$$\text{then} \qquad \mathbf{R} = \begin{bmatrix} R_x(0,0) & R_x(2,-1) & R_x(1,-1) & R_x(0,-1) \\ R_x(-2,1) & R_x(0,0) & R_x(-1,0) & R_x(-2,0) \\ R_x(-1,1) & R_x(1,0) & R_x(0,0) & R_x(-1,0) \\ R_x(0,1) & R_x(2,0) & R_x(1,0) & R_x(0,0) \end{bmatrix}$$

and $\mathbf{Ra} = \mathbf{r}$ is the normal equation.

7.3

From Ch. 7, section 7.2,

$$\sum_{(l_1,l_2)\in\mathcal{R}_h} h_{l_1,l_2} R_{xy}(k_1 - l_1, k_2 - l_2) = R_{xy}(k_1, k_2) \qquad \text{for all}(k_1,k_2) \in \mathcal{R}_h$$

$$\begin{aligned}
\text{Here } \mathcal{R}_h &= \{[-2,-2], ..., [+2,+2]\} \\
&= \text{a } 5 \times 5 \text{ square centered on the origin.} \\
R_{yy}(m_1,m_2) &= 25R_{xx}(m_1,m_2) + 9R_{ww}(m_1,m_2) \\
&= 25\sigma_x^2 \rho^{|m_1|+|m_2|} + 9\sigma_w^2 \delta(m_1,m_2) \\
R_{xy}(m_1,m_2) &= R_{xx}(m_1,m_2) \quad \text{since} \quad x \perp w \\
&= 5\sigma_x^2 \rho^{|m_1|+|m_2|}.
\end{aligned}$$

7.4

For infinite extent observation plane, we can write Wiener filter in the 2-D frequency domain as

$$\begin{aligned}
H(\omega_1,\omega_2) &= \frac{S_{xy}(\omega_1,\omega_2)}{S_{yy}(\omega_1,\omega_2)}, \text{ where} \\
S_{xy}(\omega_1,\omega_2) &= \mathbf{FT}\Big\{ R_{xy}(m_1,m_2) \Big\} \\
\text{and } S_{yy}(\omega_1,\omega_2) &= \mathbf{FT}\Big\{ R_{yy}(m_1,m_2) \Big\} \\
&= \mathbf{FT}\Big\{ 25\sigma_x^2 \rho^{|m_1|+|m_2|} + 9\sigma_w^2 \delta(m_1,m_2) \Big\} \\
&= 25\sigma_x^2 \mathbf{FT}\Big\{ \rho^{|m_1|+|m_2|} \Big\} + 9\sigma_w^2
\end{aligned}$$

Since $\rho^{|m_1|+|m_2|}$ is separable, we first calculate

$$\begin{aligned}
\mathbf{FT}\left\{\rho^{|m|}\right\} &= 1+\sum_{m=1}^{\infty}\rho^{m}e^{-j\omega m}+\sum_{m=-\infty}^{-1}\rho^{-m}e^{-j\omega m}\\
&= \frac{1-\rho^2}{1+\rho^2-2\rho\cos\omega}\\
\text{so } S_{yy}(\omega_1,\omega_2) &= \frac{25\sigma_x^2(1-\rho^2)^2}{(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}+9\sigma_w^2\\
\text{and } S_{xy}(\omega_1,\omega_2) &= \frac{5\sigma_x^2(1-\rho^2)^2}{(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}.\\
\text{Thus}\quad & H(\omega_1,\omega_2)=\frac{5\sigma_x^2(1-\rho)^2}{(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}\times\\
&\frac{(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}{25\sigma_x^2(1-\rho^2)^2+9\sigma_w^2(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}\\
&= \frac{5\sigma_x^2(1-\rho)^2}{25\sigma_x^2(1-\rho^2)^2+9\sigma_w^2(1+\rho^2-2\rho\cos\omega_1)(1+\rho^2-2\rho\cos\omega_2)}
\end{aligned}$$

7.5

From Th. 7.2-1,

$$\begin{aligned}
S_x(\omega_1,\omega_2) &= \sigma^2 B_{\oplus+}(\omega_1,\omega_2)B_{\ominus-}(\omega_1,\omega_2)\\
\text{and } B_{\ominus-}(z_1,z_2) &= B_{\oplus+}(z_1^{-1},z_2^{-1})\\
\text{Here}\quad & B_{\oplus+}(z_1,z_2)=1+0.8z_1^{-1}+0.1z_2^{-1}\\
\text{so}\quad & B_{\ominus-}(z_1,z_2)=1+0.8z_1+0.1z_2.
\end{aligned}$$

$$\begin{aligned}
& B_{\oplus+}(\omega_1,\omega_2)B_{\ominus-}(\omega_1,\omega_2)\\
=\;& (1+0.8e^{-j\omega_1}+0.1e^{-j\omega_2})(1+0.8e^{+j\omega_1}+0.1e^{+j\omega_2})\\
=\;& 1+0.8e^{-j\omega_1}+0.1e^{-j\omega_2}+0.8e^{+j\omega_1}+0.64+0.08e^{+j(\omega_1-\omega_2)}\\
& +0.1e^{+j\omega_2}+0.08e^{-j(\omega_1-\omega_2)}+0.01\\
=\;& (1.0+0.64+0.01)+0.1(e^{+j\omega_2}+e^{-j\omega_2})+0.8(e^{+j\omega_1}+e^{-j\omega_1})\\
& +0.08(e^{+j(\omega_1-\omega_2)}+e^{-j(\omega_1-\omega_2)})\\
=\;& 1.65+0.2\cos\omega_2+1.6\cos\omega_1+0.16\cos(\omega_1-\omega_2).
\end{aligned}$$

Thus verifying the theorem with $\sigma^2=1.0$.

7.6

a)

$$\begin{aligned} E\Big[|e(\boldsymbol{n}|^2\Big] &= E\Big[(\hat{x}-x)(\boldsymbol{n})e^*(\boldsymbol{n})\Big] \\ &= -E\Big[x(\boldsymbol{n})e^*(\boldsymbol{n})\Big] \end{aligned}$$

since $\hat{x}(\boldsymbol{n}) = \sum h_{k_1,k_2} y(n_1 - k_1, n_2 - k_2)$ and measurements $y(\mathbf{n}-\mathbf{k})$ are orthogonal to the error $e(\mathbf{n})$.

$$\begin{aligned} \text{Next } e^*(\boldsymbol{n}) &= \hat{x}^*(\boldsymbol{n}) - x^*(\boldsymbol{n}), \\ \text{so we have, } E\Big[|e(\boldsymbol{n})|^2\Big] &= E\Big[|x(\boldsymbol{n}|^2\Big] - E\Big[x(\boldsymbol{n})\hat{x}^*(\boldsymbol{n})\Big] \\ &= R_{xx}(0,0) - E\Big[x(n_1,n_2)\sum_{k_1,k_2\in\mathcal{R}_h} h^*_{k_1,k_2} y^*(n_1-k_1,n_2-k_2)\Big] \\ &= \sigma_x^2 - \sum_{k_1,k_2\in\mathcal{R}_h} h^*_{k_1,k_2}\underbrace{E\Big[x(n_1,n_2)y^*(n_1-k_1,n_2-k_2)\Big]}_{R_{xy}(k_1,k_2)} \end{aligned}$$

where we have assumed homogeneity and zero mean.

b) Putting these normal equations in matrix-vector form, we have

$$\begin{aligned} \boldsymbol{R}_{yy}\boldsymbol{h} &= \boldsymbol{r}_{xy} \\ \text{so } \boldsymbol{h} &= \boldsymbol{R}_{yy}^{-1}\boldsymbol{r}_{xy} \end{aligned}$$

From, part a), in matrix-vector form, we have

$$\begin{aligned} \sigma_e^2 &= \sigma_x^2 - \boldsymbol{h}^{*T}\boldsymbol{r}_{xy} \\ \text{so } \boldsymbol{h}^{*T} &= (\boldsymbol{R}_{yy}^{-1}\boldsymbol{r}_{xy})^{*T} \\ &= \boldsymbol{r}_{xy}^{*T}\boldsymbol{R}_{yy}^{-1} \end{aligned}$$

thus finally,

$$\begin{aligned} \sigma_e^2 &= \sigma_x^2 - \boldsymbol{r}_{xy}^{*T}\boldsymbol{R}_{yy}^{-1}\boldsymbol{r}_{xy} \\ &= \sigma_x^2 - \boldsymbol{r}_{xy}^{T}\boldsymbol{R}_{yy}^{-1}\boldsymbol{r}_{xy}^{*} \quad \text{(since real valued).} \end{aligned}$$

7.7

a) RUKF is motivated to omit updates far from the observation because, due to the stable AR signal model, there should be a low "correlation" (really covariance) between these two values, i.e. current observation and distant value of signal. $\mathcal{U}_{\oplus+}(n_1, n_2)$ is the update region for the observation located at (n_1, n_2).

b) The approximate RUKF also omits further away error-covariance updates. $\mathcal{J}_{\oplus+}(n_1, n_2)$ is the error covariance update region. Overall we have $\mathcal{U}_{\oplus+} \subseteq \mathcal{J}_{\oplus+} \subseteq \mathcal{S}_{\oplus+}$, the global state region.
c) Stability of the AR (Markov) signal model would be a necessary condition for an RUKF steady-state to exist, since far from the boundaries, we would expect that boundary conditions should not affect the estimates that mucg, since the underlying signals should have little 'correlation' (really covariance) with these distant boundary values.