Online Optimization, Learning, and Games (O2LG)
Lesson 3: Game dynamics

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ACSYON
Master of Applied Mathematics

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Motivation

- Recall, in a congestion game:
  - each driver chooses a route to minimize its own travel time,
  - lead to congestion on the whole network,
  - increase everyone's travel time.
- Finding the Nash equilibria of a game: rather complicated.
  - require a great deal of global calculations,
  - even in the case of potential games.
- $\Longrightarrow$ Interest to see whether there are simple and distributed learning schemes
  - allow players to arrive at a reasonably stable solution.

How?

- By adapting their behavior in response to the behavior of other players.

Learning scheme

Input: a finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$.

Repeat for each epoch $t = 0, 1, 2, \ldots$, for all players $i \in \mathcal{N}$,

- Choose mixed strategy $x_i(t) \in \mathcal{X}_i$.
- Receive payoff vector $v_i(x(t))$.
- Observe mixed payoff vector $u_i(x(t)) = \langle v_i(x(t)), x_i(t) \rangle$.

Until end

Exponential weights

How to choose mixed strategy $x_i(t) \in \mathcal{X}_i$?

- Score each action based on its cumulative payoff over time:

$$y_{i,a_i}(t) = \int_0^t v_{i,a_i}(x(s))ds.$$

- Choose an action with probability exponentially proportional to its score

$$x_{i,a_i}(t) \propto \exp(y_{i,a_i}(t)).$$

Exponential weights dynamics

Exponential Weights Dynamics (EWD)

$$\dot{y}_{i,a_i}(t) = v_{i,a_i}(t).$$

$$x_{i,a_i}(t) = \frac{\exp(y_{i,a_i}(t))}{\sum_{a_i' \in \mathcal{A}_i} \exp(y_{i,a_i'}(t))}.$$

How do mixed strategies x_i evolve under (EWD)?
In other words, how to evaluate $\dot{x}_{i,a_i}$ in terms of x_{i,a_i} and u_i?

Replicator dynamics

Replicator dynamics (RD) (Taylor et al. 1978)

$$\begin{aligned}\dot{x}_{i,a_i} &= x_{i,a_i}\left[v_{i,a_i}(x) - \sum_{a_i' \in \mathcal{A}_i} x_{i,a_i'} v_{i,a_i'}(x)\right] \\ &= x_{i,a_i}\left[u_i(a_i, x_{-i}) - u_i(x)\right].\end{aligned}$$

Proposition 1

Solution orbits of (EWD) $\Longleftrightarrow$ Interior orbits of (RD).

Example 1: Return to The Prisoner's Dilemma

Player 1 = Thomas; Player 2 = Julian.
Action 1 = C(ooperate); Action 2 = D(efect).

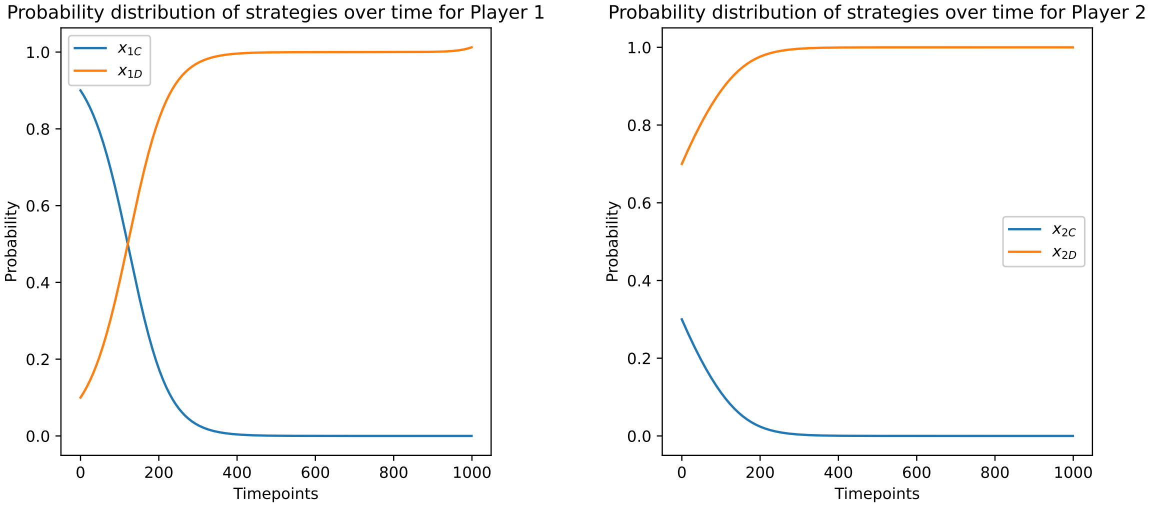
		Julian	
		C	D
Thomas	C	$(-1, -1)$	$(-5, 0)$
	D	$(0, -5)$	$(-3, -3)$

From Lesson 1, (D, D) is a Nash equilibrium of this game.

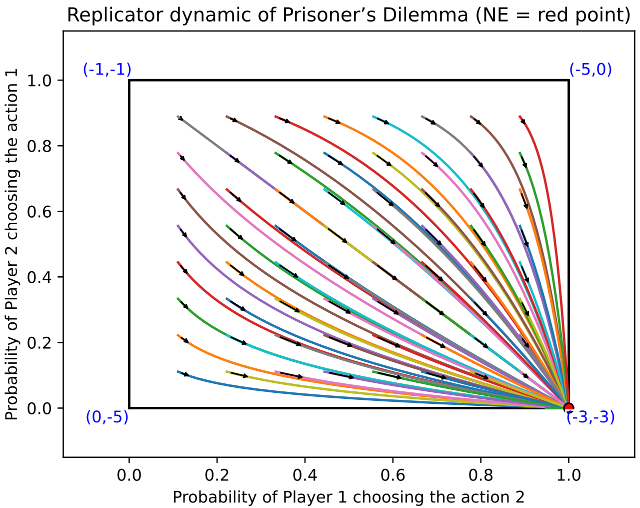
Task

Find replicator dynamics for this game.

What do the mixed strategies look like? Suppose that $x_1(0) = (0.9, 0.1)^\top$ and $x_2(0) = (0.3, 0.7)^\top$.



What do the dynamics look like?



Example 2: Congestion Game

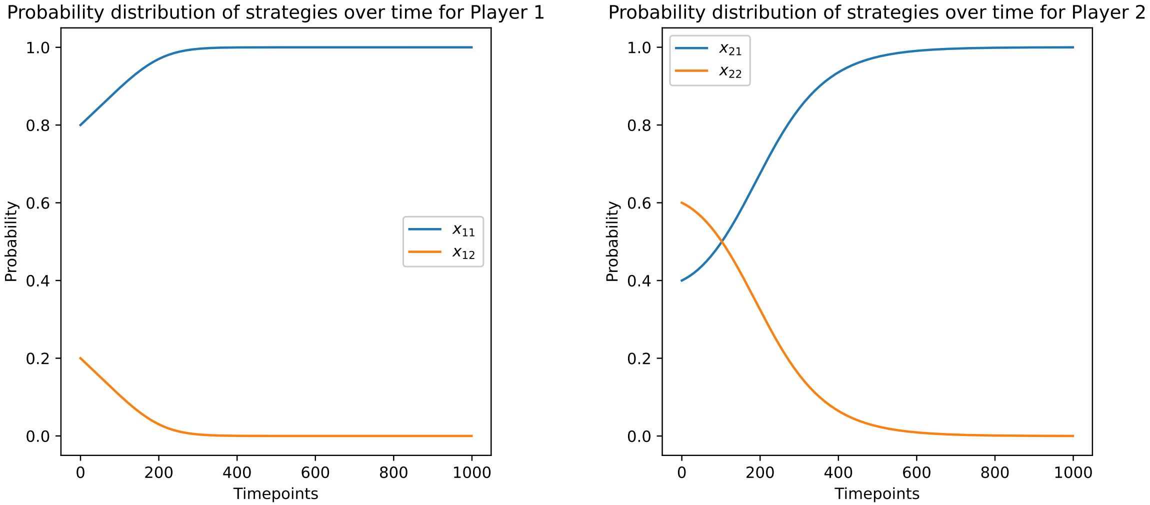
Player 2

Player 1	1	2
1	$(3, 1)$	$(0, 0)$
2	$(0, 0)$	$(1, 3)$

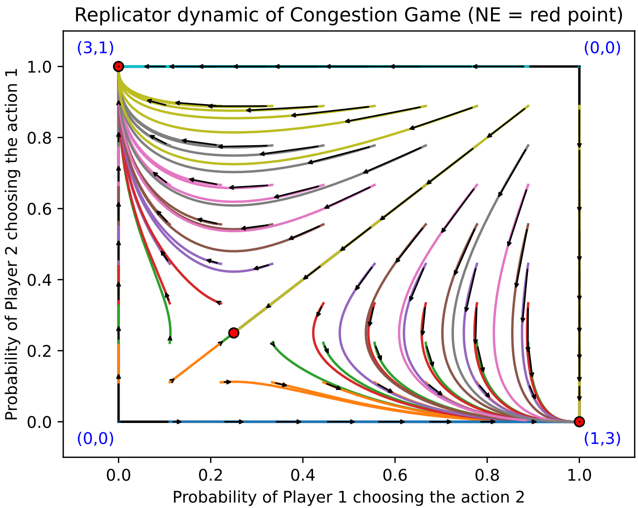
Task

- Show that $[[0, 1], [0, 1]]$, $[[1, 0], [1, 0]]$, $[[0.75, 0.25], [0.25, 0.75]]$ are Nash equilibria.
- Find replicator dynamics for this game.

What do the mixed strategies look like? Suppose that $x_1(0) = (0.8, 0.2)^\top$ and $x_2(0) = (0.4, 0.6)^\top$.



What do the dynamics look like?



Example 3: Matching pennies game

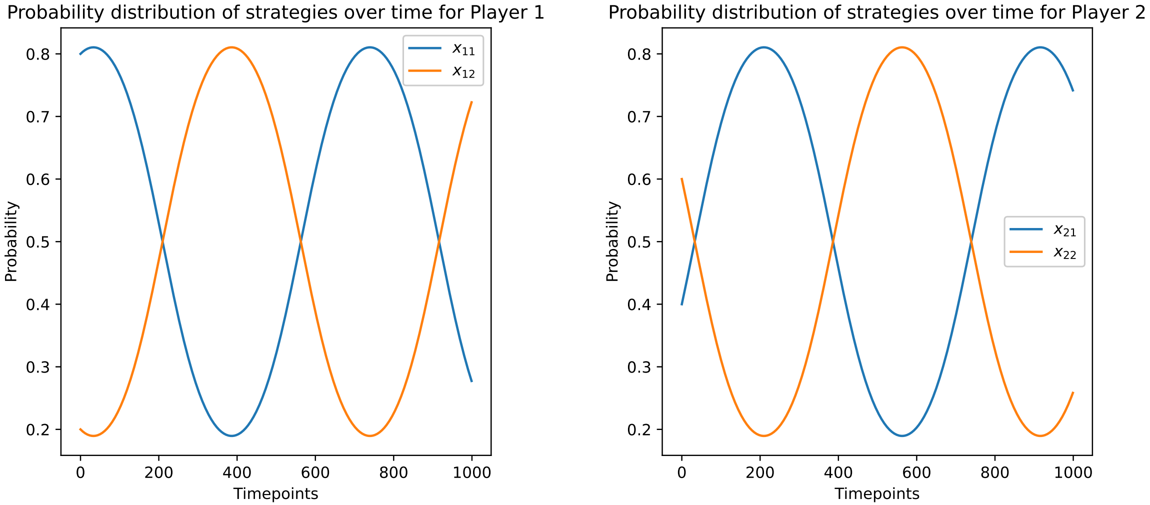
Player 2

Player 1	1	2
1	$(-1, 1)$	$(1, -1)$
2	$(1, -1)$	$(-1, 1)$

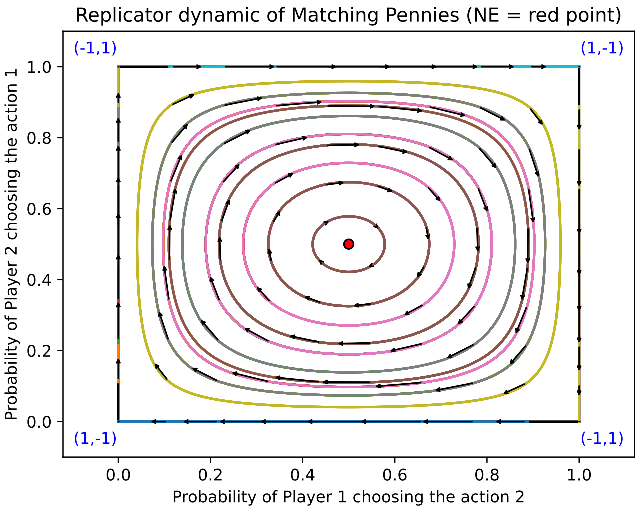
Task

- Show that $[[0.5, 0.5], [0.5, 0.5]]$ is a Nash equilibrium.
- Find replicator dynamics for this game.

What do the mixed strategies look like? Suppose that $x_1(0) = (0.8, 0.2)^\top$ and $x_2(0) = (0.4, 0.6)^\top$.



What do the dynamics look like?



Properties

Basic properties of (EWD)/(RD):

1. **Well-posedness**: every initial condition $x \in \mathcal{X}$ admits a **unique** solution trajectory $x(t)$ that exists for all time.
2. **Consistent**: $x(t) \in \mathcal{X}$ for all $t \geq 0$.
3. **Faces are forward invariant** ("strategies breed true"):

$$x_{i,a_i}(0) > 0 \iff x_{i,a_i}(t) > 0 \ \text{ for all } \ t \geq 0.$$
$$x_{i,a_i}(0) = 0 \iff x_{i,a_i}(t) = 0 \ \text{ for all } \ t \geq 0.$$

Return to the example: The Prisoner's Dilemma

Player 1 = Thomas; Player 2 = Julian.
Action 1 = C(ooperate); Action 2 = D(efect).

Julian

Thomas	C	D
C	$(-1,-1)$	$(-5,0)$
D	$(0,-5)$	$(-3,-3)$

Task

Verify Properties 2 and 3 for the example of The Prisoner's Dilemma.

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Dominated strategies

Suppose that $a_i \in \mathcal{A}_i$ is dominated by $a_i' \in \mathcal{A}_i$.

1. Consistent payoff gap:
$$v_{i,a_i}(x) \leq v_{i,a_i'}(x) - \varepsilon \text{ for some } \varepsilon > 0.$$
2. Consistent difference in scores:
$$y_{i,a_i}(t) = \int_0^t v_{i,a_i}(x(s))ds \leq \int_0^t [v_{i,a_i'}(x(s)) - \varepsilon]ds = y_{i,a_i'}(t) - \varepsilon t \text{ for some } \varepsilon > 0.$$
3. Consistent difference in choice probabilities:
$$\frac{x_{i,a_i}(t)}{x_{i,a_i'}(t)} = \frac{\exp(y_{i,a_i}(t))}{\exp(y_{i,a_i'}(t))} \leq \exp(-\varepsilon t) \text{ for some } \varepsilon > 0.$$

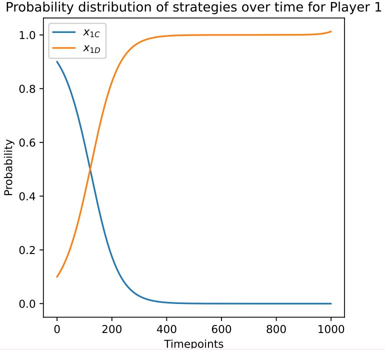
Task

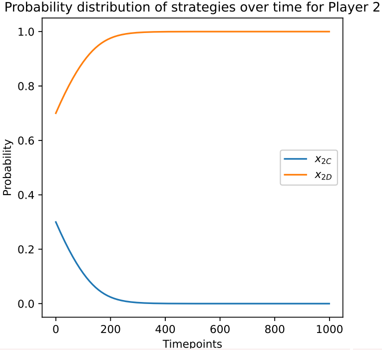
Verify Property 1 in the example of The Prisoner's Dilemma.

Dominated strategies

Theorem 1 (Samuelson et al. 1992)

Let $x(t)$ be a solution orbit of (EWD)/(RD). If $a_i \in \mathcal{A}_i$ is dominated, then $\lim_{t\to\infty} x_{i,a_i}(t) = 0$. In other words, under (EWD)/(RD), dominated strategies become extinct.





Stationarity of equilibria

Nash equilibrium x^*: $v_{i,a_i}(x^*) \geq v_{i,a_i'}(x^*)$ for all $a_i' \in \mathcal{A}_i$ and $a_i \in \text{supp}(x_i^*)$.

1. Supported strategies have equal payoffs:

$$v_{i,a_i}(x^*) = v_{i,a_i'}(x^*) \text{ for all } a_i, a_i' \in \text{supp}(x_i^*).$$

2. Mean payoff equal to equilibrium payoff:

$$u_i(x^*) = v_{i,a_i}(x^*) \text{ for all } a_i \in \text{supp}(x_i^*).$$

3. Replicator field vanishes at Nash equilibria:

$$x^*_{i,a_i}[v_{i,a_i}(x^*) - u_i(x^*)] = 0 \text{ for all } a_i \in \mathcal{A}_i.$$

Task

Verify these properties in the example of The Prisoner's Dilemma.

Stationarity of equilibria

Proposition 2 (Stationarity of Nash equilibria)

Let $x(t)$ be a solution orbit of (RD). Then

$$x(0) \text{ is a Nash equilibrium} \Longrightarrow x \text{ is stationary, i.e. } x(t) = x(0) \text{ for all } t \geq 0.$$

The converse does not hold!

Stability and equilibrium

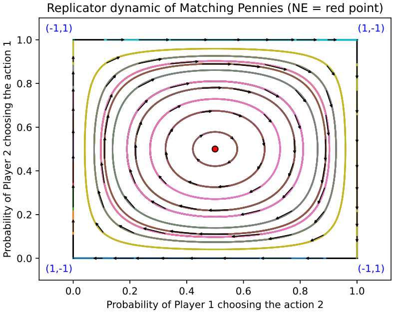
Definition 1 (Lyapunov stability)

x^* is (Lyapunov) stable if for every neighborhood of $\mathcal{U}$ of x^* in $\mathcal{X}$, there exists a neighborhood $\mathcal{U}'$ of x^* such that

$$x(0) \in \mathcal{U}' \Longrightarrow x(t) \in \mathcal{U} \quad \text{for all} \quad t \geq 0.$$

Proposition 3

Suppose that x^* is Lyapunov stable under (EWD)/(RD). Then x^* is a Nash equilibrium.



Asymptotic stability

Definition 2

- x^* is **attracting** if $\lim_{t\to\infty} x(t) = x^*$ whenever $x(0)$ is close enough to x^*.
- x^* is **asymptotically stable** if it is stable and attracting.

Proposition 4

Strict Nash equilibria are asymptotically stable under (RD).

"Folk theorem" of the replicator dynamics

Theorem 2 (Hofbauer et al. 2003)

Let Γ be a finite game. Then, under (RD), we have:

1. x^* *is a Nash equilibrium* $\Longrightarrow$ x^* *is stationary.*
2. x^* *is the limit of an interior trajectory* $\Longrightarrow$ x^* *is a Nash equilibrium.*
3. x^* *is stable* $\Longrightarrow$ x^* *is a Nash equilibrium.*
4. x^* *is asymptotically stable* $\Longleftrightarrow$ x^* *is a strict Nash equilibrium.*

Summary

This lesson

- Game dynamics
- Exponential weights and the replicator dynamics
- Rationality analysis

Next lesson

- Discrete-time models of learning
- Different types of feedback
- Rationality analysis

References

[1] Josef Hofbauer and Karl Sigmund. Evolutionary game dynamics. In: *Bulletin of the American Mathematical Society* 40.4 (2003), pp. 479–519 (cited at slide -1).

[2] Larry Samuelson and Jianbo Zhang. Evolutionary stability in asymmetric games. In: *Journal of Economic Theory* 57.2 (1992), pp. 363–391 (cited at slide -6).

[3] Peter D. Taylor and Leo B. Jonker. Evolutionary stable strategies and game dynamics. In: *Mathematical Biosciences* 40.1 (1978), pp. 145–156 (cited at slide -20).