Lecture 5:
Neural Networks

Waitlist update

I was confused about the way waitlists work on Monday =(

We have set enrollment sizes of 35 / 85 for 498 / 598

Each day overrides will be sent automatically in waitlist order to fill up to capacity

If you don’t enroll within a day of getting an override you will be dropped from the waitlist

Assignment 1

Was due on Sunday

If you use all 3 late days then you can turn it in today with no penalty

If you enrolled late, your A1 will be due **one week from the time you enrolled**

Assignment 2

Due Monday, September 30

Much longer than A1 – Start early

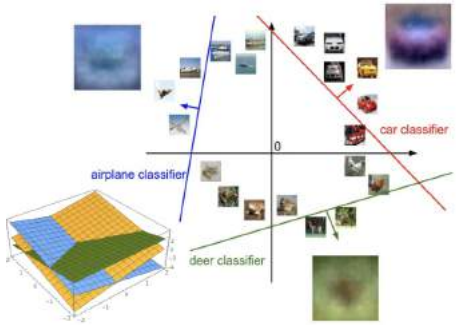
Your submission **must** pass the validation script to be graded!

We will be lenient on A1 submissions, but starting with A2 we will not grade your assignment if it does not pass the validation script

Where we are:

1. Use **Linear Models** for image classification problems

$$s = f(x; W) = Wx$$



2. Use **Loss Functions** to express preferences over different choices of weights

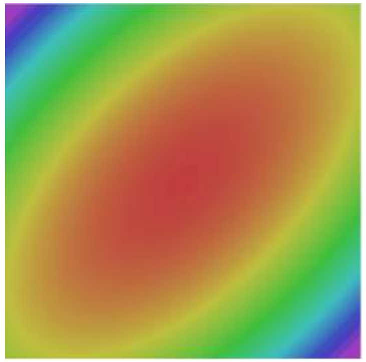
$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

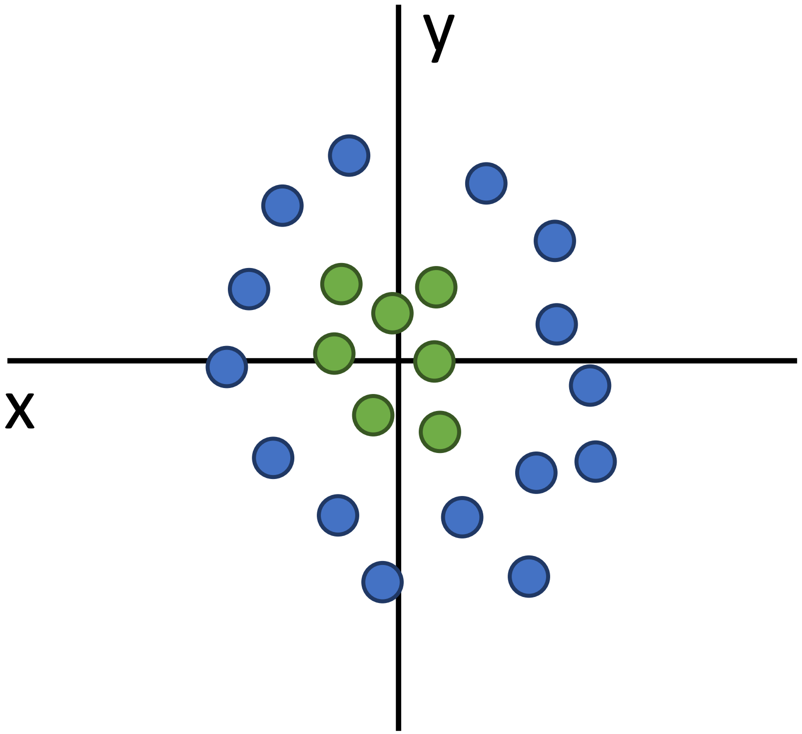
3. Use **Stochastic Gradient Descent** to minimize our loss functions and train the model

```
v = 0
for t in range(num_steps):
  dw = compute_gradient(w)
  v = rho * v + dw
  w -= learning_rate * v
```



Problem: Linear Classifiers aren't that powerful

Geometric Viewpoint

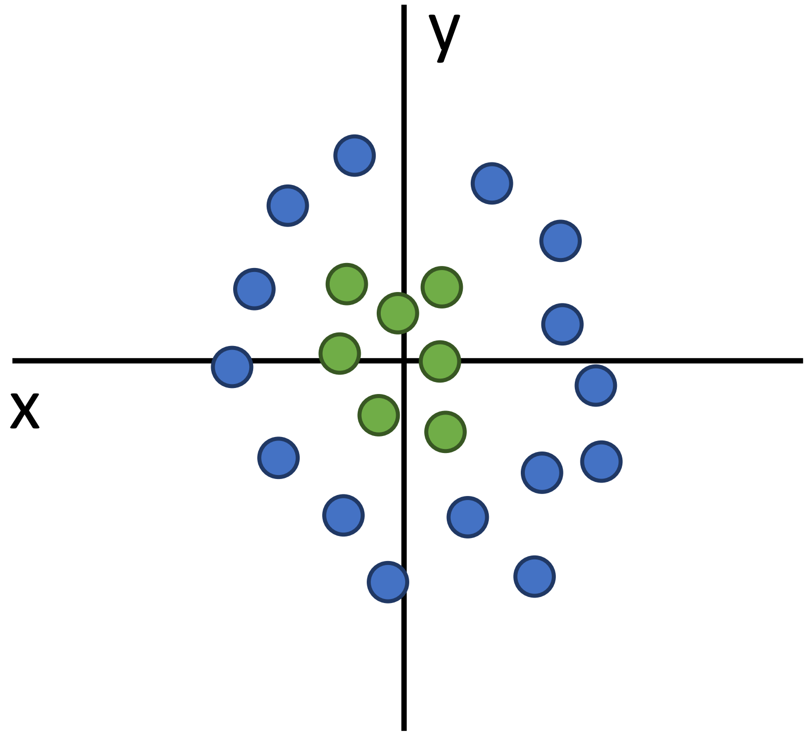


Visual Viewpoint

One template per class:
Can't recognize different
modes of a class

One solution: **Feature Transforms**

Original space

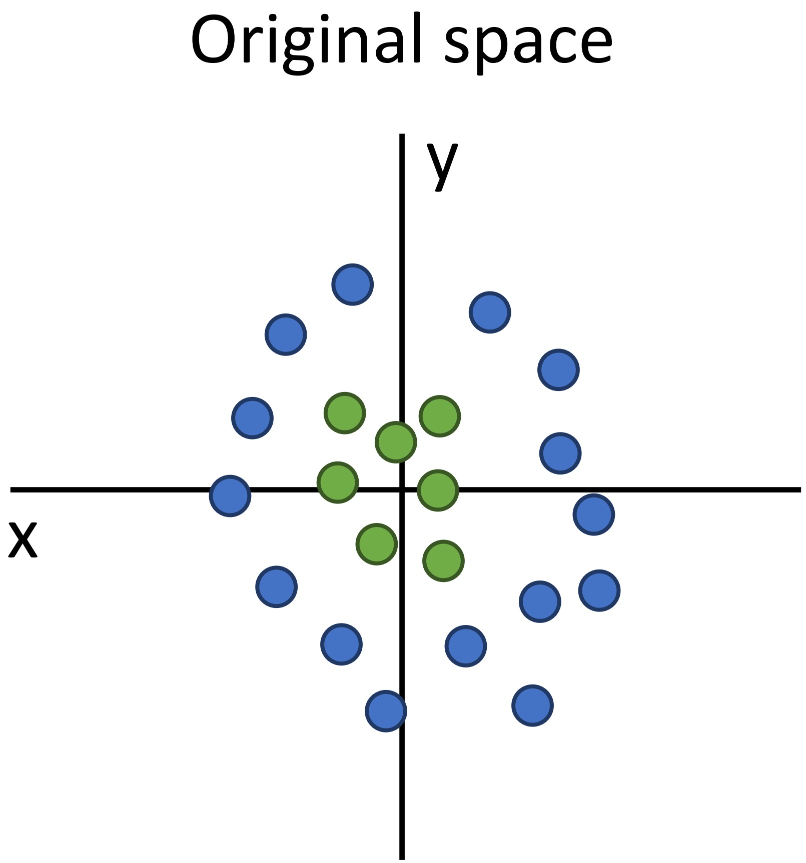


$r = (x^2 + y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$

Feature transform

One solution: **Feature Transforms**



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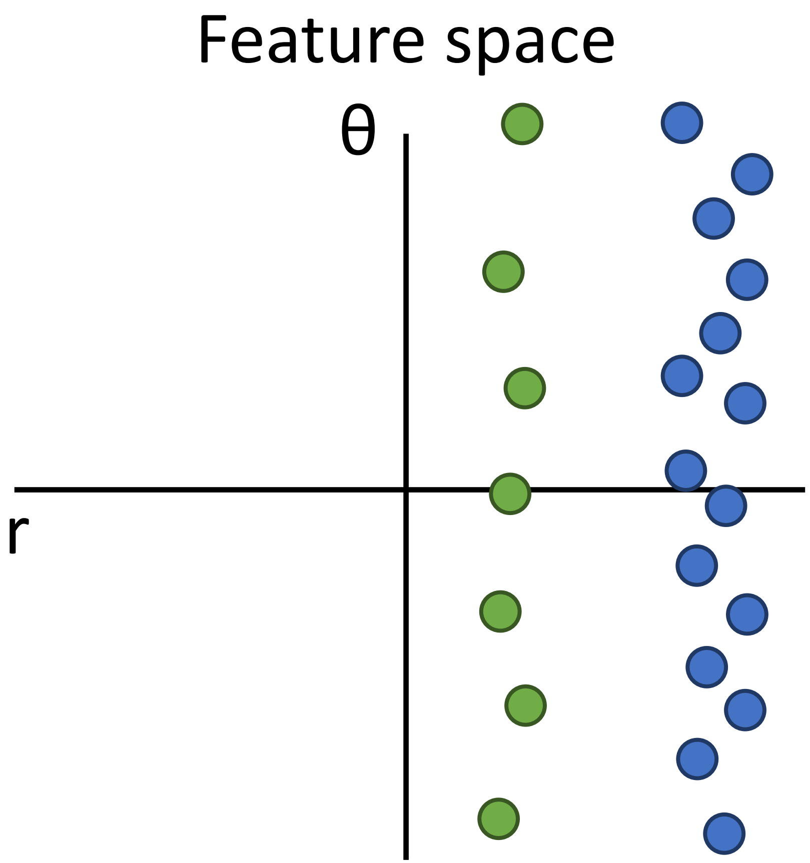

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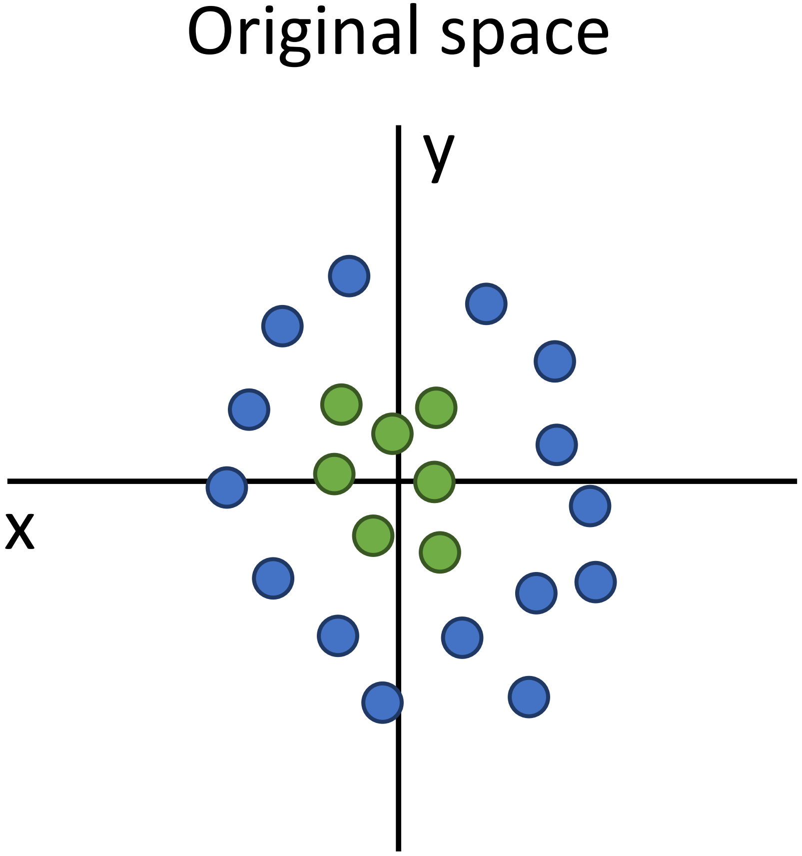

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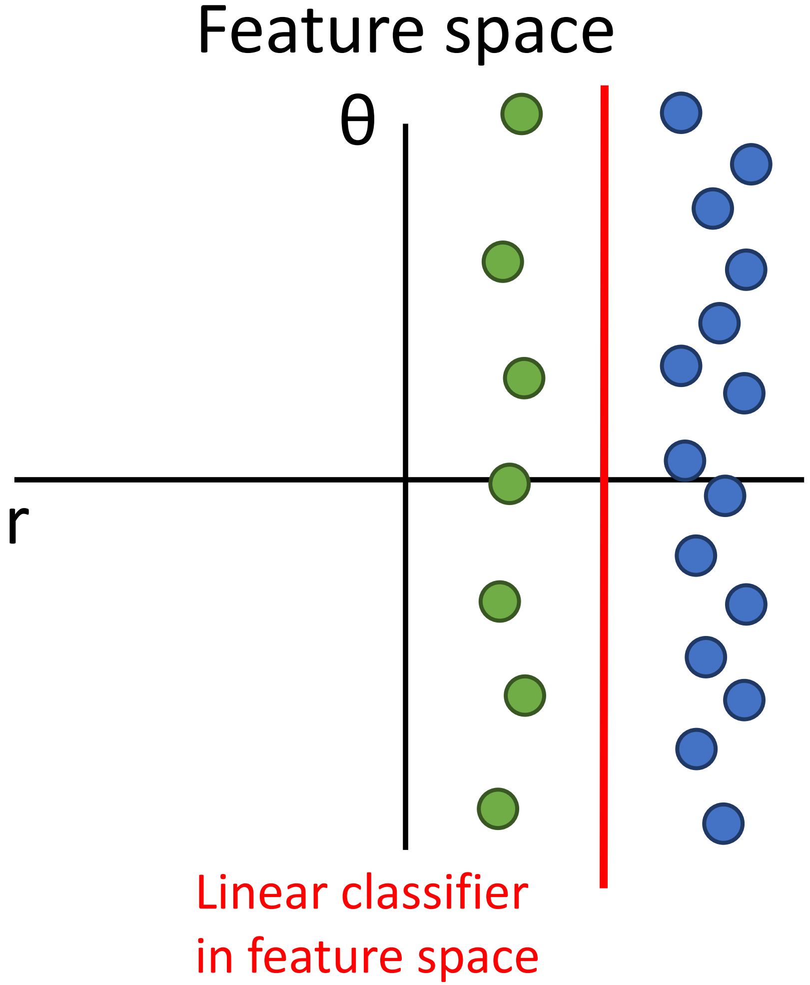
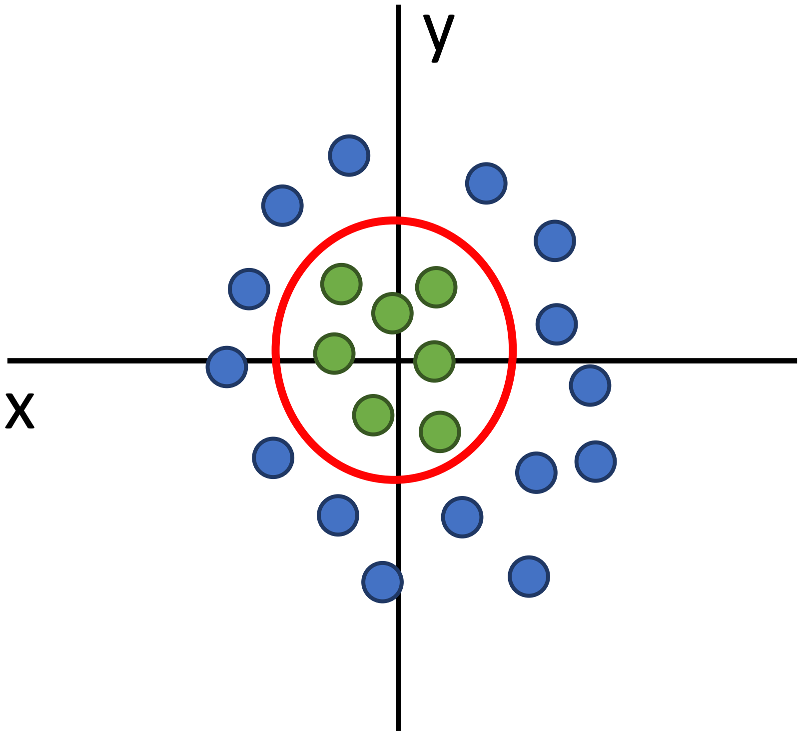


One solution: **Feature Transforms**

Original space

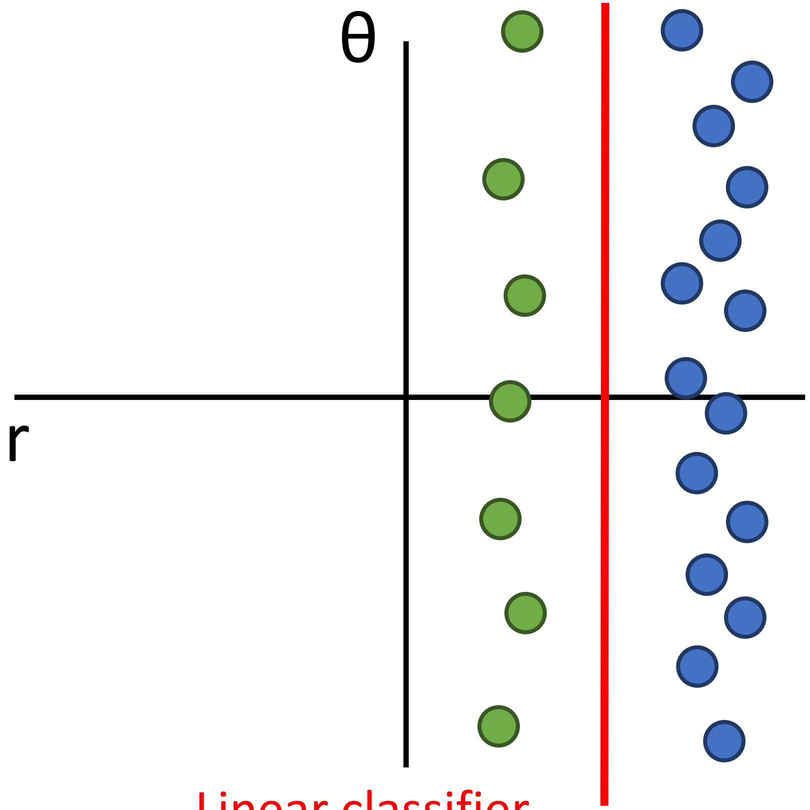


Nonlinear classifier
in original space!

$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1}(y/x)$$

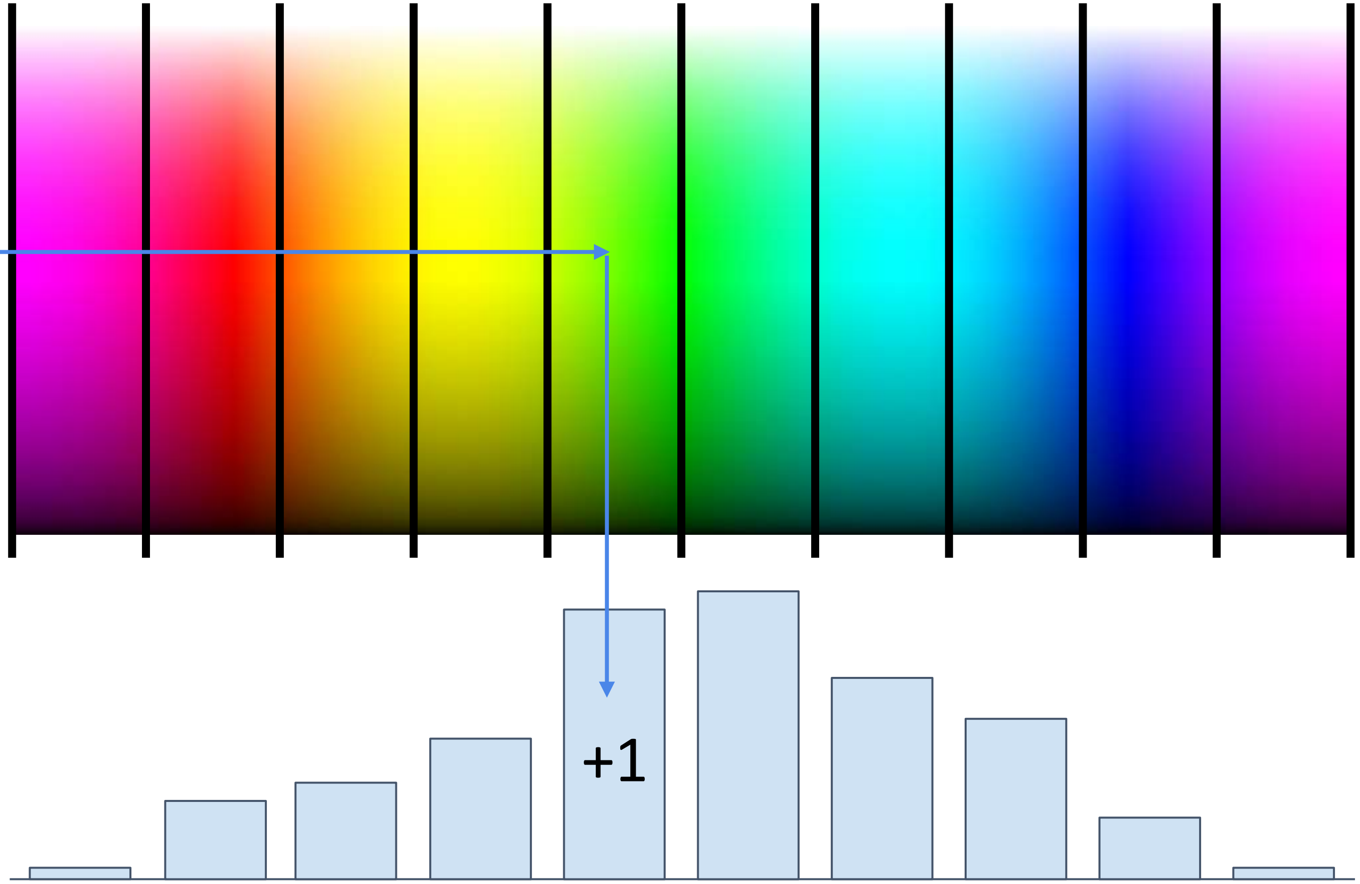
Feature transform

Feature space



Linear classifier
in feature space

Image Features: Color Histogram



Ignores texture,
spatial positions

Frog image is in the public domain

Image Features: Histogram of Oriented Gradients (HoG)

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Lowe, “Object recognition from local scale-invariant features”, ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

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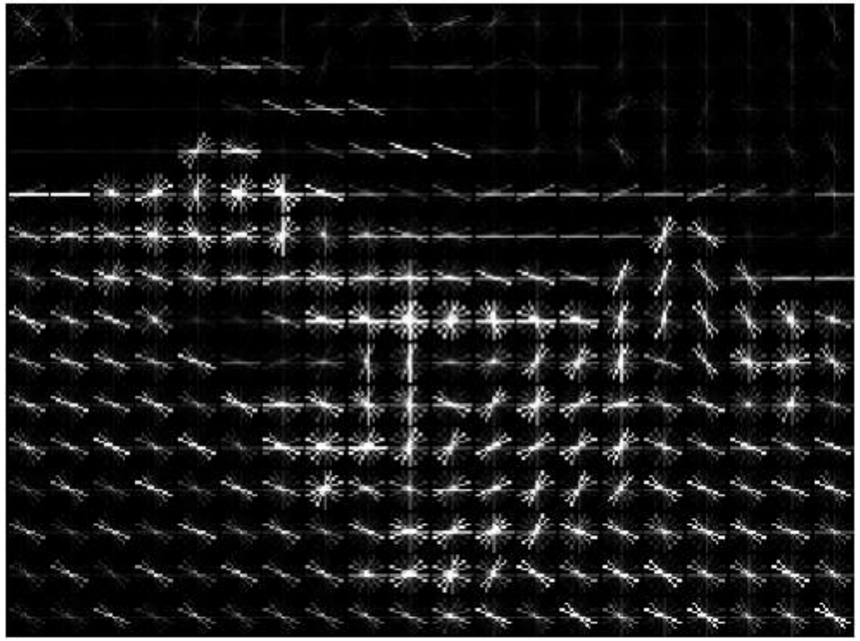

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
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Example: 320x240 image gets divided into 40x30 bins; 8 directions per bin; feature vector has 30*40*9 = 10,800 numbers

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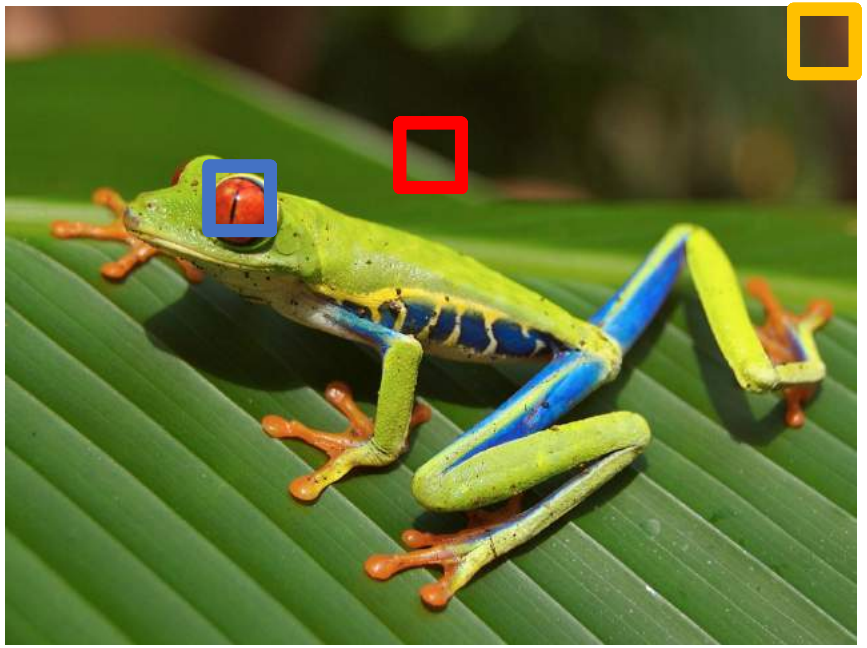
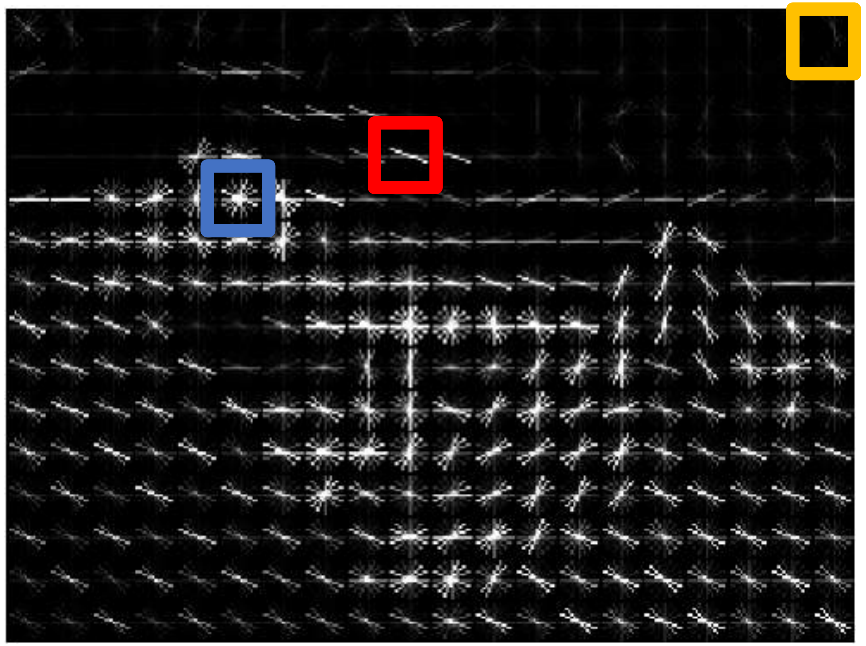


Weak edges

Strong diagonal edges

Edges in all directions



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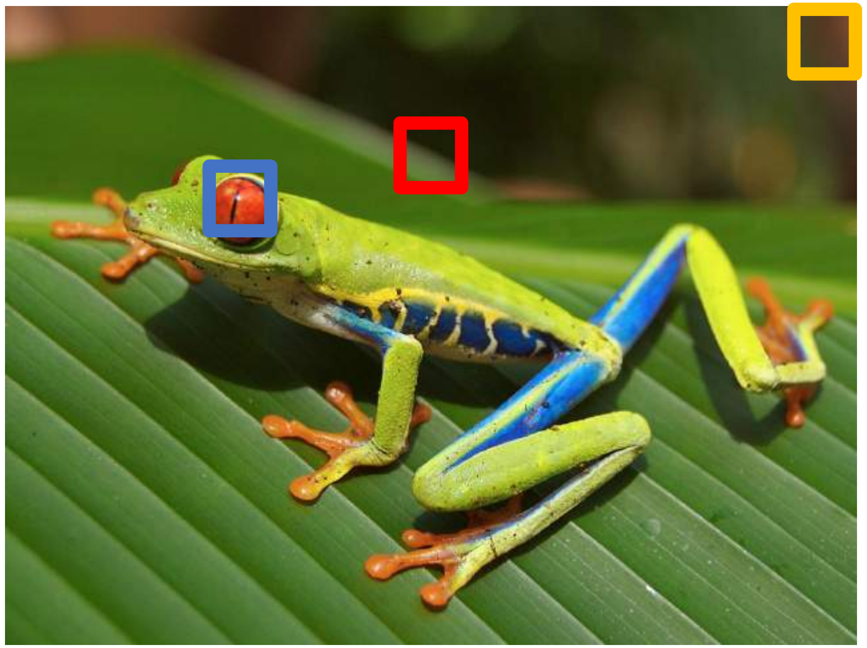

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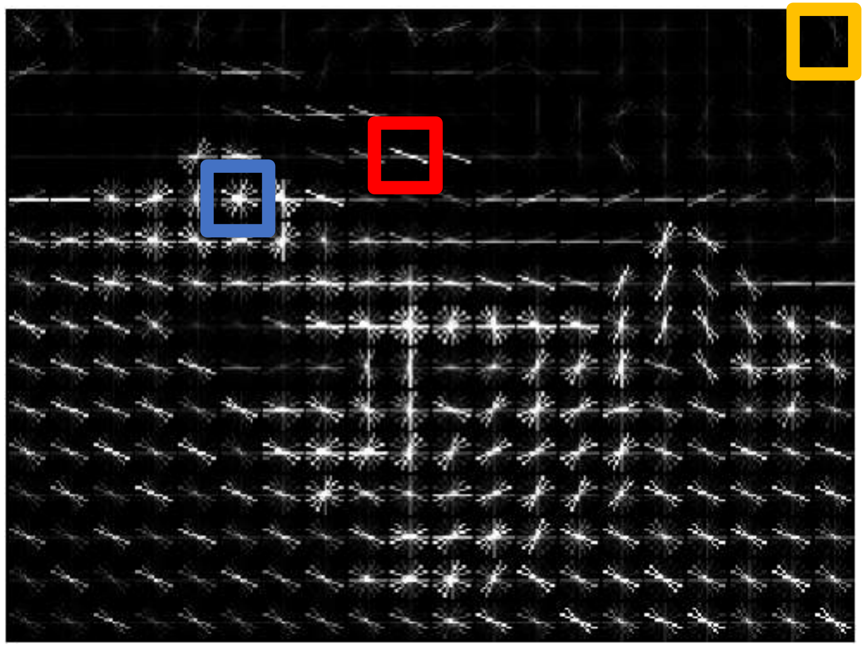
Edges in all directions



1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Captures texture and position, robust to small image changes

Example: 320x240 image gets divided into 40x30 bins; 8 directions per bin; feature vector has 30*40*9 = 10,800 numbers

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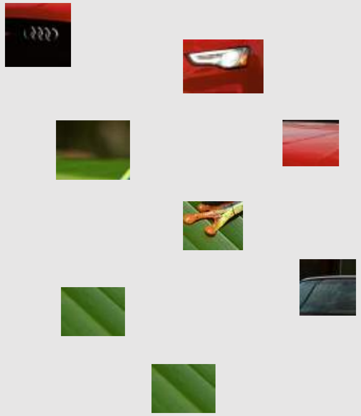
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

Image Features: Bag of Words (Data-Driven!)

Step 1: Build codebook

Extract random patches



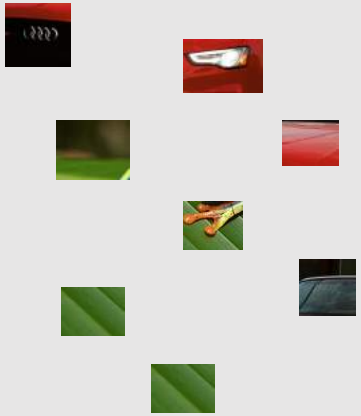
Cluster patches to form "codebook" of "visual words"

Fei-Fei and Perona, "A bayesian hierarchical model for learning natural scene categories", CVPR 2005

Image Features: Bag of Words (Data-Driven!)

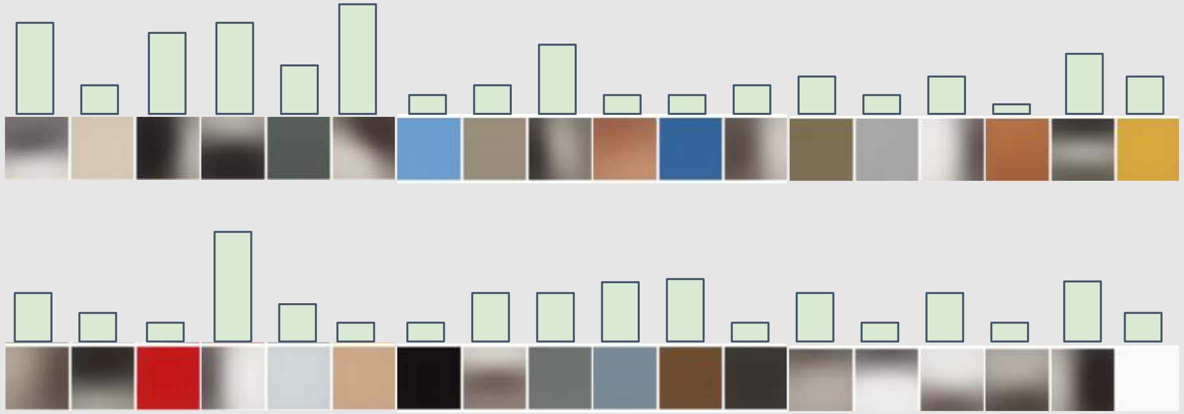
Step 1: Build codebook

Extract random patches



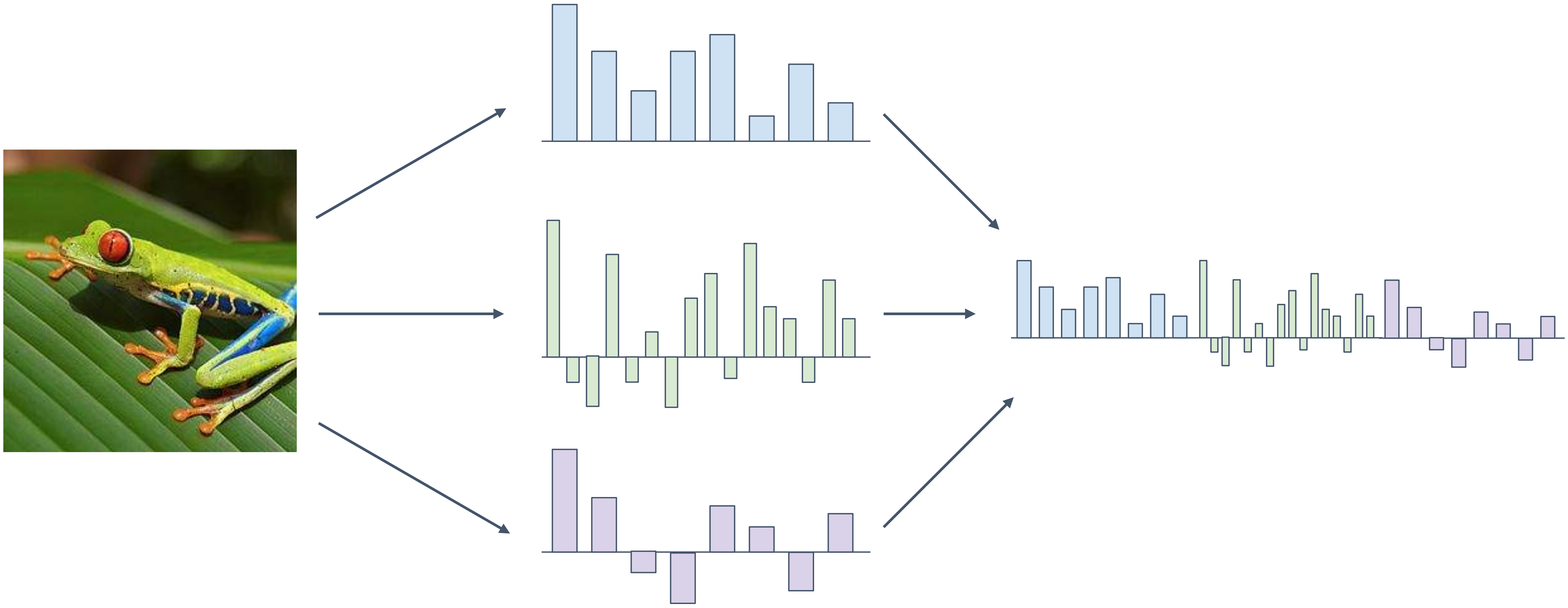
Cluster patches to form “codebook” of “visual words”

Step 2: Encode images



Fei-Fei and Perona, “A bayesian hierarchical model for learning natural scene categories”, CVPR 2005

Image Features



Example: Winner of 2011 ImageNet challenge

Low-level feature extraction ≈ 10k patches per image

- SIFT: 128-dim
- color: 96-dim

} reduced to 64-dim with PCA

FV extraction and compression:

- N=1,024 Gaussians, R=4 regions ⇨ 520K dim x 2
- compression: G=8, b=1 bit per dimension

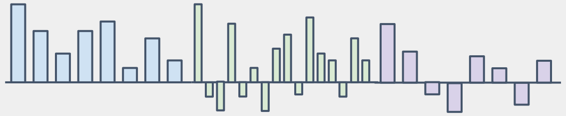
One-vs-all SVM learning with SGD

Late fusion of SIFT and color systems

F. Perronnin, J. Sánchez, “Compressed Fisher vectors for LSVRC”, PASCAL VOC / ImageNet workshop, ICCV, 2011.

Image Features

Feature Extraction



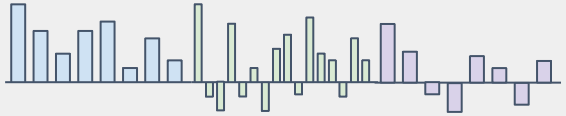
f

10 numbers giving scores for classes

training

Image Features vs Neural Networks

Feature Extraction



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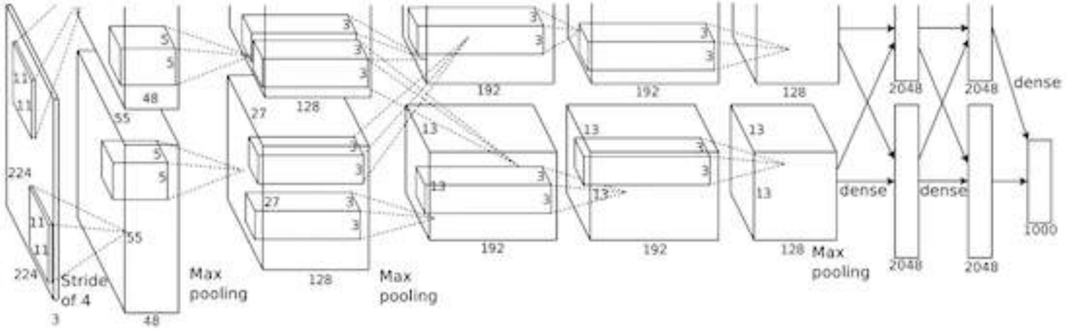

Krizhevsky, Sutskever, and Hinton, “Imagenet classification with deep convolutional neural networks”, NIPS 2012.

10 numbers giving scores for classes

training

Neural Networks

(**Before**) Linear score function: $f = Wx$

$$x \in \mathbb{R}^D, W \in \mathbb{R}^{C \times D}$$

Neural Networks

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

$$W_2 \in \mathbb{R}^{C \times H} \quad W_1 \in \mathbb{R}^{H \times D} \quad x \in \mathbb{R}^D$$

(In practice we will usually add a learnable bias at each layer as well)

Neural Networks

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$
or 3-layer Neural Network

$$f = W_3 \max(0, W_2 \max(0, W_1 x))$$

$$W_3 \in \mathbb{R}^{C \times H_2} \quad W_2 \in \mathbb{R}^{H_2 \times H_1} \quad W_1 \in \mathbb{R}^{H_1 \times D} \quad x \in \mathbb{R}^D$$

(In practice we will usually add a learnable bias at each layer as well)

Neural Networks

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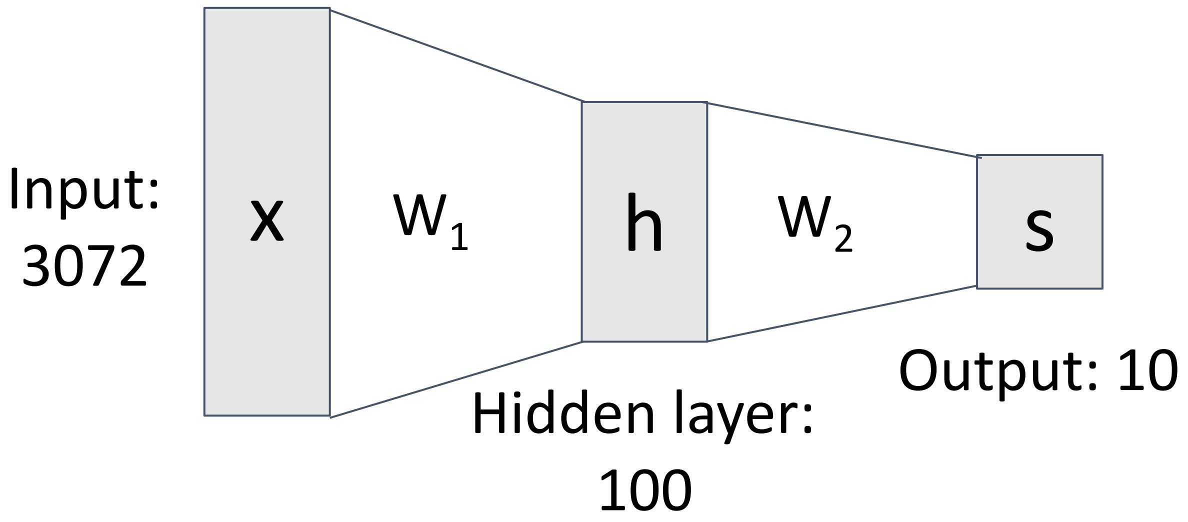


$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

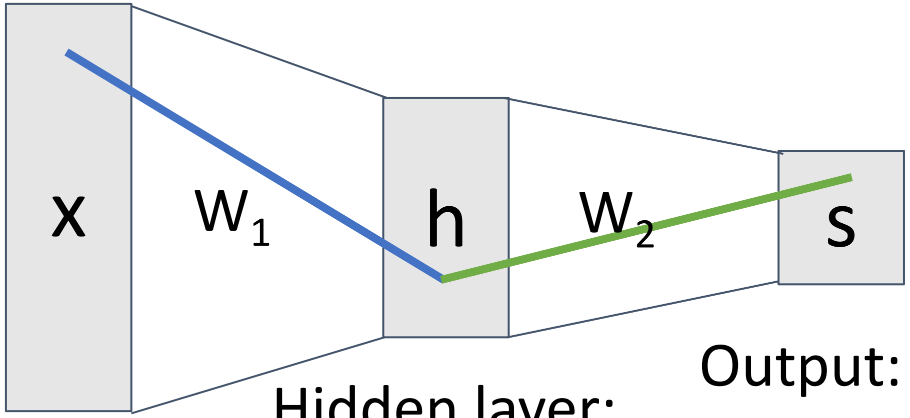
Neural Networks

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

Element (i, j) of W_1 gives the effect on h_i from x_j

Input: 3072



Hidden layer: 100

Output: 10

Element (i, j) of W_2 gives the effect on s_i from h_j

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

(**Before**) Linear score function: $$f = Wx$$

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Element (i, j) of W_1 gives the effect on h_i from x_j

All elements of x affect all elements of h

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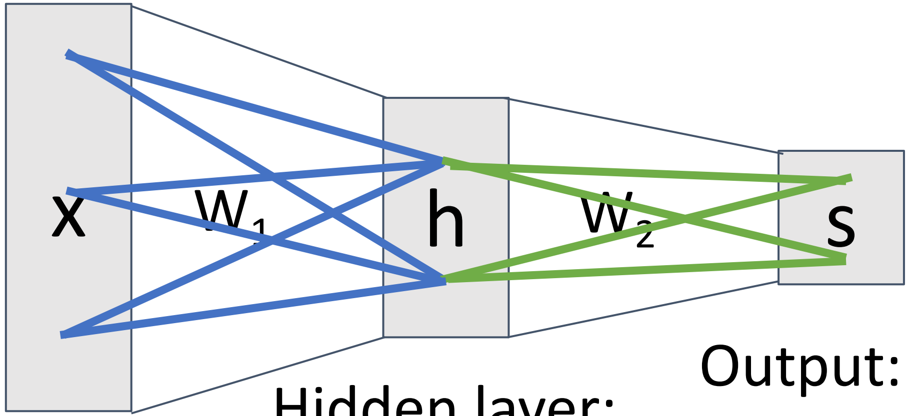


Hidden layer: 100

Output: 10

Element (i, j) of W_2 gives the effect on s_i from h_j

All elements of h affect all elements of s

Fully-connected neural network

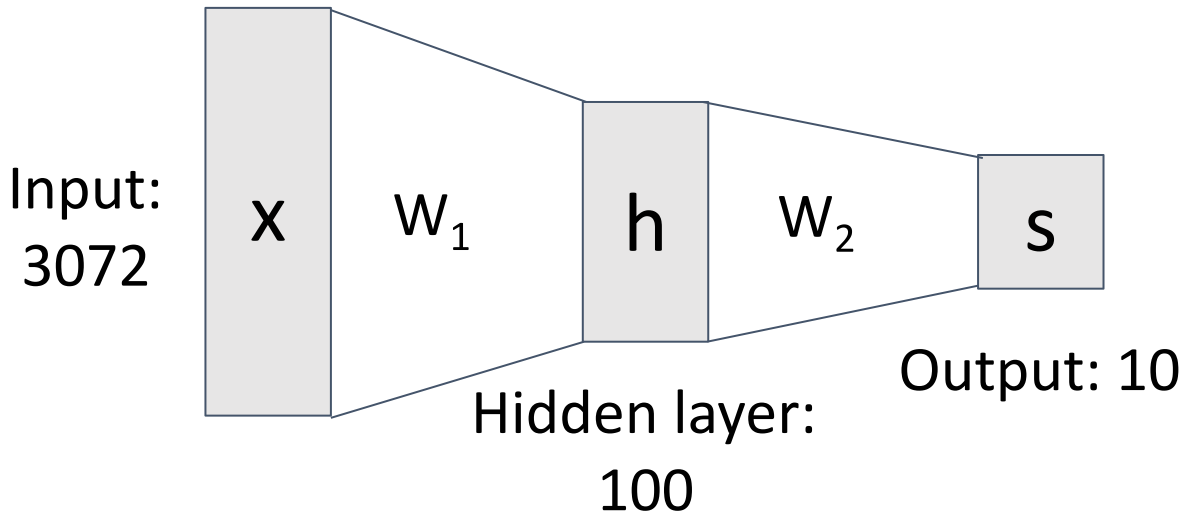
Also “Multi-Layer Perceptron” (MLP)

Neural Networks

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network

Linear classifier: One template per class



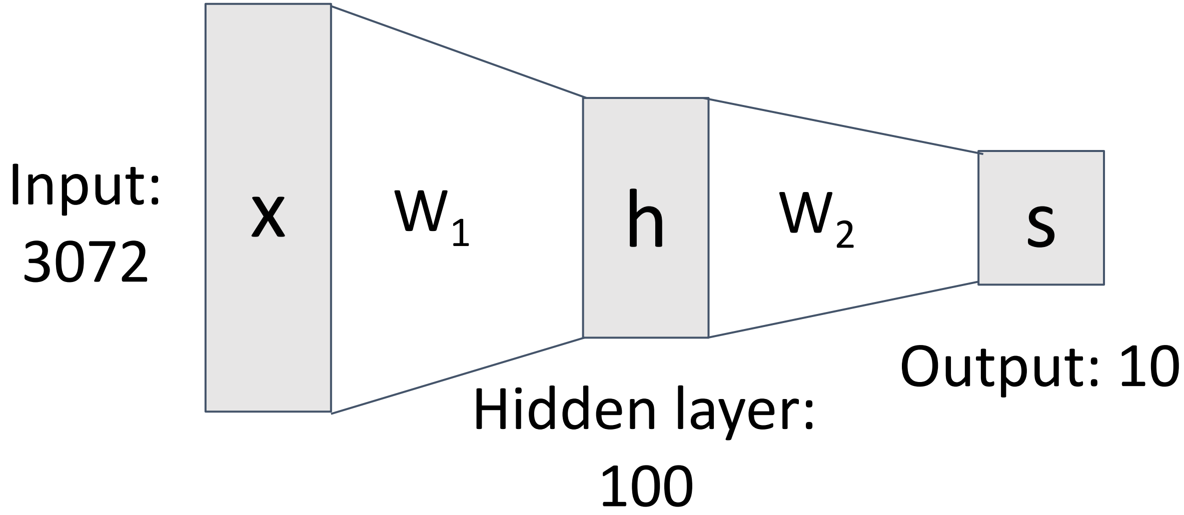
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Neural net: first layer is bank of templates;
Second layer recombines templates

(**Before**) Linear score function:

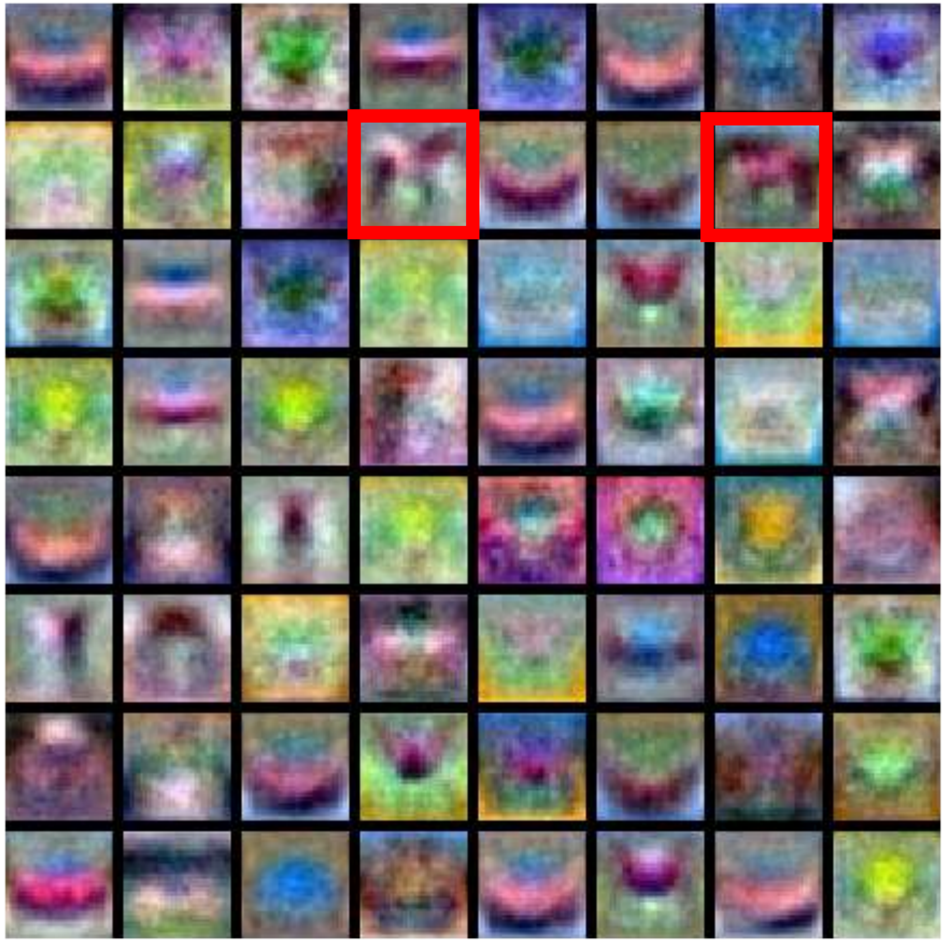
(**Now**) 2-layer Neural Network



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

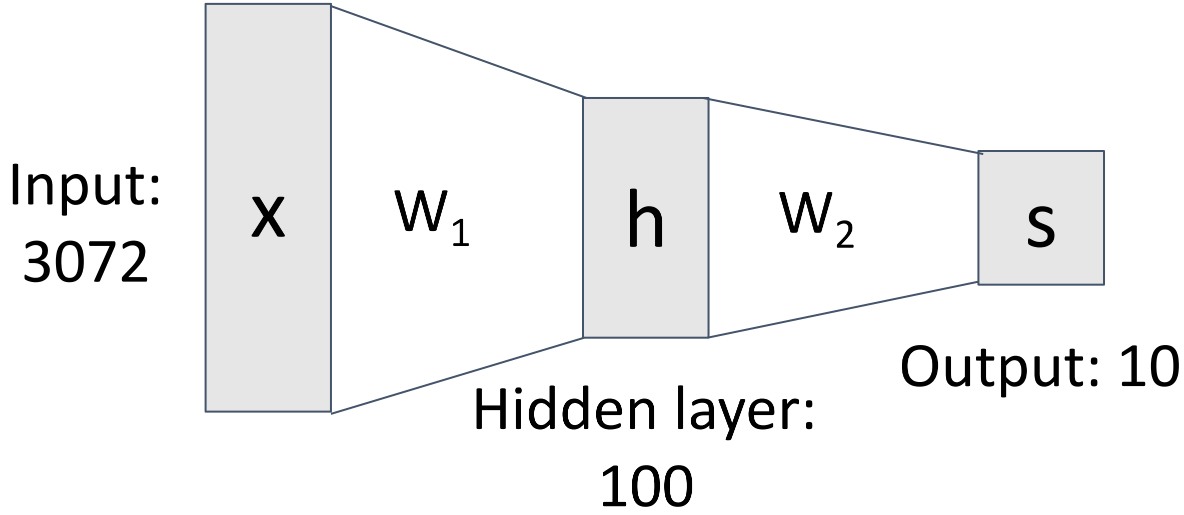
Neural Networks

Can use different templates to cover multiple modes of a class!



(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



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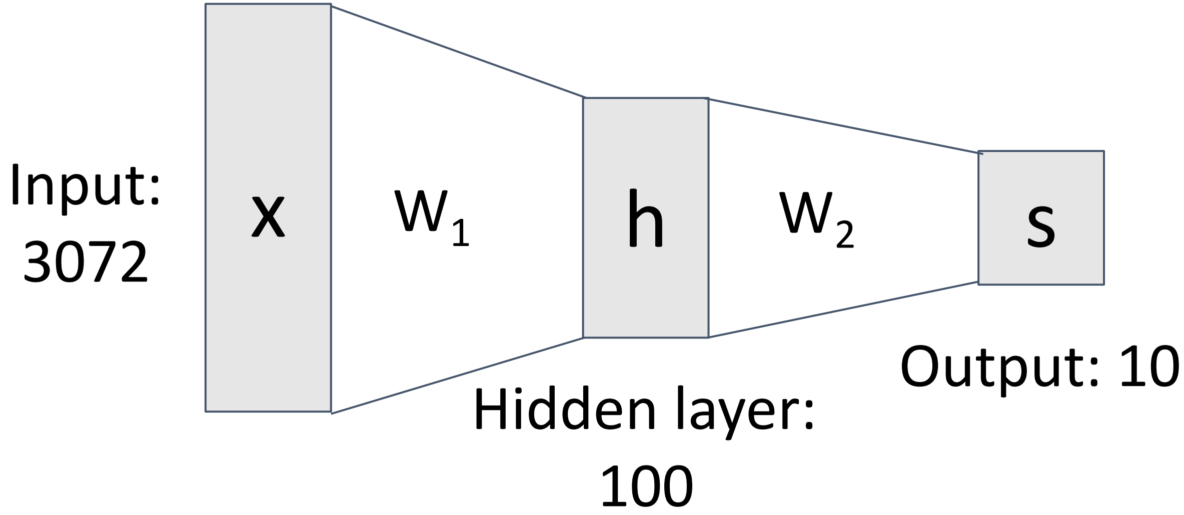
Neural Networks

"Distributed representation":
Most templates not interpretable!

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



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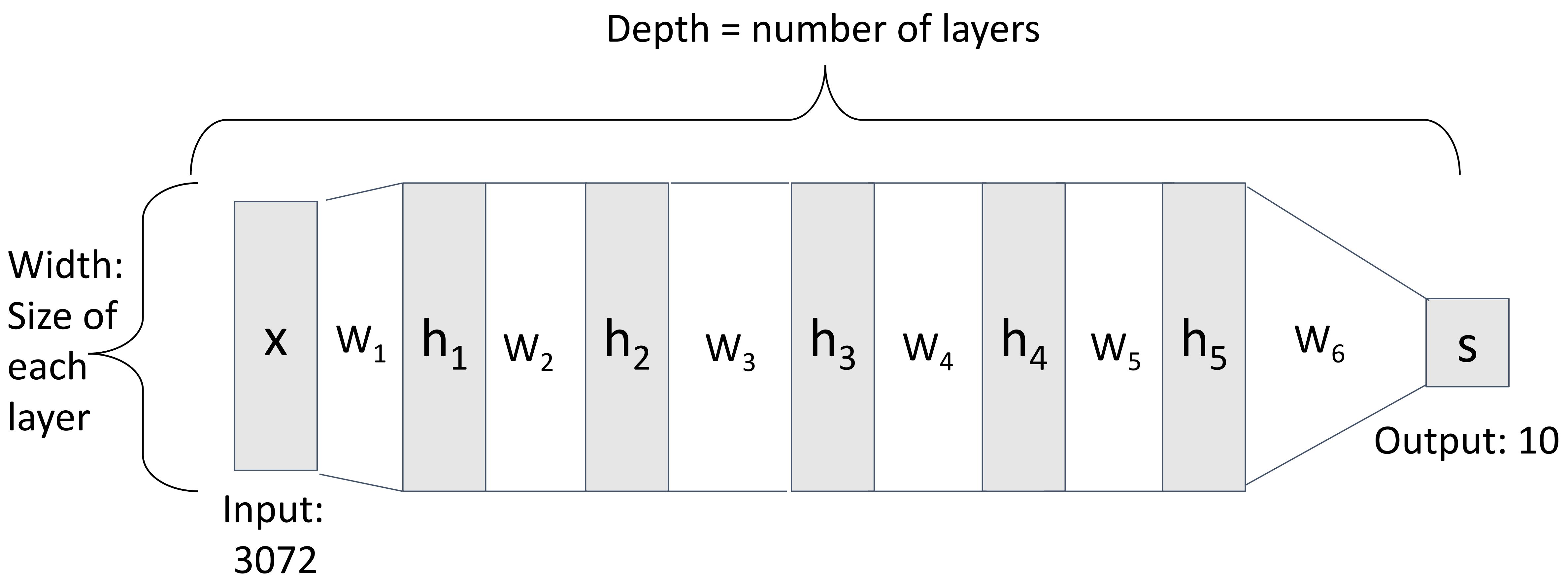
Deep Neural Networks



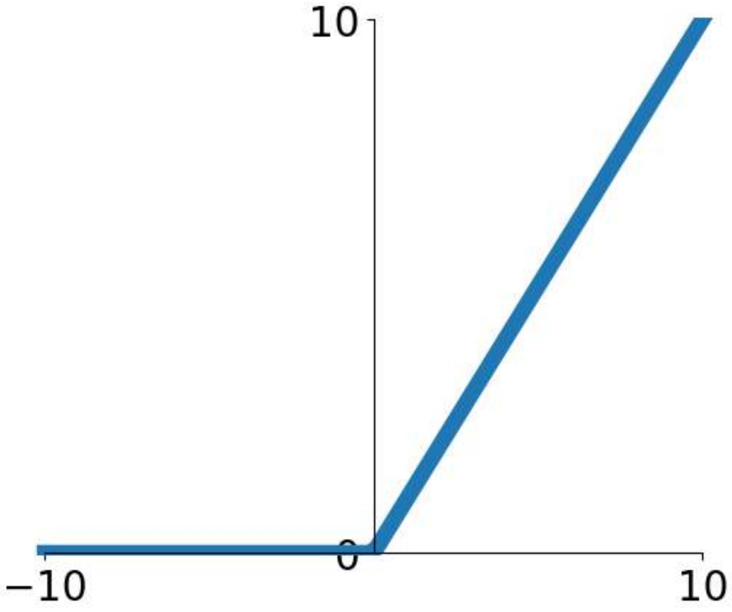
$$s = W_6 \max(0, W_6 \max(0, W_5 \max(0, W_4 \max(0, W_3 \max(0, W_2 \max(0, W_1 x))))))$$

Activation Functions

2-layer Neural Network

$$f = W_2 \boxed{\max(0,} W_1 x)$$

The function $ReLU(z) = max(0, z)$ is called "Rectified Linear Unit"



This is called the **activation function** of the neural network

Activation Functions

2-layer Neural Network

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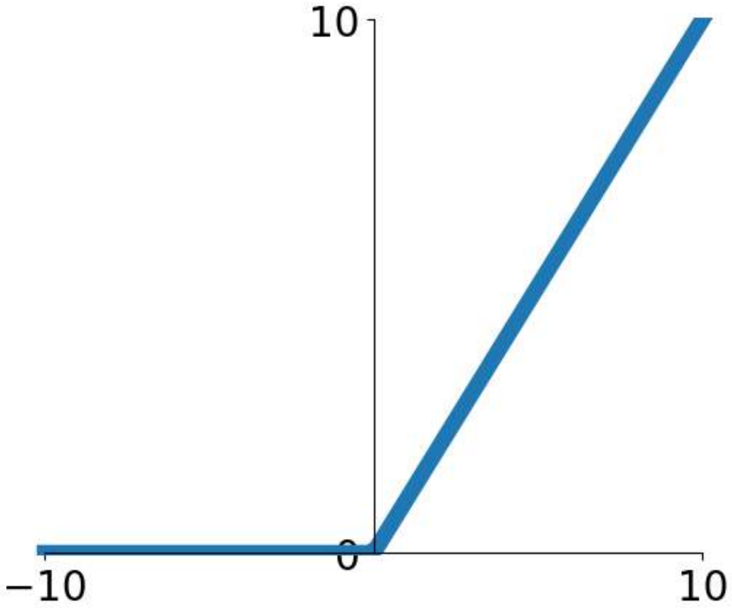

This is called the **activation function** of the neural network

Q: What happens if we build a neural network with no activation function?

$$s = W_2 W_1 x$$

Activation Functions

2-layer Neural Network

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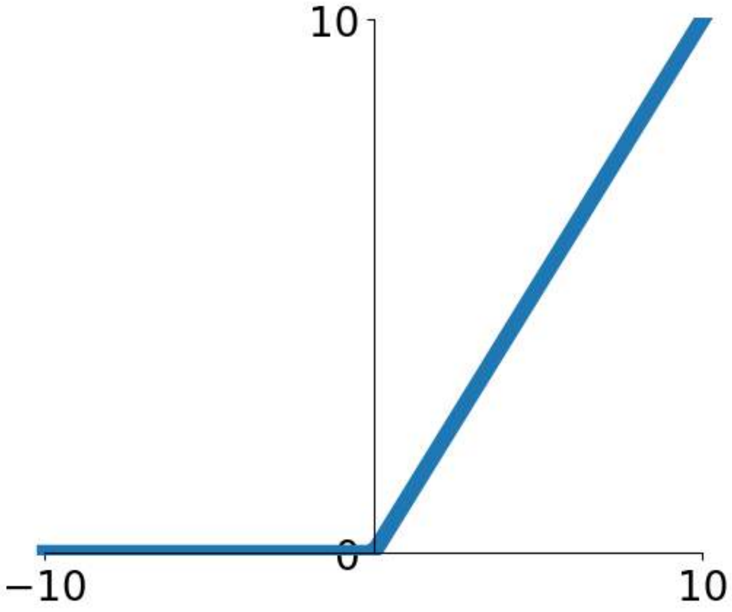


This is called the **activation function** of the neural network

Q: What happens if we build a neural network with no activation function?

$$s = W_2 W_1 x$$

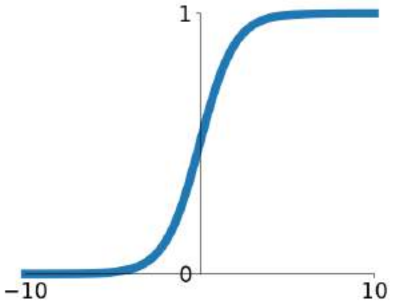
$$W_3 = W_2 W_1 \in \mathbb{R}^{C \times H} \qquad s = W_3 x$$

A: We end up with a linear classifier!

Activation Functions

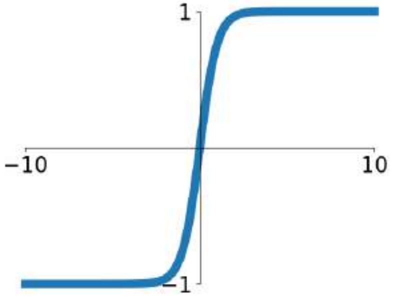
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



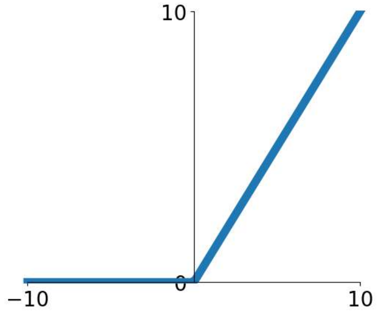
tanh

$$\tanh(x)$$



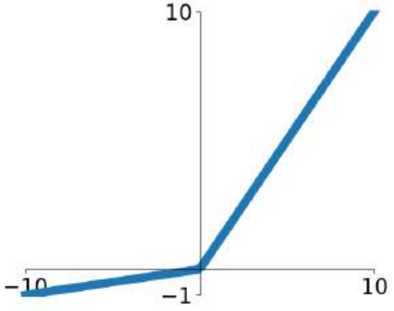
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$

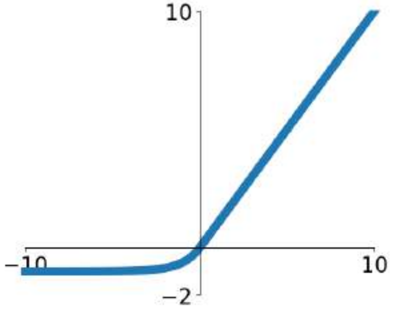


Maxout

$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$

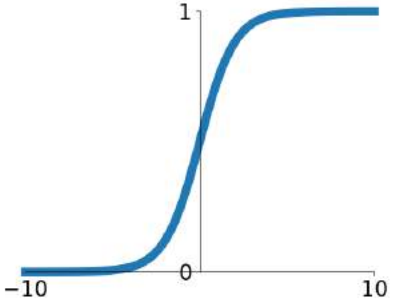


Activation Functions

ReLU is a good default choice for most problems

Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



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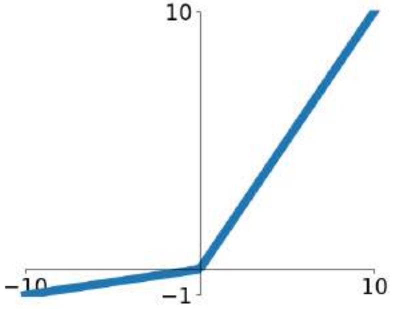
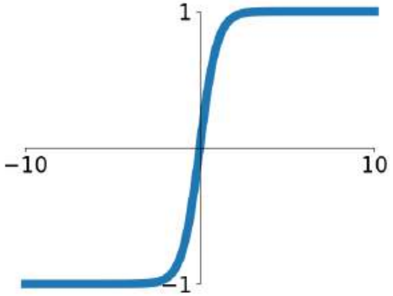


tanh

$$\tanh(x)$$



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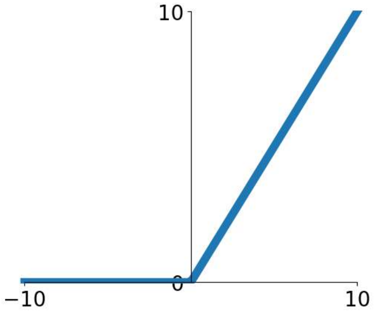

ELU

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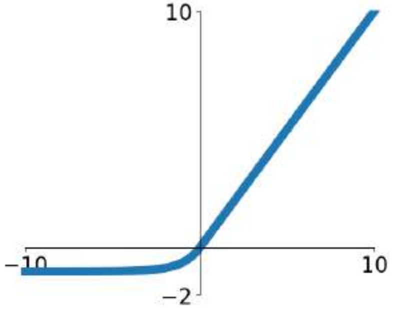
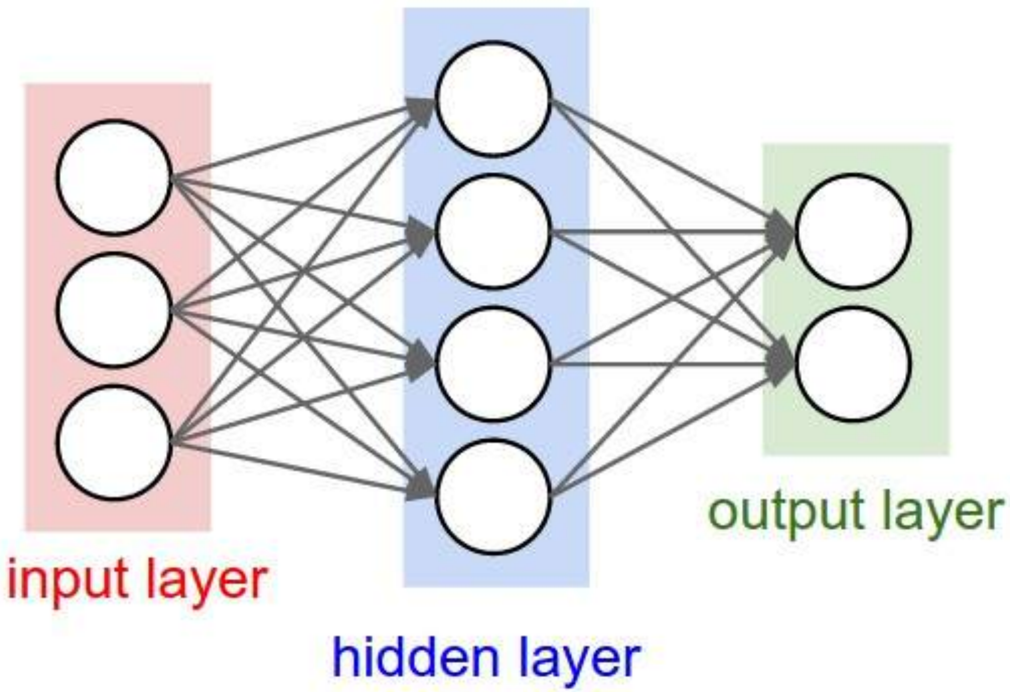


Neural Net in <20 lines!




```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

Neural Net in <20 lines!

Initialize weights and data

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
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  y_pred = h.dot(w2)
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  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```
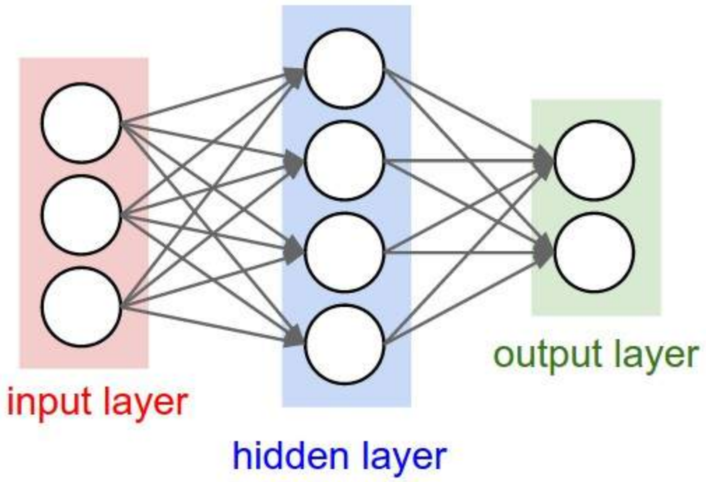
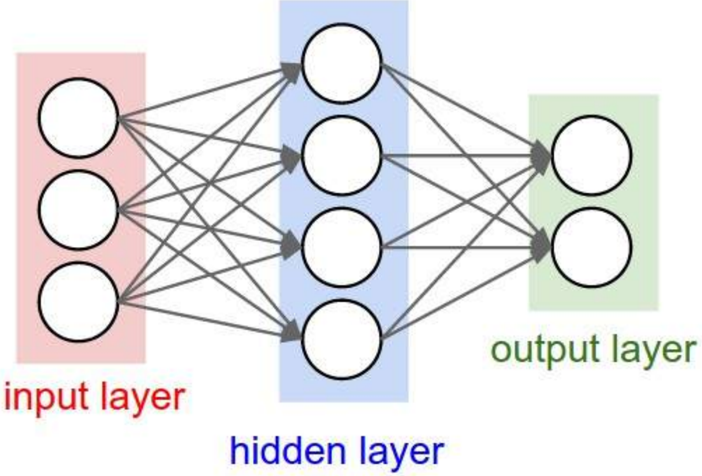



Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
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```
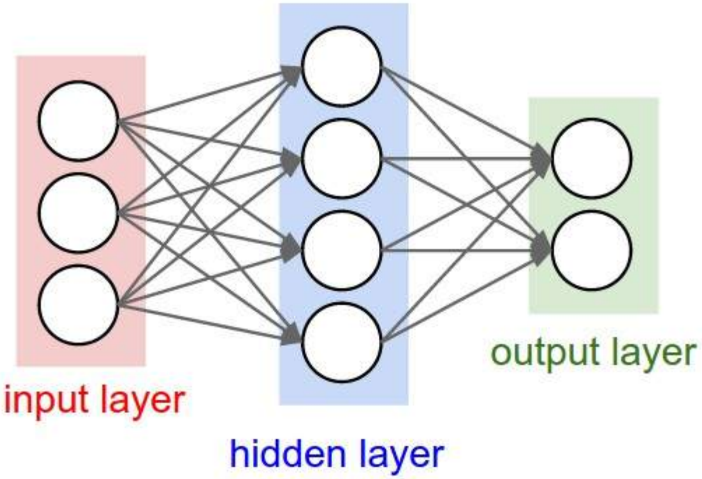

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

Compute gradients

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
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  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
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  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```
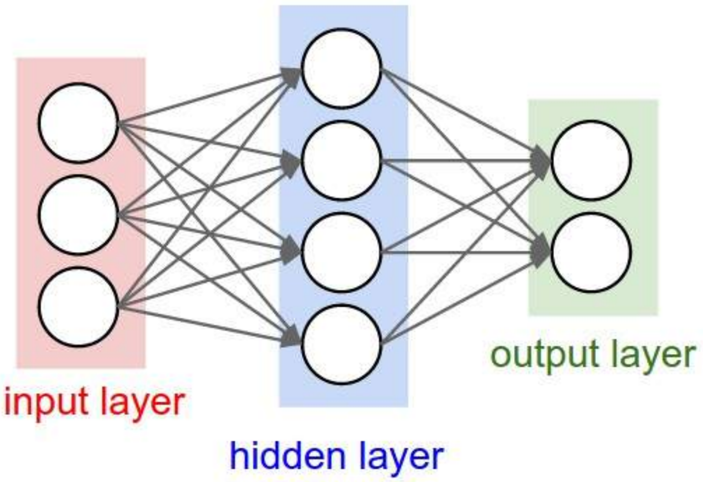

Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

Compute gradients

SGD step

```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
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  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```
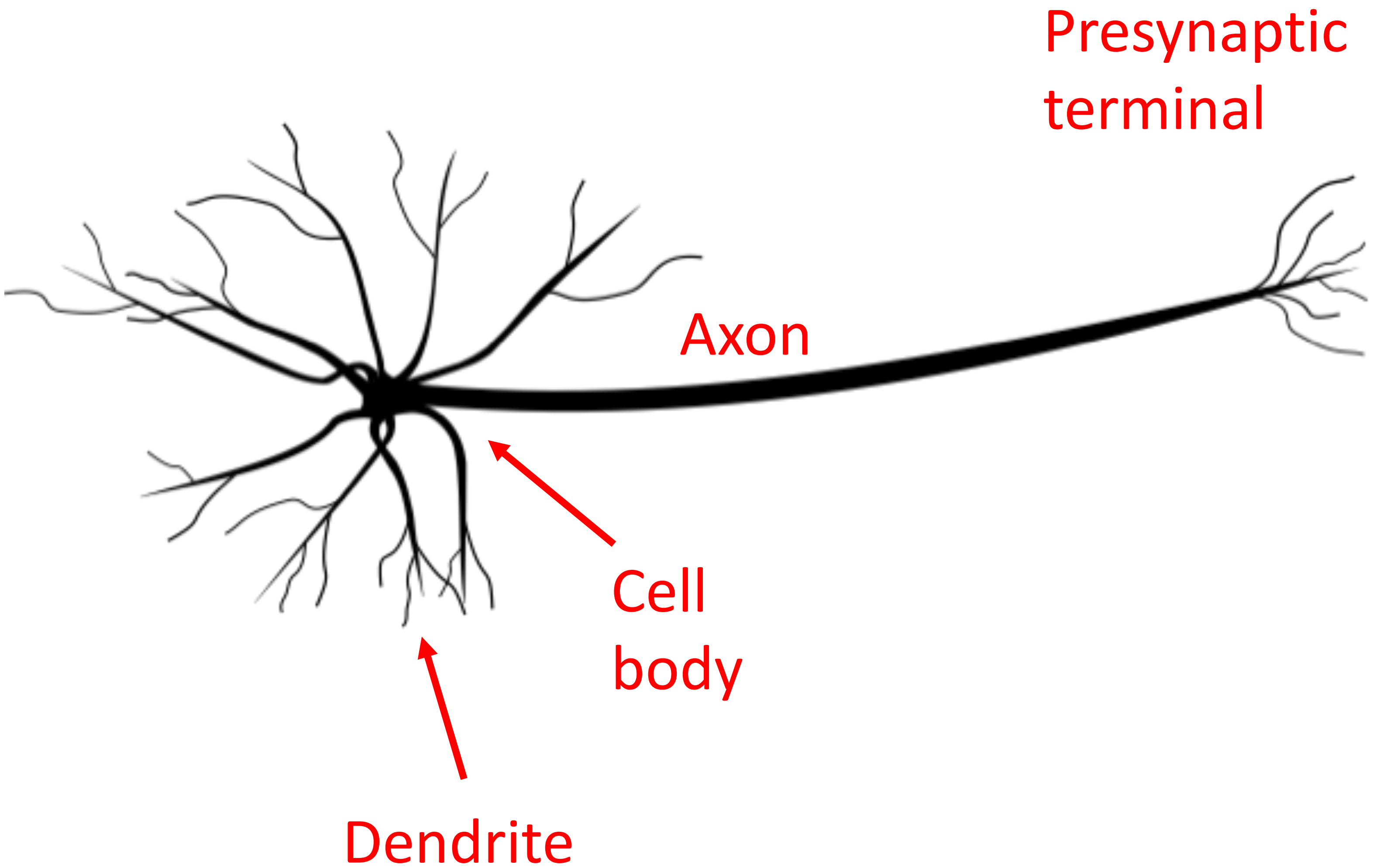
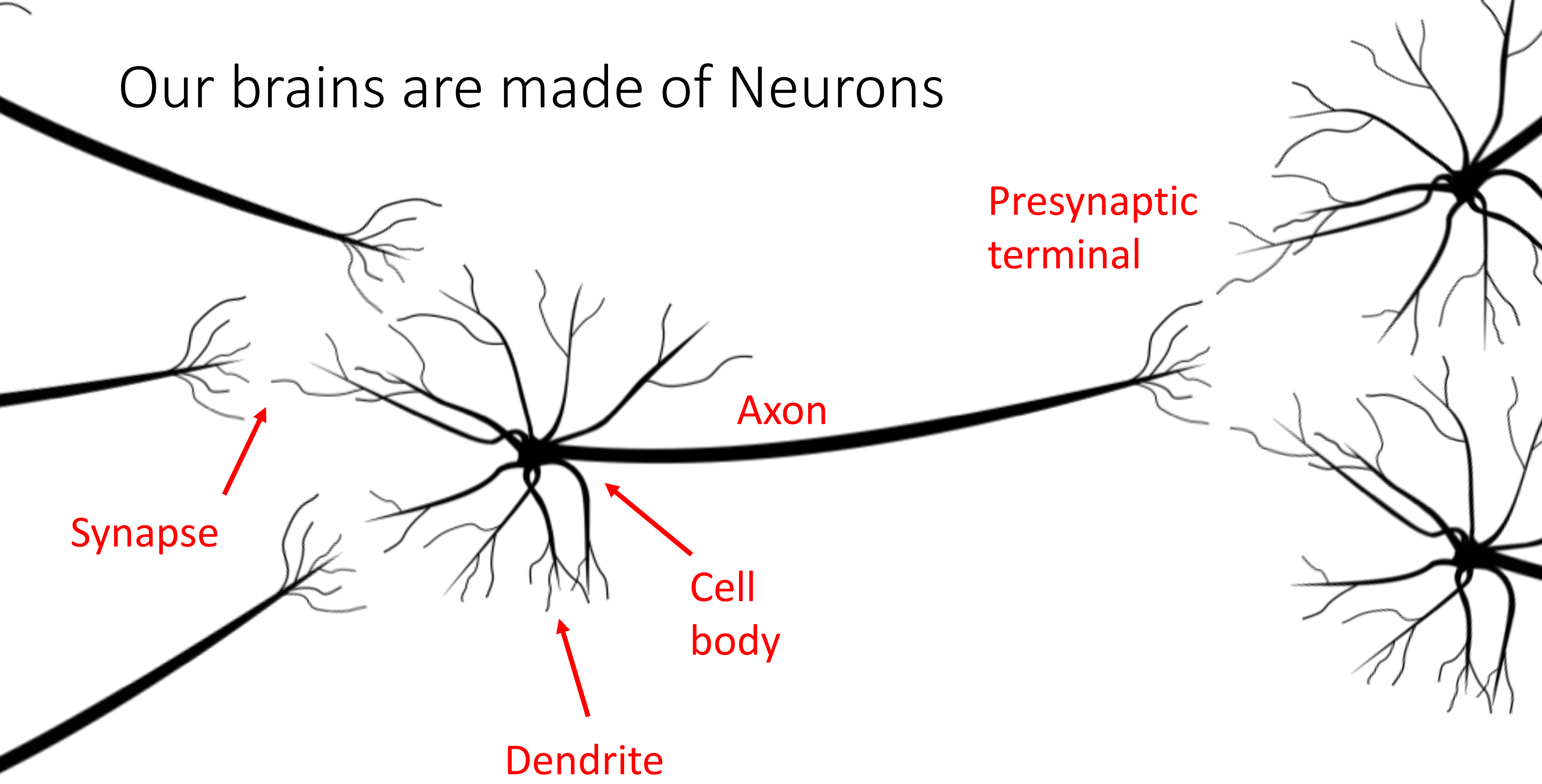
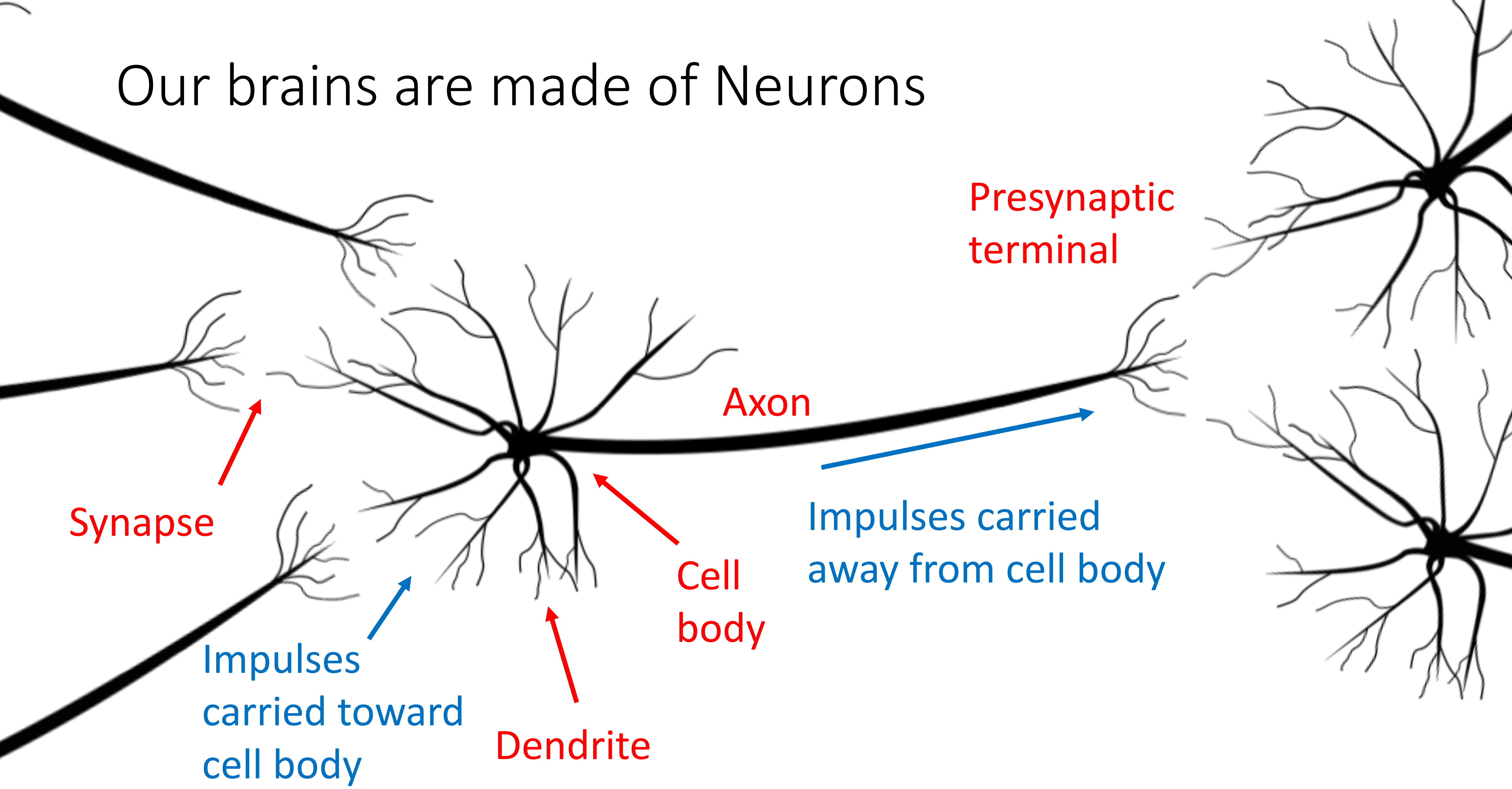

Our brains are made of Neurons



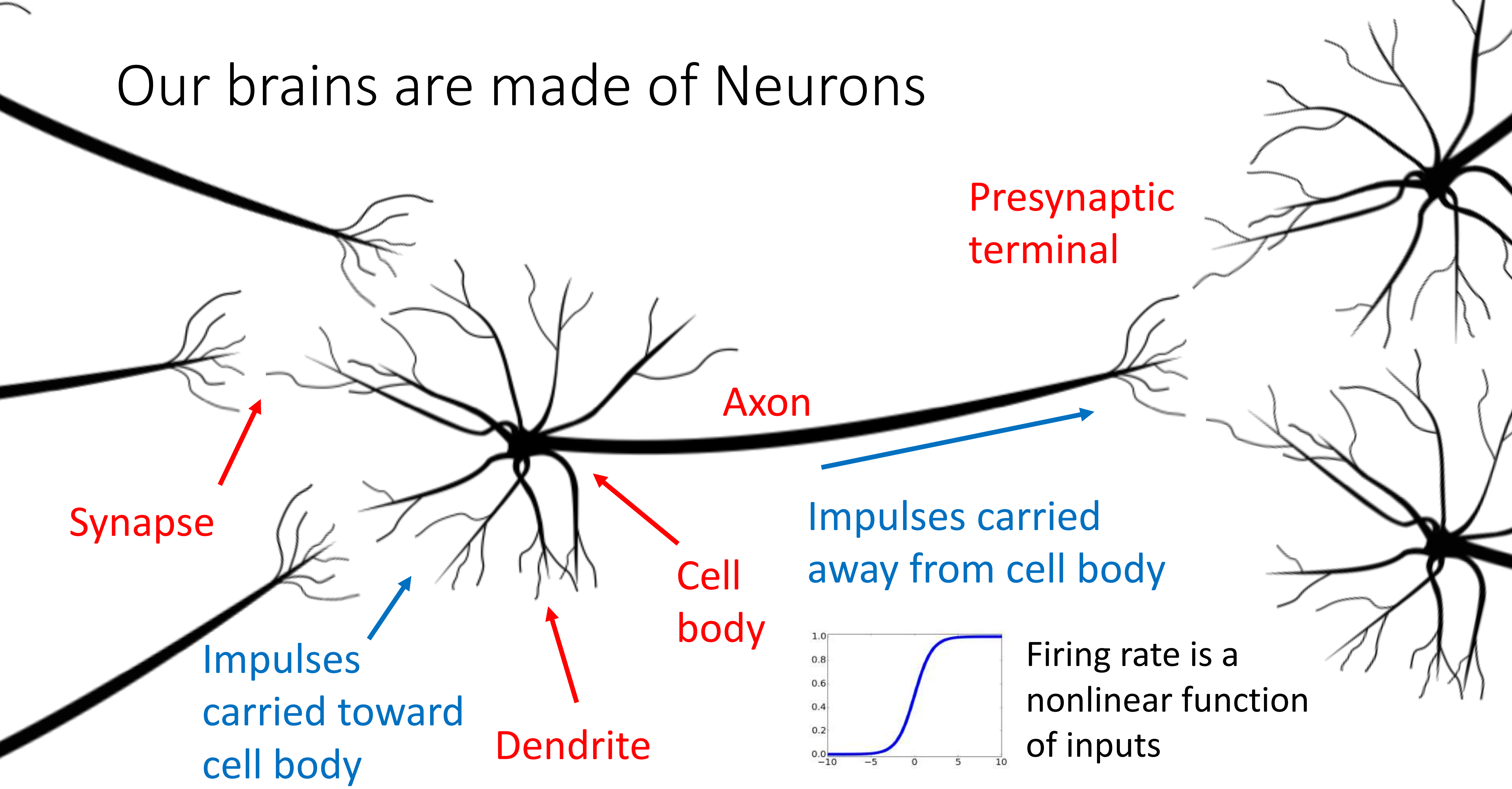
Our brains are made of Neurons



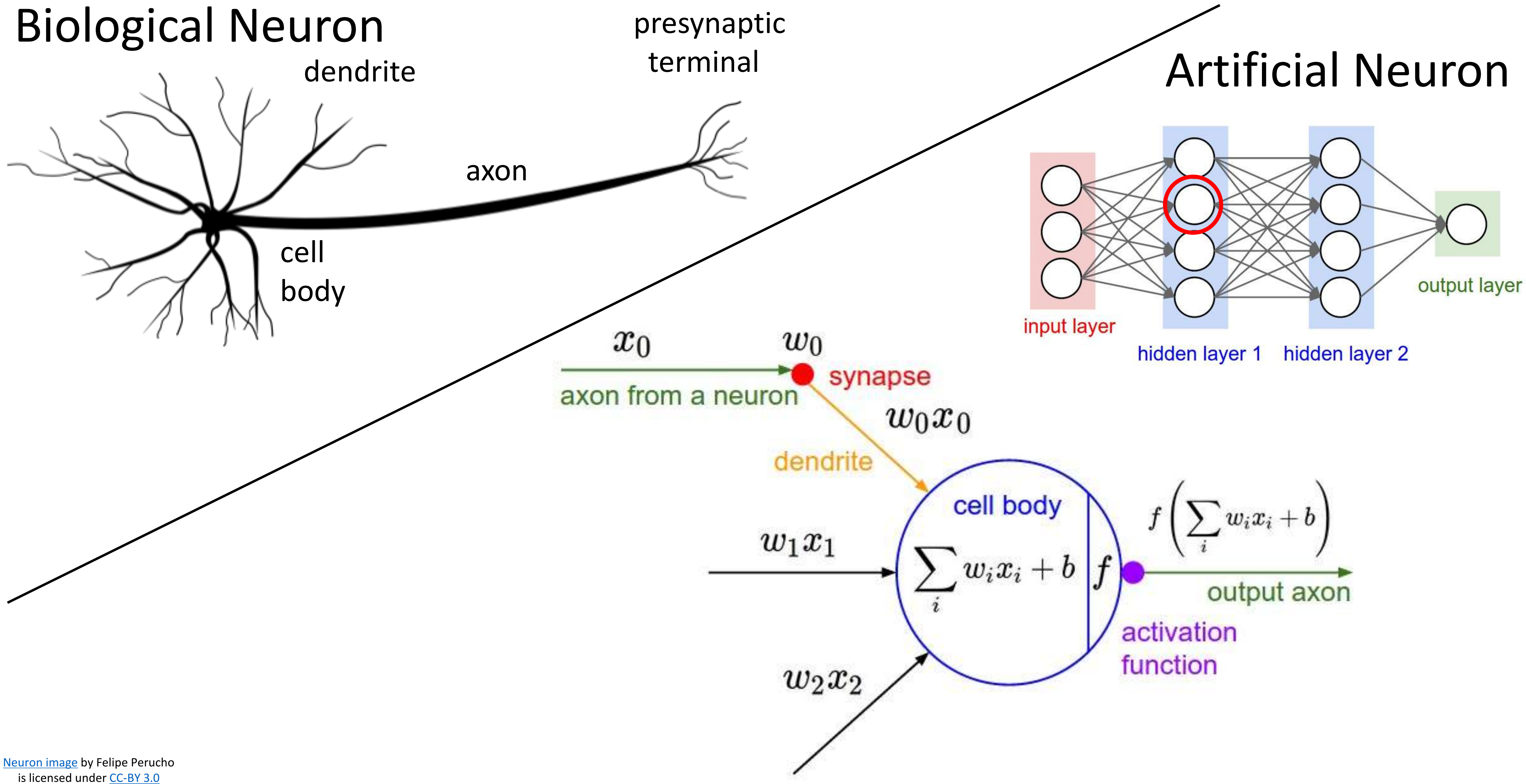
Our brains are made of Neurons



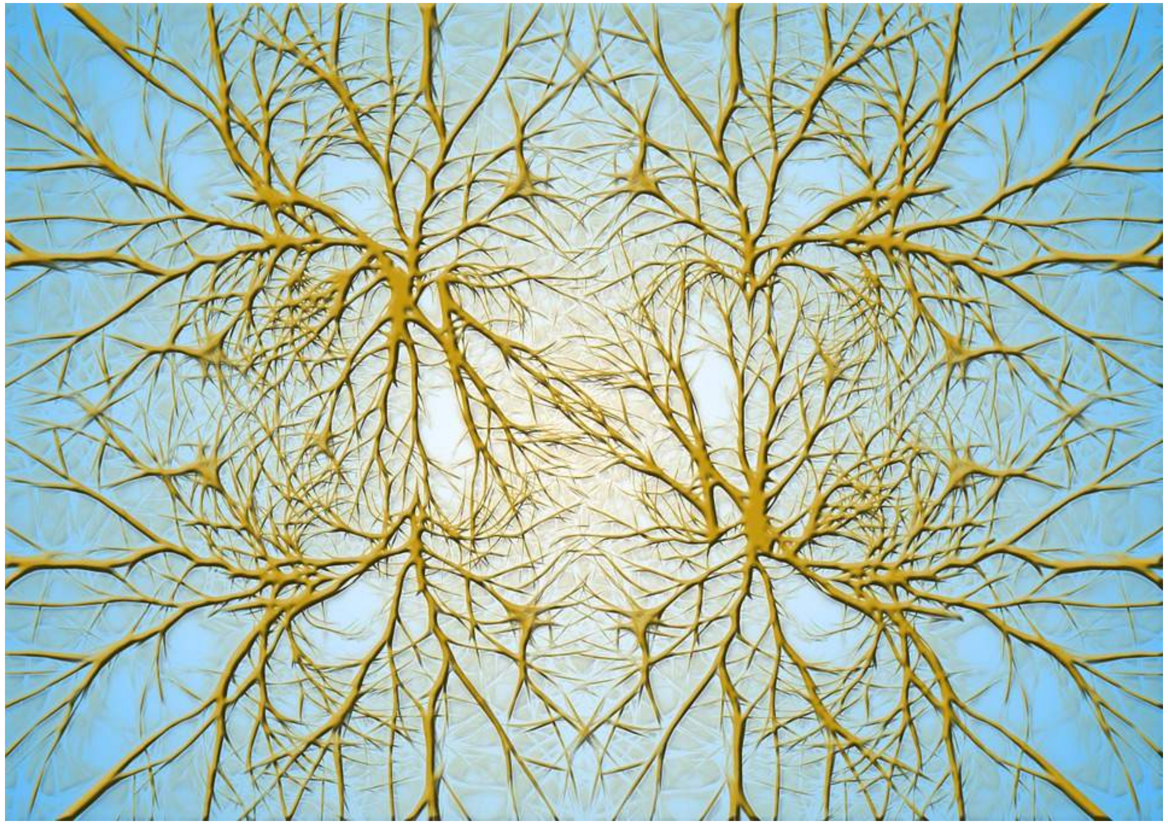
Our brains are made of Neurons



Firing rate is a nonlinear function of inputs

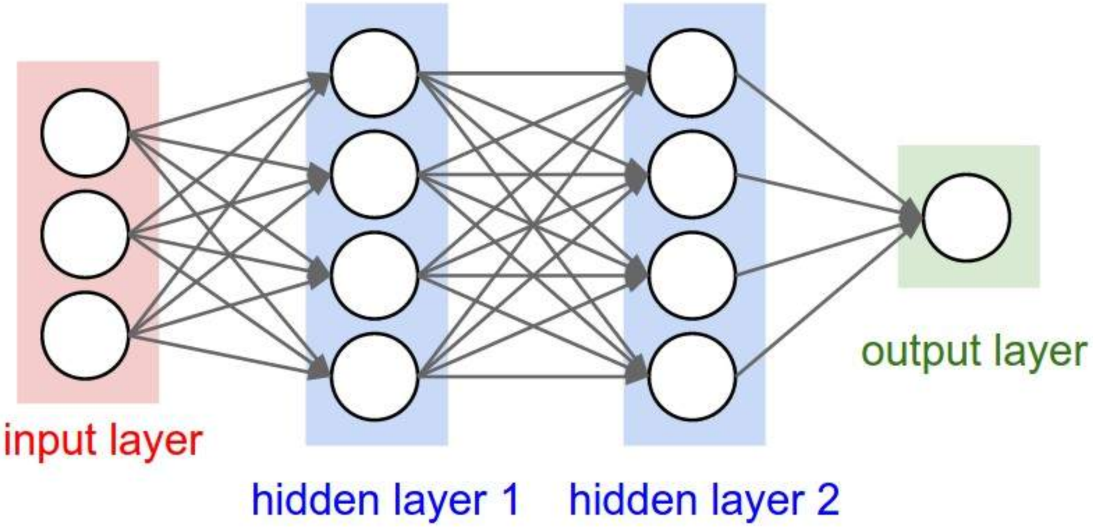

Biological Neuron
dendrite
presynaptic terminal
axon
cell body
Artificial Neuron
input layer
hidden layer 1
hidden layer 2
output layer
x_0
w_0
synapse
axon from a neuron
$w_0 x_0$
dendrite
cell body
$\sum_i w_i x_i + b$
f
$f\left(\sum_i w_i x_i + b\right)$
$w_1 x_1$
output axon
activation function
$w_2 x_2$

Biological Neurons:
Complex connectivity patterns

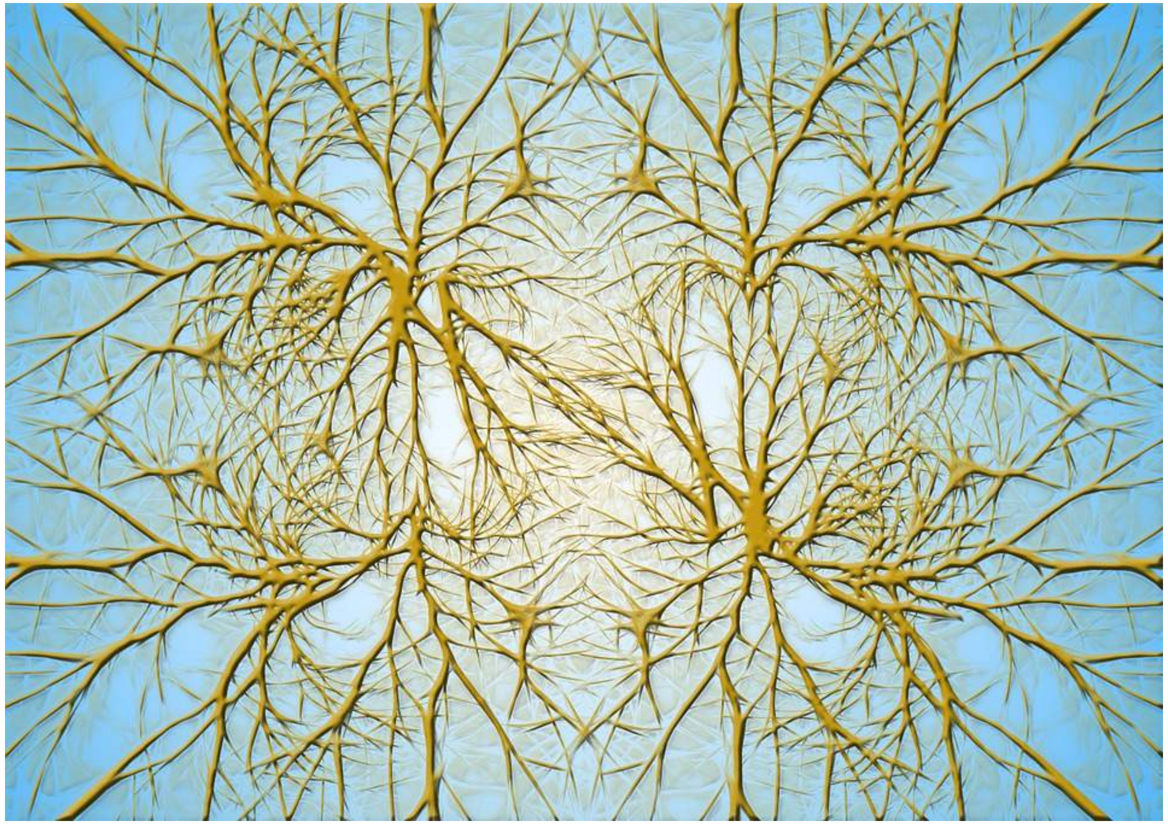


This image is CC0 Public Domain

Neurons in a neural network:
Organized into regular layers for computational efficiency

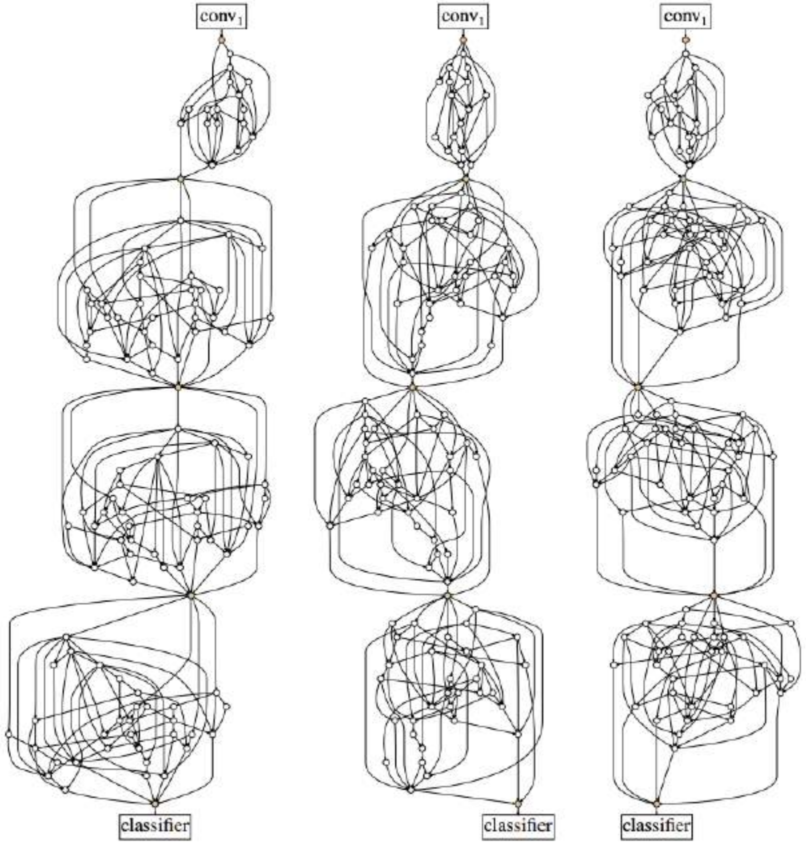


Biological Neurons:
Complex connectivity patterns



This image is CC0 Public Domain

But neural networks with random connections can work too!



Xie et al, “Exploring Randomly Wired Neural Networks for Image Recognition”, ICCV 2019

Be very careful with brain analogies!

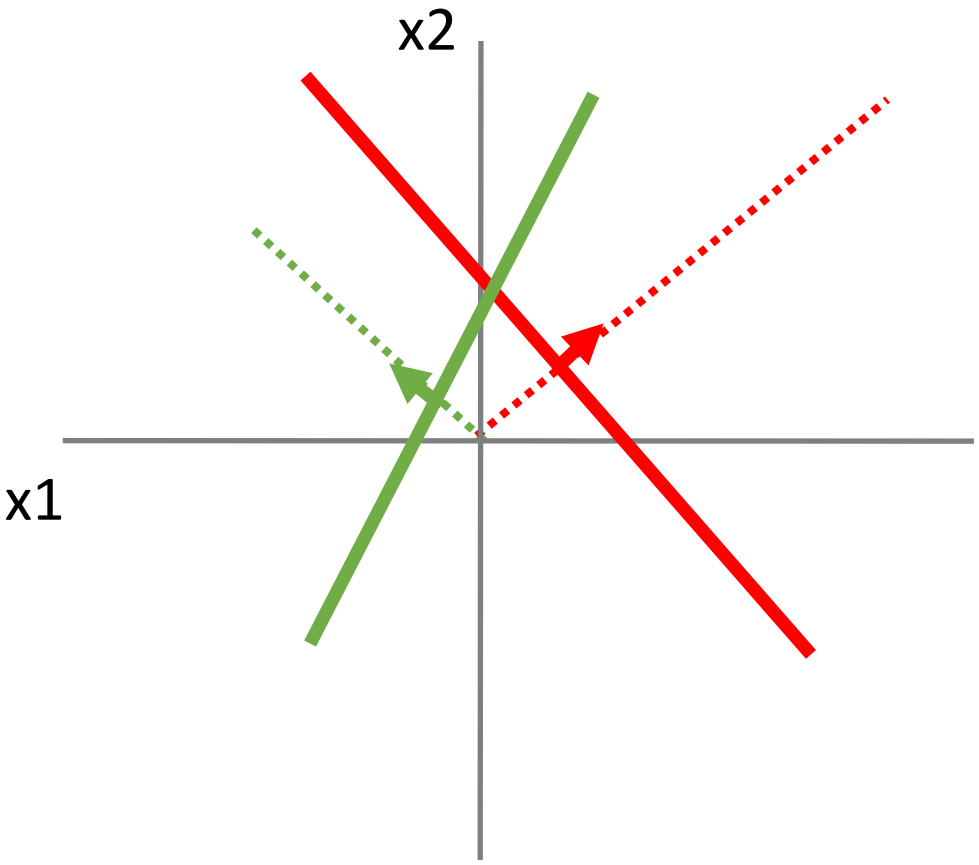
Biological Neurons:

- Many different types
- Dendrites can perform complex non-linear computations
- Synapses are not a single weight but a complex non-linear dynamical system
- Rate code may not be adequate

[Dendritic Computation. London and Hausser]

Space Warping

Consider a linear transform: h = Wx
Where x, h are both 2-dimensional



Space Warping

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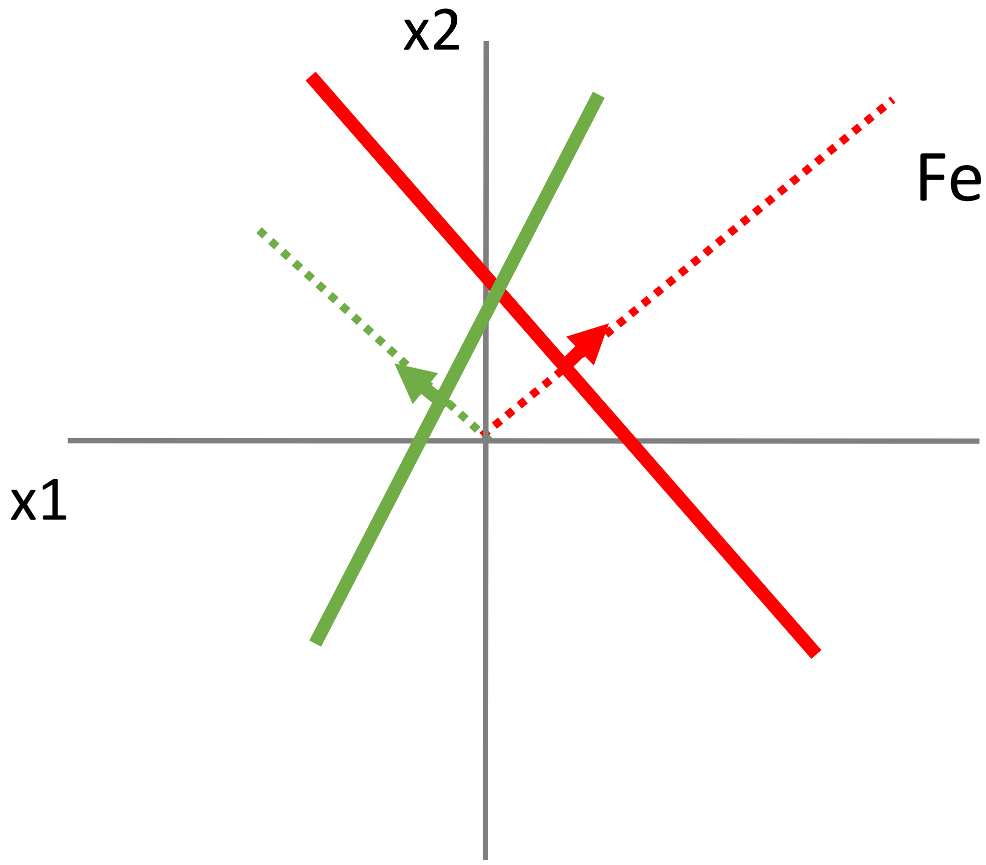
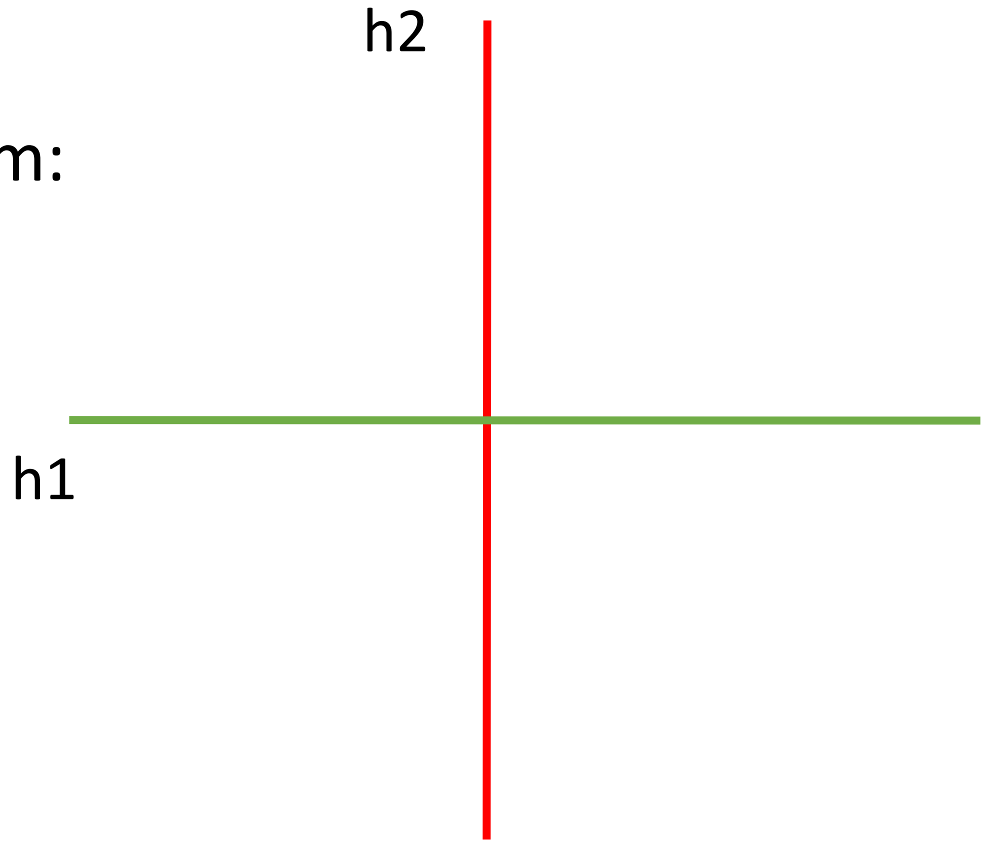
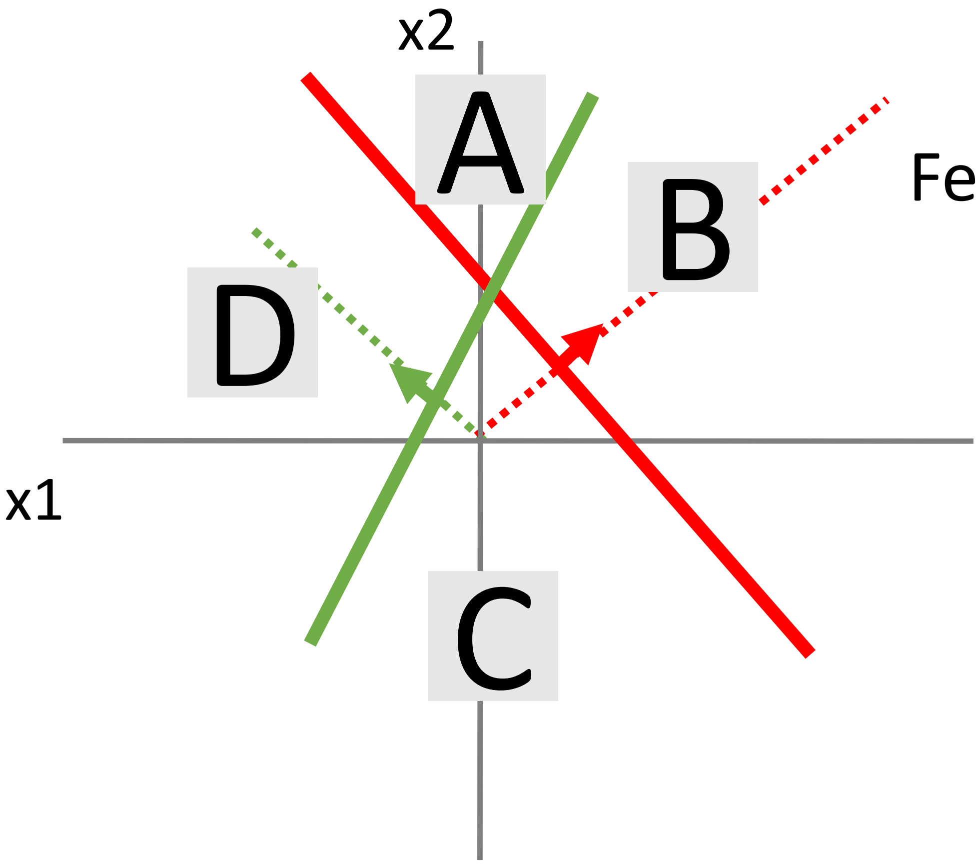


Feature transform:
h = Wx

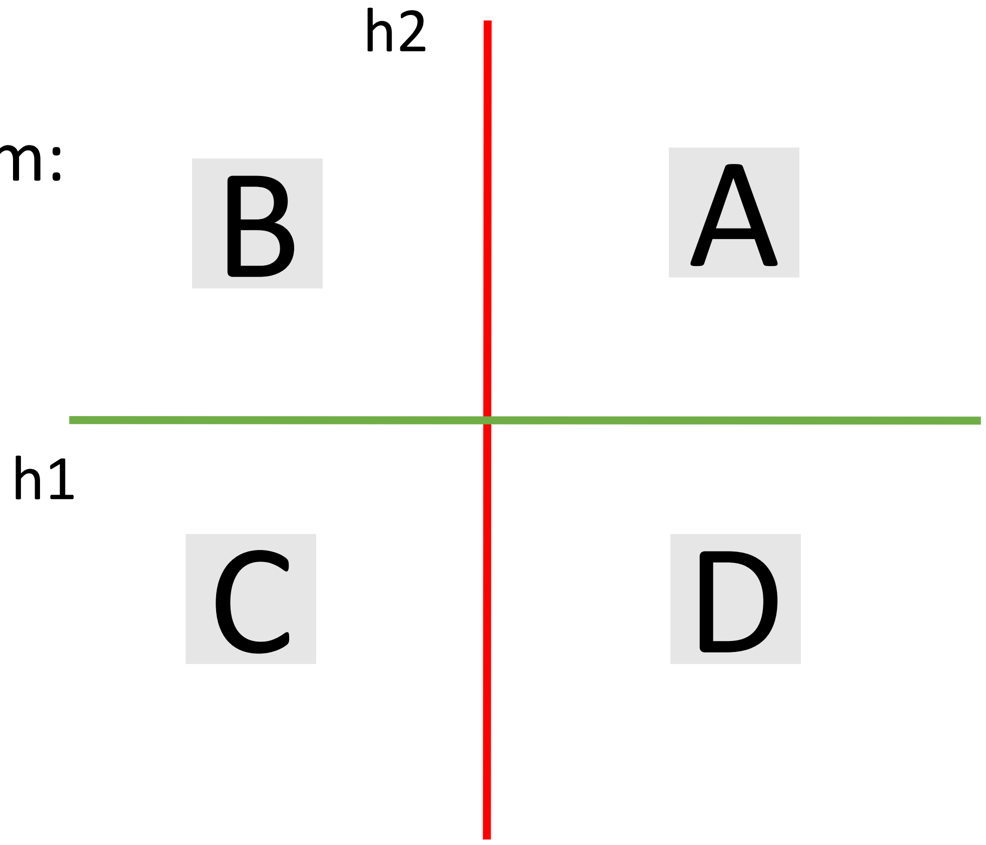


Space Warping

Consider a linear transform: $h = Wx$
Where x, h are both 2-dimensional

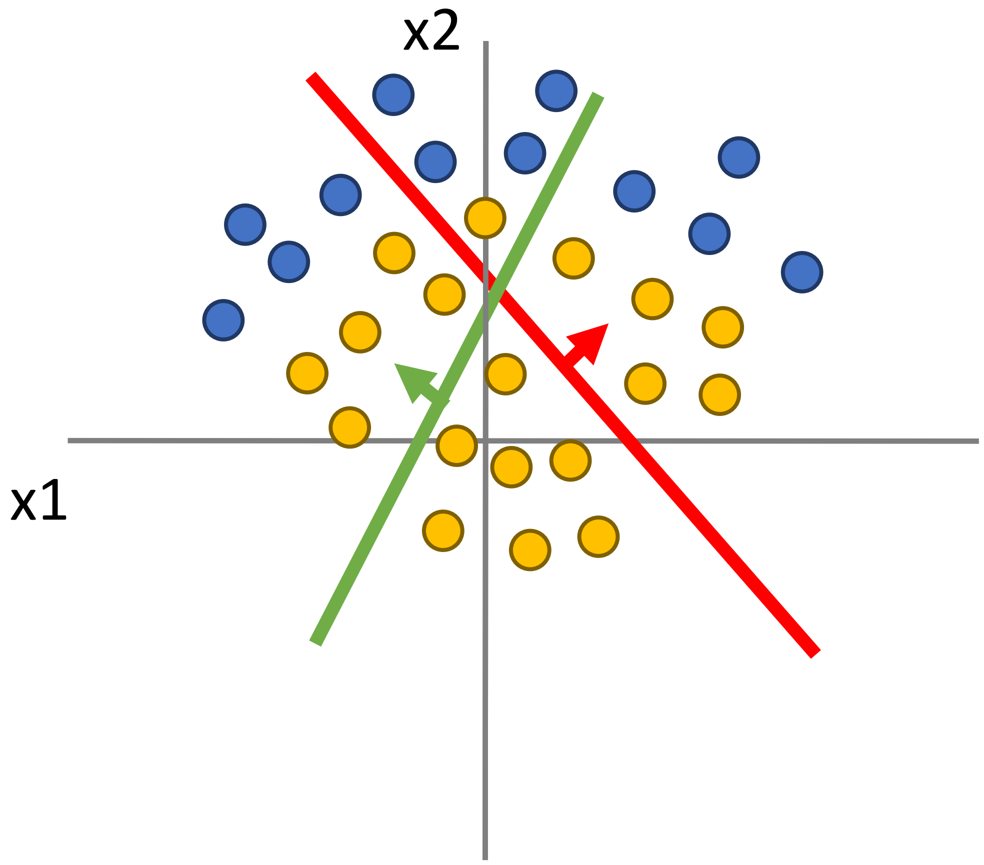


Feature transform:
$h = Wx$



Space Warping

Points not linearly separable in original space



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Space Warping

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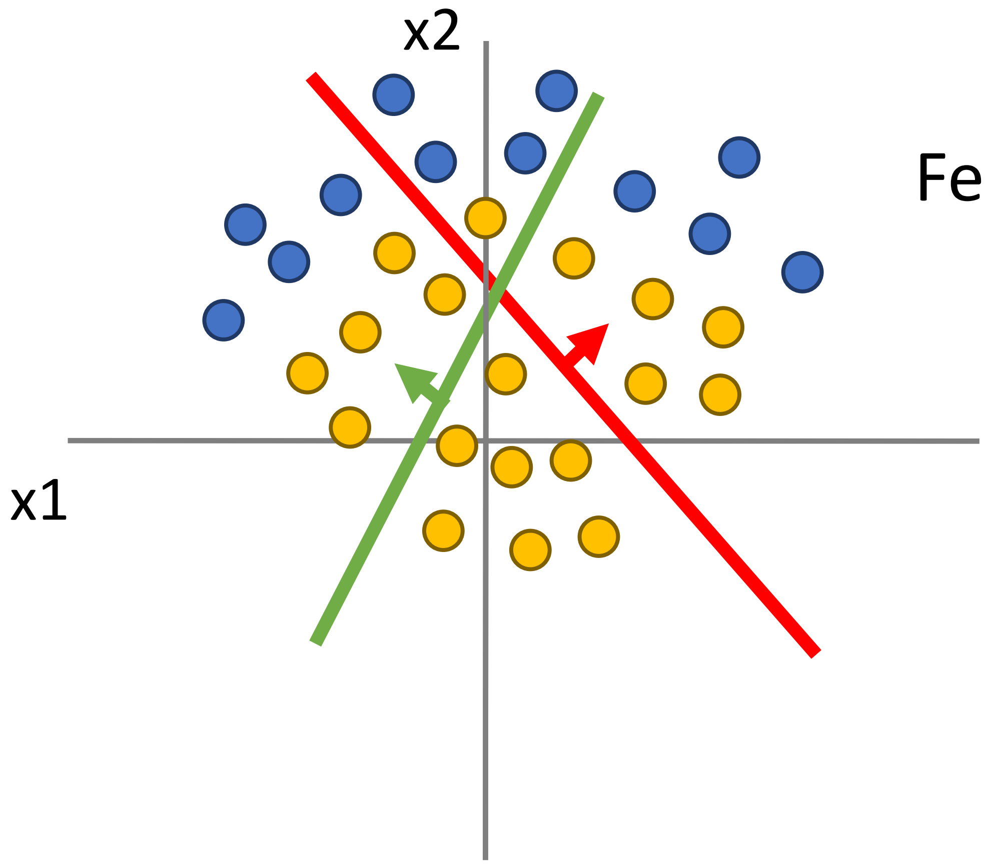
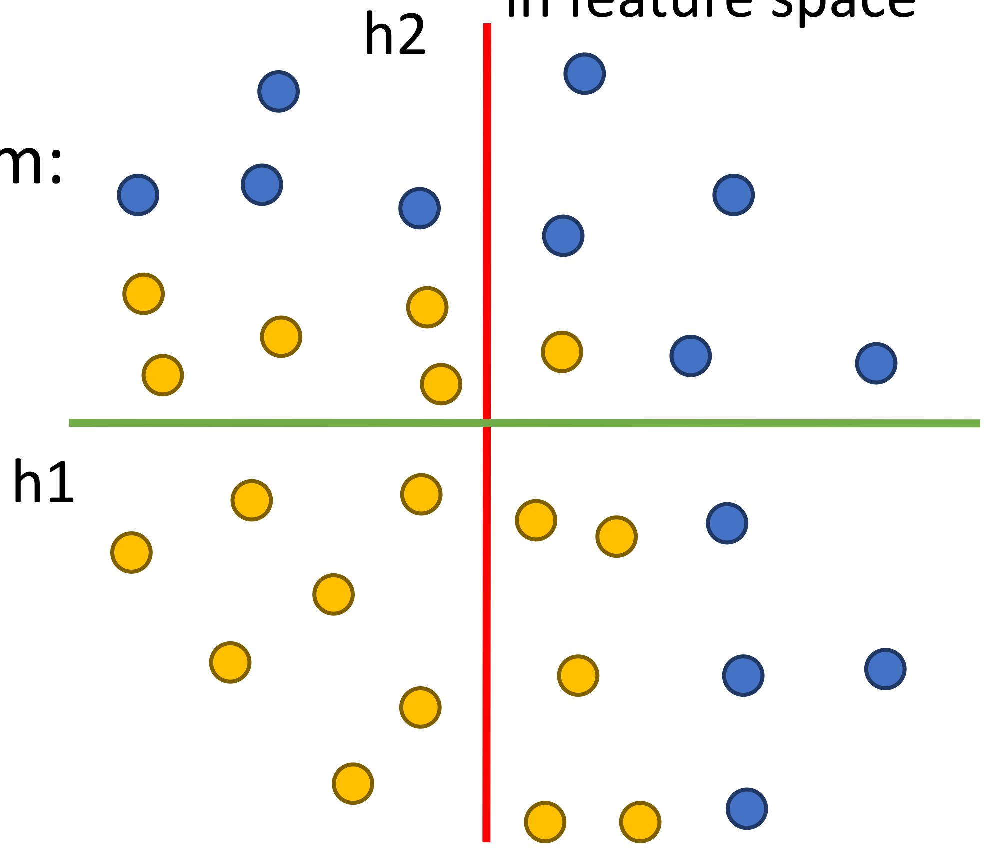
Where x, h are both 2-dimensional

Points not linearly separable in original space



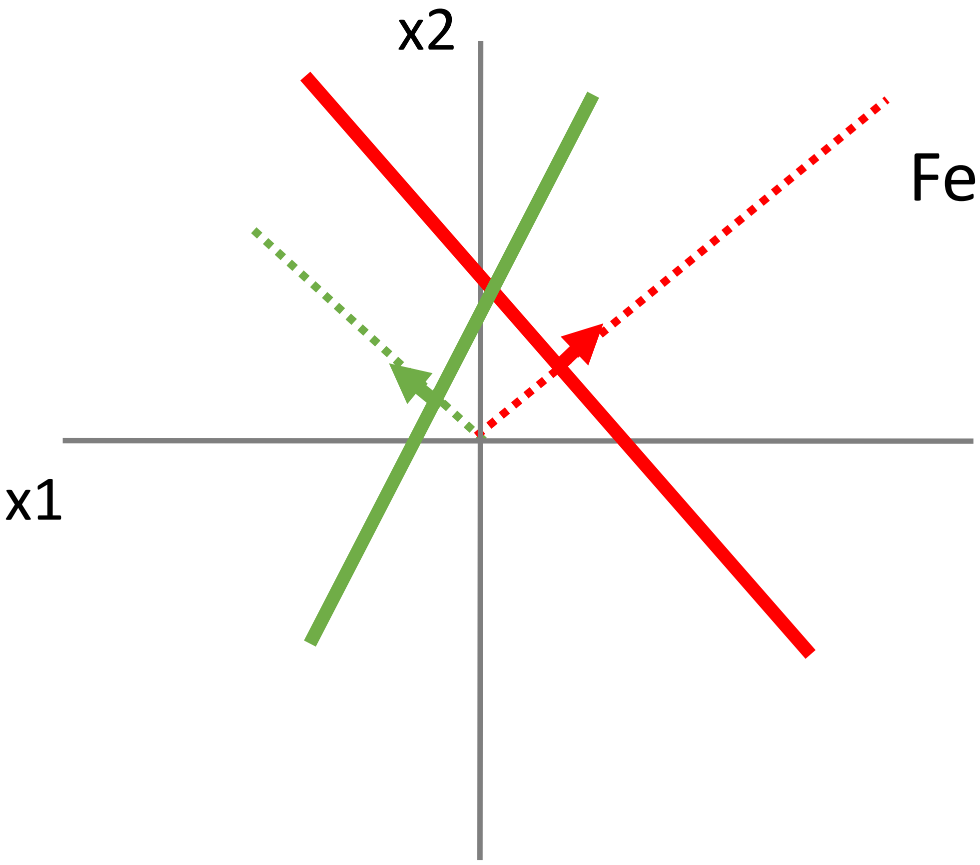
Feature transform:
h = Wx

Not linearly separable in feature space

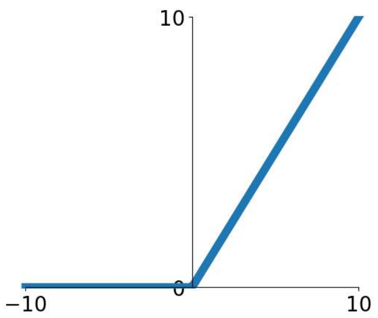


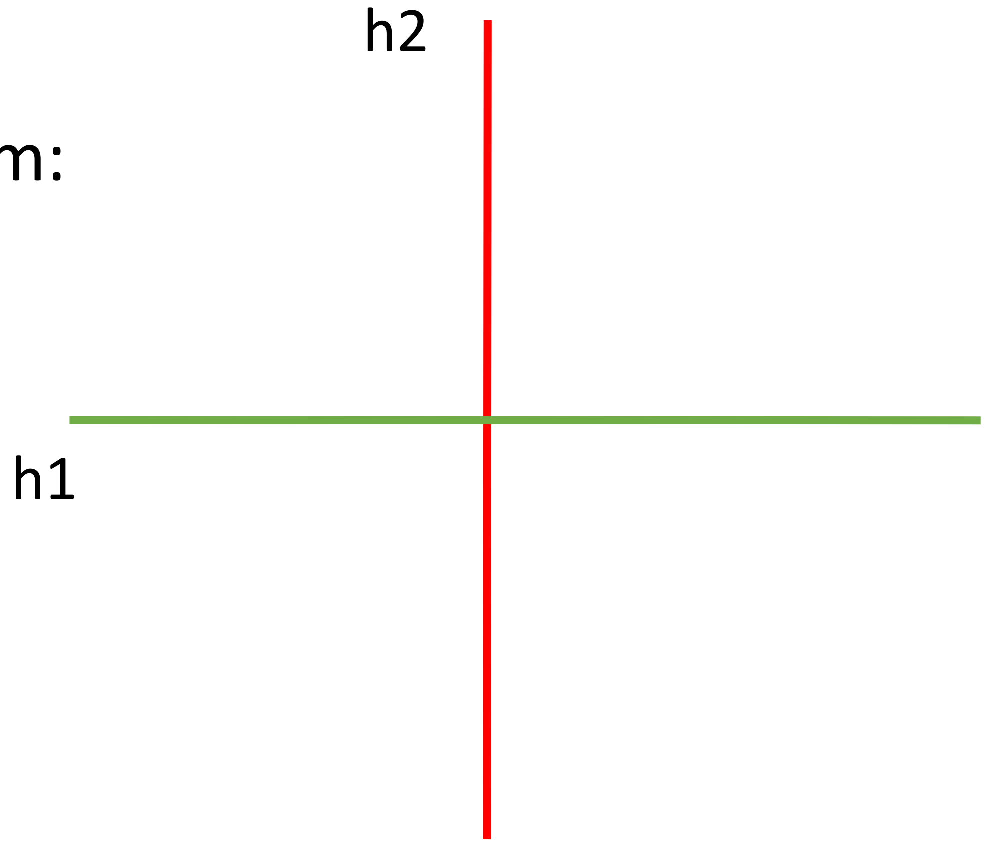
Space Warping

Consider a neural net hidden layer:
h = ReLU(Wx) = max(0, Wx)
Where x, h are both 2-dimensional



Feature transform:
h = ReLU(Wx)





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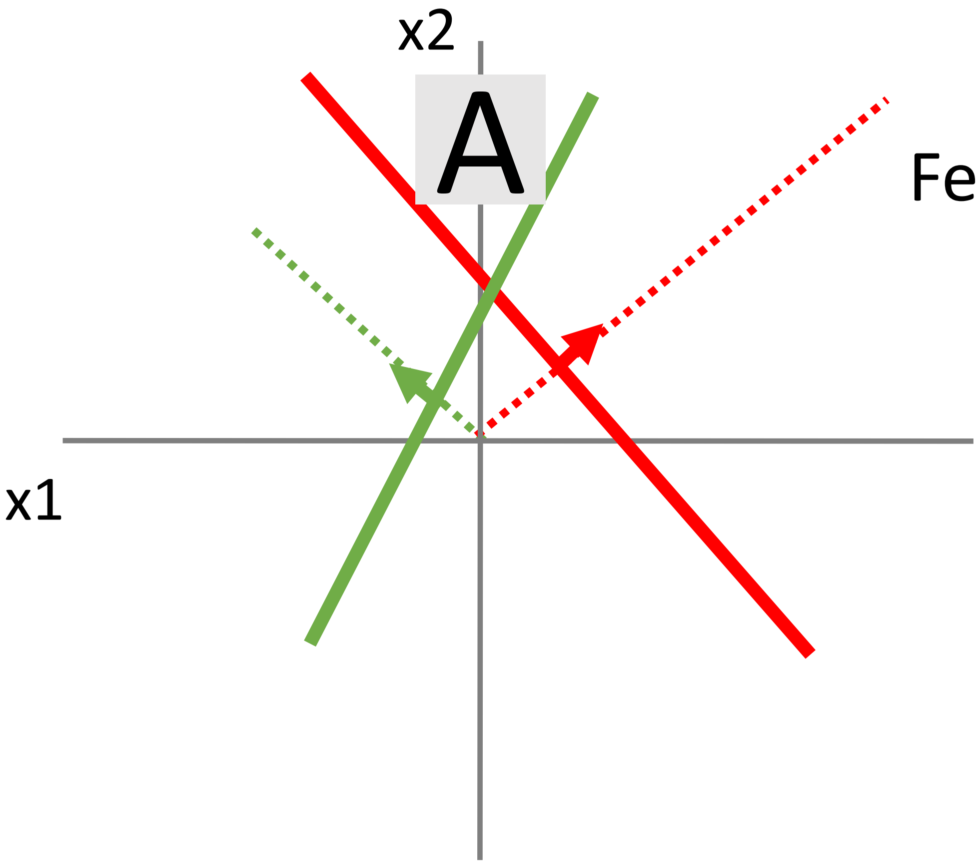

Feature transform:

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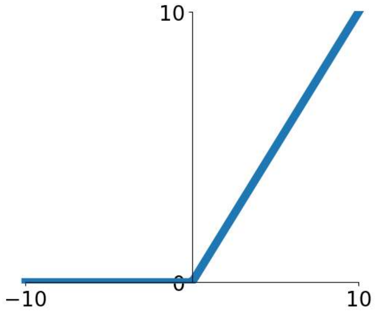

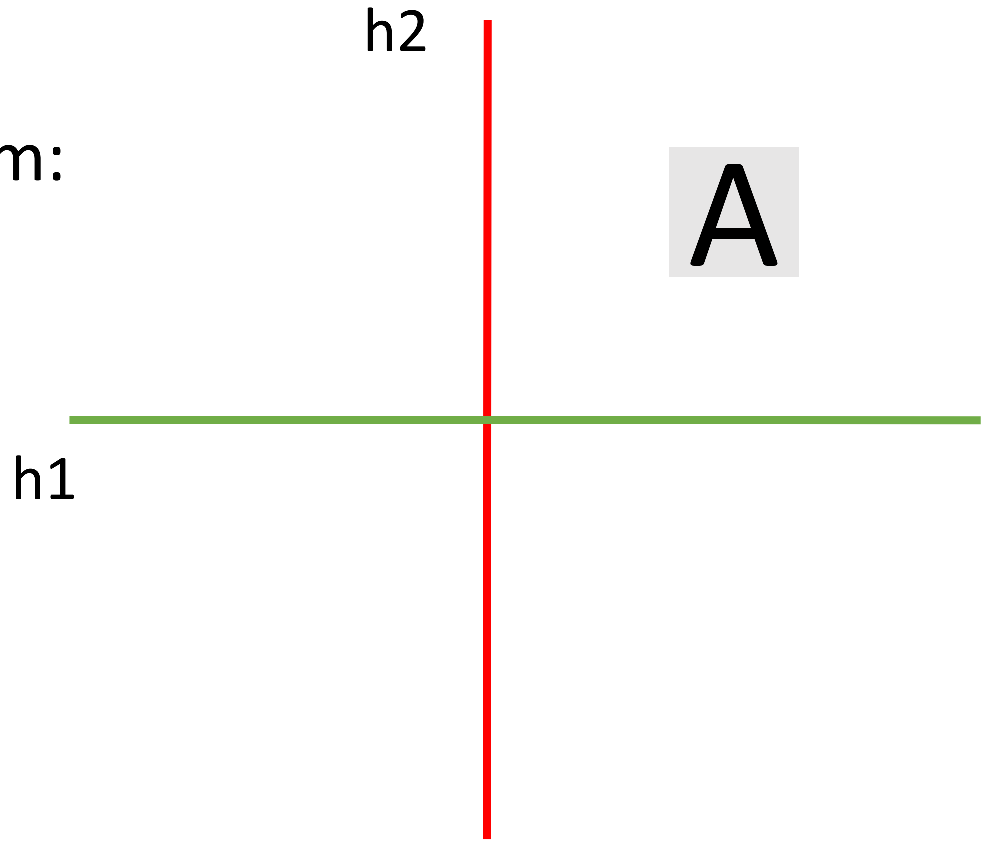

Space Warping

Consider a neural net hidden layer:
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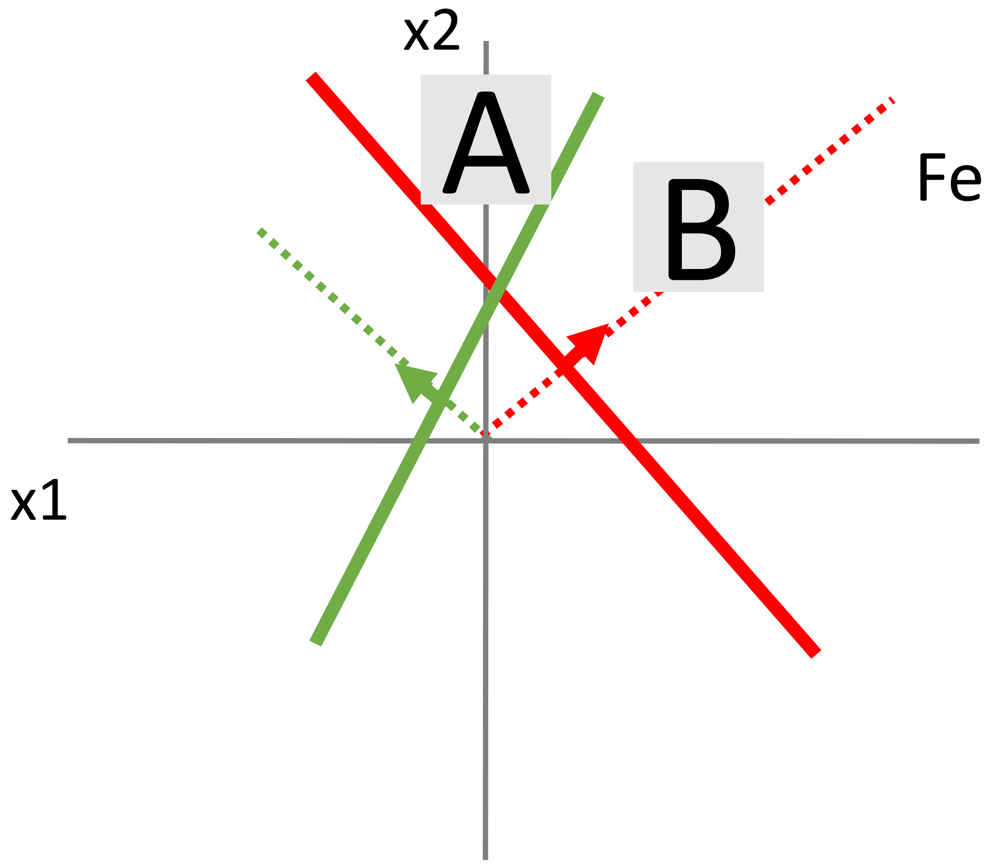
Where x, h are both 2-dimensional



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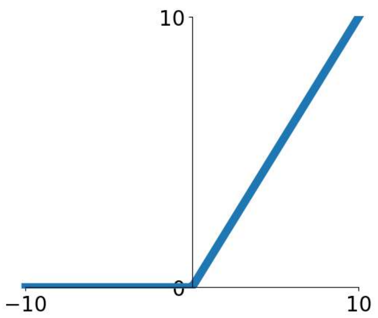

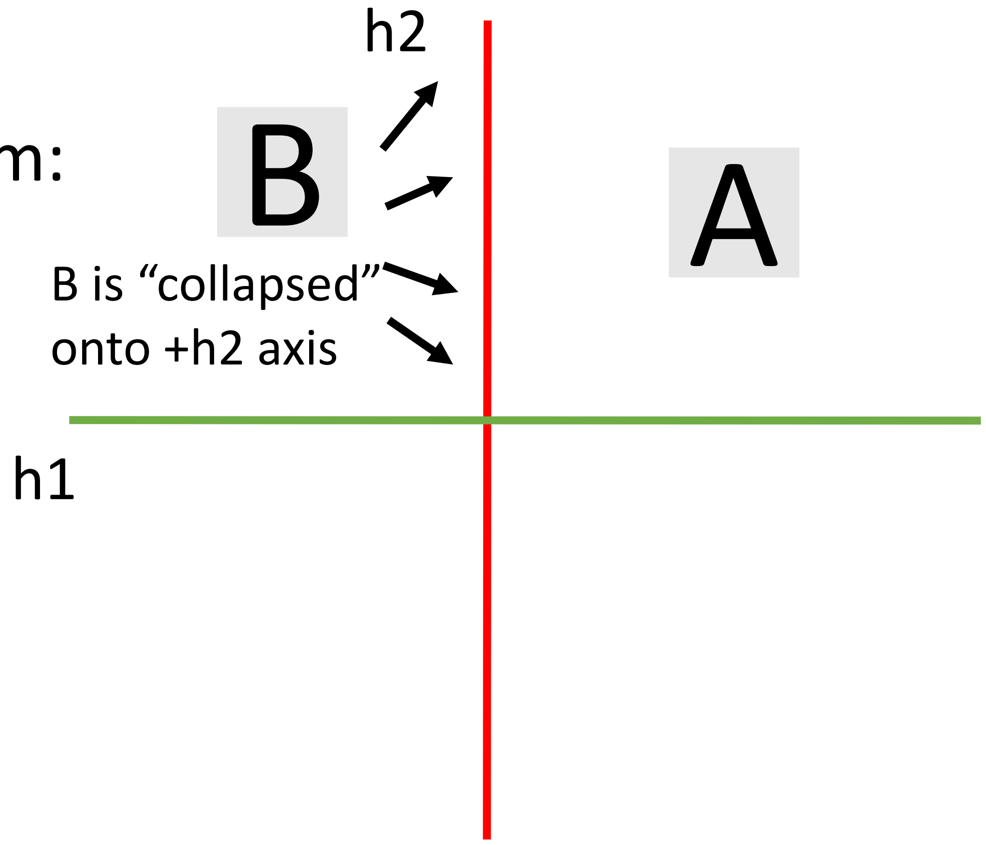

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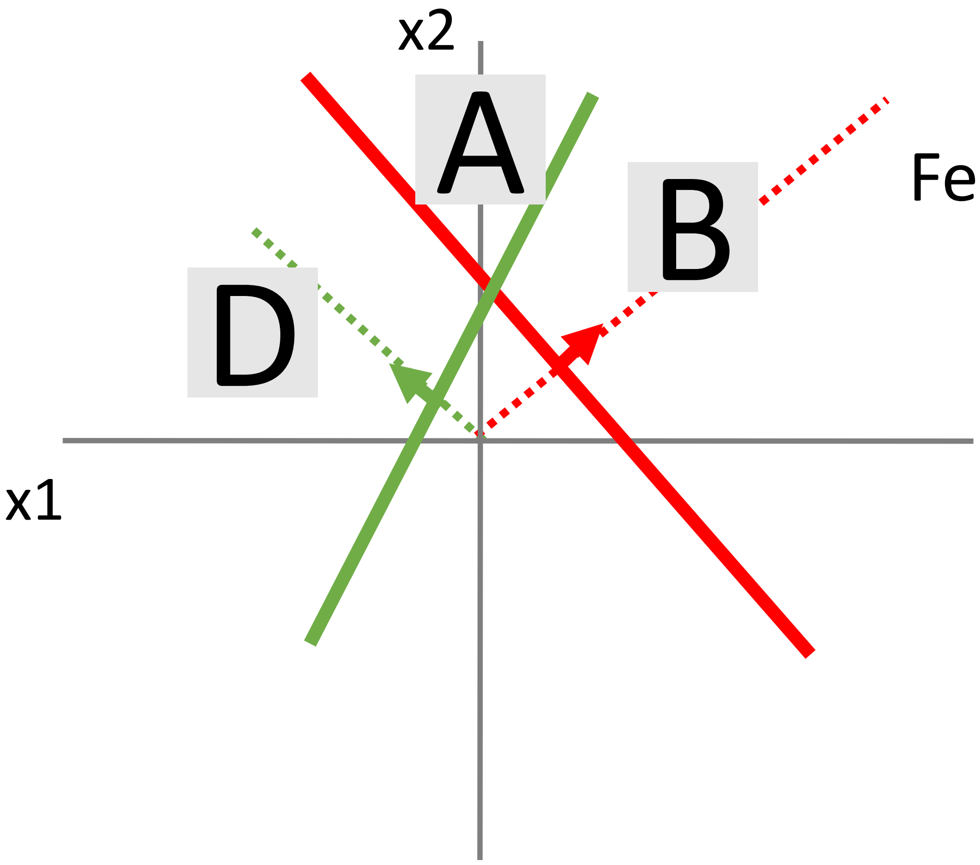
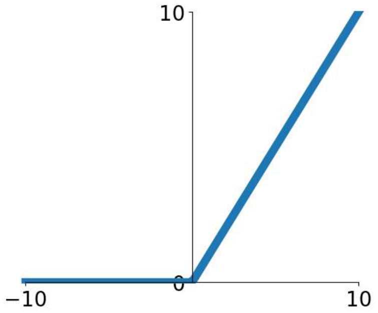
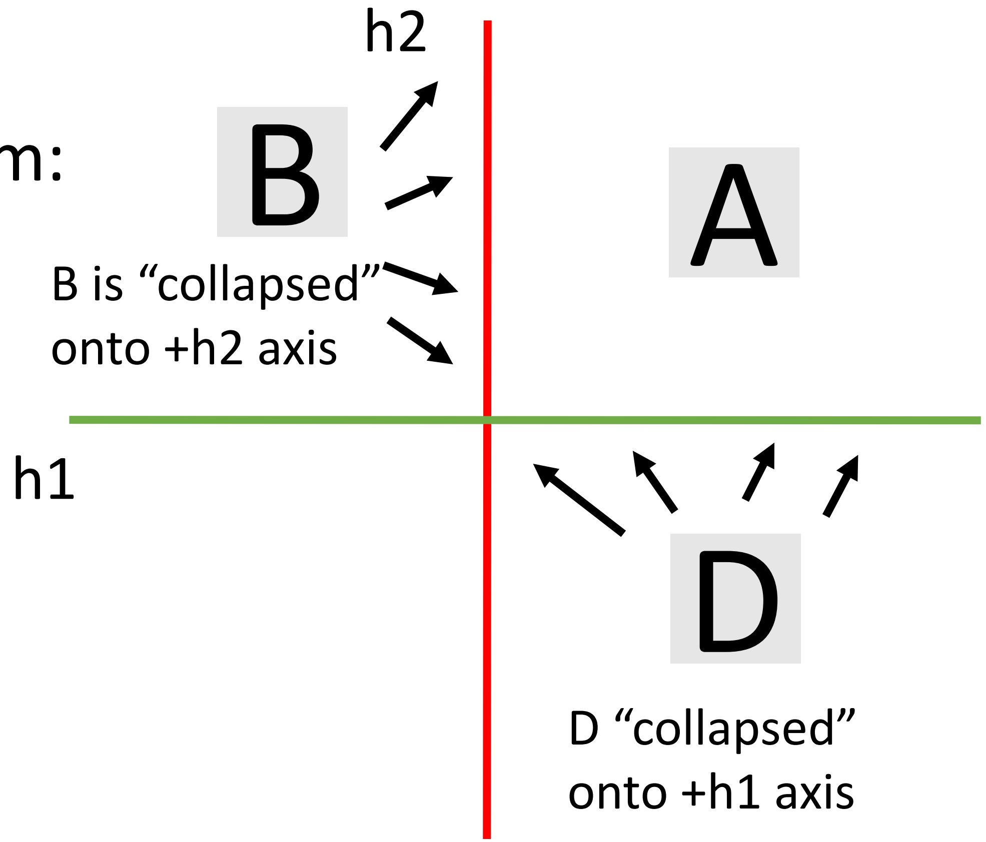


Feature transform:
h = ReLU(Wx)





Space Warping

Consider a neural net hidden layer:

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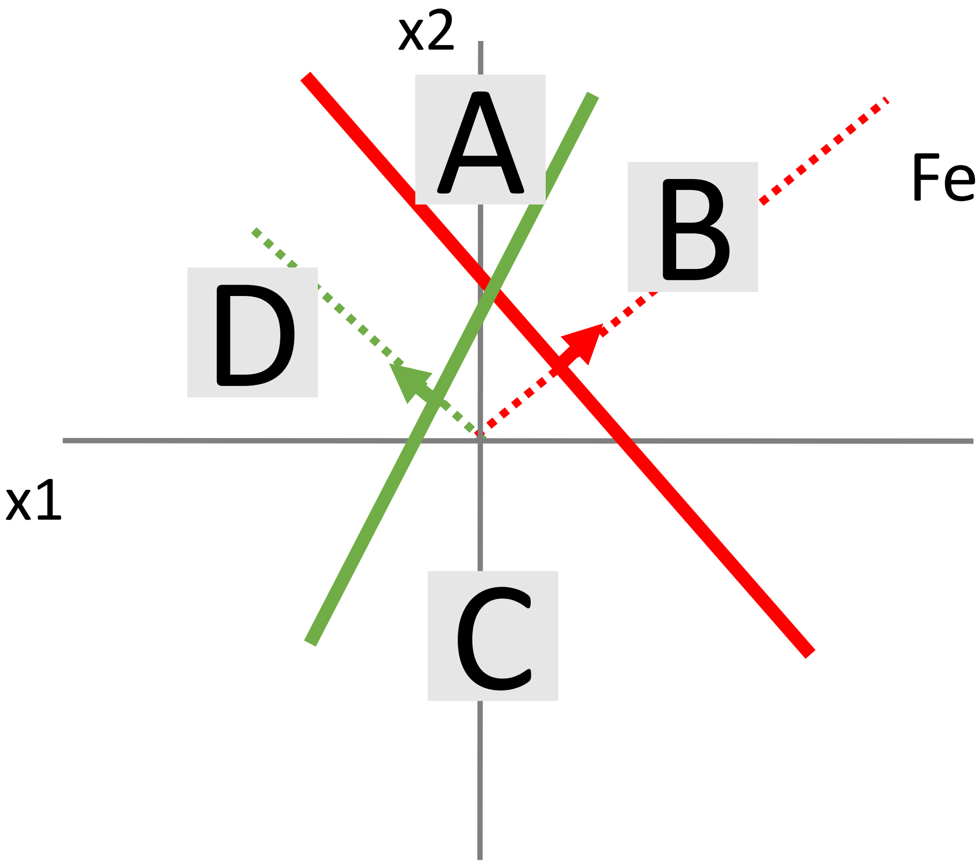
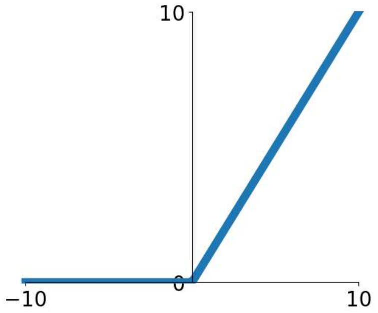
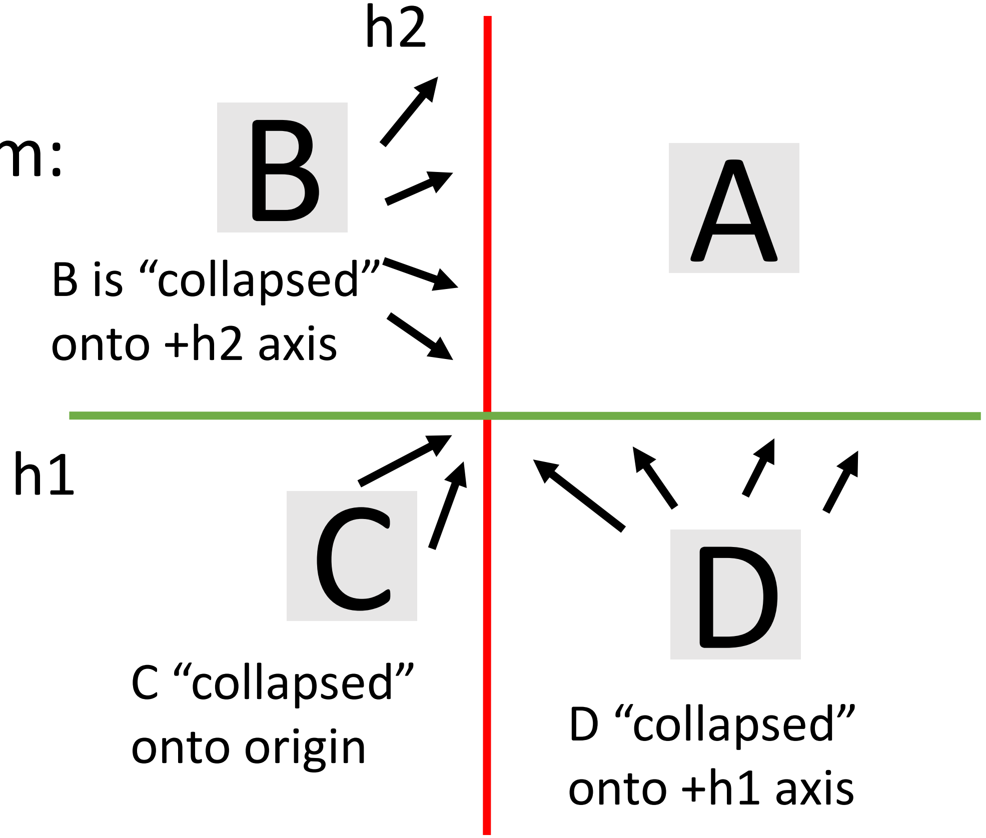
Where x, h are both 2-dimensional



Feature transform:

$h = \text{ReLU}(Wx)$





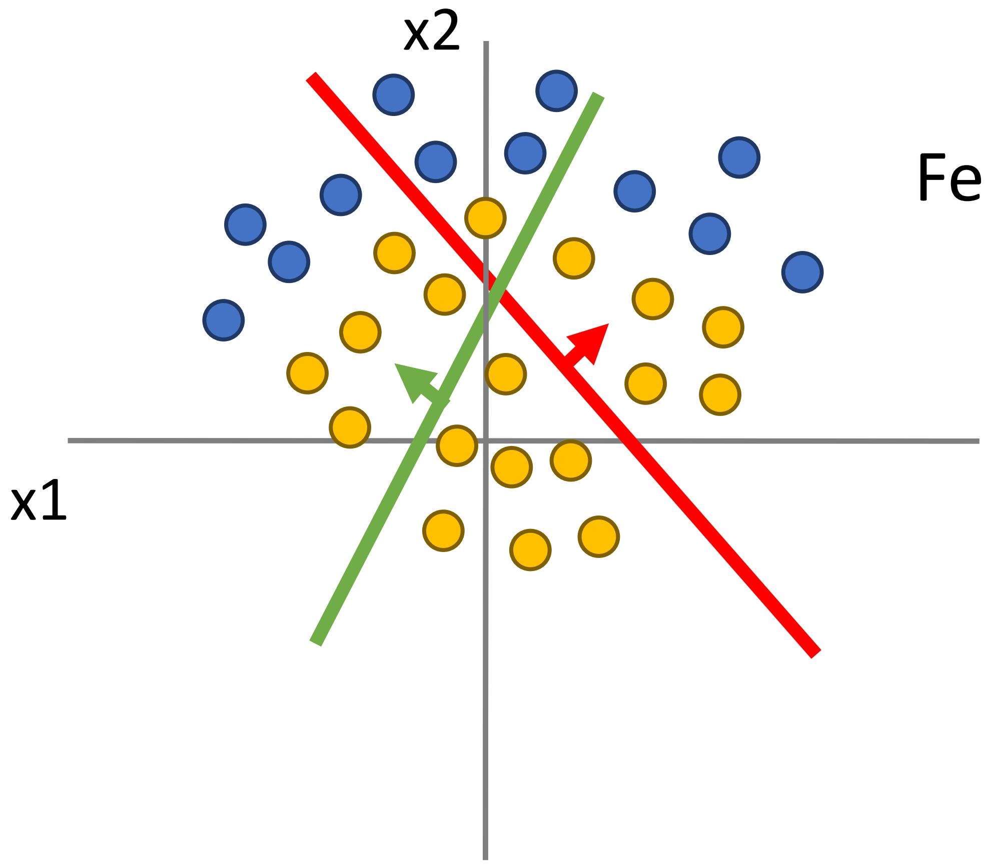
Space Warping

Points not linearly separable in original space

Consider a neural net hidden layer:

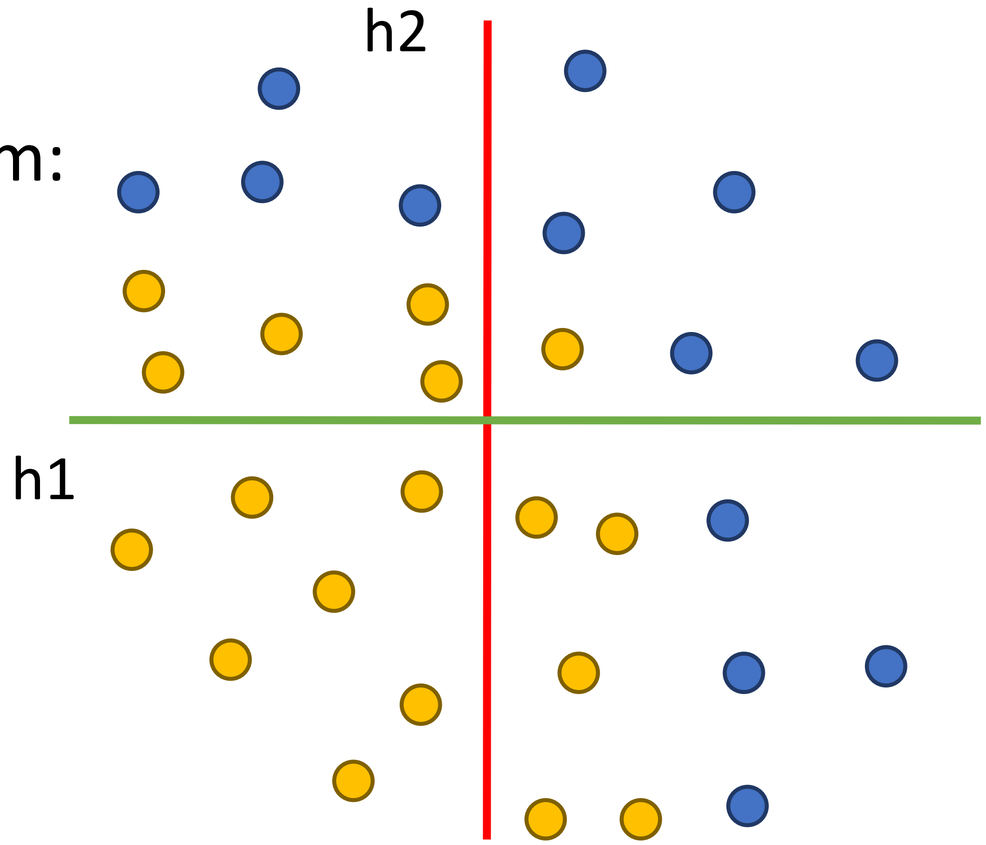
h = ReLU(Wx) = max(0, Wx)

Where x, h are both 2-dimensional



Feature transform:

h = Wx



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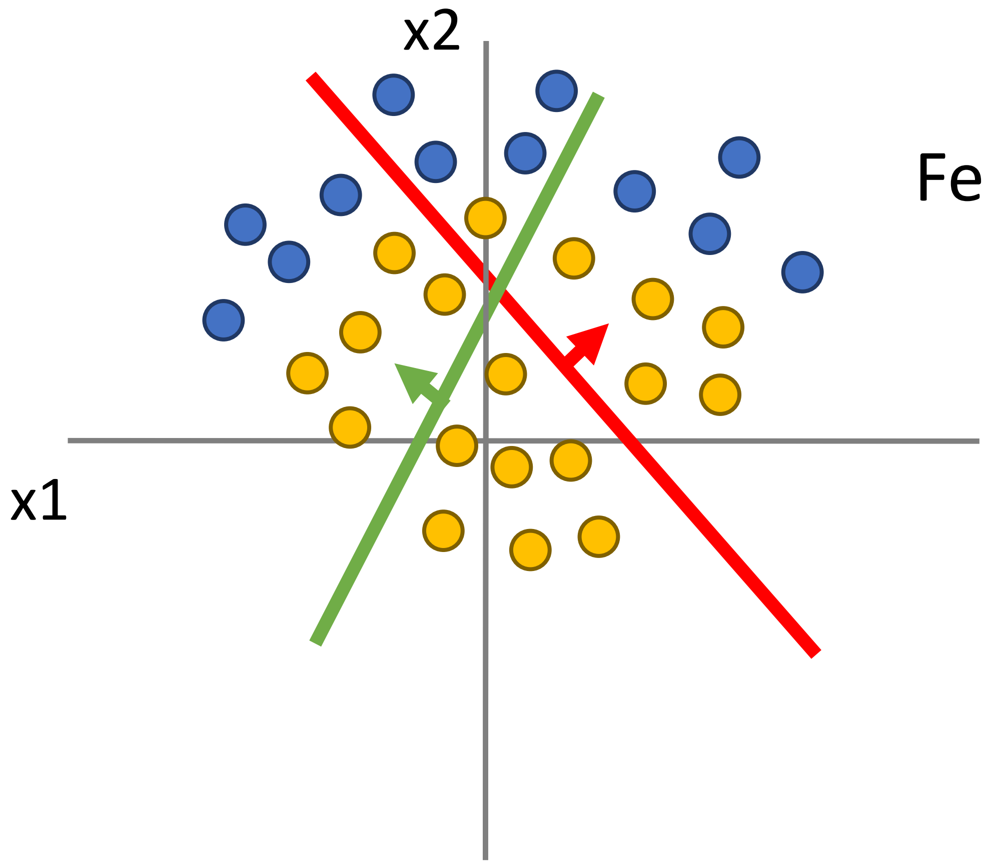


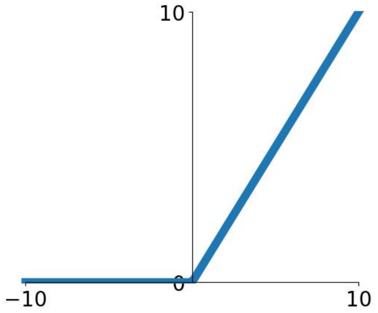
Consider a neural net hidden layer:

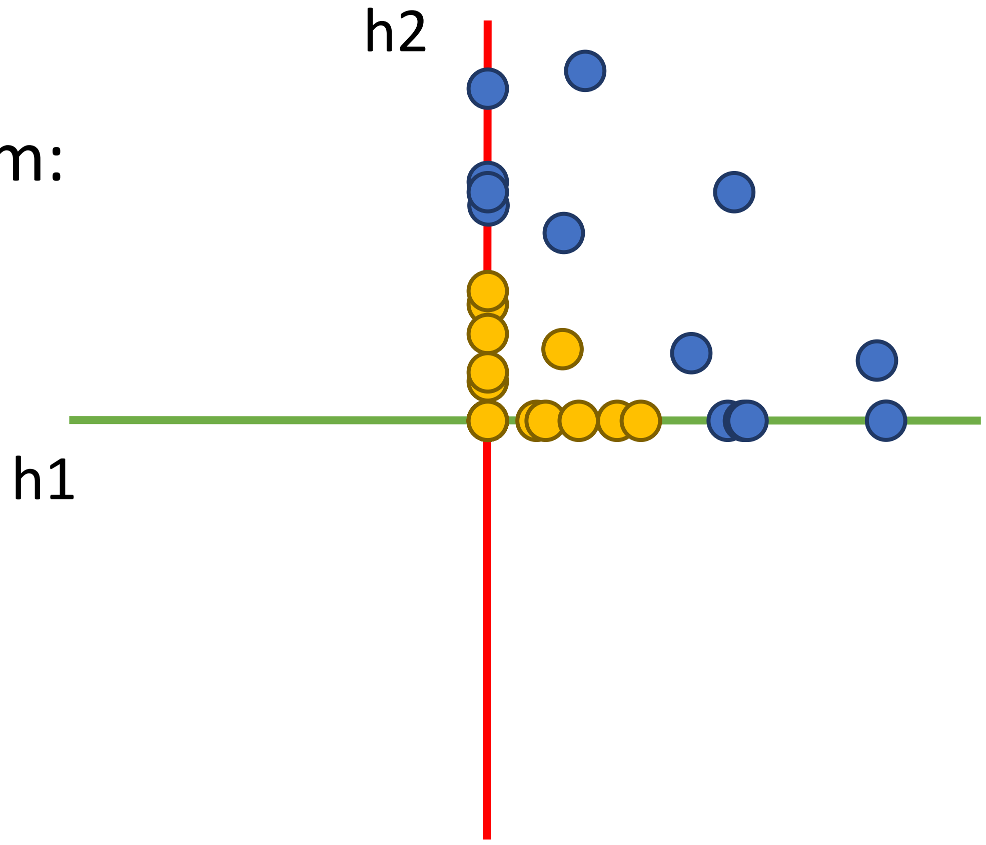
$h = \text{ReLU}(Wx) = \max(0, Wx)$

Where x, h are both 2-dimensional

Feature transform:

$h = \text{ReLU}(Wx)$





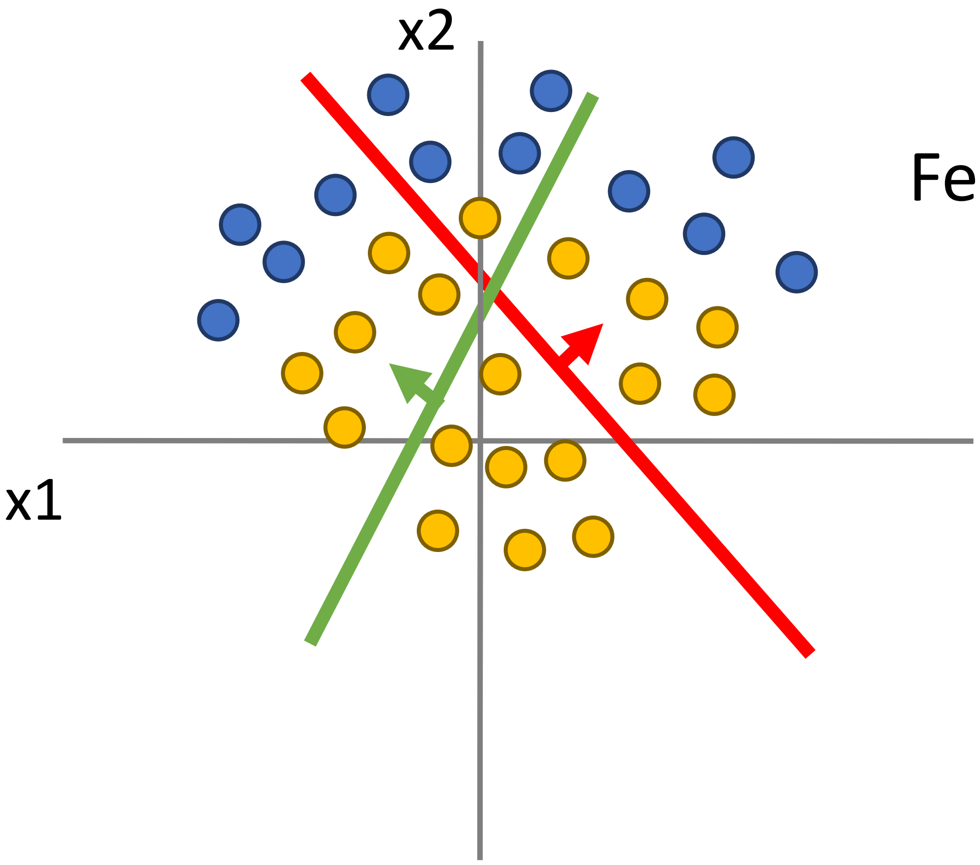
Space Warping

Consider a neural net hidden layer:

$h = \text{ReLU}(Wx) = \max(0, Wx)$

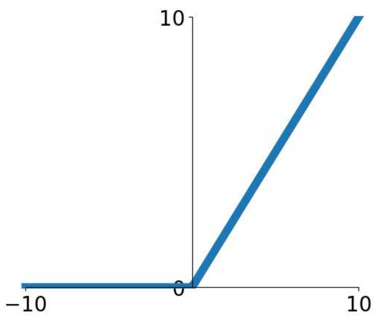
Where x, h are both 2-dimensional

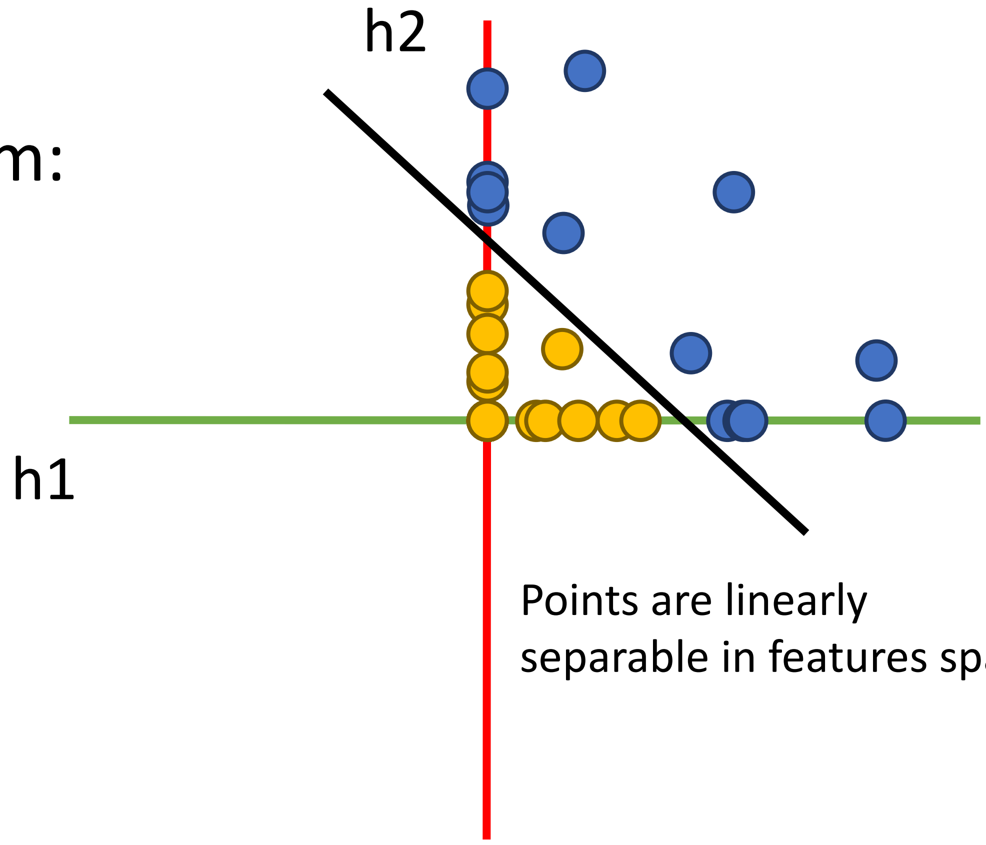
Points not linearly separable in original space



Feature transform:

$h = \text{ReLU}(Wx)$

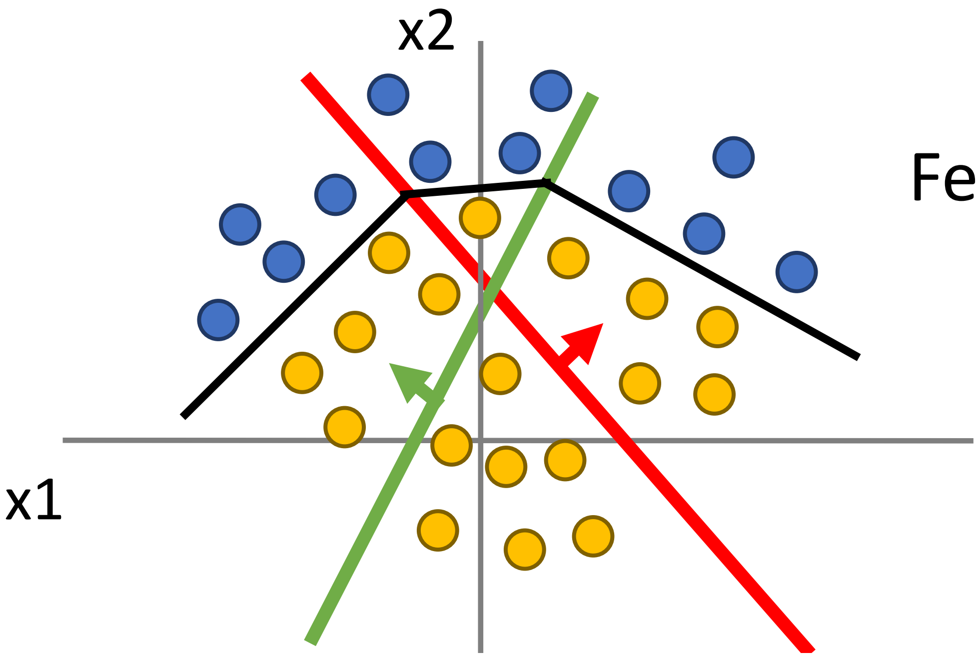




Points are linearly separable in features space!

Space Warping

Points not linearly separable in original space



Linear classifier in feature space gives nonlinear classifier in original space

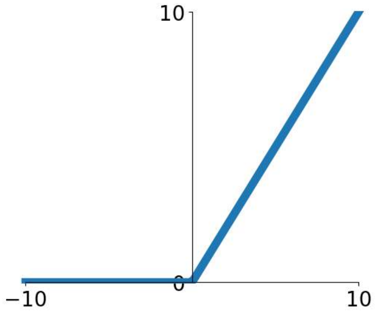
Consider a neural net hidden layer:

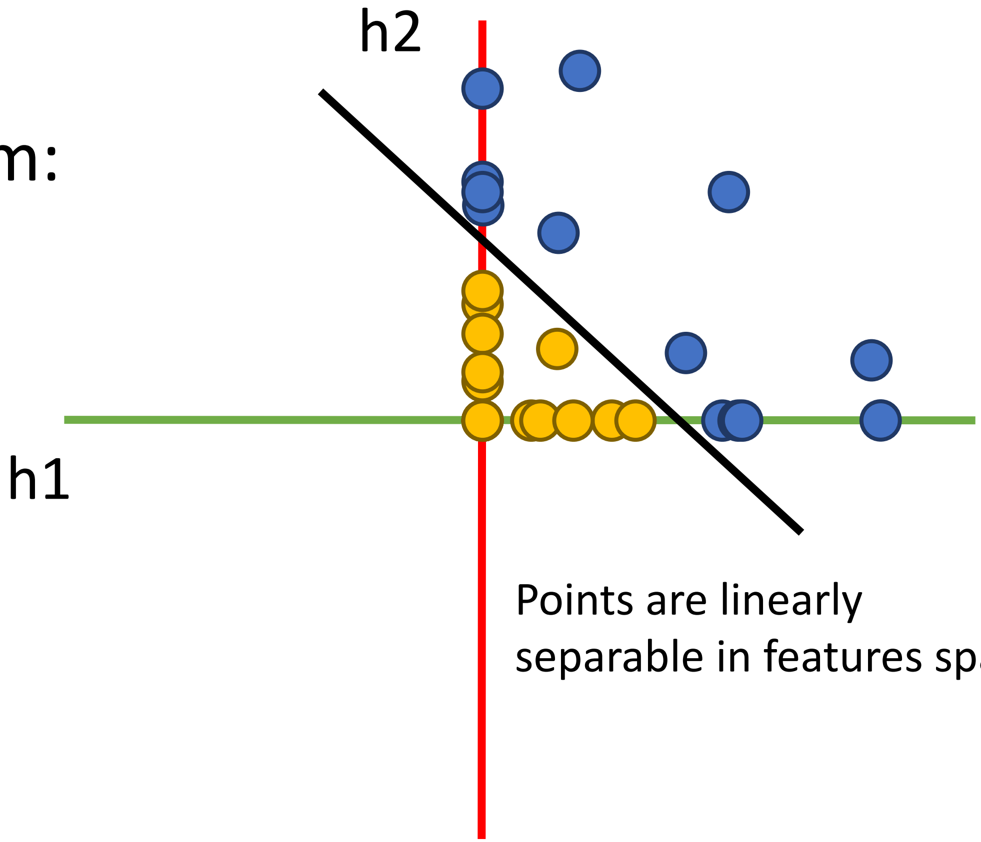
h = ReLU(Wx) = max(0, Wx)

Where x, h are both 2-dimensional

Feature transform:

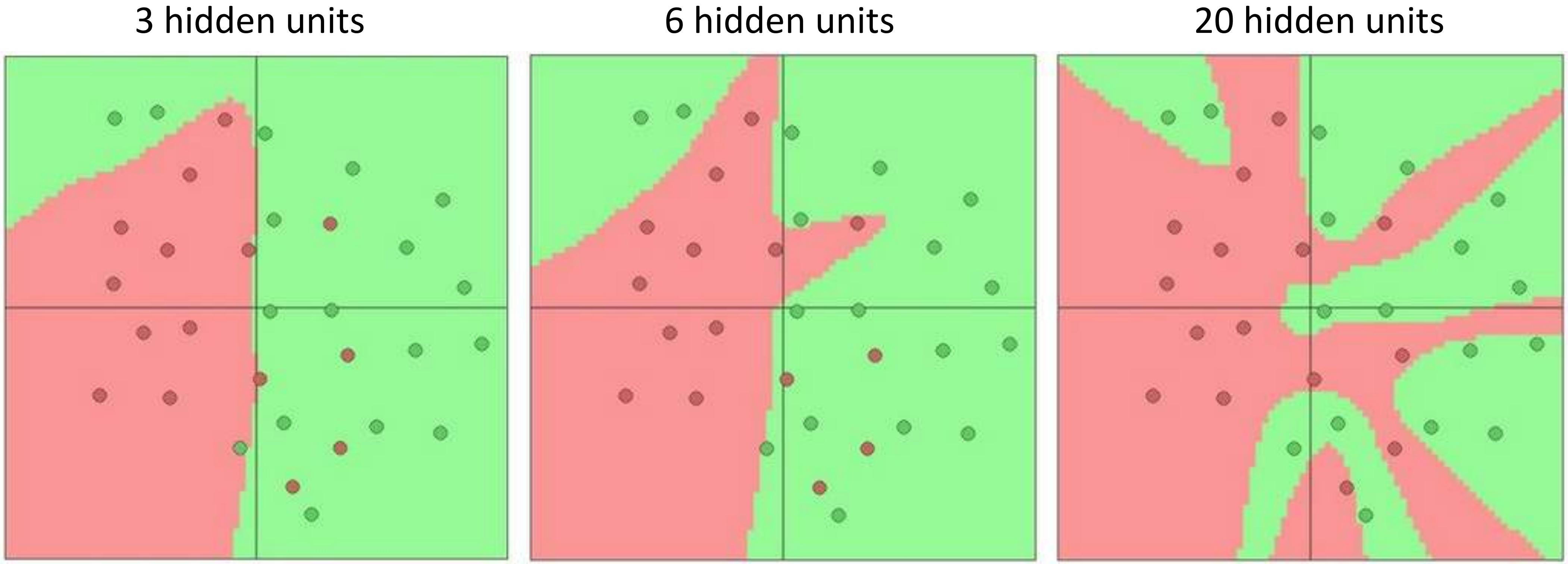
h = ReLU(Wx)





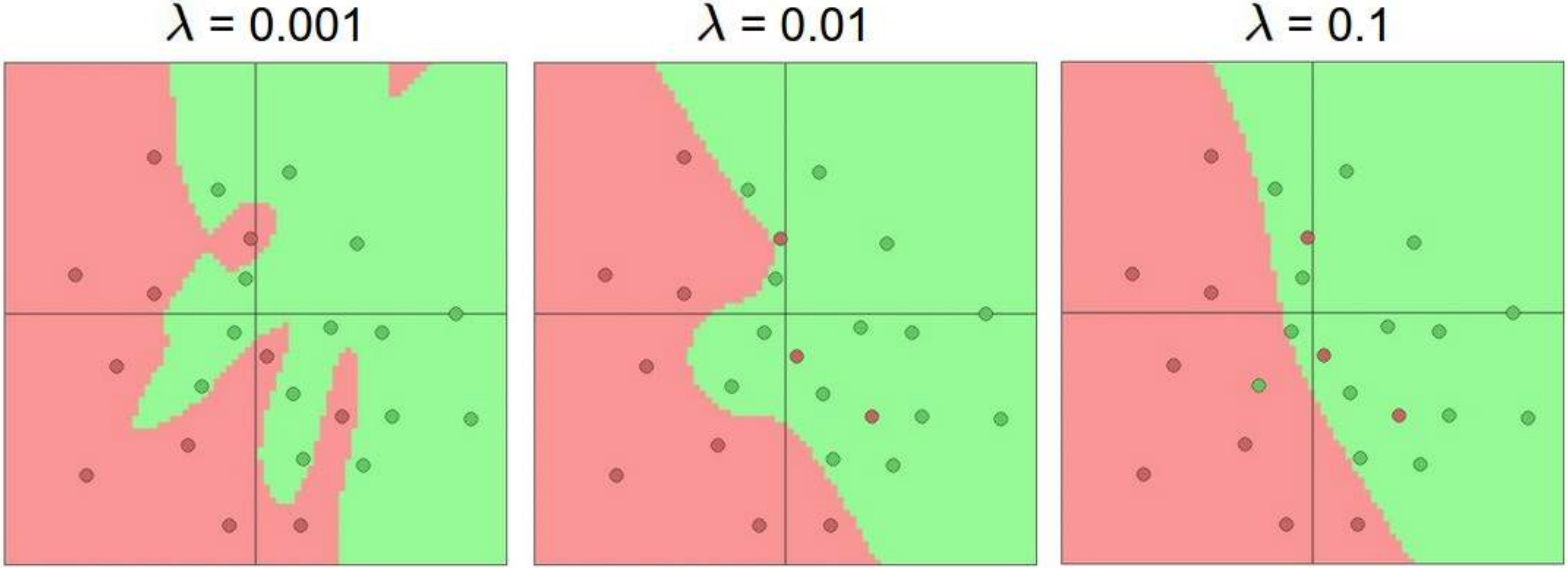
Points are linearly separable in features space!

Setting the number of layers and their sizes



More hidden units = more capacity

Don’t regularize with size; instead use stronger L2



(Web demo with ConvNetJS: http://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html)

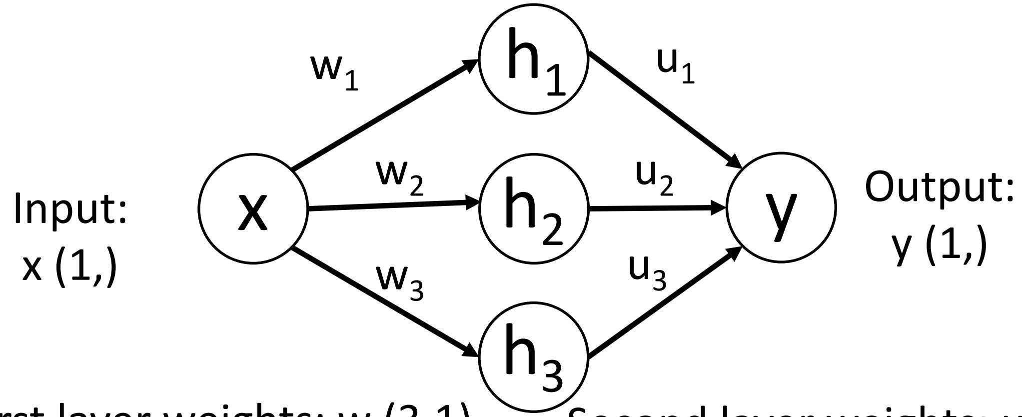
Universal Approximation

A neural network with one hidden layer can approximate any function f: R^N -> R^M with arbitrary precision*

*Many technical conditions: Only holds on compact subsets of R^N; function must be continuous; need to define "arbitrary precision"; etc

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

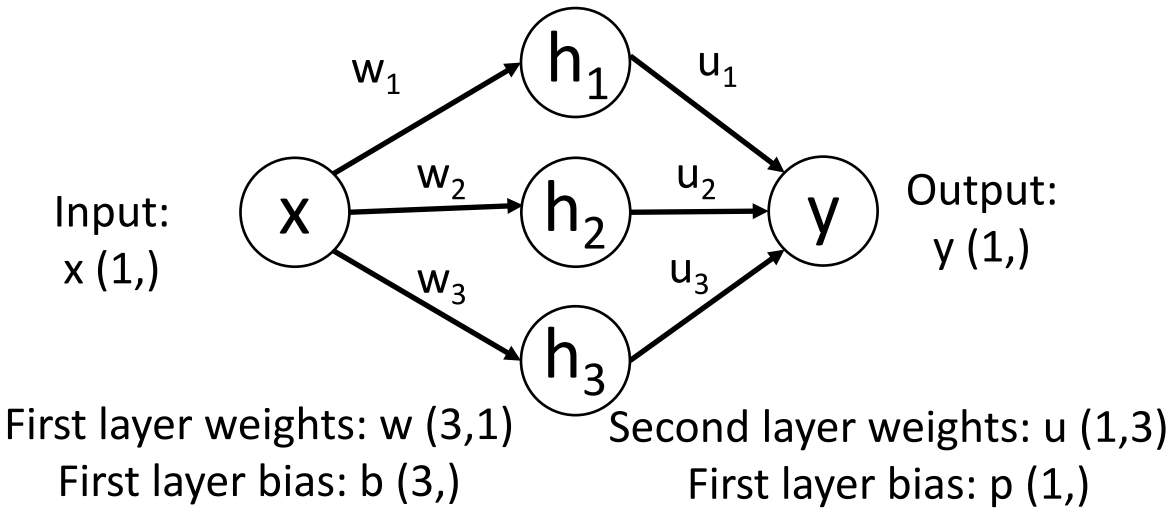


First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

Universal Approximation

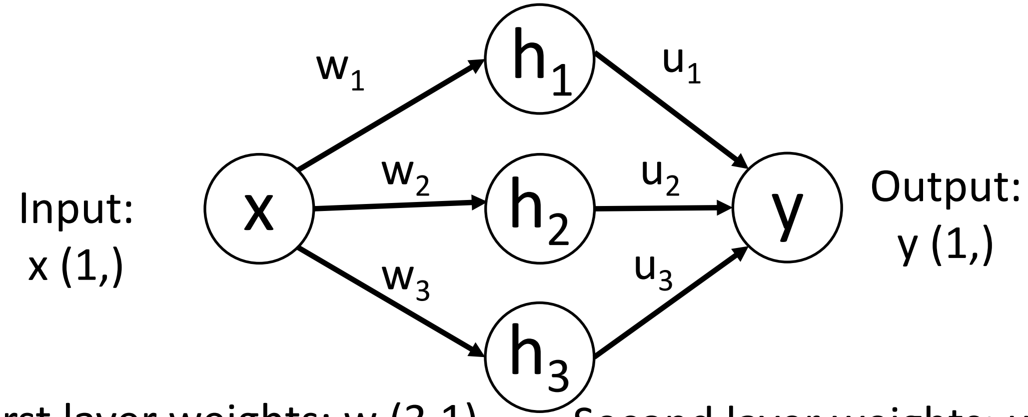
Example: Approximating a function f: R -> R with a two-layer ReLU network



h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

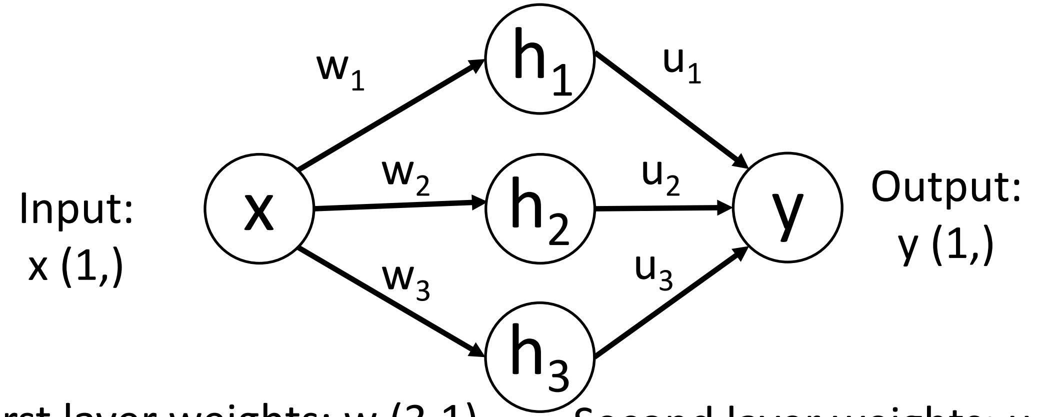
Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



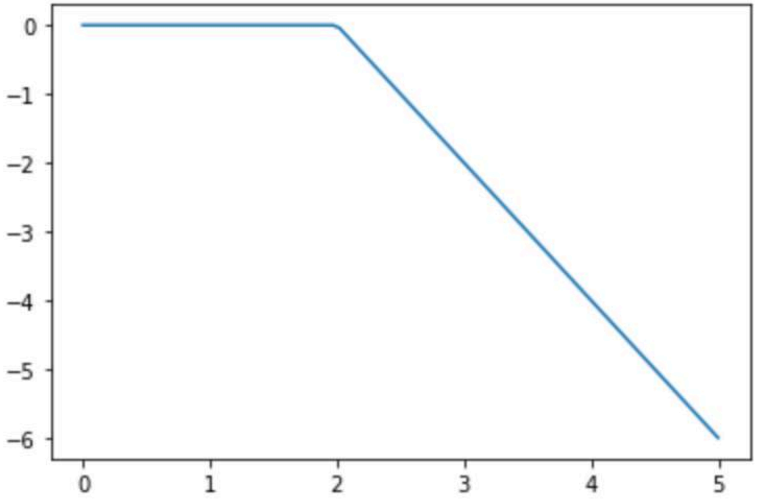
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

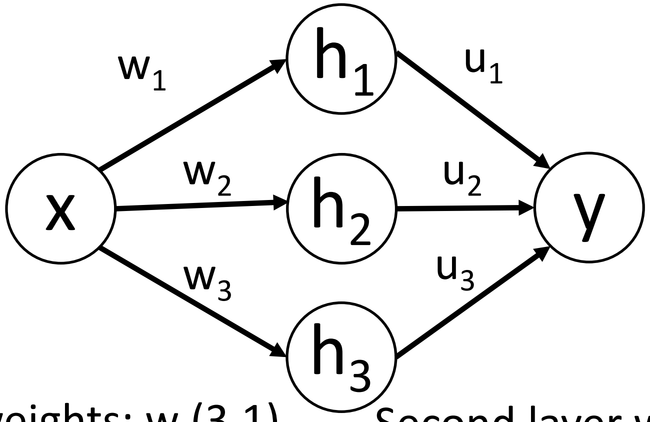
Output is a sum of shifted, scaled ReLUs:



Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

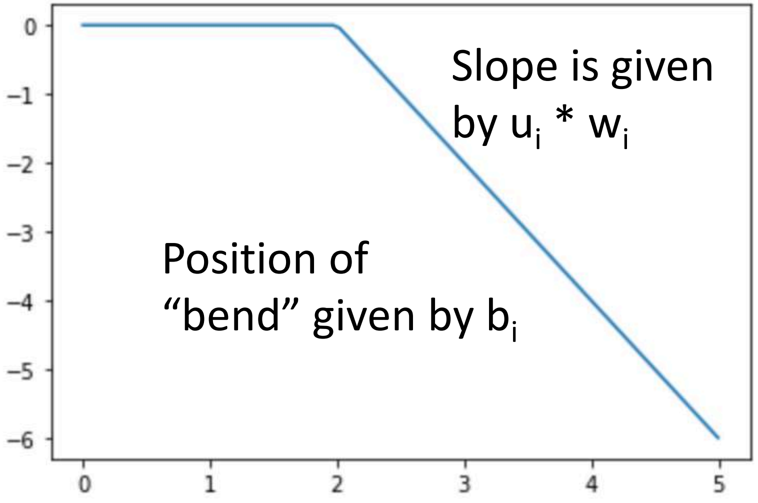
Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

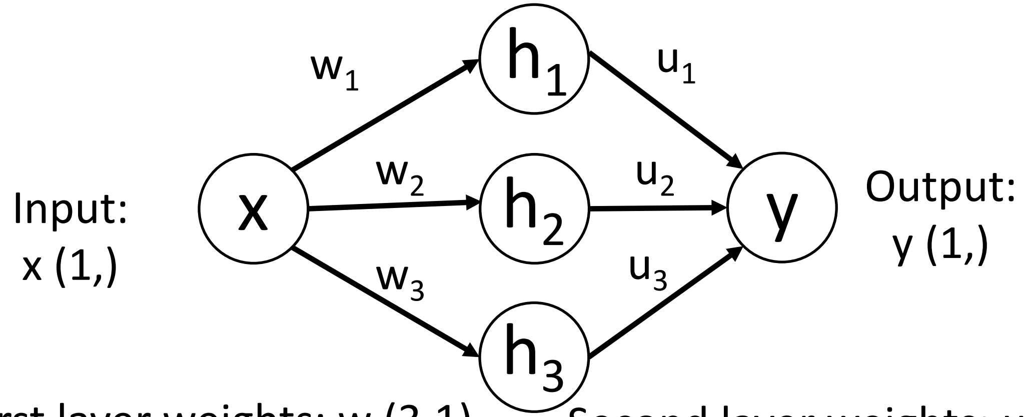
Output is a sum of shifted, scaled ReLUs:

Flip left / right based on sign of w_i



Universal Approximation

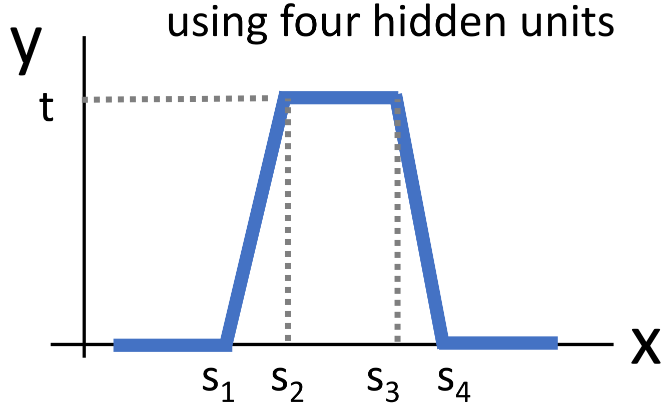
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units

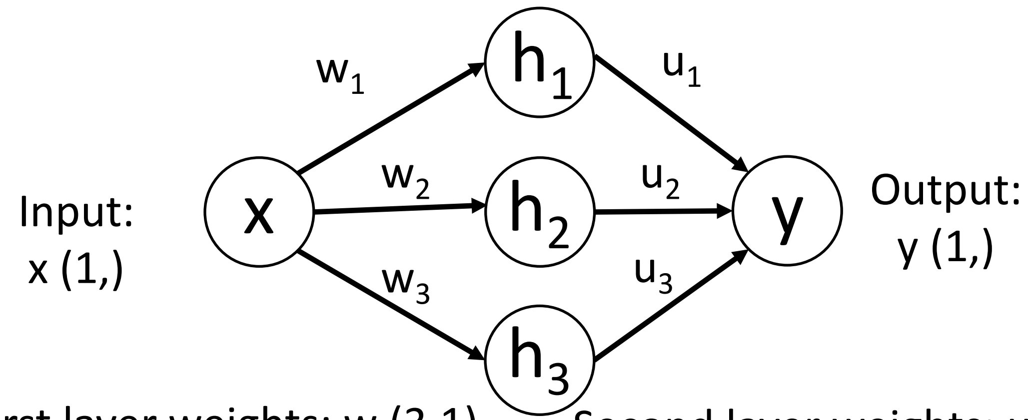


h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

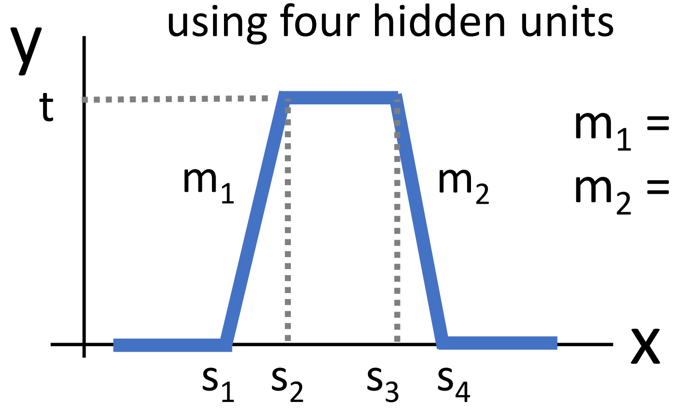
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

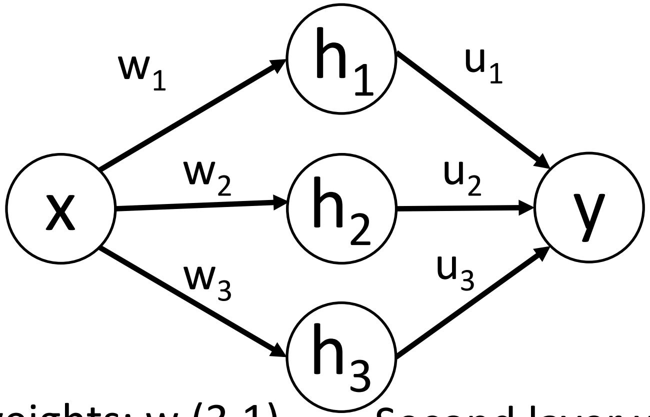
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)

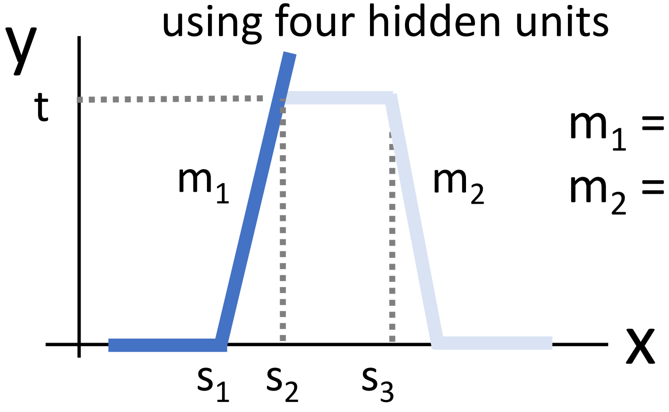


Output:
y (1,)

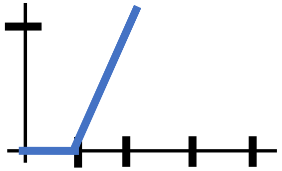
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$



$m_1 * \max(0, x - s_1)$

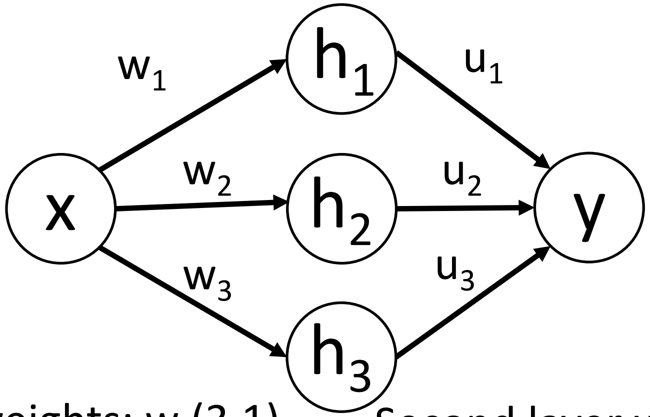
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

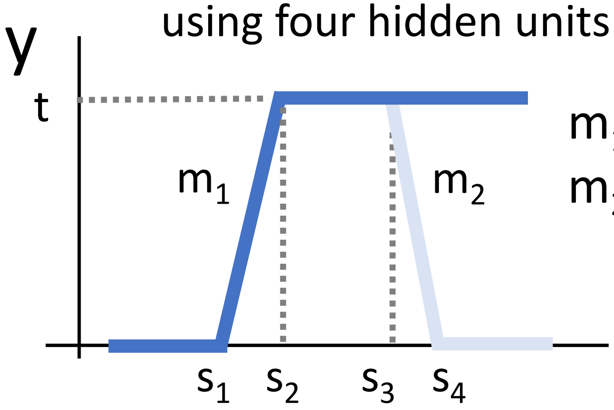
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

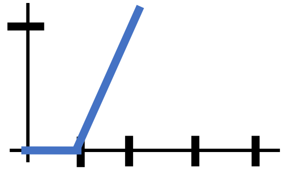
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

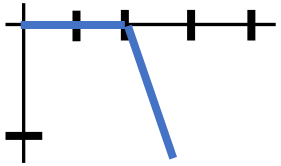
We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$



$m_1 * \max(0, x - s_1)$

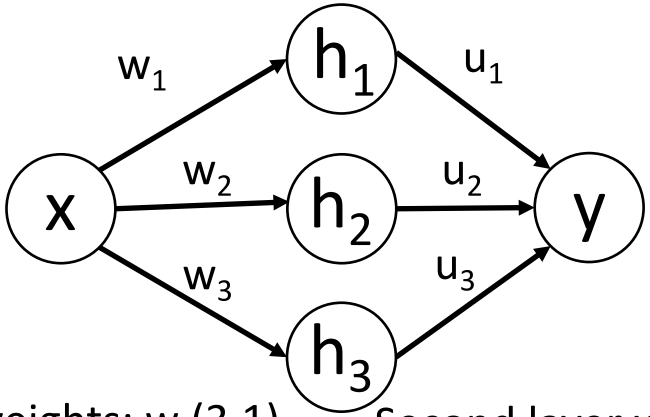


$-m_1 * \max(0, x - s_2)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

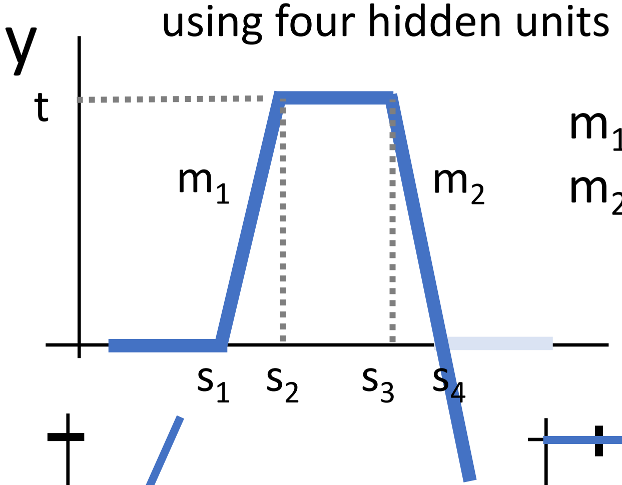
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

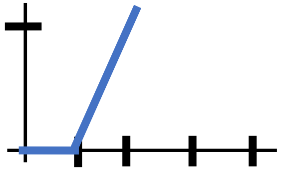
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

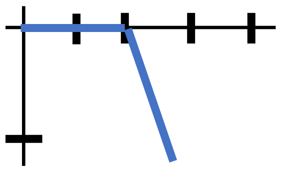
We can build a "bump function" using four hidden units



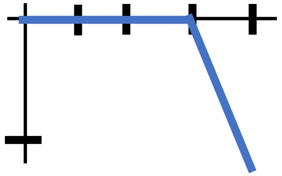
$m_1 = t / (s_2 – s_1)$
$m_2 = t / (s_4 – s_3)$



m_1 * max(0, x – s_1)



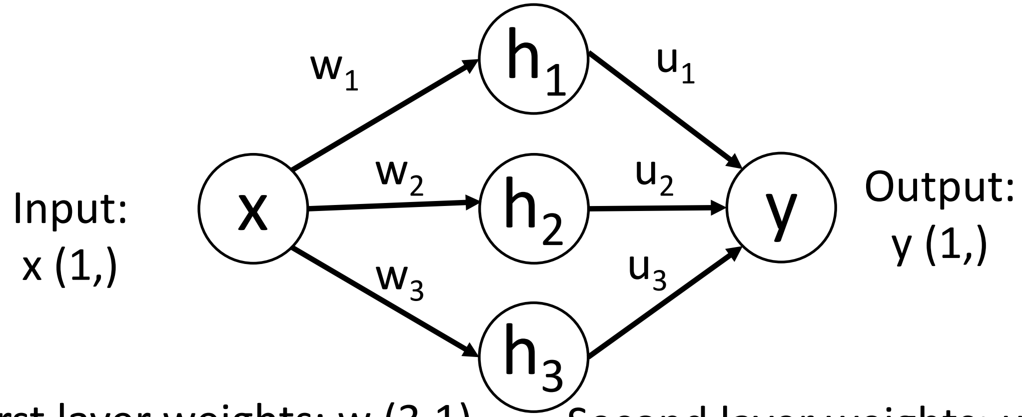
$-m_1$ * max(0, x – s_2)



$-m_2$ * max(0, x – s_3)

Universal Approximation

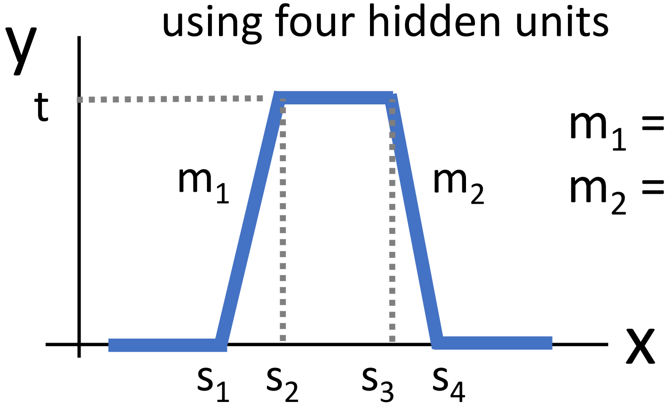
Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

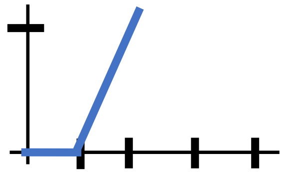
We can build a "bump function" using four hidden units



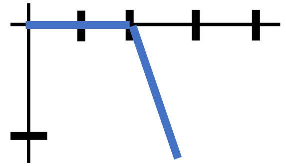
$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

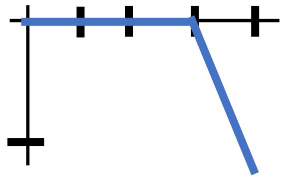
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p



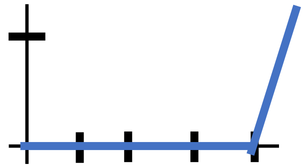
$m_1 * \max(0, x - s_1)$



$-m_1 * \max(0, x - s_2)$



$-m_2 * \max(0, x - s_3)$

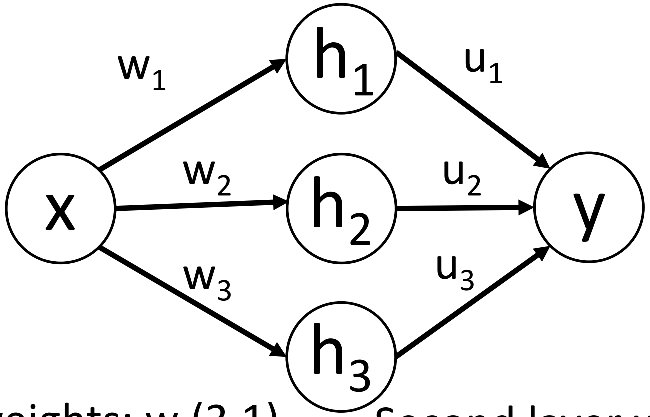


$m_2 * \max(0, x - s_4)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

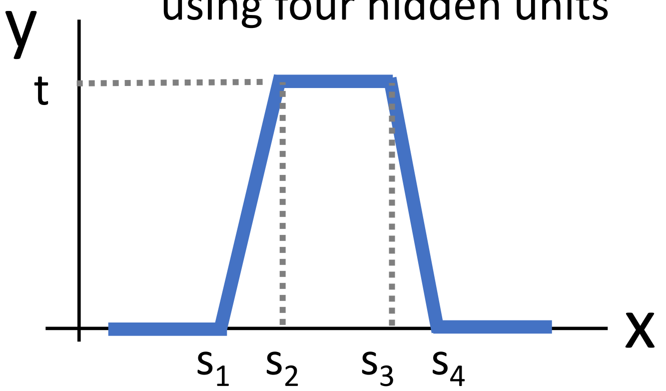
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

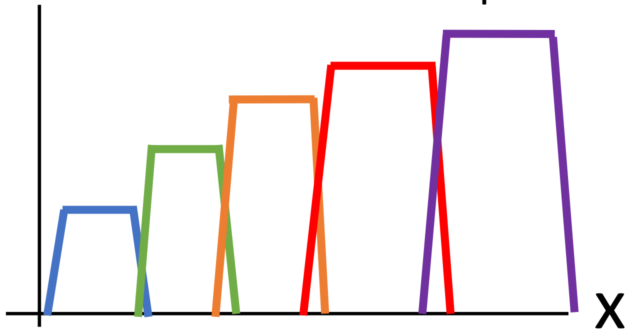
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

We can build a "bump function" using four hidden units



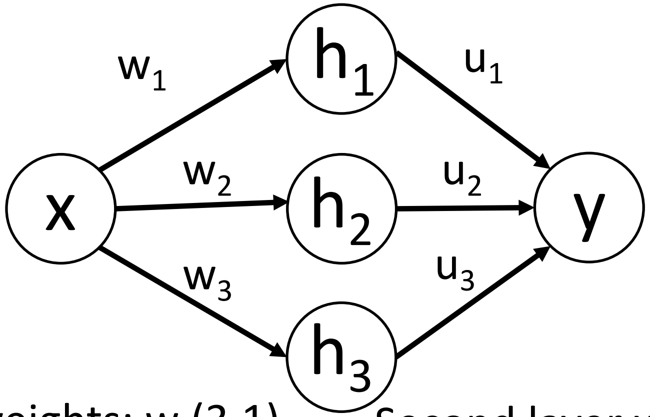
With 4K hidden units we can build a sum of K bumps



Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

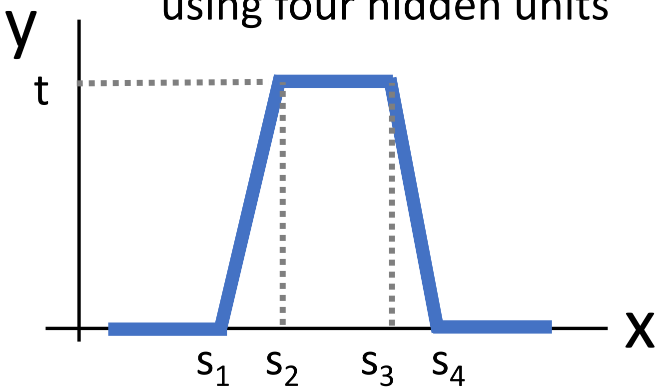
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

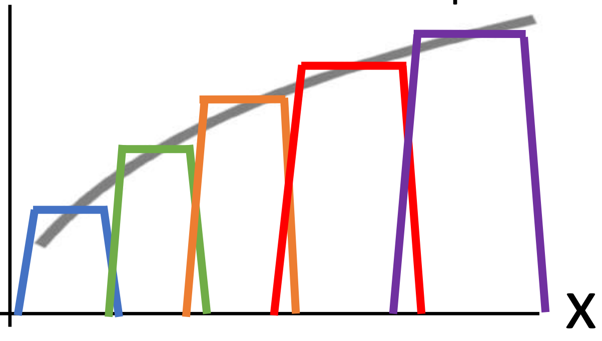
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

We can build a "bump function" using four hidden units



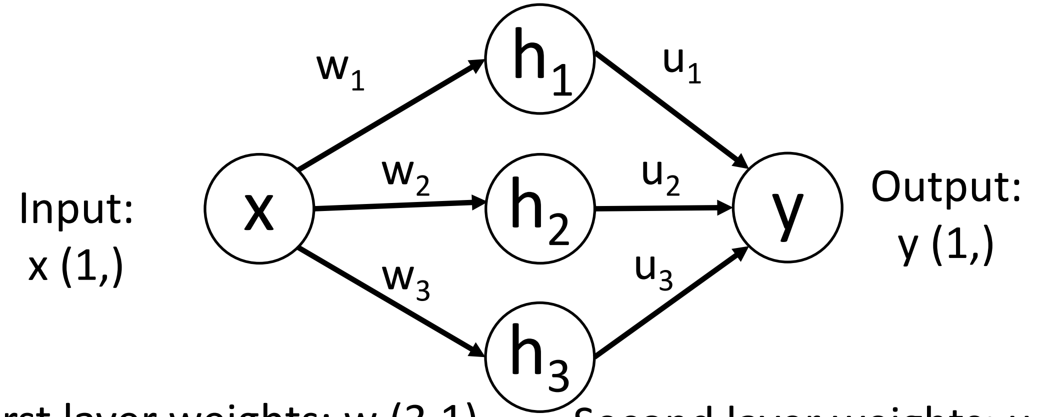
With 4K hidden units we can build a sum of K bumps



Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

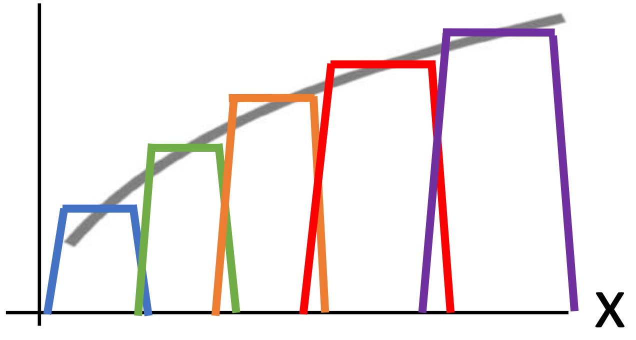
What about…

- Gaps between bumps?
- Other nonlinearities?
- Higher-dimensional functions?

See [Nielsen, Chapter 4]

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

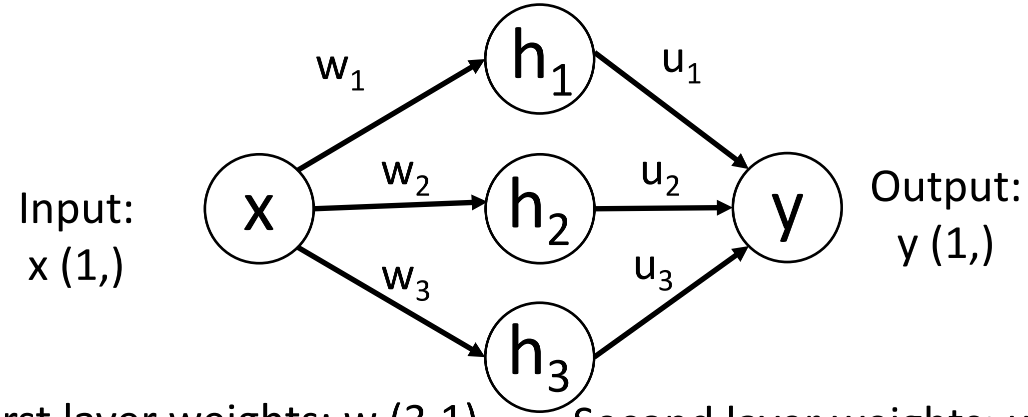
y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p



Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



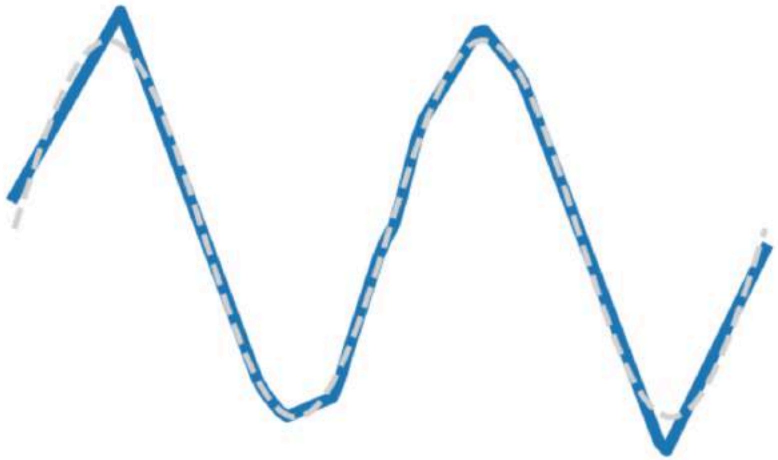
First layer weights: w (3,1)
First layer bias: b (3,)

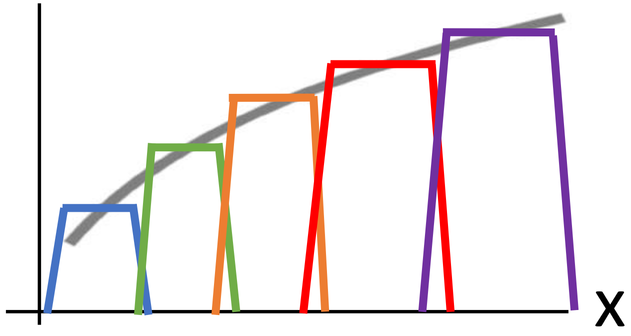
Second layer weights: u (1,3)
First layer bias: p (1,)

$$
\begin{aligned}
h1 &= \max(0, w1 * x + b1) \\
h2 &= \max(0, w2 * x + b2) \\
h3 &= \max(0, w3 * x + b3) \\
y &= u1 * h1 + u2 * h2 + u3 * h3 + p
\end{aligned}
$$

$$
\begin{aligned}
y = u1 &* \max(0, w1 * x + b1) \\
+ u2 &* \max(0, w2 * x + b2) \\
+ u3 &* \max(0, w3 * x + b3) \\
+ p&
\end{aligned}
$$

Reality check: Networks don't really learn bumps!

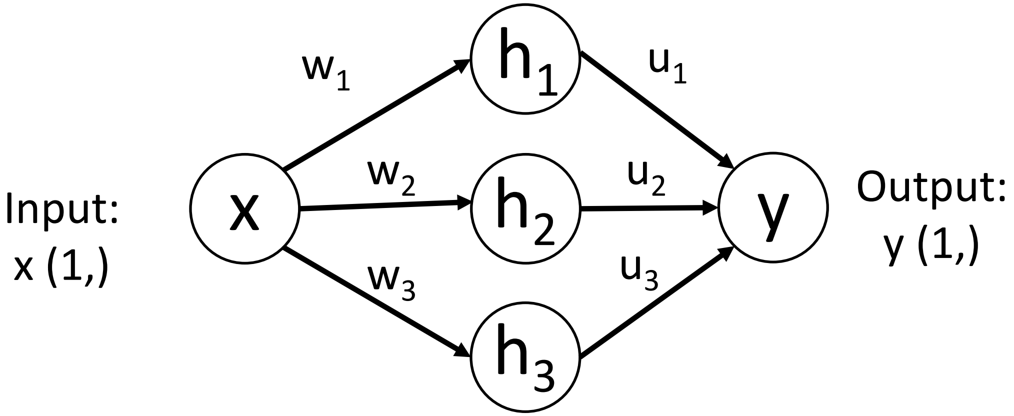




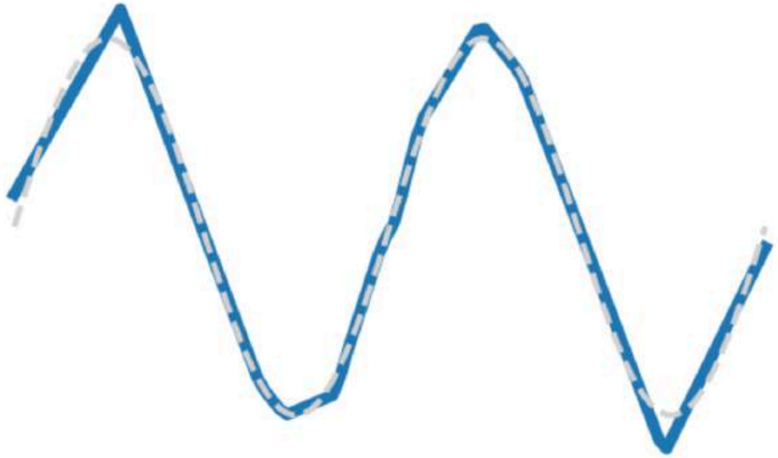
Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



Reality check: Networks don't really learn bumps!



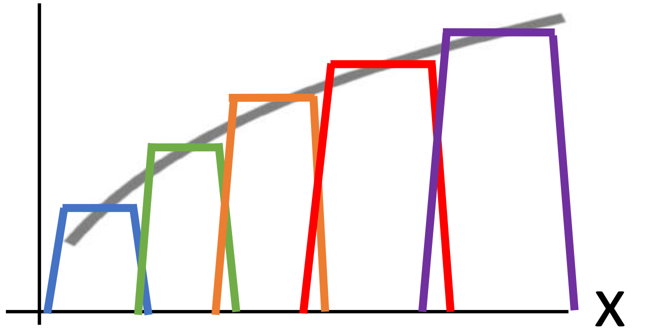
Universal approximation tells us:

- Neural nets can represent any function

Universal approximation DOES NOT tell us:

- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!



Approximate functions with bumps!

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Convex Functions

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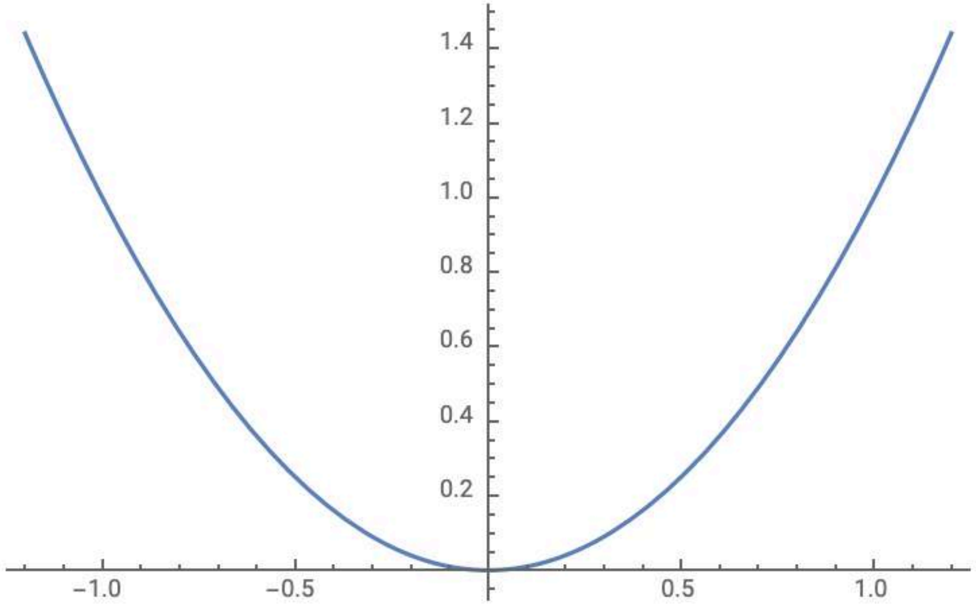
Example: $f(x) = x^2$ is convex:



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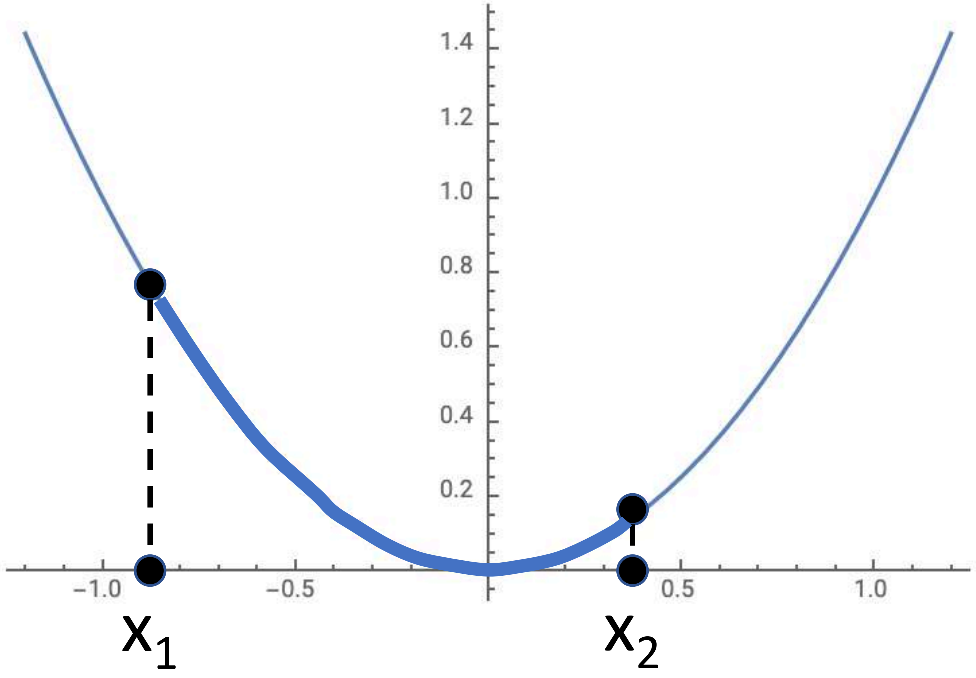

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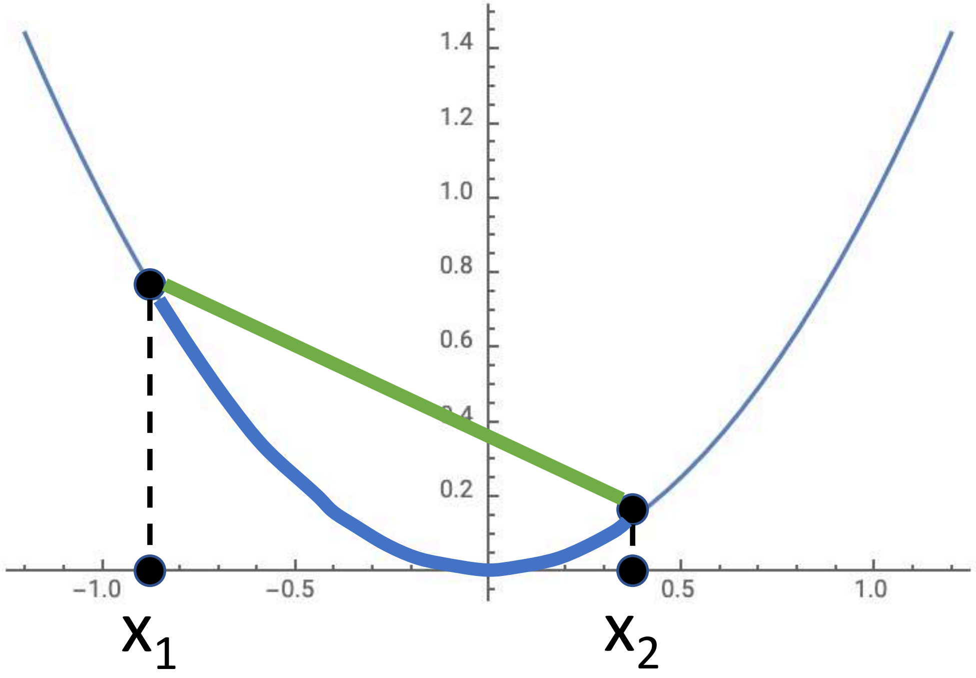

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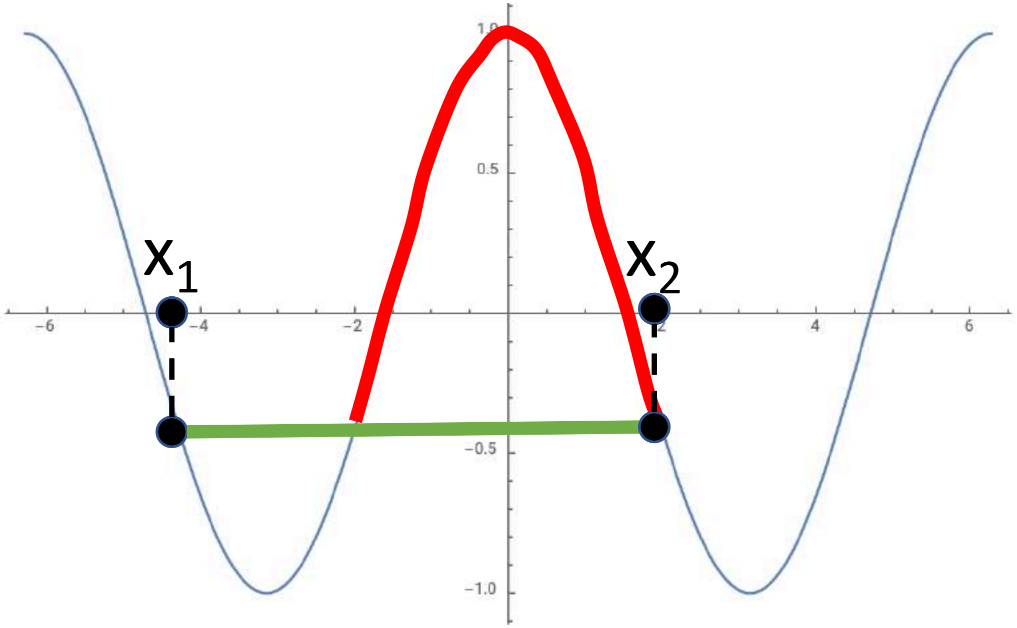
Example: $f(x) = \cos(x)$
is <u>not</u> convex:



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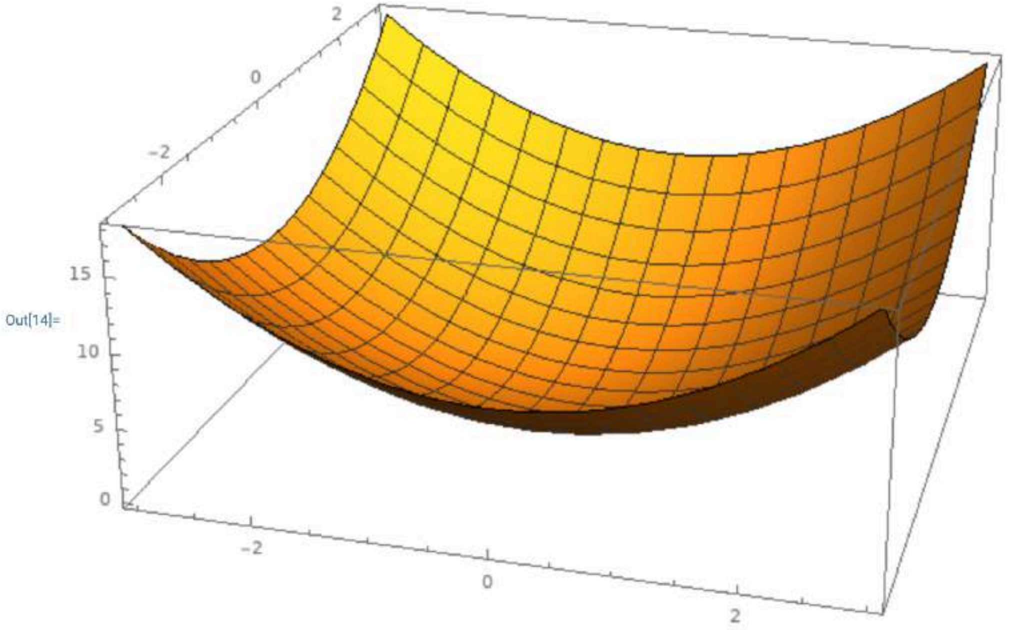
Intuition: A convex function is a (multidimensional) bowl



*Many technical details! See e.g. IOE 661 / MATH 663

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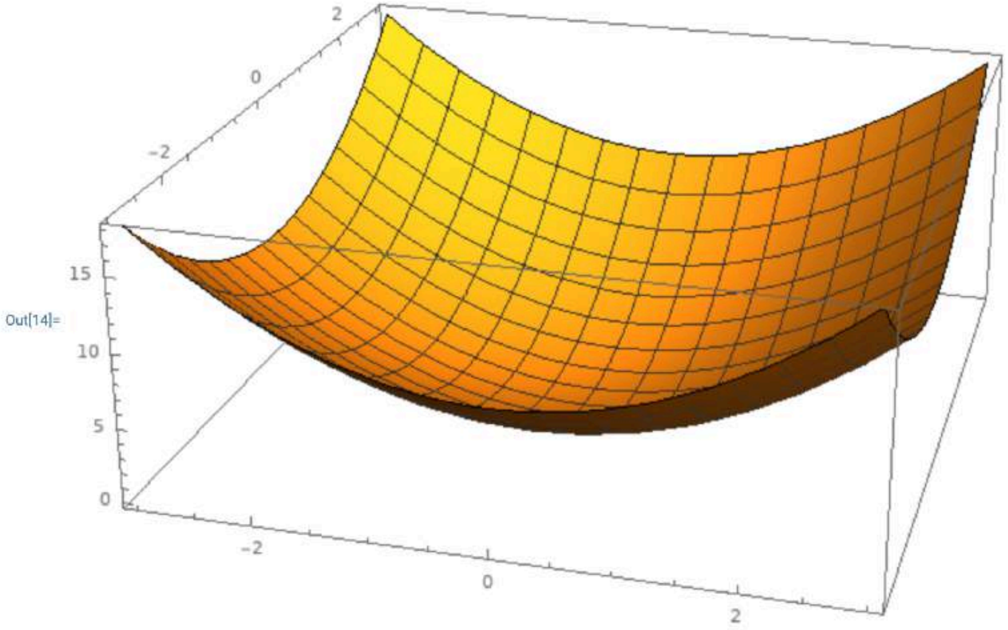
Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***



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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \text{ Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \text{ SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W)$$

R(W) = L2 or L1 regularization

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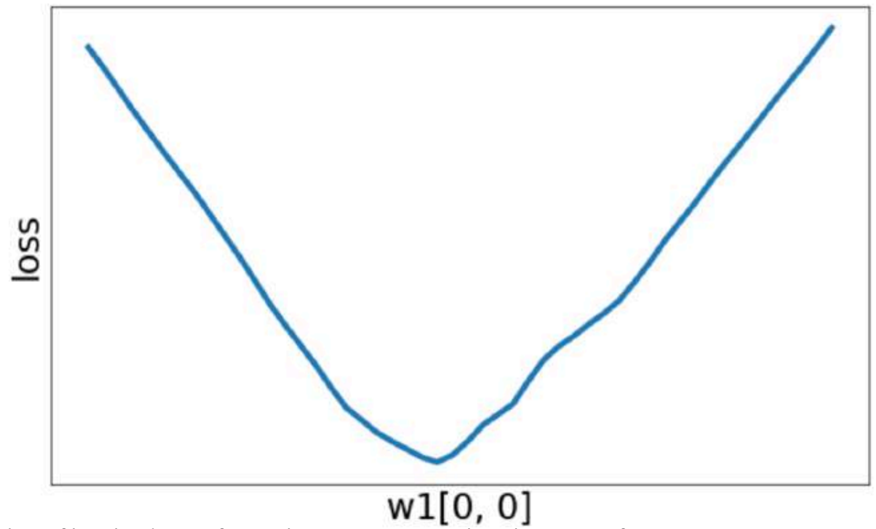
Neural net losses sometimes look convex-ish:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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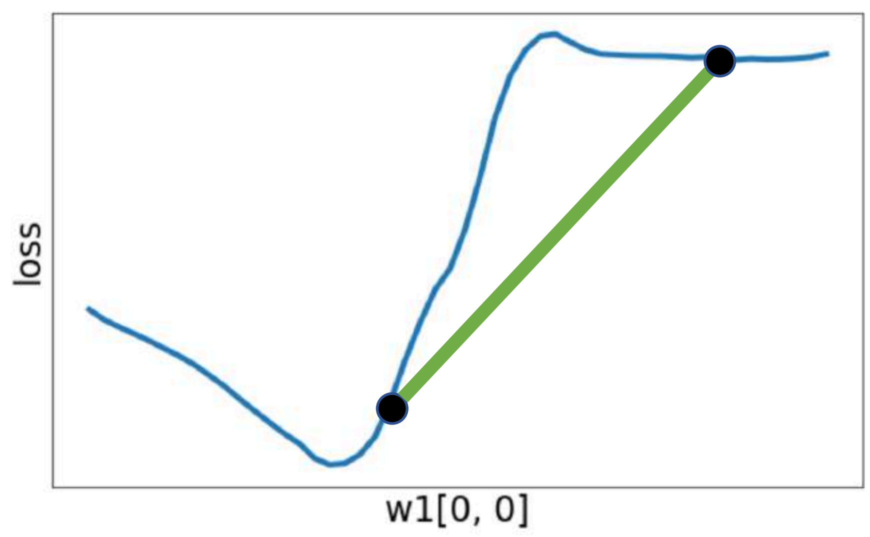
But often clearly nonconvex:



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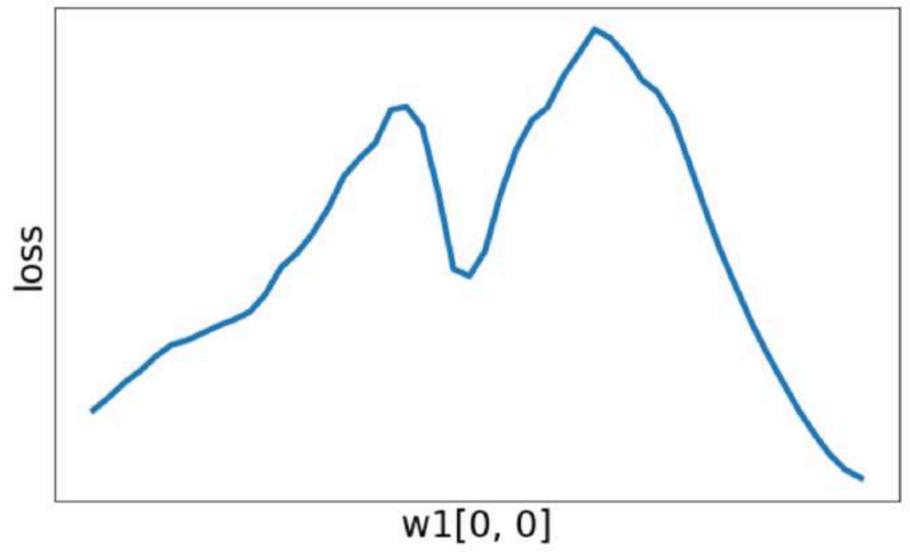
With local minima:



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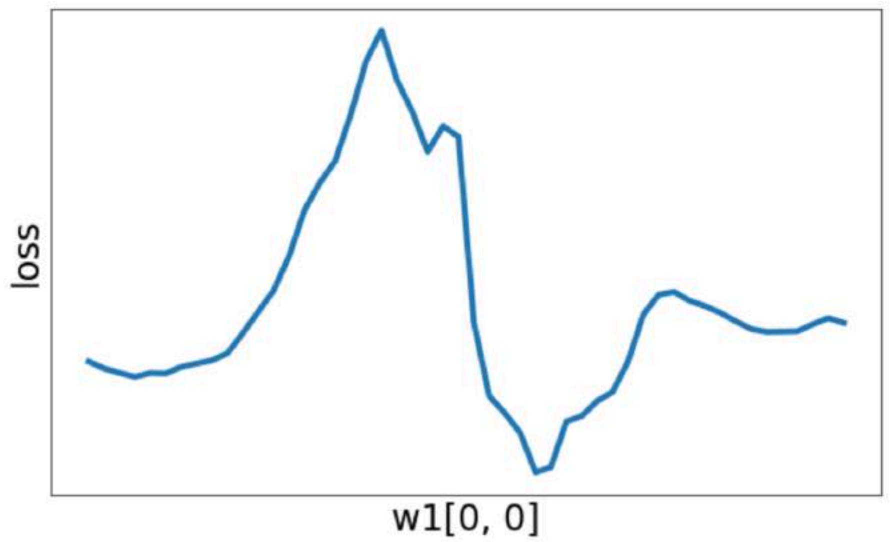
Can get very wild!



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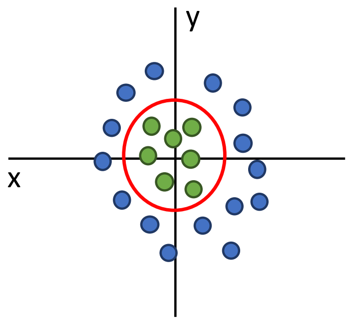
Most neural networks need **nonconvex optimization**
- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

*Many technical details! See e.g. IOE 661 / MATH 663

Summary

Feature transform + Linear classifier allows nonlinear decision boundaries

Original space

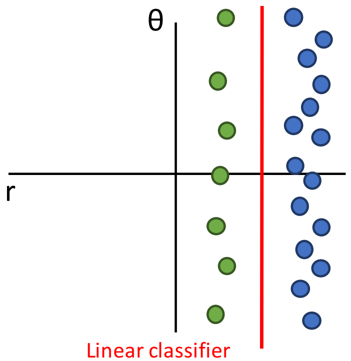


$r = (x^2 + y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$

Feature transform

Feature space

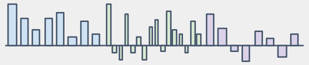


Nonlinear classifier in original space!

Linear classifier in feature space

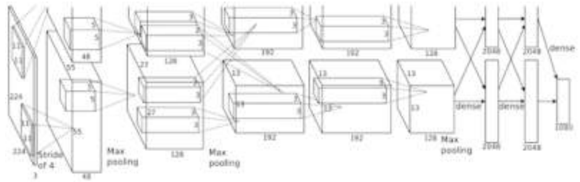
Neural Networks as learnable feature transforms

Feature Extraction



10 numbers giving scores for classes

training



Krizhevsky, Sutskever, and Hinton, "Imagenet classification with deep convolutional neural networks", NIPS 2012. Figure copyright Krizhevsky, Sutskever, and Hinton, 2012. Reproduced with permission.

10 numbers giving scores for classes

training

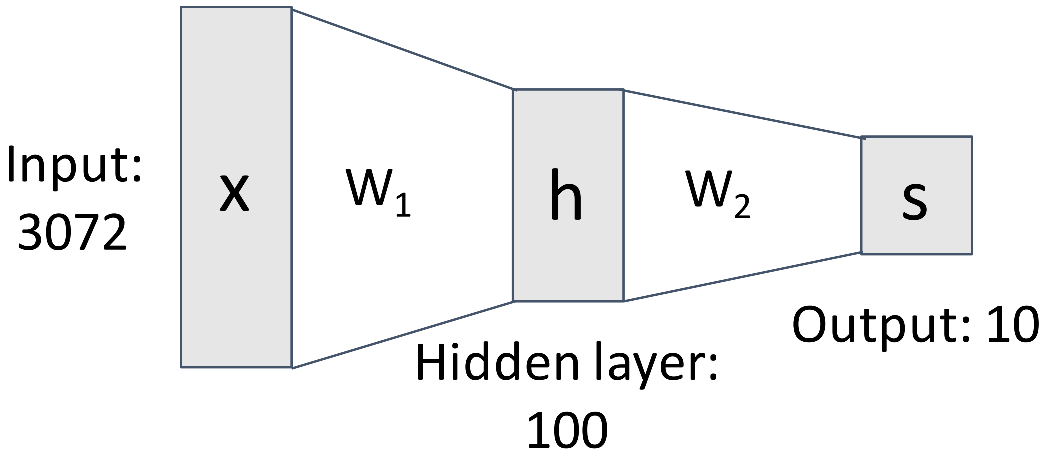
Summary

Linear classifier: One template per class

Neural networks: Many reusable templates

From linear classifiers to
fully-connected networks

$$f = W_2 \max(0, W_1 x)$$



Summary

From linear classifiers to
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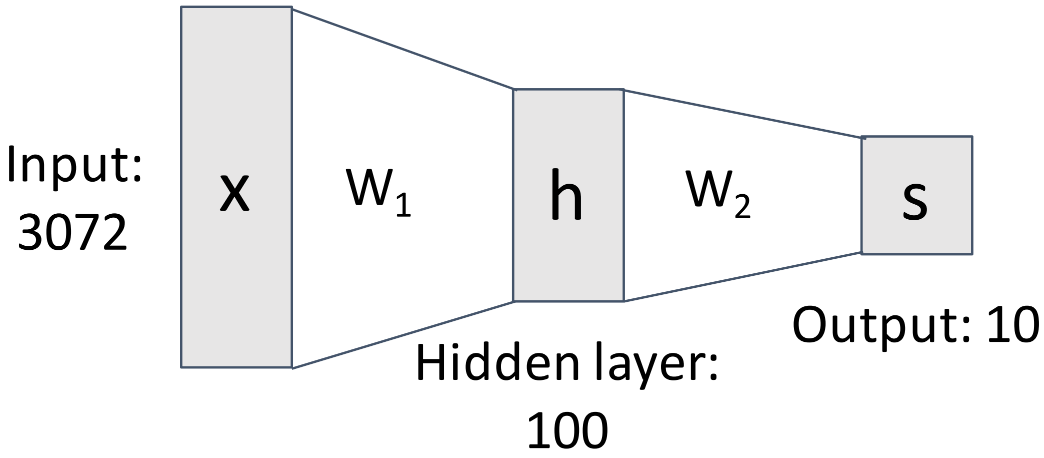
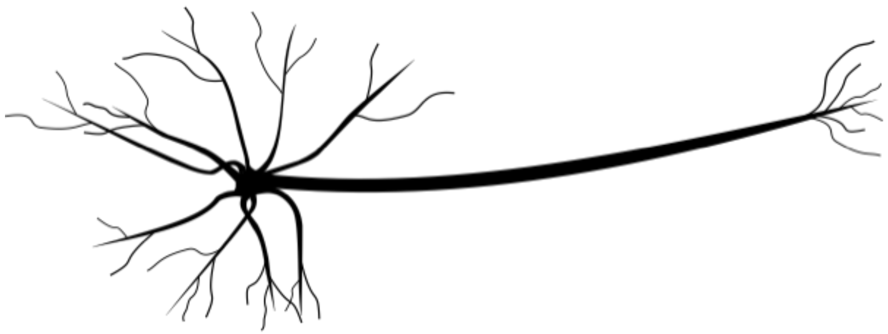


Neural networks loosely inspired by biological neurons but be careful with analogies



Summary

From linear classifiers to
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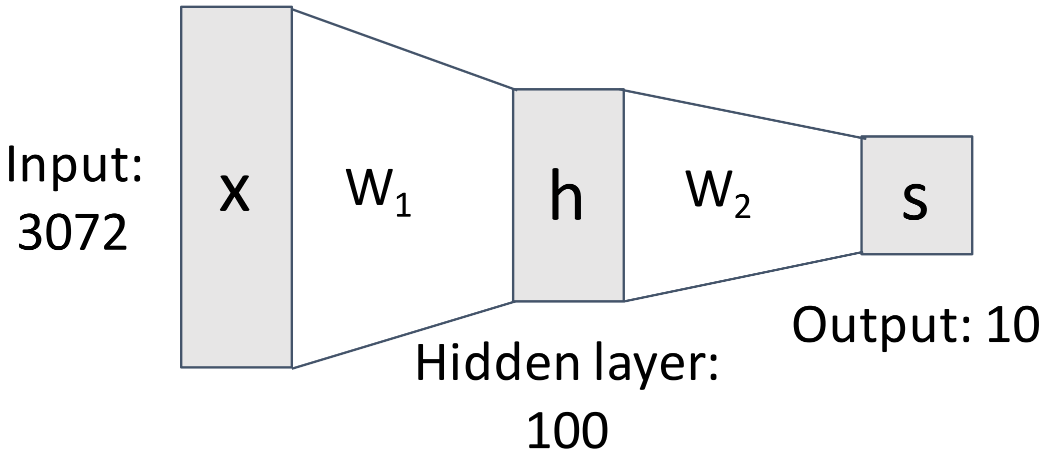
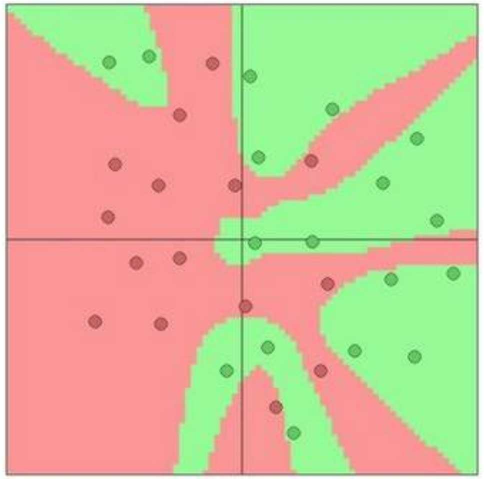
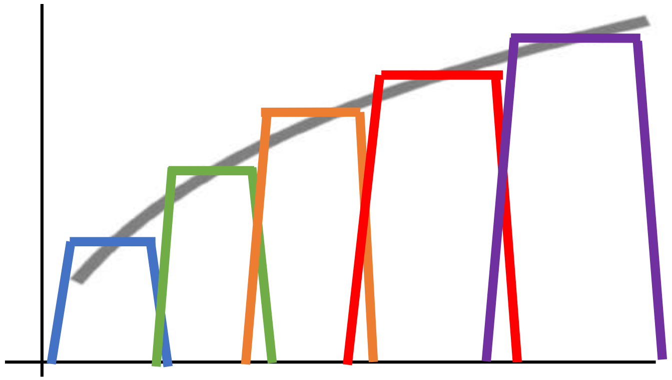


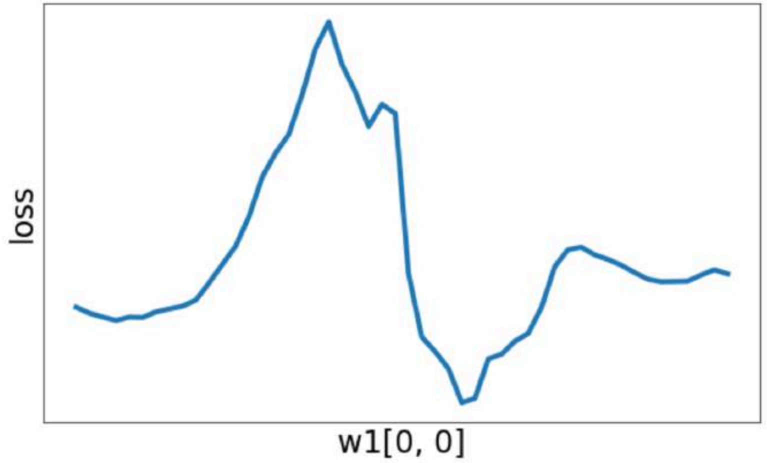
Space Warping



Universal Approximation



Nonconvex



Problem: How to compute gradients?

$$s = f(x; W_1, W_2) = W_2 \max(0, W_1 x)$$ Nonlinear score function

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM Loss on predictions

$$R(W) = \sum_k W_k^2$$ Regularization

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$$ Total loss: data loss + regularization

If we can compute $\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}$n we can learn W_1 and W_2

Next time:
Backpropagation