Online Optimization, Learning, and Games (O2LG)

Lesson 9: Learning with Stochastic Gradients

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Recall: Follow The Regularized Leader (FTRL)

$$x_{n+1} = \arg\min_{x \in \mathcal{X}} \left\{ \sum_{k=1}^{n} \ell_k(x) + \underbrace{\lambda h(x)}_{\text{“}\ell_0(x)\text{”}} \right\} \tag{FTRL}$$

where

- regularization function $h \colon \mathcal{X} \to \mathbb{R}$ is strongly convex. # $h - (K_h/2)\|\cdot\|^2$ convex for some $K_h > 0$
- regularization weight $\lambda > 0$ can be tuned.

Theorem 1 (Shalev-shwartz et al. 2006, Shalev-Shwartz 2012)

Assume: h is K_h-strongly convex; each ℓ_n is G-Lipschitz continuous; $\lambda = G\sqrt{T/(2K_h R_h)}$ with $R_h = \max h - \min h$.

$$\mathrm{Reg}(T) \leq G\sqrt{(2R_h/K_h)\,T} = \mathcal{O}(\sqrt{T}).$$

Linearization and gradient descent

Follow the linearized leader

$$x_{n+1} = \arg\min_{x \in \mathcal{X}} \left\{ \eta \sum_{k=1}^{n} \langle g_k, x \rangle + h(x) \right\}. \tag{FTLL}$$

Dual averaging

$$y_{n+1} = y_n - \eta g_n, \quad x_{n+1} = Q(y_{n+1}), \tag{DA}$$

where $Q(y) = \arg\max_{x \in \mathcal{X}} \{\langle y, x \rangle - h(x)\}$ is the mirror map associated to h.

Online Gradient Descent (OGD)

Special case when $h(x) = (1/2)\|x\|_2^2$: # lazy version

$$y_{n+1} = y - \eta g_n, \quad x_{n+1} = \Pi_{\mathcal{X}}(y_{n+1}). \tag{OGD}$$

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Recall: Oracle feedback

The oracle model

A stochastic first-order oracle (SFO) model of g_n is a random vector $\hat{g}_n$ of the form

$$\hat{g}_n = g_n + U_n + b_n, \tag{SFO}$$

where U_n is zero-mean and $b_n = \mathbb{E}[\hat{g}_n \mid \mathcal{F}_n] - g_n$ is the bias of $\hat{g}_n$.

Assumptions 1

- Bias: $\|b_n\| \leq B_n$.
- Variance: $\mathbb{E}[\|U_n\|^2 \mid \mathcal{F}_n] \leq \sigma_n^2$.
- Second moment: $\mathbb{E}[\|\hat{g}_n\|^2 \mid \mathcal{F}_n] \leq M_n^2$.

Stochastic Gradient Descent (SGD)

Algorithm 1 Stochastic Gradient Descent (SGD) # OGD with stochastic feedback

Require: convex action set $\mathcal{X} \subseteq \mathcal{V}$; convex loss functions $\ell_n \colon \mathcal{X} \to \mathbb{R}$, $n = 1, 2, \ldots$

Initialize: $y_1 \in \mathbb{R}^{\mathcal{A}}$.

for all $n = 1, 2, \ldots$ **do**

play $x_n \leftarrow \Pi(y_n)$. ▷ action selection

incur $c_n = \ell_n(x_n)$. ▷ incur cost

observe an estimate $\hat{g}_n$ of $g_n \in \partial \ell_n(x_n)$. ▷ SFO feedback

set $y_{n+1} \leftarrow y_n - \eta_n \hat{g}_n$. ▷ update state

end for

Regret under SGD

- Gradient trick: # linear model

$$\ell_n(x_n) - \ell_n(p) \leq \langle g_n, x_n - p \rangle \quad \text{for all } p \in \mathcal{X}.$$

- Energy function: # as before

$$F_n = \tfrac{1}{2}\|y_n - p\|^2 - \tfrac{1}{2}\|y_n - x_n\|^2.$$

- Energy inequality: # $\hat{g}_n$ instead of g_n

$$F_{n+1} \leq F_n - \eta\langle \hat{g}_n, x_n - p \rangle + \frac{\eta^2}{2}\|\hat{g}_n\|^2.$$

- Expand and rearrange: # How?

$$\langle g_n, x_n - p \rangle \leq \frac{F_n - F_{n+1}}{\eta} - \langle U_n, x_n - p \rangle - \langle b_n, x_n - p \rangle + \frac{\eta}{2}\|\hat{g}_n\|^2.$$

- How to bound the regret?

Regret of SGD

Theorem 2

☞ *Assume: feedback of the form* (SFO) *and* $\eta = \mathrm{diam}(\mathcal{X}) \Big/ \sqrt{\sum_{n=1}^{T} M_n^2}$.

✓ *Then: for all* $p \in \mathcal{X}$, *the SGD algorithm enjoys the bound*

$$\mathbb{E}[\mathrm{Reg}_p(T)] \leq \mathrm{diam}(\mathcal{X}) \sum_{n=1}^{T} B_n + \mathrm{diam}(\mathcal{X}) \sqrt{\sum_{n=1}^{T} M_n^2}$$

where the diameter of a set, $\mathrm{diam}(\mathcal{X})$, *is the supremum of distances between its points.*

Remarks:

- $\mathcal{O}(\sqrt{T})$ regret if feedback is unbiased ($b_n = 0$) and has finite variance ($M_n \leq M$).
- This bound is tight in T.

➻ Abernethy et al. 2008

References

[1] Jacob Abernethy et al. Optimal Stragies and Minimax Lower Bounds for Online Convex Games. In: Jan. 2008, pp. 415–424 (cited at slide 0).

[2] Shai Shalev-Shwartz. Online Learning and Online Convex Optimization. In: *Foundations and Trends® in Machine Learning* 4.2 (2012), pp. 107–194 (cited at slide -6).

[3] Shai Shalev-shwartz and Yoram Singer. Convex Repeated Games and Fenchel Duality. In: *Advances in Neural Information Processing Systems*. Ed. by B. Schölkopf, J. Platt, and T. Hoffman. Vol. 19. MIT Press, 2006 (cited at slide -6).