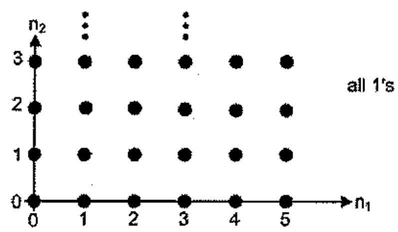
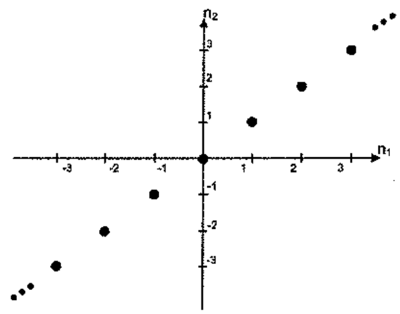
Chapter 1 Solutions

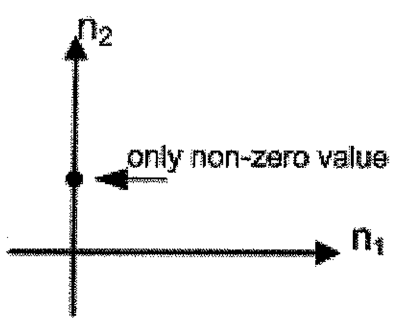
1.1



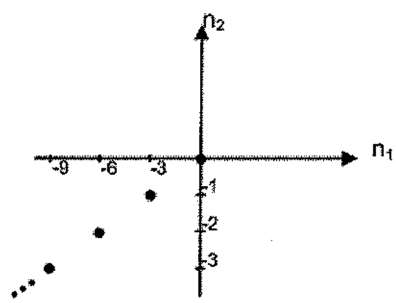
a)



b)



c)



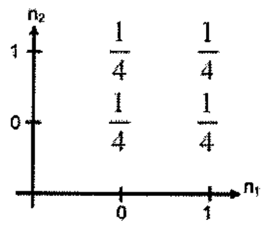
d)

case.)

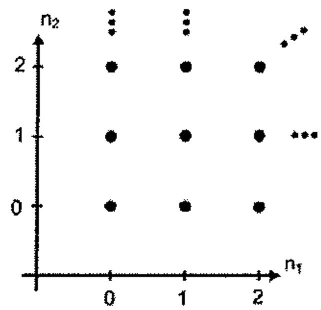
NB: The dashed lines in the above figures are determined by planes that are equidistant from the period anchor points. The region inside all these planes must be specified to determine the signal.

1.4

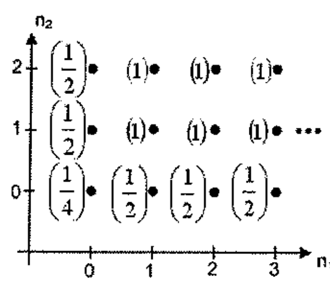
h=



u++=



so h*u++=



Expressed in an equation, we have,

$$(h * u_{++})(n_1, n_2) = \begin{cases} \frac{1}{4} & (n_1, n_2) = (0, 0); \\ \frac{1}{2} & (n_1, n_2) = (0, n_2 > 0); \\ \frac{1}{2} & (n_1, n_2) = (n_1 > 0, 0); \\ 1 & (n_1, n_2) = (n_1 \geq 1, n_2 \geq 1); \\ 0 & \text{else.} \end{cases}$$

1.5

a) $y(n_1, n_2) = 3x(n_1, n_2) - x(n_1 - 1, n_2) = h * x$

with $h(n_1, n_2) = 3\delta(n_1, n_2) - \delta(n_1 - 1, n_2)$, so it's LSI. Also it's clearly stable, since

$$h \in \ell^1, \text{i.e.} \sum_{all n_1, n_2} |h(n_1, n_2)| < \infty$$

b) $y(n_1, n_2) = 3x(n_1, n_2) - y(n_1 - 1, n_2)$.

This system of equations has to be solved over a region with prescribed boundary conditions. Say we start with zero initial conditions, initial rest, and solve in the $+n_1$ direction for each output row, and then in the $+n_2$ direction, increment n_2 by 1, and then solve in the $+n_1$ direction again, and so forth (called the 'raster scan'). Then there will be a unique solution, and so we have defined a system. Is it linear? Say we have two solutions to two separate inputs, then

$y_1(n_1, n_2) = 3x_1(n_1, n_2) - y_1(n_1 - 1, n_2)$

$y_2(n_1, n_2) = 3x_2(n_1, n_2) - y_2(n_1 - 1, n_2)$.

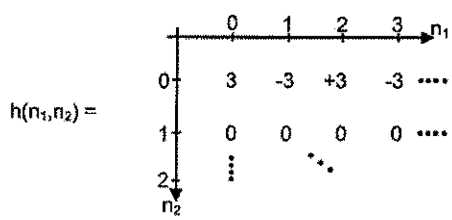
Consider the input $x(n_1, n_2) = a_1x_1(n_1, n_2) + a_2x_2(n_1, n_2)$,

clearly we have

$$(a_1y_1(n_1, n_2) + a_2y_2(n_1, n_2)) = 3(a_1x_1(n_1, n_2) + a_2x_2(n_1, n_2)) \\ -(a_1y_1(n_1 - 1, n_2) + a_2y_2(n_1 - 1, n_2)),$$

for all (n_1, n_2). So if the initial rest solution is unique, the solution must be $y(n_1, n_2) = a_1y_1(n_1, n_2) + a_2y_2(n_1, n_2)$.

So the system is linear.
Since the system is linear, lets find the response to an impulse $\delta(n_1, n_2)$ as input. Assuming $y = 0$ for $n_1 < 0$ or $n_2 < 0$, we get



so $\sum |h(n_1, n_2)| = \infty$, hence not stable. Shifting the impulse right-left, up-down, does not change the character of the output, merely shifting it. Hence,

$x = \delta(n_1 - m_1, n_2 - m_2)$

$\Rightarrow y = h(n_1 - m_1, n_2 - m_2)$,

and so the system is shift invariant. More on these recursive systems later.
c) This system is the same as convolving with the impulse response,
$h(n_1, n_2) = \delta(n_1, n_2) + \delta(n_1 - 1, n_2) + \delta(n_1, n_2 - 1) + \delta(n_1 - 1, n_2 - 1)$
and so, is LSI and stable.

d) Here $y(n_1,n_2) = x(n_1-n_1, n_2-n_2) + x(n_1-(n_1-1), n_2-n_2) + x(n_1-n_1, n_2-(n_2-1)) + x(n_1-(n_1-1), n_2-(n_2-1))$

$= x(0,0) + x(1,0) + x(0,1) + x(1,1)$
so not LSI, but clearly stable.

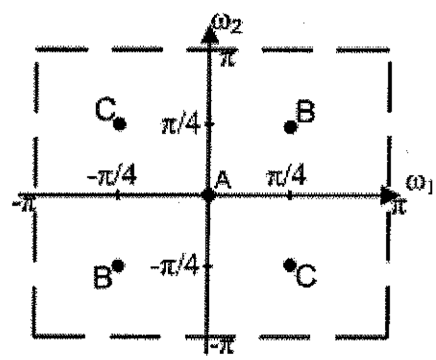
1.6

$$x(\mathbf{n}) = 4 + e^{+j\frac{\pi}{4}(n_1+n_2)} + e^{-j\frac{\pi}{4}(n_1+n_2)} + e^{+j\frac{\pi}{4}(n_1-n_2)} + e^{-j\frac{\pi}{4}(n_1-n_2)}$$

Performing Fourier transform on $x(\mathbf{n})$ we get,

$$\begin{aligned} X(\omega) &= 4(2\pi)^2\delta(\omega) + (2\pi)^2\delta\left(\omega_1 - \frac{\pi}{4}, \omega_2 - \frac{\pi}{4}\right) + (2\pi)^2\delta\left(\omega_1 + \frac{\pi}{4}, \omega_2 + \frac{\pi}{4}\right) \\ &\quad + (2\pi)^2\delta\left(\omega_1 - \frac{\pi}{4}, \omega_2 + \frac{\pi}{4}\right) + (2\pi)^2\delta\left(\omega_1 + \frac{\pi}{4}, \omega_2 - \frac{\pi}{4}\right) \end{aligned}$$

for $\omega \in [-\pi, +\pi]^2$, and periodic elsewhere.



The impulses at A-C in the figure have *areas*: $A = 16\pi^2$, $B = 4\pi^2$, and $C = 4\pi^2$.

1.7

Let $s_x(n_1,n_2) = x(n_1)$, then $S_x(\omega_1,\omega_2) = X(\omega_1)2\pi\delta(\omega_2)$

Similarly, if $s_y(n_1,n_2) = y(n_2)$, we get $S_y(\omega_1,\omega_2) = 2\pi\delta(\omega_1)Y(\omega_2)$

Performing the periodic convolution,

Chapter 2 Solutions

2.1

a) $s_c(x_1, x_2) = 100 + 20\cos 6\pi x_1 + 40\sin(10\pi x_1 + 6\pi x_2)$
so $\Omega_{c1} = 10\pi$ and $\Omega_{c2} = 6\pi$ is rectangular bandwidth

b)

$$\begin{aligned}
\text{Set } \Delta_1 &= \frac{1}{20} \text{ and } \Delta_2 = \frac{1}{10}, \text{ then} \\
s(n_1, n_2) &\triangleq s_c\left(\frac{1}{20}n_1, \frac{1}{10}n_2\right) \\
&= 100 + 20\cos\frac{3\pi}{10}n_1 + 40\sin\left(\frac{\pi}{2}n_1 + \frac{3\pi}{5}n_2\right).
\end{aligned}$$

Since sampling spacing (Δ_1, Δ_2) satisfies:

$$\Delta_1 < \frac{\pi}{\Omega c_1} = \frac{1}{10} \text{ and } \Delta_2 < \frac{\pi}{\Omega c_2} = \frac{1}{6},$$

there is no aliasing.

c)

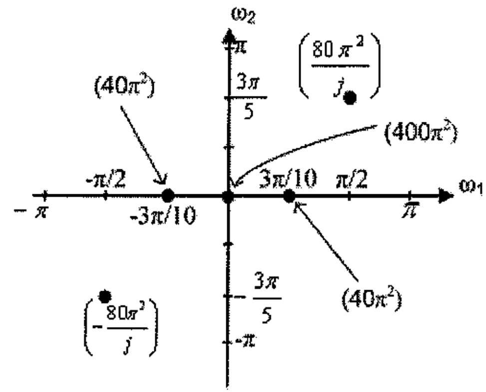
$$\begin{aligned}
\text{We use } \mathbf{FT}\{c\} &= (2\pi)^2 c\delta(\boldsymbol{\omega}), \\
\mathbf{FT}\{e^{j\boldsymbol{\omega}^0 \cdot \boldsymbol{n}}\} &= (2\pi)^2\delta(\boldsymbol{\omega} - \boldsymbol{\omega}^0), \\
\mathbf{FT}\{\cos\boldsymbol{\omega}^0 \cdot \boldsymbol{n}\} &= \frac{(2\pi)^2}{2}[\delta(\boldsymbol{\omega} - \boldsymbol{\omega}^0) + \delta(\boldsymbol{\omega} + \boldsymbol{\omega}_0)] \\
\mathbf{FT}\{\sin\boldsymbol{\omega}^0 \cdot \boldsymbol{n}\} &= \frac{(2\pi)^2}{2}[\delta(\boldsymbol{\omega} - \boldsymbol{\omega}^0) - \delta(\boldsymbol{\omega} + \boldsymbol{\omega}_0)]
\end{aligned}$$

all for $\boldsymbol{\omega} \in [-\pi, +\pi]^2$ and periodic repeated elsewhere.

So

$$\begin{aligned}
S(\omega_1, \omega_2) &= (2\pi)^2\Bigg[100\delta(\boldsymbol{\omega}) + 10\left(\delta(\omega_1 - \frac{3\pi}{10}, \omega_2\right) + \delta\left(\omega_1 + \frac{3\pi}{10}, \omega_2\right)\Bigg) \\
&\quad + \frac{20}{j}\left(\delta\left(\omega_1 - \frac{\pi}{2}, \omega_2 - \frac{3\pi}{5}\right) - \delta\left(\omega_1 + \frac{\pi}{2}, \omega_2 + \frac{3\pi}{5}\right)\right)\Bigg]
\end{aligned}$$

Here is a sketch of $S(\omega)$



2.2

a) By the 2-D sampling theorem for

$$
\begin{aligned}
y(n_1,n_2) &= x(n_1N_1,n_2N_2) = x_c(n_1N_1T_1,n_2N_2T_2)\\
Y(\omega_1,\omega_2) &= \frac{1}{N_1T_1N_2T_2}\sum_{l_1',l_2'=-\infty}^{+\infty} X_c\left(\frac{\omega_1-2\pi l_1'}{N_1T_1},\frac{\omega_2-2\pi l_2'}{N_2T_2}\right)\\
&= \frac{1}{N_1T_1N_2T_2}\sum_{l_1,l_2=-\infty}^{+\infty}\sum_{k_1,k_2=0}^{N_1-1,N_2-1} X_c\left(\frac{\omega_1-2\pi k_1-2\pi l_1N_1}{N_1T_1},\frac{\omega_2-2\pi k_2-2\pi l_2N_2}{N_2T_2}\right)
\end{aligned}
$$

(Using the representation $l_i' = k_i + l_iN_i$ and
$k_c = 0,\ldots,N_i-1,\ \ -\infty < l_i < +\infty$)

$$
= \frac{1}{N_1N_2}\sum_{k_1,k_2=0}^{N_1-1,N_2-1} X\left(\frac{\omega_1-2\pi k_1}{N_1},\frac{\omega_2-2\pi k_2}{N_2}\right)
$$

by interchanging the double sums, since by the 2-D sampling theorem for x,we have

$$
\begin{aligned}
X(\omega_1,\omega_2) &= \frac{1}{T_1T_2}\sum_{l_1,l_2=-\infty}^{+\infty} X_c\left(\frac{\omega_1-2\pi l_1}{T_1},\frac{\omega_2-2\pi l_2}{T_2}\right), \text{ so}\\
X\left(\frac{\omega_1-2\pi k_1}{N_1},\frac{\omega_2-2\pi k_2}{N_2}\right) &= \frac{1}{T_1T_2}\sum_{l_1,l_2=-\infty}^{+\infty} X_c\left(\frac{\omega_1-2\pi k_1-2\pi l_1N_1}{N_1T_1},\frac{\omega_2-2\pi k_2-2\pi l_2N_2}{N_2T_2}\right)
\end{aligned}
$$

so

$$\begin{aligned} x_c(n_1T_1, n_2T_2) &= T_1T_2x(n_1,n_2)\frac{1}{T_1T_2} \\ &= x(n_1,n_2) \quad \text{as it should} \end{aligned}$$

2.4

a) By the 1-D sampling theorem,

$$\begin{aligned} X(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\Big(\frac{\omega-2\pi k}{T}\Big) \\ &= \sum_{k=-\infty}^{+\infty}\sqrt{\frac{\alpha}{\pi}}\exp\Big(-\frac{1}{4\alpha}(\omega-2\pi k)^2\Big) \quad \text{since } T=1. \end{aligned}$$

b) At $k=+1$ and $\omega=+\pi$, the error term is

$$\begin{aligned} \sqrt{\frac{\alpha}{\pi}}e^{-\frac{1}{4\pi}(\pi-2\pi)^2} &= 10^{-3}\sqrt{\frac{\alpha}{\pi}} \\ \Rightarrow \frac{\pi^2}{4\alpha} &= 3\ln 10 \\ \text{or } \alpha &= \frac{\pi^2}{12\ln 10} = 0.36 \end{aligned}$$

As α increases, the bandwidth spreads, so this is an upperbound on α.

c)

$$\begin{aligned} X(\omega_1,\omega_2) &= \frac{1}{T_1T_2}\sum_{k_1,k_2} X_c\Big(\frac{1}{T_1}(\omega_1-2\pi k_1), \frac{1}{T_2}(\omega_2-2\pi k_2)\Big) \\ \text{or} \quad & \sum_{k_1,k_2}\frac{\alpha}{\pi}\exp\Big(-\frac{1}{4\alpha}[(\omega_1-2\pi k_1)^2+(\omega_2-2\pi k_2)^2]\Big) \end{aligned}$$

d) At $(k_1,k_2)=(1,0)$ and $\omega=(\pi,0)$, we get the error

$$\begin{aligned} \frac{\alpha}{\pi}e^{-\frac{\pi^2}{4\alpha}} &= 10^{-3}\frac{\alpha}{\pi} \\ \Rightarrow \alpha &= \frac{\pi^2}{12\ln 10}, \quad \text{(same as answer to part b).} \\ &= \frac{9.87}{27.63} = 0.36. \end{aligned}$$

2.5

a) From the shown figure, the FT support fits in the rectangle $[-\Omega_{c1}, +\Omega c1] \times [-\Omega_{c2}, +\Omega_{c2}]$. We can't reduce the size of this rectangle without aliasing, so we must use sample spacing $\Delta_1 \leq \pi/\Omega_{c1}$ and $\Delta_2 \leq \pi/\Omega_{c2}$, with alias repeat or anchor points at $(\pm 2\Omega_{c1}, \pm 2\Omega_{c2})$. We thus have

$$\mathbf{V} = \begin{bmatrix} \frac{\pi}{\Omega_{c1}} & 0 \\ 0 & \frac{\pi}{\Omega_{c2}} \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 2\Omega_{c1} & 0 \\ 0 & 2\Omega_{c2} \end{bmatrix}$$

b) Because of the diagonal support of this Fourier transform, closer alias repeat points will work (i.e. avoid alias overlap). We can use either

$$\mathbf{U} = \begin{bmatrix} \Omega_{c1} & 0 \\ 0 & 2\Omega_{c2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2\Omega_{c1} & 0 \\ 0 & \Omega_{c2} \end{bmatrix}.$$

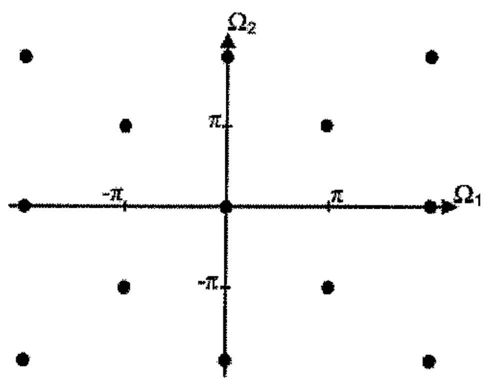
Both will cover the plane without overlap, what is known as *tiling the plane.* Via $\mathbf{U}^T\mathbf{V} = 2\pi\mathbf{I}$, we get

$$\mathbf{V} = \begin{bmatrix} 2\pi/\Omega_{c1} & 0 \\ 0 & \pi/\Omega_{c2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pi/\Omega_{c1} & 0 \\ 0 & 2\pi/\Omega_{c2} \end{bmatrix}.$$

Both these (rectangular) sampling matrices will give a $\times 2$ lower sample rate and still avoid alias overlap for this particular 2-D signal.

2.6

By Fig. 2.16, for $T = 1$ and $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we get $\mathbf{U} = \begin{bmatrix} \pi & \pi \\ \pi & -\pi \end{bmatrix}$ and frequency repeat points as



By connecting the nearest alias points to the origin by straight lines, and then bisecting these lines, we can determine a maximal alias-free unit cell, as shown below.