JENNRICH'S ALGORITHM - A MATLAB IMPLEMENTATION

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1 Jennrich's Algorithm

Let there be a three-order tensor $\mathcal{X} \in \mathbb{R}^{m\times n\times p}$. Suppose that $\mathcal{X}$ has a canonical polyadic decomposition

$$\mathcal{X} = [[\mathbf{A}, \mathbf{B}, \mathbf{C}]] = \sum_{i=1}^{r} \mathbf{a}_i \otimes \mathbf{b}_i \otimes \mathbf{c}_i, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times r}$, $\mathbf{B} \in \mathbb{R}^{n\mathbf{C}r}$ and $\mathbf{C} \in \mathbb{R}^{p\times r}$. Choose a unit vector $\mathbf{x}_i \in \mathbb{R}^p$ uniformly. We have

$$\mathbf{M}_x = \sum_{i=1}^{p} a_i \mathcal{X}_{:,:,i} = \sum_{j=1}^{r} \langle \mathbf{c}_j, \mathbf{x}\rangle \mathbf{a}_j \mathbf{b}_j^\top = \mathbf{A}\mathrm{diag}(\langle \mathbf{c}_1, \mathbf{x}\rangle, \dots, \langle \mathbf{c}_r, \mathbf{x}\rangle)\mathbf{B}^\top. \tag{2}$$

Let $\mathbf{D}_x = \mathrm{diag}(\langle \mathbf{c}_1, \mathbf{x}\rangle, \dots, \langle \mathbf{c}_r, \mathbf{x}\rangle)$ for brevity. We write

$$\mathbf{M}_x = \mathbf{A}\mathbf{D}_x\mathbf{B}^\top. \tag{3}$$

Similarly, we can choose a unit vector $\mathbf{b}$ uniformly and construct the matrix

$$\mathbf{M}_y = \mathbf{A}\mathbf{D}_y\mathbf{B}^\top. \tag{4}$$

If $\mathbf{B}$ is full column rank, we have $\mathbf{B}^\top(\mathbf{B}^\top)^\dagger = \mathbf{I}_r$. Therefore,

$$\mathbf{M}_x\mathbf{M}_y^\dagger = \mathbf{A}\mathbf{D}_x\mathbf{B}^\top(\mathbf{B}^\top)^\dagger\mathbf{D}_y^\dagger\mathbf{A}^\dagger = \mathbf{A}(\mathbf{D}_x\mathbf{D}_y^\dagger)\mathbf{A}^\dagger, \tag{5}$$

where $\mathbf{D}_x\mathbf{D}_y^\dagger = \mathrm{diag}\left(\frac{\langle \mathbf{C}_1, \mathbf{a}\rangle}{\langle \mathbf{C}_1, \mathbf{b}\rangle}, \dots, \frac{\langle \mathbf{C}_r, \mathbf{a}\rangle}{\langle \mathbf{C}_r, \mathbf{b}\rangle}\right)$. Since $\mathbf{x}$ and $\mathbf{y}$ are chosen uniformly, the elements of $\mathbf{D}_x\mathbf{D}_y^\dagger$ are distinct with probability 1. Hence, the columns of $\mathbf{A}$ are eigenvectors of $\mathbf{M}_x\mathbf{M}_y^\dagger$. Similarly, if $\mathbf{A}$ is full column rank, then

$$\mathbf{M}_x^\top(\mathbf{M}_y^\top)^\dagger = \mathbf{B}(\mathbf{D}_x\mathbf{D}_y^\dagger)\mathbf{B}^\dagger, \tag{6}$$

which means that the columns of $\mathbf{B}$ are eigenvectors of $\mathbf{M}_x^\top(\mathbf{M}_y^\top)^\dagger$. Finally, we recover $\mathbf{C}$ using

$$\mathcal{X}_{(3)} = \mathbf{C}(\mathbf{B} \odot \mathbf{A})^\top. \tag{7}$$