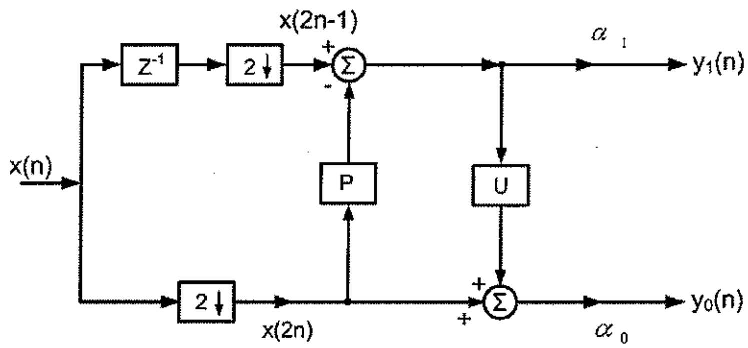
5.12



Note: First note that the outputs were inadvertently switched in Fig. 5.33 in early printing of the textbook. Also the filter $h_1(n)$ in the $\frac{5}{3}$ SWT in part(c) must be centered at $n = +1$, not zero as $h_0(n)$ is centered.

a) Calling the conventional SWT filter outputs $w_i(n)$, $i = 1, 2$, for the Haar $\frac{2}{2}$ filter set, we have

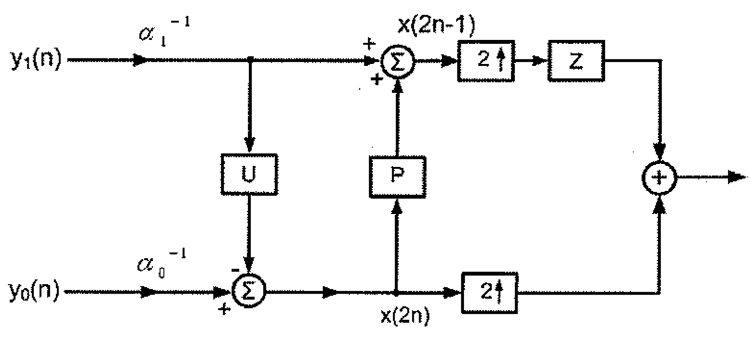
$$\begin{aligned} y_1(n) &= w_1(2n) = x(2n) - x(2n-1) \\ y_0(n) &= w_0(2n) = x(2n) + x(2n-1) \end{aligned}$$

with $P = 1$ and $\alpha_1 = -1$, we achieve the $y_1(n)$ output in the above lifted SWT structure. Then we need

$$y_0(n) = \alpha_0[x(2n) + U(x(2n-1) - x(2n))].$$

Setting $U = \frac{1}{2}$ and $\alpha_0 = 2$, we satisfy the $y_0(n)$ lowpass output too.

b) The ISWT can be constructed as follows, proceeding from left-to-right, we undo each SWT lifting stage from right-to-left. We first undo the α_i scaling. Then subtract, rather than add, the result of the update U operator. Finally add, rather than subtract the output of the predictor P operator. Please see diagram below.



Note:

c) The help reference for this part should have been [13] or [17]. Also, while

$h_0(n)$ is zero-phase (centered at zero), we must take $h_1(n)$ as centered at $n = +1$. Proceeding as in (a), we get

$$\begin{aligned} w_1(n) &= x(n-1) - \frac{1}{2}(x(n) + x(n-2)) \\ \text{or } y_1(n) &= x(2n-1) - \frac{1}{2}(x(2n) + x(2n-2)). \end{aligned}$$

So we set operator $P(z) = \frac{1}{2}(1 + z^{-1})$ to achieve this output with the lifting structure.

For the y_0 output, we first compute

$$\begin{aligned} w_0(n) &= x(n) + \frac{1}{4}[x(n-1) + x(n+1)] - \frac{1}{8}[x(n-2) + x(n+2)] \text{ or} \\ y_0(n) &= w_0(2n) \\ &= x(2n) + \frac{1}{4}[x(2n-1) + x(2n+1)] - \frac{1}{8}[x(2n-2) + x(2n+2)] \end{aligned}$$

which we must construct as

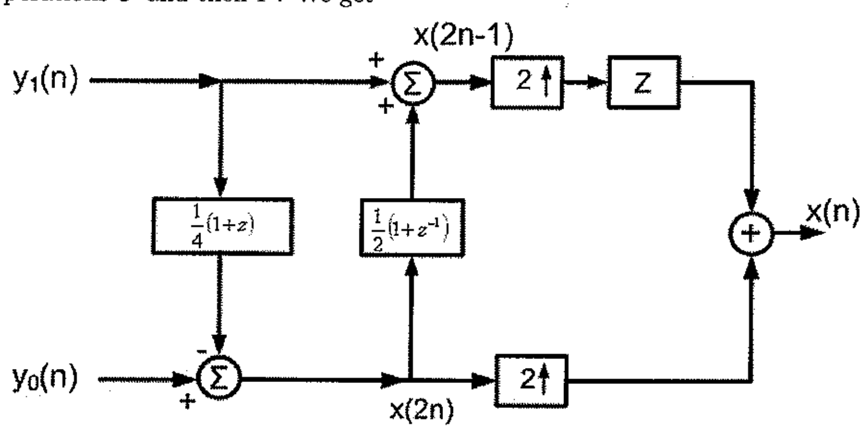
$$\begin{aligned} &= x(2n) + U[y_1(n)] \\ &= x(2n) + U\left[x(2n-1) - \frac{1}{2}\left(x(2n) + x(2n-1)\right)\right] \end{aligned}$$

The choice $U(z) = \frac{1}{4}(1 + z)$ works to get

$$\begin{aligned} y_0(n) &= x(2n) + \frac{1}{4}\left[y_1(n) + y_1(n+1)\right] \\ &= x(2n) + \frac{1}{4}\left[x(2n-1) - \frac{1}{2}\left(x(2n) + x(2n-2)\right) + x(2n+1) - \frac{1}{2}\left(x(2n+2) + x(2n)\right)\right] \\ &= \left(\frac{3}{4}\right)x(2n) + \frac{1}{4}\left[x(2n-1) + x(2n+1)\right] - \frac{1}{8}\left[x(2n+2) + x(2n-2)\right], \end{aligned}$$

which is the desired 5/3 SWT lowpass output.

The inverse SWT may be constructed same as in part a). First undo operations U and then P. We get



Checking:

$$
\begin{aligned}
& y_0(n) - \frac{1}{4}\Big(y_1(n) + y_1(n+1)\Big) \\
= \;& \frac{3}{4}x(2n) + \frac{1}{4}[x(2n-1) + x(2n+1)] - \frac{1}{8}\Big[x(2n-2) + x(2n+2)\Big] \\
& -\frac{1}{4}\Big[x(2n-1) - \frac{1}{2}\Big[x(2n) + x(2n-2)\Big] + y_1(n+1)\Big] \\
= \;& \left(\frac{3}{4} + \frac{1}{8}\right)x(2n) + \frac{1}{4}x(2n+1) - \frac{1}{8}x(2n+2) - \frac{1}{4}y_1(n+1) \\
= \;& \left(\frac{7}{8} + \frac{1}{8}\right)x(2n) + \frac{1}{4}x(2n+1) - \frac{1}{8}x(2n+2) - \frac{1}{4}x(2n+1) + \frac{1}{8}x(2n+2). \\
= \;& x(2n) \quad \surd
\end{aligned}
$$

ENEE 631 HW3

Yi-Chen Chen

$\boxed{\text{Problem 1}}$

(a) $E[y[n]] = E[\sum_k h[k]x[n-k]] = \sum_k h[k]E[x[n-k]] = \sum_k h[k]\mu_x$

($\because x[n]$ WSS, $E[x[n]] \triangleq \mu_x$ independent of n)

Hence mean of $y[n]$ is independent of n.

Also, $K_Y[n,m] = Cov(y[n], y[m])$ — assume $E[y[n]] \triangleq \mu_y$, which is independent of n.

$$= E[(y[n]-\mu_y)(y[m]-\mu_y)]$$

$$= E[(\sum_k h[k]x[n-k] - \sum_k h[k]\mu_x)(\sum_\ell h[\ell]x[m-\ell] - \sum_\ell h[\ell]\mu_x)]$$

$$= E[(\sum_k h[k](x[n-k]-\mu_x))(\sum_\ell h[\ell](x[m-\ell]-\mu_x))]$$

$$= E[\sum_k\sum_\ell h[k]h[\ell](x[n-k]-\mu_x)(x[m-\ell]-\mu_x)]$$

$h[k]$ is causal $$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} h[k]h[\ell]E[(x[n-k]-\mu_x)(x[m-\ell]-\mu_x)]$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} h[k]h[\ell]\, Cov(x[n-k], x[m-\ell])$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} h[k]h[\ell]K_x[n-k, m-\ell]$$

$\because x[n]$ is WSS, $K_x[m,n] = K_x[m-n]$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} h[k]h[\ell]K_x[n-m-(k-\ell)]$$

i.e., $K_Y[r,s] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h[m]h[n]K_x[r-s-(m-n)]$

$k = r-s$ $\Rightarrow K_Y[k] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h[m]h[n]K_x[k-(m-n)]$

($K_Y[r,s]$ is a function of $(r-s)$ only) *

(b) $\mu_x = 0 \Rightarrow \mu_y = \sum_k h[k]\cdot 0 = 0$.

From (a), we have $K_{XY}[r,s] = E[x[r]y[s]]$

$$= E[x[r]\sum_k h[k]x[s-k]]$$

$$= \sum_k h[k]E[x[r]x[s-k]]$$

$x[n]$ is WSS

$$= \sum_k h[k]K_x[r-s+k]$$

$m = r-s$ $\Rightarrow K_{XY}[m] = \sum_k h[k]K_x[m+k]$

($h[k]$ causal) $$= \sum_{k=0}^{\infty} h[k]K_x[m+k]$$

(c) $x[n] \begin{cases} \text{WSS}, & \text{if } n \geq 0 \\ 0, & n<0 \end{cases}$, Let $E[x[n]] \triangleq \mu_x$ for $n \geq 0$.

$$E[Y[n]] = E\left[\sum_{k=0}^{\infty} h[k]\, x[n-k]\right]$$

$$= \sum_{k=0}^{\infty} h[k]\, E[x[n-k]] = \sum_{k=0}^{\infty} h[k]\, \mu_x\, u[n-k] = \mu_x \sum_{k=0}^{\infty} h[k]\, u[n-k], \text{ where}$$

$$u[m] = \begin{cases} 1, & m \geq 0 \\ 0, & m<0 \end{cases}$$

$h[k]$ causal

case ①: $h[k]$ is causal FIR filter

Let l^* be the smallest integer such that $h[k]=0 \ \forall k > l^*$

We have: $E[Y[n]] = \mu_x \sum_{k=0}^{K} h[k]$, where $K = \min(l^*, n)$

Case ②: $h[k]$ is causal IIR filter

we have $E[Y[n]] = \mu_x \sum_{k=0}^{n} h[k]$

In both case ① and ②, $E[Y[n]]$ depends on n.

$E[Y[n]] = 0, \ \forall n < 0$

Problem 3.

(problem 2 is on the last page(s)).

We use $T(\cdot)$ to represent the given spatially invariant system

i.e., $y(m,n) = T(x(m,n))$ where $x(m,n)$ is the input and $y(m,n)$ is the output,

Let $G(\cdot)$ be the inverse of $T(\cdot)$, i.e., $G \circ T(x(m,n)) = x(m,n)$

Now, since $T(\cdot)$ is spatially invariant, we have

$$y(m-m', n-n') = T(x(m-m', n-n')), \ \forall m,n,m',n'$$

$$\Rightarrow G(y(m-m', n-n')) = G \circ T(x(m-m', n-n'))$$

$$= x(m-m', n-n'), \ \forall m,n,m',n'$$

Hence $G(\cdot)$ is also spatially invariant.

Problem 4

The following proof is done for 1-D case and can easily be generalized to 2-D or higher dimension cases.

Let $x[n]$ be a WSS random process.

$R_x[m]$ be the autocorrelation function, $R_x[m] = E[x[n+m]\, x^*[n]]$

$S_x(\omega) = \sum_{m=-\infty}^{\infty} R_x[m]\, e^{-j\omega m}$ be the power spectral density.

$\boxed{\text{Problem 4}}$ continued

(1) $S_X(\omega)$ is real:

We have: $R_X[-m] = E[\,X[n-m]X^*[n]\,]$

$$\Rightarrow R_X^*[-m] = E[\,X^*[n-m]\,X[n]\,] = E[\,X[n]X^*[n-m]\,] = R_X[m] \quad \ldots (*)$$

$$\text{Hence } S_X^*(\omega) = \sum_{m=-\infty}^{\infty} R_X^*[m]e^{j\omega m} = \sum_{n=-\infty}^{\infty} R_X^*[-n]e^{-j\omega n} \underset{(*)}{=} \sum_{n=-\infty}^{\infty} R_X[n]e^{-j\omega n} = S_X(\omega)$$

i.e. $S_X(\omega)$ is real ※

(2) $S_X(\omega)$ is nonnegative: (Use Periodogram Estimate for $S_X(\omega)$ to show $S_X(\omega) \geq 0$ — ENEE 620 notes)

Let $\tilde{X}_K(\omega) \triangleq \sum_{n=0}^{K-1} X[n]e^{-j\omega n}$, and $\tilde{P}_K(\omega) = \frac{1}{K}|\tilde{X}_K(\omega)|^2$

Claim $E[\tilde{P}_K(\omega)] \longrightarrow S_X(\omega)$ as $K \to \infty$

proof: $E[\tilde{P}_K(\omega)] = \frac{1}{K}E[\,|\tilde{X}_K(\omega)|^2\,]$

$$= \frac{1}{K}E\left[\left(\sum_{n=0}^{K-1} X[n]e^{-j\omega n}\right)\left(\sum_{m=0}^{K-1} X^*[m]e^{+j\omega m}\right)\right]$$

$$= \frac{1}{K}\sum_{n=0}^{K-1}\sum_{m=0}^{K-1} R_X[n-m]e^{-j\omega(n-m)}$$

$\left\{\begin{array}{l} u = n-m \\ v = n+m \end{array}\right.$ auxilliary

$$= \sum_{u=-(K-1)}^{K-1}\left(1-\frac{|u|}{K}\right)R_X[u]e^{-j\omega u} \xrightarrow[K\to\infty]{} \sum_{u=-\infty}^{\infty} R_X[u]e^{-j\omega u}$$

$$= S_X(\omega)$$

$\because\ \tilde{P}_K(\omega) = \frac{1}{K}|\tilde{X}_K(\omega)|^2 \geq 0, \ \forall \omega$

$\therefore\ E[\tilde{P}_K(\omega)] \geq 0, \ \forall \omega$

$\therefore\ S_X(\omega) = \lim_{K\to\infty} E[\tilde{P}_K(\omega)] \geq 0 \quad \forall \omega$ ※

Note: another way is that we consider a LTI system $h[n]$:

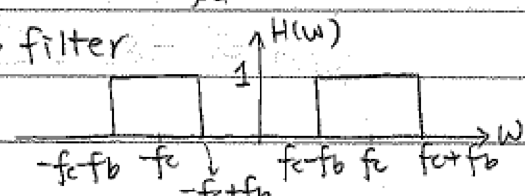
$X[n] \to \boxed{h[n]} \to Y[n]$

It can be shown $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$.

Since $R_Y(0) \geq 0$, we have $0 \leq R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\,d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega)\,d\omega$$

we can choose $H(\omega)$ to be ideal bandpass filter



$\Rightarrow \frac{2}{2\pi}\int_{f_c-f_b}^{f_c+f_b} S_X(\omega)\,d\omega \geq 0 \Rightarrow S_X(\omega) \geq 0 \ \forall \omega$ ※

assume continuous

for any f_c, and $f_b > 0$