Random Walks for Image Segmentation

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Content of the Paper

- An user interactive K-way Image Segmentation Algorithm
- Users define K seeds belonging to K different objects
- Apply an algorithm similar to random walk
- Solve the corresponding Dirichlet problem
- Some theoretical and behavioral properties of the algorithm

Random walk or Drunkard's Walk

A mathematical formalisation of a trajectory that consists of taking successive random steps

Penny Matching Game

- Say John and Peter have 5 pennies
- On each match Peter **loose or gain a penny with probability 0.5**
- When Peter have all the five or zero penny the game ends
- Find **p(x) (probability of Peter's win) with x initial pennies**

- $p(0) = 0$
- $p(N) = 1$
- $p(x) = \frac{p(x-1)+p(x+1)}{2}$

How $p(x) = \frac{p(x-1)+p(x+1)}{2}$?

E: Probability that **Peter wins if it starts with x** pennies

F: The first step is to **loose a penny**

G: The first step is to **gain a penny**

P(E)=P(E/F)P(F)+P(E/G)P(G)

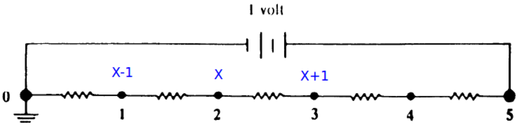
Now, P(E)=p(x) and P(F)=P(G)=$\frac{1}{2}$

P(E/F)=p(x-1) and p(E/G)=p(x+1)

So,

$p(x) = \frac{p(x-1)+p(x+1)}{2}$

Electric Circuit Analogue



By applying Ohm's law and Kirchoff's law: $v(x) = \frac{v(x-1)+v(x+1)}{2}$

If resistors between x and $x+1$ is R_x:

$$v(x) = \frac{\frac{1}{R_x - 1}}{\frac{1}{R_x - 1} + \frac{1}{R_x}} v(x-1) + \frac{\frac{1}{R_x}}{\frac{1}{R_x - 1} + \frac{1}{R_x}} v(x+1)$$

Harmonic Function

Is a function that satisfies Laplace's equation: $\nabla^2 u = 0$

In 1-dimensional:

Let, $S = \{0, 1, 2, ..., N\}$, $D = \{1, 2, ..., N-1\}$, $B = \{0, N\}$

A function $f(x)$ defined on S is harmonic if: $f(x) = \frac{f(x-1)+f(x+1)}{2}$

Dirichlet Problem: Find a harmonic function given it's boundary values.

Uniqueness Principle: There can't be two different harmonic functions having same boundary values.

Qualities of Interactive Segmentation

- Fast Computation
- Fast Editing
- Limited interaction
- Intuitive segmentation

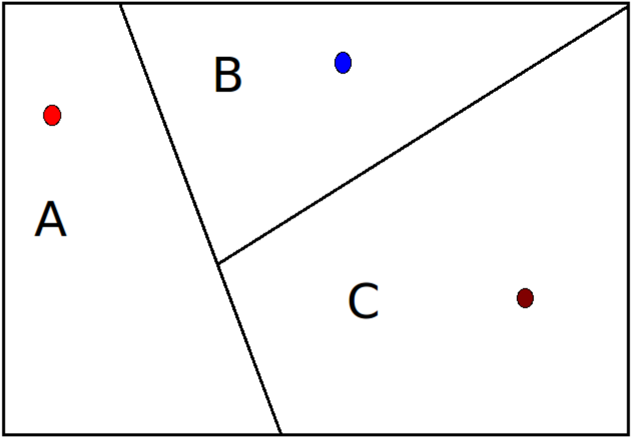
This algorithm has:

- All four of these properties
- **noise robustness**
- **avoids small cut**.

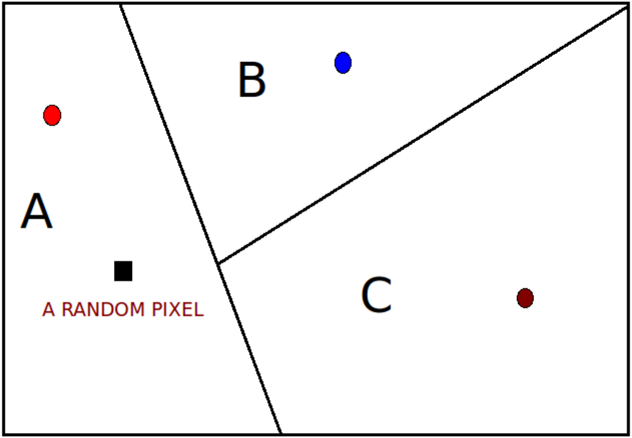
Graph on Image

- We can define a **connected** and **undirected** graph on the image $G = (V, E)$
- A vertex is v_i, an edge between v_i and v_j is e_{ij}
- Degree of a vertex v_i is $d_i = \sum w(e_{ij})$
- $w_{ij} = exp(-\beta(intensity(g_i) - intensity(g_j)))$

Random walk on a finite graph is a path of length k, which starts at some node O and continue selecting the next node randomly from it's neighbours, until it has k-1 transitions.



An Image



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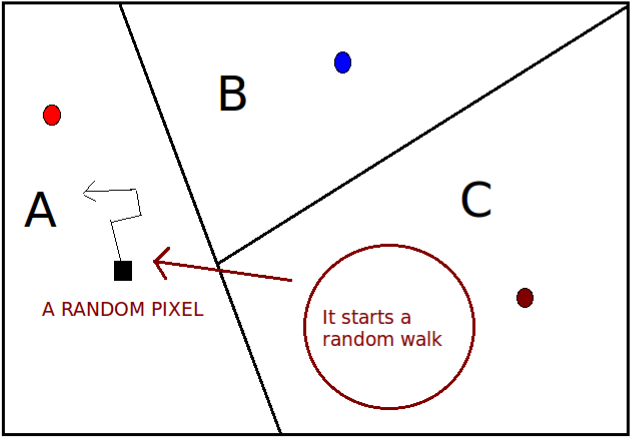
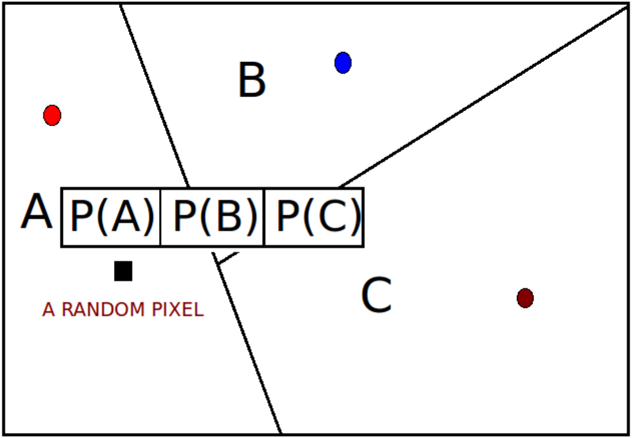


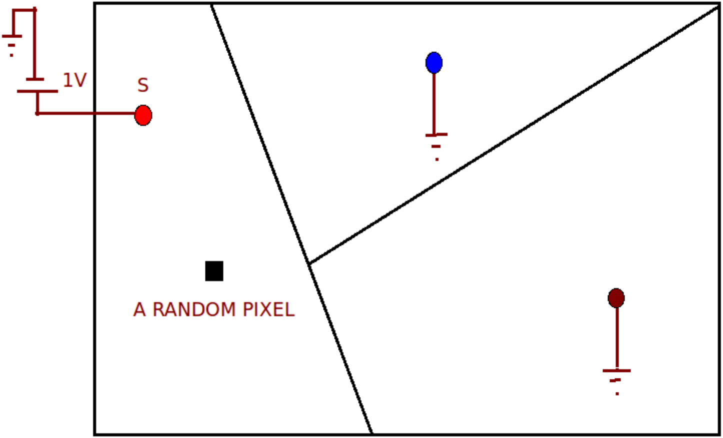
An Image

$P(A)+P(B)+P(C)=1$



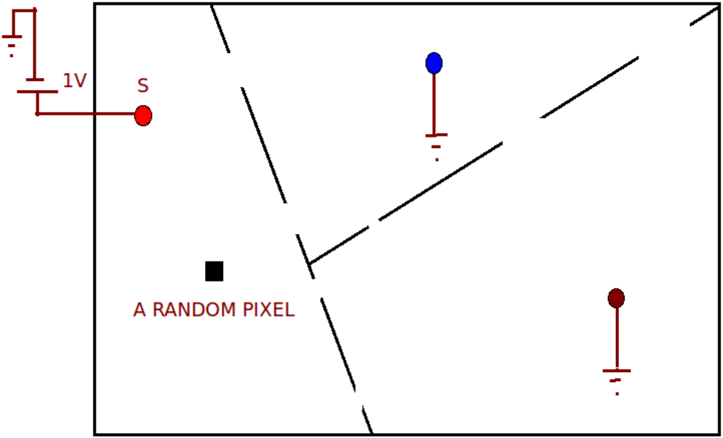
An Image

ELECTRIC ANALOGUE AGAIN !!!



An Image

What if boundaries of image are not well defined ?



An Image

Some Definitions

Dirichlet Integral:

$D[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 d\Omega$

Combinatorial Laplacian Matrix:

$L_{ij}=$

$$\left\{ \begin{array}{cc} d_i, & \text{if i=j} \\ -w_{ij}, & \text{if } v_i \text{ and } v_j \text{ are adjacent nodes} \\ 0, & \text{otherwise} \end{array} \right\}$$

$m \times n$ **edge node Incidence Matrix:**

$A_{e_{ij}v_k}=$

$$\left\{ \begin{array}{cc} +1 & \text{if i=k} \\ \text{-1} & \text{if j=k} \\ 0, & \text{otherwise} \end{array} \right\}$$

Some more definitions

Constitutive Matrix (C)

Diagonal matrix with weights of each edge along the diagonal.

As an operator:

A: **Gradiaent** Operator

A^T: Combinatorial **Divergence** Operator

So, Combinatorial **Laplacian**: $L = A^T A$ and

Combinatorial **Laplace-Beltrami** Operator: $L = A^T C A$

Dirichlet Again

$$D[x] = \frac{1}{2}(Ax)^T C(AX) = \frac{1}{2}x^T L x = \frac{1}{2}\sum_{e_{ij}\in E} w_{ij}(x_i - x_j)^2$$

V_M: Marked Nodes
V_U: Unmarked Nodes

$$D[x_U] = \frac{1}{2}\left[x_M^T x_U^T\right]\begin{bmatrix} L_M & B \\ B^T & L_U \end{bmatrix}\begin{bmatrix} x_M \\ x_U \end{bmatrix} =$$
$$\frac{1}{2}\left(x_M^T L_M x_M + 2x_U^T B^T x_M + x_U^T L_U x_U\right)$$

Differentiating $D[x_U]$ with respect to x_U and finding the critical points yield:

$L_U x_U = -B^T x_M$

x_i^s: **Probability/potential assumed at node v_i for each label s**

$Q(v_j) = s$, for $v_j \in V_M$, $s \in \mathbb{Z}$, $0 \leq s \geq K$

Define a column vector m_s of length $|V_M|$ for a label s and $v_j \in V_M$

$m_j^s =$

$$\left\{ \begin{array}{ll} 1 & \text{if } Q(v_j) = s \\ 0 & \text{if } Q(v_j) \neq s \end{array} \right\}$$

So for label s, the solution for Combinatorial Dirichlet problem is found by solving

$L_U x^s = -B^T m^s$ for one label or,

$L_U X = -B^T M$

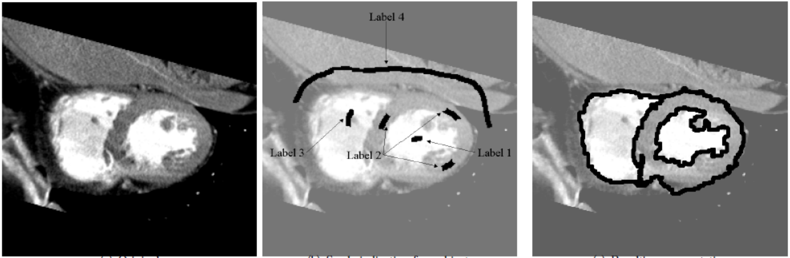
for all labels, where X and M have the columns x^s and m^s respectively.

We also have the constraint:

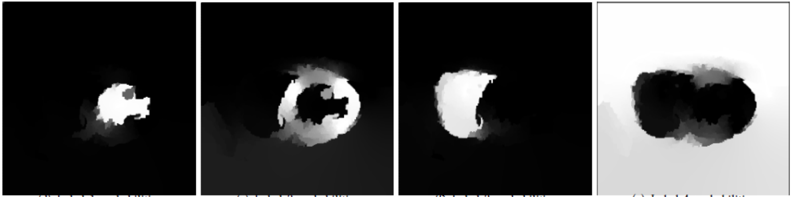
$\sum_s x_i^s = 1$, For all $v_i \in V$,

So we have K-1 sparse linear equations to solve !!!

Experimental Results



(a) Original (b) Seeds indicating four objects (c) Resulting segmentation



(d) Label 1 probabilities (e) Label 2 probabilities (f) Label 3 probabilities (g) Label 4 probabilities

Comments and Conclusion

- A segmentation algorithm with user interaction
- Random walk has been used very nicely
- There are some nice discussions on theoretical and behavioral proerties of the algorithm
- Not sure why there is no experiment with natural images
- The algorithm works well with weak boundaries and for noisy images