2.9

a)

$$\begin{aligned} x_d(n_1,n_2) &= x(2n_1,2n_2) \\ &= 4+2\cos\Big(\frac{\pi}{2}(n_1+n_2)\Big)+2\cos\Big(\frac{\pi}{2}(n_1-n_2)\Big), \end{aligned}$$

so

$$\begin{aligned} X_d(\omega) &= 4(2\pi)^2\delta(\omega)+(2\pi)^2\delta\Big(\omega_1-\frac{\pi}{2},\omega_2-\frac{\pi}{2}\Big)+(2\pi)^2\delta\Big(\omega_1+\frac{\pi}{2},\omega_2+\frac{\pi}{2}\Big) \\ &\quad +(2\pi)^2\delta\Big(\omega_1-\frac{\pi}{2},\omega_2+\frac{\pi}{2}\Big)+(2\pi)^2\delta\Big(\omega_1+\frac{\pi}{2},\omega_2-\frac{\pi}{2}\Big) \end{aligned}$$

b) We have to verify that:

$$\begin{aligned} X_d(\omega_1,\omega_2) = \tfrac{1}{4}\Big[&\overbrace{X\Big(\frac{\omega_1}{2},\frac{\omega_2}{2}\Big)}^{1}+\overbrace{X\Big(\frac{\omega_1-2\pi}{2},\frac{\omega_2}{2}\Big)}^{2}+\overbrace{X\Big(\frac{\omega_1}{2},\frac{\omega_2-2\pi}{2}\Big)}^{3} \\ &+\underbrace{X\Big(\frac{\omega_1-2\pi}{2},\frac{\omega_2-2\pi}{2}\Big)}_{4}\Big] \end{aligned}$$

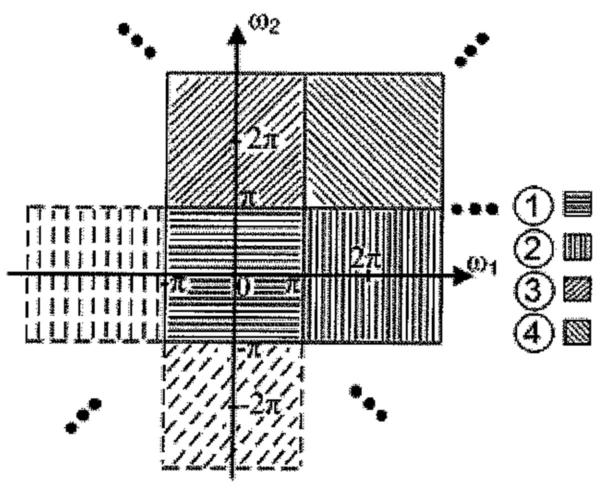
where we have numbered the terms 1 thru 4.
First, we note the effect of dividing by $(2,2)$ in the frequency variables is to spread out its period from $(2\pi,2\pi)$ to $(4\pi,4\pi)$, with term 1 centered on $(0,0)$, term 2 centered on $(2\pi,0)$, term 3 centered on $(0,2\pi)$, and term 4 centered on $(2\pi,2\pi)$, each with period $(4\pi,4\pi)$. So, added together the period becomes $(2\pi,2\pi)$, since the terms 2 through 4 just "fill in the holes" left by term 1 alone.
Now, the bandwidth of $X(\omega_1,\omega_2)$ is $[-\frac{\pi}{4},+\frac{\pi}{4}]^2$in its unit cell, so the bandwidth of $X(\frac{\omega_1}{2},\frac{\omega_2}{2})$ is
contained in $[-\frac{\pi}{2},+\frac{\pi}{2}]^2$, and hence there is no aliasing.
So, we only really have
to verify $X_d(\omega_1,\omega_2)=\frac{1}{4}X(\frac{\omega_1}{2},\frac{\omega_2}{2})$ on $[-\pi,+\pi]^2$ and let periodicity *take care of the rest.*
For the impulse or DC term, a simple change of variable shows that
$\delta(\frac{\omega_1}{2},\frac{\omega_2}{2})=4\delta(\omega_1,\omega_2)$,
so we get

$$\begin{aligned} 16\pi^2\delta(\omega_1,\omega_2) &= \tfrac{1}{4}\cdot 4\cdot(16\pi^2)\delta(\omega_1,\omega_2) \\ &= 16\pi^2\delta(\omega_1,\omega_2) \qquad \checkmark \end{aligned}$$

The other four shifted impulses behave similarly, eg.

$$\begin{aligned}
(2\pi)^2\delta(\omega_1 - \frac{\pi}{2}, \omega_2 - \frac{\pi}{2}) &= \frac{1}{4}(2\pi)^2\delta(\frac{\omega_1}{2} - \frac{\pi}{4}, \frac{\omega_2}{2} - \frac{\pi}{4}) \\
&= \frac{1}{4}(2\pi)^2\delta(\frac{\omega_1}{2} - \frac{\pi}{4}, \frac{\omega_2}{2} - \frac{\pi}{4}) \\
&= \frac{1}{4}(2\pi)^2\delta(\frac{1}{2}(\omega_1 - \frac{\pi}{2}), \frac{1}{2}(\omega_2 - \frac{\pi}{2})) \\
&= \frac{1}{4}(2\pi)^2 4\delta(\omega_1 - \frac{\pi}{2}, \omega_2 - \frac{\pi}{2}) \\
&= (2\pi)^2\delta(\omega_1 - \frac{\pi}{2}, \omega_2 - \frac{\pi}{2}) \qquad \checkmark
\end{aligned}$$

A sketch of the layout of the terms 1→4 would be as follows



N.B: Sketch not requested.

2.11

a) Directly sampling with sampling matrix MV gives

$x_d(\mathbf{n})$ with Fourier transform

$$X_d(\omega) = \frac{1}{|\det\mathbf{MV}|}\sum_{\text{all } k}[(\mathbf{MV})^{-T}(\omega - 2\pi \boldsymbol{k})]$$

so with alternative definition we have

$$\begin{aligned} X'_d(\omega) &= X_d((\mathbf{MV})^T\omega) \\ &= \frac{1}{|\det\mathbf{MV}|}\sum_{\text{all } k} X_c[\omega - 2\pi(\mathbf{MV})^{-T}\boldsymbol{k}] \\ &= \frac{1}{|\det\mathbf{V}|\cdot|\det\mathbf{V}|}\sum_{\text{all } k} X_c[\omega - \mathbf{M}^{-T}\mathbf{U}\boldsymbol{k}], \quad (1) \\ &\quad \text{where } 2\pi\mathbf{I} = \mathbf{U}^T\mathbf{V}. \end{aligned}$$

b) Clearly if $x(n) = x_c(\mathbf{V}n)$ and $x_d(\mathbf{n}) = x(\mathbf{Mn})$ then $x_d(\mathbf{n}) = x_c(\mathbf{MVn})$. Thus the above $X_d(\omega)$ must also be the alternative FT of $x_d(\mathbf{n})$.

c) If we sample using just $\mathbf{V}$, we get

$$X(\omega) = \frac{1}{|\det\mathbf{V}|}\sum_{allk} X_c(\mathbf{V}^{-T}(\omega - 2\pi\boldsymbol{k}))$$

and alternative FT

$$\begin{aligned} X'(\omega) &= \frac{1}{|det\mathbf{V}|}\sum_{allk} X_c(\omega - 2\pi\mathbf{V}^{-T}\boldsymbol{k}) \\ &= \frac{1}{|det\mathbf{V}|}\sum_{allk} X_c(\omega - \mathbf{U}\boldsymbol{k}) \quad (2) \end{aligned}$$

Comparing (1) and (2), we see that the repeat points $\{\mathbf{U}\boldsymbol{k}\}$ in (2) form a sublattice to the repeat points $\{\mathbf{M}^{-T}\mathbf{U}\boldsymbol{k}\}$ in (1), i.e. $\{\mathbf{U}\boldsymbol{k}\} \subset \{\mathbf{M}^{-T}\mathbf{U}\boldsymbol{k}\}$,

so if we set

$$X'_d(\omega) = \frac{1}{|\det\mathbf{M}|}\sum_{k} X'(\omega - 2\pi\mathbf{M}^{-T}\boldsymbol{k}) \quad (3)$$

where the sum tries to pick up the missing repeat points in $\{\mathbf{M}^{-T}\mathbf{U}\boldsymbol{k}\}$ that are not in $\{\mathbf{U}\boldsymbol{k}\}$, but for one period only.

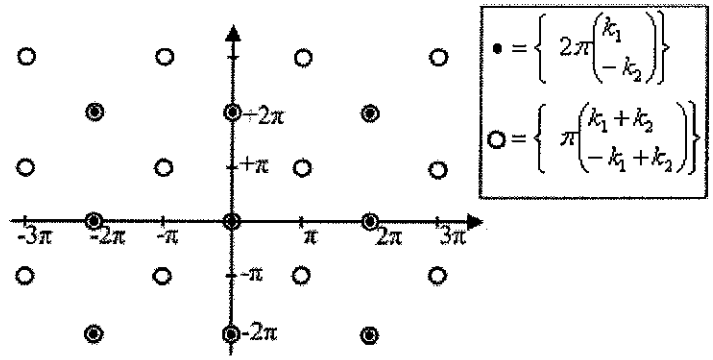
d) Take the simpler case where $\mathbf{V} = \mathbf{I}$. Then (2) and (3) become

$$\begin{aligned} X'(\omega) &= \sum_{\text{all } \mathbf{k}} X_c(\omega - 2\pi\boldsymbol{k}) \\ X'_d(\omega) &= \frac{1}{|\det\mathbf{M}|}\sum_{\substack{\mathbf{k} \\ \text{only certain } k}} X'(\omega - 2\pi\mathbf{M}^{-T}\boldsymbol{k}) \end{aligned}$$

So $\{2\pi \boldsymbol{k}\}$ is the sublattice and $\{2\pi \mathrm{M}^{-T}\boldsymbol{k}\}$ is the big lattice. In example 2.4-2, we have

$$\mathrm{M}^{-T} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \text{ so } 2\pi\mathrm{M}^{-T}\mathbf{k} = \pi\begin{pmatrix} k_1 + k_2 \\ -k_1 + k_2 \end{pmatrix}.$$

Thus the full lattice is $\left\{\pi\begin{pmatrix} k_1 + k_2 \\ -k_1 + k_2 \end{pmatrix}\right\}$., we sketch both below:



In that example we chose the sum over *certain* $\boldsymbol{k}$ to include just two points $\mathbf{k} = (0,0)^T$ and $\mathbf{k} = (1,0)^T$. Then the term $\boldsymbol{X}'(\omega)$ already includes all the repeat points $\{2\pi\mathbf{k}\}$ and the additional term

$$\boldsymbol{X}'\left(\omega - \pi\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

brings in all the remaining points in the full lattice $\left\{\pi\begin{pmatrix} k_1 + k_2 \\ -k_1 + k_2 \end{pmatrix}\right\}$.

This shows that for diamond subsampling,

$$\boldsymbol{X}'_d(\omega) = \tfrac{1}{2}\left(\boldsymbol{X}'(\omega) + \boldsymbol{X}'(\omega - \pi\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

4.4

a) It's a periodic (bi-)sequence, so we take the DFS to obtain

$$\widetilde{X}(k_1,k_2) = X\left(\frac{2\pi k_1}{n_1}, \frac{2\pi k_2}{N_2}\right).$$

b) It's periodic, and same as in part a) since sequences in sum do not overlap

$$\widetilde{X}(k_1,k_2) = X\left(\frac{2\pi k_1}{n_1}, \frac{2\pi k_2}{N_2}\right).$$

c)

$$x((N_1-n_1)_{N_1},(N_2-n_2)_{N_2})$$
$$\widetilde{X}(-k_1,-k_2) = X\left(-\frac{2\pi k_1}{N_1}, -\frac{2\pi k_2}{N_2}\right).$$

d) $\text{supp}\{x(N_1-1-n_1, N_2-1-n_2)\} = [0,N_1-1]\times[0,N_2-1]$ so DFT is appropriate here.

$$X(k_1,k_2) = \sum_{n_1,n_2=0}^{N_1-1,N_2-1} x(N_1-1-n_1,N_2-1-n_2)W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2}$$

Then let $n_i' \triangleq N_i - 1 - n_i$ to get

$$\begin{aligned}
& \sum_{n_1'=N_1-1}^{0}\ \sum_{n_2'=N_2-1}^{0} x(n_1',n_2')W_{N_1}^{(N_1-1-n_1')k_1}W_{N_2}^{(N_2-1-n_2')k_2} \\
= & \ X\left(-\frac{2\pi k_1}{N_1}, -\frac{2\pi k_2}{N_2}\right)W_{N_1}^{-k_1}W_{N_2}^{-k_2}.
\end{aligned}$$

4.5

Take $N_1 \geq 3$ and $N_2 \geq 1$. We have

$$\begin{aligned}
X(k_1,k_2) &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2)W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2} \\
&= 1\cdot W_{N_1}^0 W_{N_2}^0 + 2W_{N_1}^{k_1} + W_{N_2}^{2k_2} \\
&= 1 + 2e^{-\frac{j2\pi k_1}{N_1}} + e^{-\frac{-j4\pi k_2}{N_2}}
\end{aligned}$$

4.6

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2) W_{N_1}^{n_1k_1} W_{N_2}^{n_2k_2}$$

Now since x is real valued, we can say, by taking conjugate and substituting $N_i - k_i$ for k_i,

$$\begin{aligned}
X^*(N_1-k_1, N_2-k_2) &= \sum_{n_1}\sum_{n_2} x(n_1,n_2) W_{N_1}^{-(N_1-k_1)n_1} W_{N_2}^{-(N_1-k_2)n_2} \\
&= \sum_{n_1}\sum_{n_2} x(n_1,n_2) W_{N_1}^{n_1k_1} W_{N_2}^{n_2k_2} \\
&= X(k_1,k_2) \qquad \text{, since for integer } l,\ W_{N_i}^{-N_il} = 1.
\end{aligned}$$

4.7

$$\begin{aligned}
&\sum_{n_1,n_2} x(n_1,n_2) y^*(n_1,n_2) \\
&= \sum_{n_1,n_2}\left[\frac{1}{N_1N_2}\sum_{k_1,k_2} X(k_1,k_2) W_{N_1}^{-n_1k_1} W_{N_2}^{-n_2k_2}\right] y^*(n_1,n_2) \\
&\qquad\qquad \hookrightarrow \text{using IDFT formula} \\
&= \frac{1}{N_1N_2}\sum_{k_1,k_2} X(k_1,k_2)\left[\sum_{n_1,n_2} y^*(n_1,n_2) W_{N_1}^{-n_1k_1} W_{N_2}^{-n_2k_2}\right] \\
&= \qquad '' \qquad '' \qquad \left[\sum_{n_1,n_2} y(n_1,n_2) W_{N_1}^{n_1k_1} W_{N_2}^{n_2k_2}\right]^* \\
&= \frac{1}{N_1N_2}\sum_{k_1,k_2} X(k_1,k_2) Y^*(k_1,k_2) \quad \rightarrow \text{ using DFT formula.}
\end{aligned}$$

All sums above are over $[0, N_1 - 1] \times [0, N_2 - 1]$.

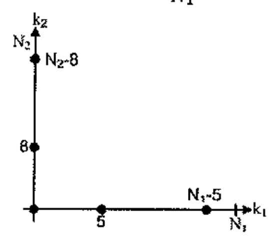
4.8

$$\begin{aligned}
x(n_1,n_2) &= 10 + 2\cos\left(\frac{2\pi 5 n_1}{N_1}\right) + 5\sin\left(\frac{2\pi 8 n_2}{N_2}\right) \\
&= \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} X(k_1,k_2) W_{N_1}^{-k_1n_1} W_{N_2}^{-k_2n_2} \\
&= 10 W_{N_1}^0 W_{N_2}^0 + W_{N_1}^{-5n_1} + W_{N_1}^{+5n_1} + \frac{5}{2j} W_{N_2}^{-8n_2} - \frac{5}{2j} W_{N_2}^{+8n_2}
\end{aligned}$$

so we need

$$\begin{aligned} N_1N_2X(k_1,k_2) = 10\delta(k_1,k_2) &+ 1\cdot\delta(k_1-5,k_2) + 1\cdot\delta(k_1-(N_1-5),k_2) \\ &+ \frac{5}{j2}\delta(k_1,k_2-8) - \frac{5}{j2}\delta(k_1,k_2-(N_2-8)), \end{aligned}$$

with use of $W_{N_2}^{+8n_2} = W_{N_2}^{-(N_2-8)n_2}$ and $W_{N_1}^{+5n_1} = W_{N_1}^{-(N_1-5)n_1}$.



4.9

Prove periodic convolution (property 2) from circular shift (property 5)
Taking DFT of circular convolution, we have

$$\begin{aligned} &\sum_{n_1,n_2}\left(\sum_{l_1,l_2} x(l_1,l_2)y((n_1-l_1)_{N_1},(n_2-l_2)_{N_2})W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2}\right) \\ = &\sum_{l_1,l_2} x(l_1,l_2)\left[\sum_{n_1,n_2} y((n_1-l_1)_{N_1},(n_2-l_2)_{N_2})W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2}\right] \\ = &\sum_{l_1,l_2} x(l_1,l_2)Y(k_1,k_2)e^{-j2\pi(\frac{l_1k_1}{N_1}+\frac{l_2k_2}{N_2})}, \quad \text{by circular shift property of DFT,} \\ = &\; Y(k_1,k_2)\left[\sum_{l_1,l_2} x(l_1,l_2)W_{N_1}^{l_1k_1}W_{N_2}^{l_2k_2}\right] \\ = &\; Y(k_1,k_2)X(k_1,k_2) \text{ as was to be shown.} \end{aligned}$$

d) For MATLAB solution in 2-D case, run `DCT2Deg1.m` from the book CD-ROM. Set $\alpha = 1.0$ for unmodified signal. The 1-D case is also there as `DCTeg1.m`

4.13

a) If $y(n) = x(n) + x(2N-1-n)$

$$\begin{pmatrix} n & : & 0 & . & . & . & N-1|N & . & . & 2N-1 \\ y & : & x(0) & . & . & . & x(N-1)|x(N-1) & . & . & x(0) \end{pmatrix}$$

Then

$$X_c(k) = W_{2N}^{\frac{k}{2}} Y_{2N}(k) \; for \; k \in [0, N-1]$$

where $2N$ is $2N$-point DFT of $y(n)$.

Now let $x'(n) = x(N-1-n)$ then

$$\begin{aligned} y'(n) &= x'(n) + x'(2N-1-n) \\ &= x(N-1-n) + x(n-N) \end{aligned}$$

$$\begin{pmatrix} n & : & 0 & . & . & . & N-1|N & . & . & 2N-1 \\ y' & : & x(N-1) & . & . & . & x(0)|x(0) & . & . & x(N-1) \end{pmatrix}$$

and $X_c'(k) = W_{2N}^{\frac{k}{2}} Y_{2N}'(k)$ for $k \in [0, N-1]$

Use DFT-DFS analogy: Define $\widetilde{y}'(n) \triangleq y'((n)_{2N})$

$$\widetilde{y}(n) \triangleq y((n)_{2N})$$

Then $\widetilde{y}'(n) = \widetilde{y}(n+N)$ as seen above, so

$$\begin{aligned} \mathrm{DFT}_{2N}\{y'(n)\} &= \mathrm{DFT}_{2N}\{y(n)\} \cdot W_{2N}^{-Nk} \\ \text{or } Y_{2N}'(k) &= W_{2N}^{-Nk} Y_{2N}(k) \\ &= (-1)^k Y_{2N}(k) \end{aligned}$$

Thus finally

$$\begin{aligned} X_c'(k) &= W_{2N}^{\frac{k}{2}} Y_{2N}'(k) \\ &= (-1)^k W_{2N}^{\frac{k}{2}} Y_{2N}(k) \\ &= (-1)^k X_c(k), \quad 0 \le k \le N-1 \end{aligned}$$

Elsewhere $X_c'(k) = 0$.

b) Follow through as above, but with 2-D functions,

$x(n_1, n_2)$ and $x'(n_1, n_2) = x(N-1-n_1, N-1-n_2)$. Then

$$\begin{aligned} y(n_1, n_2) &= x(n_1, n_2) + x(2N-1-n_1, n_2) + x(n_1, 2N-1-n_2) \\ &\quad + x(2N-1-n_1, 2N-1-n_2) \end{aligned}$$

with support $[0, 2N-1]^2$.

Let $Y(k_1, k_2) = \mathrm{DFT}_{2N\times 2N}\{Y\}$

Then $Y'(k_1, k_2) = \mathrm{DFT}_{2N\times 2N}\{Y'\}$

with $y'(n_1, n_2) = x(N-1-n_1, N-1-n_2) + x(n_1-N, N-1-n_2)$

$$+x(N-1-n_1, n_2-N) + x(n_1-N, n_2-N).$$

We see $\widetilde{y}'(n_1, n_2) = \widetilde{y}(n_1+N, n_2+N)$

so $\widetilde{Y}'(k_1, k_2) = W_{2N}^{-Nk_1} W_{2N}^{-Nk_2} \widetilde{Y}(k_1, k_2)$,

hence $Y'(k_1, k_2) = W_{2N}^{-Nk_1} W_{2N}^{-Nk_2} Y(k_1, k_2)$, for $(k_1, k_2) \in [0, 2N-1]^2$.

We thus get

$$\begin{aligned} X_c'(k_1, k_2) &= W_{2N}^{\frac{k_1}{2}} W_{2N}^{\frac{k_2}{2}} Y'(k_1, k_2) \\ &= (-1)^{k_1+k_2} X_c(k_1, k_2) \\ &\quad \text{for } (k_1, k_2) \in [0, N-1]^2. \end{aligned}$$

4.14

a) Consider the symmetric 2N point extension of x(n),

$$x'(n) = \begin{cases} x(n), & 0 \le n \le N-1; \\ x(2N-1-n), & N \le n \le 2N-1 \end{cases}$$

Define "symmetric convolution" as circular convolution of the zero phase $h(n)$ and $x'(n)$. Note that the result will then be a $2N$ point sequence symmetric about $n = N - \frac{1}{2}$, as is $x'(n)$. So we can use the DCT to implement as follows: Letting $H(k)$ be the $2N$-point DFT of the zero-phase sequence $h(n)$,

$(h \to \widetilde{h} \to \widetilde{h}R_{2N}(n)$ first$)$,

we have $Y'(k) = H(k)X'(k), k = 0, \ldots, 2N-1$,

and $X_c(k) = W_{2N}^{\frac{k}{2}} X'(k), k = 0, \ldots, N-1$,

so $Y_c(k) = W_{2N}^{\frac{k}{2}} H(k)X'(k) = \left(W_{2N}^{\frac{k}{2}} X'(k)\right) H(k) = X_c(k)H(k)$,

$k = 0, \ldots, N-1$.

b) If support of $x(n)$ is sufficiently small part of $[0, N-1]$, we will be able to get linear convolution out of this symmetric convolution, as shown in the figure