

## Summary of Topics for the Final Exam

### STA 371G, Spring 2015

Listed below are the major topics covered in class that are likely to be in the Final Exam. Good Luck!

- Mean (expectation), variance and standard deviation of a random variable.

$$\mathbb{E}[X] = \sum_{i=1}^n x_i P(X = x_i), \quad \text{Var}[X] = \sum_{i=1}^n (x_i - \mathbb{E}[X])^2 P(X = x_i), \quad \text{sd}[X] = \sqrt{\text{Var}[X]}$$

- Add a constant to a random variable, multiply a random variable by a constant.

If  $Y = a + bX$ , then

$$\mathbb{E}[Y] = a + b\mathbb{E}[X], \quad \text{Var}[Y] = b^2 \text{Var}[X], \quad \text{sd}[Y] = |b| \times \text{sd}[X].$$

- Conditional, joint and marginal probabilities.

$$P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

$$P(Y = y, X = x) = P(Y = y|X = x)P(X = x)$$

$$P(Y = y) = \sum_x P(Y = y, X = x)$$

- Independent random variables, sum of independent random variables.

- Two random variables  $X$  and  $Y$  are independent if  $P(Y = y|X = x) = P(Y = y)$  for all possible  $x$  and  $y$ .
- If  $X$  and  $Y$  are independent, then  $P(Y = y, X = x) = P(Y = y)P(X = x)$ .
- If  $Y = a_0 + a_1X_1 + a_2X_2 + \cdots + a_nX_n$ , then

$$\mathbb{E}[Y] = a_0 + a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2] + \cdots + a_n\mathbb{E}[X_n].$$

If  $X_i$  and  $X_j$  are independent for  $i \neq j$ , then we further have

$$\text{Var}[Y] = a_1^2 \text{Var}[X_1] + a_2^2 \text{Var}[X_2] + \cdots + a_n^2 \text{Var}[X_n].$$

- Normal distribution  $X \sim \mathcal{N}(\mu, \sigma^2)$ , where  $\mu$  is the mean,  $\sigma^2$  is the variance, and  $\sigma$  is the standard deviation.

- Probability density function: area under the curve represents probability.
- Standard normal distribution  $Z \sim \mathcal{N}(0, 1)$ .
- Standardizing a normal random variable  $Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$ .
- $P(X < x) = P\left(\frac{X - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) = P\left(Z < \frac{x - \mu}{\sigma}\right)$ .

- $P(-1 < Z < 1) \approx 0.68$ ;  $P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$ .
- $P(-2 < Z < 2) \approx 0.95$ ;  $P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$ .
- Estimate  $\mu$  and  $\sigma^2$  when  $X \sim \mathcal{N}(\mu, \sigma^2)$ .
  - Use the sample mean  $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$  to estimate  $\mu$ .
  - Use the sample variance  $s^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$  to estimate  $\sigma^2$ .
- Sampling distribution of a sample mean  $\bar{X}$ :
  - $\bar{X} \sim \mathcal{N}(\mu, \sigma_{\bar{X}}^2 = \frac{\sigma^2}{n})$ .
  - We use  $\bar{X}$  to estimate the population mean  $\mu$ .
  - As the population variance  $\sigma^2$  is usually unknown, we use the sample variance  $s^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$  to estimate  $\sigma^2$  and hence  $s^2/n$  to estimate  $\sigma_{\bar{X}}^2$ .
  - Standard error of  $\bar{X}$ :  $s_{\bar{X}} = \sqrt{\frac{s^2}{n}}$ .
  - 95% confidence interval of  $\mu$  (approximately):  $\bar{X} \pm 2\sqrt{\frac{s^2}{n}}$ .
- Binomial distribution and its normal approximation
  - $X \sim \text{Binomial}(n, p)$  can be approximated with  $X \sim \mathcal{N}(np, np(1-p))$  if  $n$  is large enough and  $p$  is not too close to 0 or 1.
  - Estimate the population proportion  $p$  when  $X \sim \text{Binomial}(n, p)$ , where  $n$  is the sample size.
  - Binomial distribution, its normal approximation, and Sampling distribution of a sample proportion  $\hat{p}$ :
    - \* Use the sample proportion  $\hat{p} = \frac{X}{n}$  to estimate  $p$ .
    - \* Approximately, we have  $\hat{p} \sim \mathcal{N}(p, \frac{\hat{p}(1-\hat{p})}{n})$ .
    - \* 95% confidence interval of  $p$ :  $\hat{p} \pm 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ .
- Simple Linear Regression
  - Least squares estimation: given  $n$  observations  $(x_1, y_1), \dots, (x_n, y_n)$ , we estimate the intercept  $b_0$  and slope  $b_1$  by finding a straight line  $\hat{y}_i = b_0 + b_1 x_i$  that minimizes the sum of squared residuals (SSE)

$$\sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n [y_i - (b_0 + b_1 x_i)]^2.$$

- Sample means of  $X$  and  $Y$

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}, \quad \bar{y} = \frac{\sum_{i=1}^n y_i}{n}.$$

- Sample covariance

$$\text{Cov}(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{n - 1}$$

- Sample correlation

$$r_{xy} = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{s_x^2 s_y^2}} = \frac{\text{Cov}(X, Y)}{s_x s_y}.$$

$$s_x^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}, \quad s_x = \sqrt{s_x^2}$$

$$s_y^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n - 1}, \quad s_y = \sqrt{s_y^2}$$

- Estimating the true regression line.
- Least squares estimation:

$$b_0 = \bar{y} - b_1 \bar{x}, \quad b_1 = r_{xy} \frac{s_y}{s_x}$$

- Interpreting covariance, correlation and regression coefficients.
- SST, SSR, SSE

$$\text{SST} = \sum_{i=1}^n (y_i - \bar{y})^2$$

$$\text{SSR} = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = \sum_{i=1}^n [(b_0 + b_1 x_i) - \bar{y}]^2$$

$$\text{SSE} = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n [y_i - (b_0 + b_1 x_i)]^2$$

$$\text{SST} = \text{SSR} + \text{SSE}$$

Note that  $\bar{y} = b_0 + b_1 \bar{x}$ ,  $\hat{y}_i = b_0 + b_1 x_i$ ,  $\bar{\hat{y}} = \bar{y}$ , and  $e_i = y_i - \hat{y}_i = (y_i - \bar{y}) - (\hat{y}_i - \bar{y})$ .

- Coefficient of determination:

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} = r_{xy}^2$$

- Regression assumptions and statistical model.

$$Y = \beta_0 + \beta_1 X + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

$$y_i \sim \mathcal{N}(\beta_0 + \beta_1 x_i, \sigma^2)$$

Assuming  $\beta_0$ ,  $\beta_1$  and  $\sigma^2$  are known, given  $x_i$ , the 95% prediction interval of  $y_i$  is

$$(\beta_0 + \beta_1 x_i) \pm 2\sigma.$$

- We estimate  $\sigma$  with the regression standard error  $s$  as

$$s = \sqrt{\frac{\sum_{i=1}^n e_i^2}{n-2}} = \sqrt{\frac{SSE}{n-2}}.$$

- Approximately we have  $b_1 \sim \mathcal{N}(\beta_1, s_{b_1}^2)$  and  $b_0 \sim \mathcal{N}(\beta_0, s_{b_0}^2)$ , where the standard errors of  $b_1$  and  $b_0$  are

$$s_{b_1} = \sqrt{\frac{s^2}{(n-1)s_x^2}}, \quad s_{b_0} = \sqrt{s^2 \left( \frac{1}{n} + \frac{\bar{X}^2}{(n-1)s_x^2} \right)}.$$

Thus approximately we have the 95% confidence intervals for  $\beta_1$  and  $\beta_0$  as

$$b_1 \pm 2s_{b_1}, \quad b_0 \pm 2s_{b_0}.$$

- Hypothesis testing:
  - \* We test the null hypothesis  $H_0 : \beta_1 = \beta_1^0$  versus the alternative  $H_1 : \beta_1 \neq \beta_1^0$ .
  - \* The  $t$ -stat  $t = \frac{b_1 - \beta_1^0}{s_{b_1}}$  measures the number of standard errors the estimate  $b_1$  is from the proposed value  $\beta_1^0$ .
  - \* The  $p$ -value provides a measure of how weird your estimate  $b_1$  is if the null hypothesis is true.
  - \* We reject the null hypothesis at the 5% significance level if  $|t| > 2$ ,  $p < 0.05$ , or  $\beta_1^0$  is not within the 95% confidence interval  $(b_1 - 2s_{b_1}, b_1 + 2s_{b_1})$ .
- Forecasting:
  - \* Given  $X_f$ , the 95% plug-in prediction interval of  $Y_f$  is  $(b_0 + b_1 X_f) \pm 2s$ .
  - \* A large predictive error variance (high uncertainty) comes from a large  $s$ , a small  $n$ , a small  $s_x$  and a large difference between  $X_f$  and  $\bar{X}$ .

## • Multiple Linear Regression

- Statistical model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

$$Y | X_1 \dots X_p \sim \mathcal{N}(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p, \sigma^2)$$

- Interpretation of regression coefficients.
- Fitted values:  $\hat{y}_i = b_0 + b_1 x_{i1} + \cdots + b_p x_{ip}$
- Least squares estimation: find  $b_0, b_1, \dots, b_p$  that minimize the sum of squared residuals  $\sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2$ .
- Regression standard error:

$$s = \sqrt{\frac{\sum_{i=1}^n e_i^2}{n-p-1}} = \sqrt{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-p-1}}.$$

- $\bar{e} = 0$ ,  $\text{Corr}(X_j, e) = 0$ ,  $\text{Corr}(\hat{Y}, e) = 0$
- $R^2 = \left( \text{Corr}(Y, \hat{Y}) \right)^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$
- Approximately we have  $b_j \sim \mathcal{N}(\beta_j, s_{b_j}^2)$ .
  - \* 95% confidence interval for  $\beta_j$ :  $b_j \pm 2s_{b_j}$
  - \* t-stat:  $t_j = \frac{b_j - \beta_j^0}{s_{b_j}}$ .
  - \*  $H_0 : \beta_j = \beta_j^0$  versus  $H_1 : \beta_j \neq \beta_j^0$ . Reject  $H_0$  at the 5% significance level if  $|t_j| > 2$ ,  $p$ -value  $< 0.05$ , or  $\beta_j^0$  is not within the 95% confidence interval  $(b_j - 2s_{b_j}, b_j + 2s_{b_j})$
- F-test of overall significance.
  - \*  $H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0$  versus  $H_1$ : at least one  $\beta_j \neq 0$ .
  - \*  $f = \frac{R^2/p}{(1-R^2)/(n-p-1)} = \frac{SSR/p}{SSE/(n-p-1)}$
  - \* If  $H_0$  is true, then  $f > 4$  is very significant (happens with low probability) in general.
  - \* If  $f$  is large ( $p$ -value is small), we reject  $H_0$ .
- Understanding multiple linear regression
  - \* Correlation is not causation
  - \* Multiple linear regression allows us to control all important variables by including them into the regression model
  - \* Dependencies between the explanatory variables ( $X$ 's) will affect our interpretation of regression coefficients
  - \* Dependencies between the explanatory variables ( $X$ 's) will inflate the standard errors of regression coefficients

$$s_{b_j}^2 = \frac{s^2}{\text{variation in } X_j \text{ not associated with other } X\text{'s}}$$

- Dummy Variables and Interactions

- Dummy variables
  - \* Gender: Male, Female; Education level: High-school, Bachelor, Master, Doctor; Month: Jan, Feb,  $\dots$ , Dec
  - \* A variable of  $n$  categories can be included into multiple linear regression using (no more than)  $n - 1$  dummy variables
  - \* Representing a variable of  $n$  categories with  $n$  dummy variables will lead to the problem of “perfect multicollinearity”
  - \* Interpretation: the same slope but different intercepts
- Interactions
  - \* Interpretation: different intercepts and slopes

- Diagnostics and Transformations

– Diagnostics

- \* Model assumptions:
  - Statistical model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

- The mean of  $Y$  is a linear combination of the  $X$ 's
- The errors  $\epsilon_i$  (deviations from the true mean) are independent, and identically normal distributed as  $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$
- \* Understanding the consequences of violating the model assumptions
- \* Detecting and explaining common model assumption violations using the residual plots.

– Modeling non-linearity with polynomial regression

- \* Statistical model:

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \beta_m X^m + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

- \* We can always increase  $m$  if necessary, but  $m = 2$  is usually enough.
- \* Be very careful about over-fitting and doing prediction outside the data range, especially if  $m$  is large.
- \* For  $Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \epsilon$ , the marginal effect of  $X$  on  $Y$  is

$$\frac{\partial \mathbb{E}[Y|X]}{\partial X} = \beta_1 + 2\beta_2 X,$$

which means the slope is a function of  $X$  (no longer a constant).

– Handling non-constant variance with Log-Log transformation

- \* Statistical model:

$$\log(Y) = \beta_0 + \beta_1 \log(X) + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

$$Y = e^{\beta_0} X^{\beta_1} e^{\epsilon}, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

- \* Interpretation: about  $\beta_1\%$  change in  $Y$  per 1% change in  $X$ .
- \* Example: price elasticity
- \* 95% plug-in prediction interval of  $\log(Y)$

$$(\beta_0 + \beta_1 \log(X)) \pm 2s$$

- \* 95% plug-in prediction interval of  $Y$

$$\left( e^{\beta_0 + \beta_1 \log(X) - 2s}, e^{\beta_0 + \beta_1 \log(X) + 2s} \right) = \left( e^{\beta_0 - 2s} X^{\beta_1}, e^{\beta_0 + 2s} X^{\beta_1} \right)$$

– Log transformation of  $Y$

- \* Statistical model:

$$\log(Y) = \beta_0 + \beta_1 X + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

$$Y = e^{\beta_0} e^{\beta_1 X} e^{\epsilon}, \quad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

- \* Interpretation: about  $(100\beta_1)\%$  change in  $Y$  per unit change in  $X$  (if  $\beta_1$  is small).
- \* Example: exponential growth

- Time Series

- Trend, seasonal, cyclical, and random components of a time series
- Fitting a trend

- \* Linear trend:

$$Y_t = \beta_0 + \beta_1 t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- \* Exponential trend:

- Model:  $\log(Y_t) = \beta_0 + \beta_1 t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$

- Interpretation:  $Y_t$  increases by about  $(100\beta_1)\%$  per unit time increase.

- \* Modeling non-linearity by adding  $t^2$  into the regression model: the slope changes as time changes.
- \* 95% plug-in prediction interval

- Autoregressive models

- \* Random walk model:  $Y_t = \beta_0 + Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$

- \* Autoregressive model of order 1 (AR(1)):

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- \* Autocorrelation of residuals:  $\text{Corr}(\epsilon_t, \epsilon_{t-1})$

- \* Trend+AR(1):

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- \* Logtransformation + trend + AR(1):

$$\log(Y_t) = \beta_0 + \beta_1 \log(Y_{t-1}) + \beta_2 t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- Modeling seasonality

- \* Using at most 11 dummy variables for 12 months; using at most 3 dummy variables for 4 quarters
- \* Seasonal model:

$$Y_t = \beta_0 + \beta_1 Jan + \dots + \beta_{11} Nov + \epsilon_t$$

- \* Seasonal + AR(1) + linear trend:

$$Y_t = \beta_0 + \beta_1 Jan + \dots + \beta_{11} Nov + \beta_{12} Y_{t-1} + \beta_{13} t + \epsilon_t$$

- Model for  $t$  in December:  $Y_t = \beta_0 + \beta_{12}Y_{t-1} + \beta_{13}t + \epsilon_t$
- Model for  $t$  in Jan:  $Y_t = (\beta_0 + \beta_1) + \beta_{12}Y_{t-1} + \beta_{13}t + \epsilon_t$
- Model for  $t$  in October:  $Y_t = (\beta_0 + \beta_{10}) + \beta_{12}Y_{t-1} + \beta_{13}t + \epsilon_t$
- \* Logtransformation + Seasonal + AR(1) + trend

$$\log(Y_t) = \beta_0 + \beta_1 Jan + \dots + \beta_{11} Nov + \beta_{12} \log(Y_{t-1}) + \beta_{13}t + \epsilon_t$$

- Diagnose the residual plot of a time series regression model:
  - \* Are there any clear temporal patterns?
  - \* Are the residuals autocorrelated?
  - \* What kind of model assumptions have been violated?
- Understand when and how to include log transformation, non-linearity, dummy variables, interactions, AR(1) and trend to improve a time series regression model.
- Model Selection
  - Validate a model using out-of-sample prediction
  - Model selection criteria (AIC, BIC, Adjusted  $R^2$ )
  - Forward regression, backward regression, stepwise regression
- Decision Making Under Uncertainty
  - Frequency interpretation and subjective interpretation of probability.
  - Probabilities and betting odds
  - Payoff table
  - Bayes' theorem

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \dots + P(B|A_p)P(A_p)}$$

- Simpson's paradox
- Payoffs and losses, loss table
- Nonprobabilistic decision criteria
  - \* maximin
  - \* maximax
  - \* minimax loss
- Probabilistic decision criteria: expected payoff (ER) or expected loss (EL)
- Utility
  - \* Risk avoider, risk neutral, risk taker
  - \* A typical utility function:  $U(x) = 1 - e^{-x/R}$
- Decision tree



- \* Represent a payoff table with a decision tree
  - \* Time proceeds from left to right
  - \* Folding back procedure
  - \* Risk profile
  - \* Sensitivity analysis
  - \* Decision making and Bayes' theorem
- The value of information
  - \* Value of perfect information
  - \* Expected value of perfect information (EVPI)
  - \* Value of sample information
  - \* Expected value of sample information (EVSI)
  - \* Bayes' theorem and the value of information
- Introduction to Monte Carlo Simulation
  - Uniform random numbers
  - Flip a coin, toss a die, flip two coins, toss two dice
  - Normal random numbers, Student's  $t$  random numbers
  - Understand how to simulate from a discrete distribution
  - Understand how to use simulation to estimate  $P(X < x)$ ,  $\mathbb{E}[X]$  and  $\text{Var}[X]$ , where  $X$  is a random variable following some distribution.
  - Understand how to use simulation to demonstrate Law of Large Numbers
  - Understand how to use simulation to demonstrate the sampling distribution of sample mean
  - Understand how to use simulation to demonstrate the sampling distribution of sample proportion
  - Understand how to use simulation to demonstrate the Central Limit Theorem
  - Simulation and decision making
  - Simulate an AR(1)+Trend+Logtransformation time series model
  - Using simulation to estimate the prediction intervals
  - Understand how to construct a random experiment and find relevant answers by simulate the same experiment repeatedly under identical conditions