Online Optimization, Learning, and Games (O2LG)
Lesson 2: Basic elements of Game Theory

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Recall: Definition of finite game

Definition 1 (Finite games)

A ***finite game in normal (or strategic) form*** is a tuple $(\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$, where:

- $\mathcal{N} = \{1, 2, \ldots, N\}$ is a finite set of ***players***.
- For each player $i \in \mathcal{N}$, $\mathcal{A}_i$ is a finite set of ***actions*** (or ***pure strategies***) for player i.
- For each player $i \in \mathcal{N}$, $u_i : \prod_{j\in\mathcal{N}} \mathcal{A}_j \to \mathbb{R}$ is a ***payoff function*** mapping each combination (or *profile*) of actions $(a_1, ..., a_N)$ to a real number $u_i(a_1, ..., a_N)$ that is the payoff to player i when players $1, 2, \ldots, N$ do actions $a_1, a_2, \ldots, a_N$, respectively.

Notation: finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$.

Mixed strategies

Instead of playing **pure** strategies, players could **mix** their actions:

- **Mixed strategy** of player $i \in \mathcal{N}$: probability distribution x_i on $\mathcal{A}_i$.
- **Notation**: x_{i,a_i} is the probability that the player i selects the strategy $a_i \in \mathcal{A}_i$.
- **Strategy space** of player i:

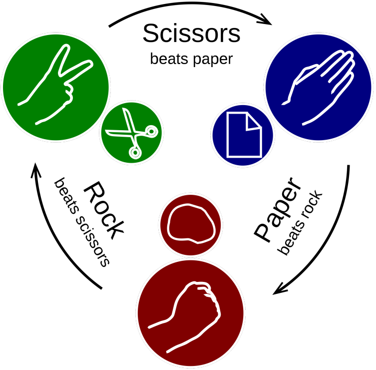
$$\mathcal{X}_i := \Delta(\mathcal{A}_i) = \left\{ x_i \in \mathbb{R}^{\mathcal{A}_i} : x_{i,a_i} \geq 0 \text{ and } \sum_{a_i \in \mathcal{A}_i} x_{i,a_i} = 1 \right\}$$

- **Support** of x_i: the set of actions played with *positive* probability under x_i.

$$\text{supp}(x_i) := \{a_i \in \mathcal{A}_i : x_{i,a_i} > 0\}$$

- x_i is **pure** when $\text{supp}(x_i)$ is a singleton.

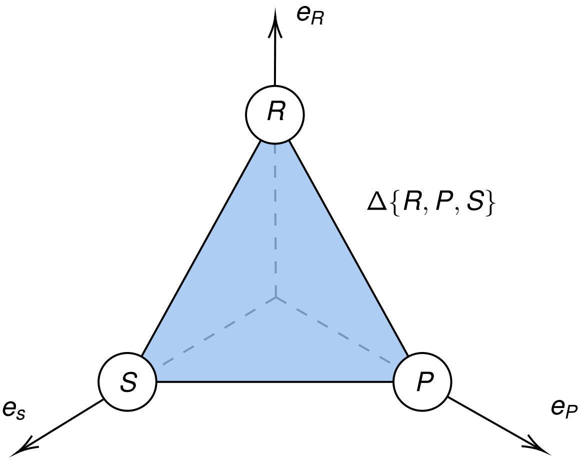
Return to the Rock-Paper-Scissors game!



Playing the Rock-Paper-Scissors game with mixed strategies

- **Players**: $\mathcal{N} = \{1, 2\}$
- **Actions**: $\mathcal{A}_i = \{R, P, S\}, i \in \mathcal{N}$
- **Mixed strategy space**: $\mathcal{X}_i = \Delta\{R, P, S\}$
  - Choose a mixed strategy $x_i \in \mathcal{X}_i$.

    For example, $x_i = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $x_{i,R} = x_{i,P} = x_{i,S} = \frac{1}{3}$.
  - Choose action $a_i \sim x_i$.



Mixed strategies: collective

When all players mix their actions:

- Each player $i \in \mathcal{N}$ uses a mixed strategy $x_i \in \mathcal{X}_i$.
- Probability of selecting the action profile $a = (a_1, \dots, a_N) \in \mathcal{A}$:
$$x_{a_1,\dots,a_N} = \prod_{j\in\mathcal{N}} x_{j,a_j}$$
- Probability of selecting the action $a_{-i} \in \mathcal{A}_{-i}$:
$$x_{-i,a_{-i}} = \prod_{j\in\mathcal{N},j\neq i} x_{j,a_j}$$
- **Mixed strategy profile:**
$$x = (x_1, \dots, x_N) \in \mathcal{X} := \prod_{i\in\mathcal{N}} \mathcal{X}_i$$
- **Mixed strategy profile of *i*'s opponents:**
$$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathcal{X}_{-i} := \prod_{j\in\mathcal{N},j\neq i} \mathcal{X}_j$$

Expected payoffs under mixed strategies

- **Expected payoff to a player** under a mixed strategy profile:

$$u_i(x) = \sum_{a \in \mathcal{A}} x_{a_1,\ldots,a_N} u_i(a_1,\ldots,a_N)$$

  or, in terms of other players' strategies:

$$u_i(x_i, x_{-i}) = \sum_{a_i \in \mathcal{A}_i} \sum_{a_{-i} \in \mathcal{A}_{-i}} x_{i,a_i} x_{-i,a_{-i}} u_i(a_i, a_{-i})$$

- **Expected payoff to a pure strategy** under a mixed strategy profile:

$$v_{i,a_i}(x) := u_i(a_i, x_{-i}) = \sum_{a_{-i} \in \mathcal{A}_{-i}} x_{-i,a_{-i}} u_i(a_i, a_{-i})$$

- **Mixed payoff vector of a player**:

$$v_i(x) = (v_{i,a_i}(x))_{a_i \in \mathcal{A}_i} = (u_i(a_i, x_{-i}))_{a_i \in \mathcal{A}_i}$$

Note that *(i)* $u_i(x) = \langle v_i(x), x_i \rangle$; *(ii)* u_i is linear in x_i; *(iii)* v_{i,a_i} and v_i are **independent** of x_i.

Playing the Rock-Paper-Scissors game with mixed strategies

- **Players**: $\mathcal{N} = \{1, 2\}$
- **Actions**: $\mathcal{A}_i = \{R, P, S\}, i \in \mathcal{N}$
- **Mixed strategies**: $x_i \in \mathcal{X}_i = \Delta\{R, P, S\}$

Mixed strategy payoffs: $u_1(x_1, x_2)$, $u_2(x_1, x_2)$?

$$u_1(x_1, x_2) = x_1^\top A x_2, \text{ where } A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

$$u_2(x_1, x_2) = -u_1(x_1, x_2).$$

Mixed extensions

Definition 2 (Mixed extensions of a finite game)

A ***mixed extension*** of a finite game $\Gamma = \Gamma(\mathcal{N}, \mathcal{A}, u)$ is a ***continuous*** game $\Delta(\Gamma)$ with

- Players $i \in \mathcal{N} = \{1, \ldots, N\}$
- Actions $x_i \in \mathcal{X}_i = \Delta(\mathcal{A}_i)$ per player $i \in \mathcal{N}$
- Payoff functions $u_i : \mathcal{X} \to \mathbb{R}$, $i \in \mathcal{N}$

Remark that

- ***Continuous game***: game with *continuous* action spaces (here $\mathcal{X}_i$ instead of $\mathcal{A}_i$).
- Without confusing Γ and $\Delta(\Gamma)$ are indistinguishable.

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Mixed best responses

Extending the notion of best-responding to mixed strategies

Definition 3 (Mixed best responses)

- A mixed strategy $x_i^* \in \mathcal{X}_i$ is a ***mixed best response*** of player i to a mixed strategy profile $x_{-i} \in \mathcal{X}_{-i}$ for the other players if
$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}) \text{ for all } x_i \in \mathcal{X}_i,$$
equivalently,
$$x_i^* \in \text{argmax}_{x_i \in \mathcal{X}_i} u_i(x_i, x_{-i}) = \text{argmax}_{x_i \in \mathcal{X}_i} \langle v_i(x), x_i \rangle.$$
- A ***mixed best-response correspondence*** of player i is a set-valued function $\text{BR}_i : \mathcal{X}_{-i} \to \mathcal{X}_i$ defined by
$$\text{BR}_i(x_{-i}) := \text{argmax}_{x_i \in \mathcal{X}_i} u_i(x_i, x_{-i})$$
- A ***collective best-response correspondence*** $\text{BR} : \mathcal{X} \to \mathcal{X}$ is defined by:
$$\text{BR}(x) := (\text{BR}_i(x_{-i}))_{i \in \mathcal{N}}.$$

Playing the Rock-Paper-Scissors game with mixed strategies

- **Players**: $\mathcal{N} = \{1, 2\}$
- **Actions**: $\mathcal{A}_i = \{R, P, S\}, i \in \mathcal{N}$
- **Mixed strategies**: $x_i^* \in \mathcal{X}_i = \Delta\{R, P, S\}$

Mixed strategy payoffs at $x^* = (x_1^*, x_2^*)$ with $x_1^* = x_2^* = (1/3, 1/3, 1/3)^\top$:

$$u_1(x_1^*, x_2^*) = u_2(x_1^*, x_2^*) = 0.$$

More generally, $u_1(x_1, x_2^*) = 0 = u_2(x_1^*, x_2)$ for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$.

In other words, $u_1(x_1, x_2^*) \leq u_1(x_1^*, x_2^*), \forall x_1 \in \mathcal{X}_1$, i.e. $x_1^* \in \mathrm{BR}_1(x_2^*) = \mathrm{BR}_1(x_{-1}^*)$. Similarly, $x_2^* \in \mathrm{BR}_2(x_{-2}^*)$. More precisely, $\mathrm{BR}_1(x_{-1}^*) = \mathcal{X}_1$, $\mathrm{BR}_2(x_{-2}^*) = \mathcal{X}_2$, and $\mathrm{BR}(x^*) = \mathcal{X}_1 \times \mathcal{X}_2 = \mathcal{X}$.

Nash equilibrium in mixed strategies

Definition 4 (Nash equilibrium)

A mixed strategy profile $x^* = (x_1^*, \ldots, x_N^*)$ is a ***Nash equilibrium*** if $x^* \in \mathrm{BR}(x^*)$, i.e.

$$x_i^* \in \mathrm{BR}_i(x_{-i}^*) \text{ for all } i \in \mathcal{N}$$

or, equivalently, if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*) \text{ for all } i \in \mathcal{N} \text{ and for all } x_i \in \mathcal{X}_i.$$

Nash's theorem

The Rock-Paper-Scissors game admits a Nash equilibrium in mixed strategies.
Is this always the case?

Theorem 1 (Nash 1950)

*Every finite game admits a Nash equilibrium in **mixed strategies**.*

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Potential games

Return to **pure** strategies:

- In **single-player** games: Nash equilibria trivially exist
- In **multi-player** games: Not true.

What is the bridge between single-player and multi-player settings?

Definition 5 (Potential games, Monderer et al. 1996)

A finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$ is a ***potential game*** if there exists a function $\Phi : \mathcal{A} \to \mathbb{R}$ such that

$$u_i(a_i', a_{-i}) - u_i(a_i'', a_{-i}) = \Phi(a_i', a_{-i}) - \Phi(a_i'', a_{-i})$$

for all $i \in \mathcal{N}$, $a_{-i} \in \mathcal{A}_{-i}$, and $a_i', a_i'' \in \mathcal{A}_i$.

Here the function Φ is called the **potential function**.

Potential games: A simple example

Consider a game Γ with payoff matrix:

Player 1

Player 2	A	B
A	$(5, 2)$	$(-5, -4)$
B	$(-1, -2)$	$(1, 4)$

Show that Γ is a potential game. What is its potential function Φ?

Player 1

Player 2 / Φ	A	B
A	4	-6
B	0	2

Potential games: Existence of equilibria

- Any ***global maximizer*** $a^* \in \operatorname{argmax} \Phi$ of Φ is a (pure) Nash equilibrium.
- Any ***unilateral maximizer*** $a^* \in \mathcal{A}$ of Φ, i.e.

$$\Phi(a^*) \geq \Phi(a_i, a^*_{-i}) \text{ for all } i \in \mathcal{N} \text{ and } a_i \in \mathcal{A}_i,$$

  is a (pure) Nash equilibrium.

Question: When does a game become potential?

Proposition 1 (Theorem 4.5, Monderer et al. 1996)

Let Γ be a game in which the strategy sets are intervals of real numbers. Suppose the payoff functions are twice continuously differentiable. Γ is a potential game if and only if

$$\nabla_{x_j} v_i(x) = \nabla_{x_i} v_j(x) \text{ for all } x \in \mathcal{X} \text{ and } i, j \in \mathcal{N},$$

where $v_i(x) = (u_i(a_i, x_{-i}))_{a_i \in \mathcal{A}_i}$ is the mixed payoff vector of player $i \in \mathcal{N}$.

Best-response dynamics

A natural updating process:

- Players may choose a new action at each stage $n = 1, 2, \ldots$
- Players select the best-responses that maximize their payoffs given the strategies of the other players.

Definition 6 (Best-response dynamics)

The ***best-response dynamics*** are defined by the recursion

$$a_{i_n,n+1} \in \begin{cases} \mathrm{BR}_{i_n}(a_{-i_n,n}) & \text{if } a_{i_n,n} \notin \mathrm{BR}_{i_n}(a_{-i_n,n}) \\ \{a_{i_n,n}\} & \text{otherwise} \end{cases}$$

where i_n is any player updating at stage n and $a_{i,n}$ is the action of player i at stage n.

Convergence of Best-response dynamics (BRD)

Does (BRD) converge?

- No - and different modes of updating do not help.

Fortunately, one has good convergence properties in potential games:

Proposition 2 (Monderer et al. 1996)

Let Γ be a finite potential game. Then the iterative version of (BRD) converges to a pure Nash equilibrium after finitely many steps.

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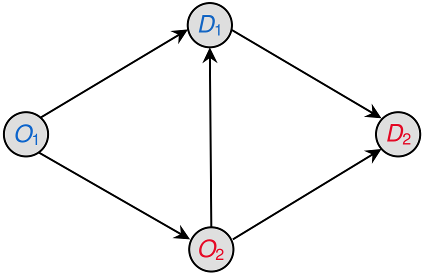
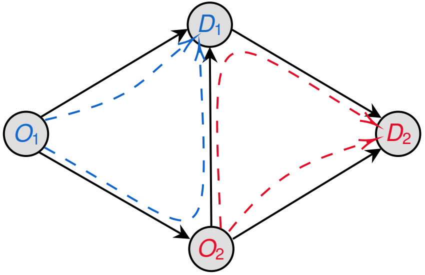
Congestion games

- **Network**: graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- $(O_i, D_i)_{i \in \mathcal{N}}$: player i travels from the origin O_i to destination D_i and induces 1 unit of traffic.
- **Paths** $\mathcal{A}_i$: (sub)set of paths joining $O_i \rightsquigarrow D_i$.
- **Path choice**: player i chooses path $a_i \in \mathcal{A}_i$.



Congestion games

- **Network**: graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- $(O_i, D_i)_{i\in\mathcal{N}}$: player i travels from the origin O_i to destination D_i and induces 1 unit of traffic.
- **Paths** $\mathcal{A}_i$: (sub)set of paths joining $O_i \rightsquigarrow D_i$.
- **Path choice**: player i chooses path $a_i \in \mathcal{A}_i$.
- **Load** $\ell_e(a) = \sum_{i\in\mathcal{N}} \mathbb{1}(e \in a_i)$: total traffic load along edge e in a strategy profile $a \in \mathcal{A}$.
- **Edge cost function** $c_e(\ell_e(a))$: cost along edge when edge load is $\ell_e(a)$ with $a \in \mathcal{A}$.
- **Player cost**: $c_i(a) = \sum_{e\in a_i} c_e(\ell_e(a))$ for $a \in \mathcal{A}$.



Congestion game: $\Gamma = (\mathcal{G}, \mathcal{N}, \mathcal{A}, c)$

- Atomic: all players have non-negligible traffic.
- Non-splittable: not split the traffic among various paths.

Rosenthal's Theorem

Theorem 2 (Rosenthal 1973)

Any congestion game admits the potential function:

$$\Phi(a) = \sum_{e \in \mathcal{E}} \sum_{k=1}^{\ell_e(a)} c_e(k) \quad \textit{for all } a \in \mathcal{A}.$$

Price of anarchy

Define $C(a) := \sum_{i \in \mathcal{N}} c_i(a)$ as the congestion game's ***social cost*** function.

Definition 7 (Social optimum)

The ***social optimum*** of a congestion game $\Gamma = (\mathcal{G}, \mathcal{N}, \mathcal{A}, c)$ is the value

$$\mathrm{Opt} = \min_{a \in \mathcal{A}} C(a).$$

Definition 8 (Price of anarchy, Koutsoupias et al. 1999)

The ***Price of Anarchy*** (PA) of a congestion game Γ is defined as

$$\mathrm{PA} = \max_{a^* \in \mathrm{Eq}(\Gamma)} \frac{C(a^*)}{\mathrm{Opt}}.$$

Here $\mathrm{Eq}(\Gamma)$ is the set of pure Nash equilibria for the game Γ.

Bound of PA with linear costs

Consider the games with the linear cost function, i.e. $c_e(\ell) = A_e\ell + b_e, \forall e$.

Theorem 3 (Christodoulou et al. 2005)

For any congestion game with linear cost functions, PA is at most $\frac{5}{2}$.

Summary

After two lessons of Game Theory

- Finite games: definitions & examples
- Strategies: pure & mixed
- Strategic dominance: strict, weak, iterated
- Best responses: pure & mixed
- Nash equilibrium: pure, mixed, existence
- Potential game
- Best-response dynamics
- Congestion game
- Price of anarchy

Next lesson

- Game dynamics
- Exponential weights and the replicator dynamics
- Rationality analysis
- ...

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