Online Optimization, Learning, and Games (O2LG)
Lesson 8: Online Convex Optimization - Part II

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Recall: Online Convex Optimization

Sequence of events: Online convex optimization (OCO)

Require: convex action set $\mathcal{X}$; convex loss functions $\ell_n \colon \mathcal{X} \to \mathbb{R}$, $n = 1, 2, \ldots$

repeat At each epoch $n = 1, 2, \ldots$ **do**

Choose action $x_n \in \mathcal{X}$. ▷ action selection

Encounter loss function $\ell_n \colon \mathcal{X} \to \mathbb{R}$. ▷ nature plays

Incur cost $c_n = \ell_n(x_n)$. ▷ reward phase

Observe entire loss function ℓ_n or gradient $g_n = \nabla \ell_n(x_n)$ or cost $c_n = \ell_n(x_n)$ ▷ feedback phase

until end

Feature: discrete time, single player, continuous actions, exogenous losses.

- Feedback: **depends** (function-based, gradient-based, loss-based, ...)

Feedback

Types of feedback

From best to worst (more to less info):

- Full information: observe entire loss function $\ell_n : \mathcal{X} \to \mathbb{R}$ # deterministic function feedback
- First-order info, exact: observe (sub)gradient $g_n \in \partial \ell_n(x_n)$ # deterministic vector feedback
- First-order info, inexact: observe noisy estimate of g_n # stochastic vector feedback
- 0th-order info (bandit): observe only incurred cost $c_n = \ell_n(x_n)$ # deterministic scalar feedback

Recall: Follow The Regularized Leader (FTRL)

Follow The Regularized Leader (FTRL)

$$x_{n+1} = \arg\min_{x\in\mathcal{X}} \left\{ \sum_{k=1}^{n} \ell_k(x) + \underbrace{\lambda h(x)}_{\text{“}\ell_0(x)\text{”}} \right\} \tag{FTRL}$$

where

- regularization function $h\colon \mathcal{X} \to \mathbb{R}$ is strongly convex. # $h - (K_h/2)\|\cdot\|^2$ convex for some $K_h > 0$
- regularization weight $\lambda > 0$ can be tuned.

Follow The Linearized Leader (FTLL)

Can we relax the full information requirement of (FTRL)?

- Replace ℓ_n with first-order surrogate $\hat{\ell}_n$ defined by:

$$\hat{\ell}_n(x) = \ell_n(x_n) + \langle g_n, x - x_n \rangle, \qquad g_n \in \partial \ell_n(x_n).$$

- Plugging into (FTRL) and setting $\lambda := 1/\eta$, yield **Follow The Linearized Leader (FTLL)**:

$$x_{n+1} = \arg\min_{x \in \mathcal{X}} \left\{ \eta \sum_{k=1}^{n} \langle g_k, x \rangle + h(x) \right\}. \tag{1}$$

How to obtain the update step (1) of (FTLL)?

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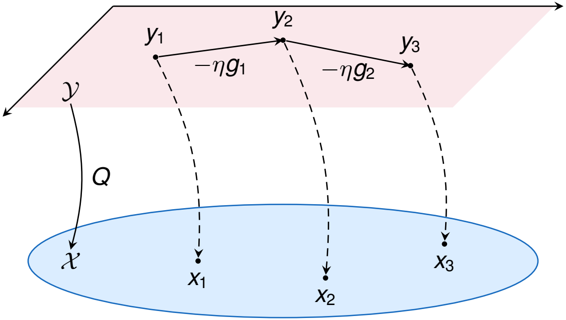
Dual Averaging (DA)

Dual Averaging (DA) formulation of (FTLL): Nesterov 2009; Xiao 2010

$$\begin{aligned} y_{n+1} &= y_n - \eta g_n, \\ x_{n+1} &= Q(y_{n+1}), \end{aligned} \tag{DA}$$

where $Q(y) = \arg\max_{x \in \mathcal{X}} \{\langle y, x \rangle - h(x)\}$ is the mirror map associated to h.



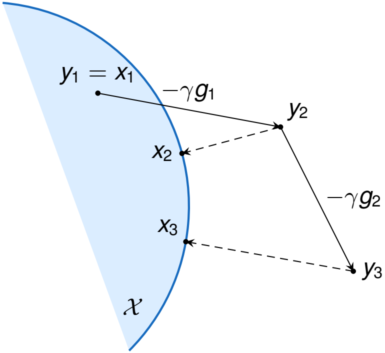
Example: Online Gradient Descent (OGD)

Special case when the regularizer $h(x) = \frac{1}{2}\|x\|_2^2 \rightsquigarrow$ **Online Gradient Descent (OGD)**

$$y_{n+1} = y_n - \eta g_n, \qquad x_{n+1} = \Pi_{\mathcal{X}}(y_{n+1}), \tag{OGD}$$

where $\Pi_{\mathcal{X}}$ is the projection of a point onto the set $\mathcal{X}$.

How to obtain the update step of (OGD)?



Online Mirror Descent (deep dive)

- Gradient signals $\{g_n\}_n$ enter (DA) unweighted / unadjusted.
- Variable weights $\rightsquigarrow$ "lazy", primal-dual variant of Online Mirror Descent (OMD):

$$\begin{aligned} y_{n+1} &= y_n - \eta_n g_n, \\ x_{n+1} &= Q(y_{n+1}). \end{aligned} \tag{OMD_{lazy}}$$

- Primal-primal ("eager") variant of (OMD_{lazy}):

$$x_{n+1} = P_{x_n}(-\eta_n g_n). \tag{OMD}$$

The **Bregman proximal mapping** P is defined as

$$P_x(w) := \arg\min_{x' \in \mathcal{X}} \{\langle w, x - x' \rangle + D(x', x)\}$$

where $D(z, v) = h(z) - h(v) - \langle \nabla h(v), z - v \rangle$ is the Bregman divergence of h.

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Regret under Dual Averaging

- **Gradient trick:** # linear model

$$\ell_n(x_n) - \ell_n(p) \leq \langle g_n, x_n - p \rangle \quad \text{for all } p \in \mathcal{X}.$$

- **Energy function:**

$$F_n = h(p) + h^*(y_n) - \langle y_n, p \rangle,$$

where $h^*(y) = \max_{x \in \mathcal{X}} \{\langle y, x \rangle - h(x)\}$ is the potential of $Q \rightsquigarrow \nabla h^* = Q$.

The function h^* enjoys the following inequality: # Pages 227-228, Nesterov 2009

$$h^*(s + \delta) \leq h^*(s) + \langle \delta, \nabla h^*(s) \rangle + \frac{1}{2K_h} \|\delta\|^2, \ \forall s, \delta \in \mathcal{Y}. \tag{2}$$

Regret under Dual Averaging

- Template inequality: # prove the inequality

$$F_{n+1} \leq F_n - \eta\langle g_n, x_n - p\rangle + \frac{\eta^2}{2K_h}\|g_n\|^2.$$

- Rearrange & telescope: # how to build the regret

$$\mathrm{Reg}(T) \leq \frac{R_h}{\eta} + \frac{\eta}{2K_h}\sum_{n=1}^{T} G_n^2,$$

where $R_h = \max h - \min h$; ℓ_n is G_n-Lipschitz; $\|g_n\| \leq G_n$ for all n. # Shalev-Shwartz 2012

Regret under Dual Averaging

Theorem 1 (Shalev-Shwartz 2012)

☞ *Assume: h is K_h-strongly convex; each ℓ_n is G-Lipschitz continuous;*

$$R_h = \max h - \min h; \quad \eta = G^{-1}\sqrt{2K_hR_h/T}.$$

✓ *Then:* (DA) *enjoys the regret bound*

$$\mathrm{Reg}(T) \leq G\sqrt{\frac{2R_hT}{K_h}}.$$

Task

Prove this theorem.

Summary

This lesson

- Learning with gradient feedback
- Dual Averaging
- Regret under Dual Averaging

Next lesson

- Learning with stochastic gradients
- ...

References

[1] Yurii Nesterov. Primal-dual subgradient methods for convex problems. In: *Mathematical Programming* 120.1 (June 2009), pp. 221–259 (cited at slides -7, -3).

[2] Shai Shalev-Shwartz. Online Learning and Online Convex Optimization. In: *Foundations and Trends® in Machine Learning* 4.2 (2012), pp. 107–194 (cited at slides -2, -1).

[3] Lin Xiao. Dual Averaging Methods for Regularized Stochastic Learning and Online Optimization. In: *Journal of Machine Learning Research* 11.88 (2010), pp. 2543–2596 (cited at slide -7).