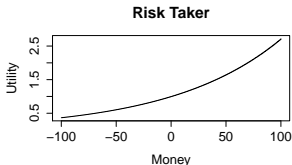
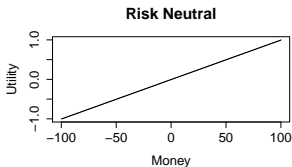
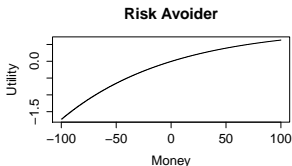
The decision maker should not drill according to the *EU* criterion.

# Utility and Money

- ▶ A Risk Avoider would be willing to pay a risk premium to avoid a fair bet.
- ▶ Risk Neutral: indifferent about a fair bet.
- ▶ A Risk Taker would be willing to pay money to take a fair bet.

When you are shopping for car insurance, are you a Risk Avoider or Taker?

When you are buying a lottery ticket, are you a Risk Avoider or Taker?



# Exponential Utility

$$U(x) = 1 - e^{-x/R}$$

- ▶  $x$  is the payoff
- ▶  $R > 0$  is the risk tolerance that specifies an individual's aversion to risk
- ▶ The risk tolerance  $R$  can be assessed as the amount that makes the decision maker indifferent between the following two lotteries:
  - ▶ Lottery I: obtain zero payoff;
  - ▶ Lottery II: win  $R$  dollars with probability  $1/2$  or lose  $R/2$  with probability  $1/2$ .
- ▶ Approximate values of  $R$  for a company: 6.4% of net sales, 124% of net income, and 15.7% of equality

## Example: Exponential Utility

Venture Limited, with net sales of \$30,000,000, need to decide:

- ▶ Invest in a sure thing with a return of \$125,000;
- ▶ Invest in a low risky venture that returns  $-\$500,000$ ,  $\$100,000$ , or  $\$1,000,000$  with probabilities 0.25, 0.50, and 0.25, respectively;
- ▶ Invest in a high risky venture that returns  $-\$1,000,000$ ,  $\$1,000,000$ , or  $\$3,000,000$  with probabilities 0.35, 0.60, and 0.05, respectively;

Your job is to:

- ▶ find the best decision in terms of expected payoff
- ▶ find the best decision in terms of expected utility