Online Optimization, Learning, and Games (O2LG)
Lesson 1: Introduction to Game Theory

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Motivation

Why game theory?

Why learning?

Why game theory?

- **Game theory** is the study of *strategic decision-making*.
- Provide a framework for understanding how agents interact with each other in competitive and cooperative environments.
- Model a wide variety of real-world situations, e.g., traffics, economics, sports, etc.
- Design algorithms that are robust to adversarial behavior.

Example: Traffic Congestion

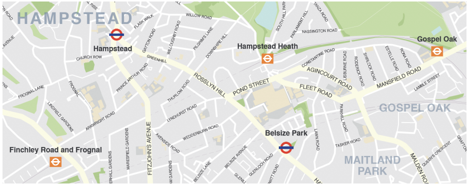
2021→2022 INRIX Global Traffic Scorecard

- **London** is the most congested city in the world.
- 148 → 156 hours lost per driver to delay.
- Congestion cost of £5.1 (1% GDP) → £5.7 billion.
- £212 (2022) fuel surcharge per driver.

How to deal with traffic congestion?

Traffic Congestion in London

Example: Traffic Congestion



Greater London map

London

- Population (2023): $\approx$ 10 million.
- 21.6 million daily trips[a] (2021)
- 831 stations & 107,000 roads [b]

[a] Travel in London Report 15

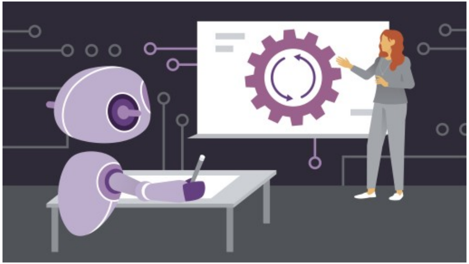
[b] Greater London map

How to model traffic congestion?

- Consider a network of roads with each road of a limited capacity.
- The *vehicles choose which roads* to travel from one station to another on the network. But if *too many vehicles* choose the *same road*, then the *road will become congested*.
- Model as a congestion game: player = vehicle; player's action = which road to take; player's payoff = travel time. Goal of each player: minimize their travel time.

A very large game!

Why learning?



- **Learning** is the ability to *improve performance* over time *by experience*.
- Key component of many online optimization and game-playing algorithms.
- **Online optimization** + **Learning**: Adapt to changes in the environment.

Example: Traffic Congestion (cont.)

Traffic Congestion

Learning can be used to deal with traffic congestion:

- **Predictive traffic modeling**: predict *traffic patterns*
  ⇒ improve traffic flow and optimize routing decisions
  ⋆ Online learning: adapt to changes in traffic patterns.
- **Dynamic routing**: provide drivers with *real-time routing information* that takes into account *traffic conditions*
  ⇒ avoid congested areas and find the fastest routes.
- **Parking optimization**: optimize *parking availability/pricing*
  ⇒ find parking more easily and reduce congestion.
- **Smart traffic lights**: *control traffic lights* in real time
  ⇒ reduce congestion and improve traffic safety.

Questions we'll try to answer

1. **How should we model player interactions?**
   - Types of situations (e.g. traffic congestion),
   - Types of players (e.g. vehicles),
   - Types of interactions (e.g. competition),. . .

2. **What is a desired operational state ?**
   - Different goals (e.g., reduce congestion),
   - Types of states (equilibrium, optimum),. . .

3. **How to compute it?**
   - Calculation (simple, well-understanding),
   - Implementation (simple, easy to control),
   - Learning (complex).

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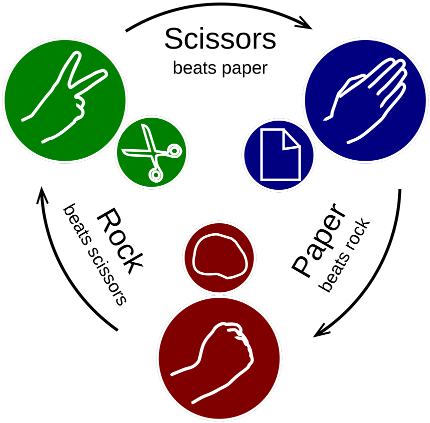
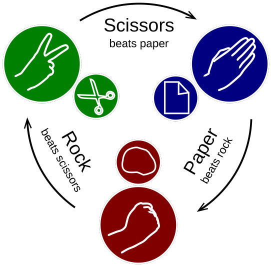
Introduction: Let’s play a game



How to model this game **mathematically**?

How to model the rock-paper-scissors game?

- **Players**: $\mathcal{N} = \{1, 2\}$
- **Actions** associated to each player: $\mathcal{A}_i = \{R, P, S\}, i \in \mathcal{N}$
- **Payoff matrix** (win: 1€; lose: -1€; tie: 0€):



$$\begin{array}{cc|ccc} & & & \text{Player 2} & \\ & & R & P & S \\ \hline \text{Player 1} & R & (0,0) & (-1,1) & (1,-1) \\ & P & (1,-1) & (0,0) & (-1,1) \\ & S & (-1,1) & (1,-1) & (0,0) \end{array}$$

- **Payoff functions**: for each player
  Player 1: $u_1 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}$ defined by $u_1(R, R) = 0$, $u_1(R, P) = -1$, $u_1(R, S) = 1$,...
  Player 2: $u_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}$ defined by $u_2(R, R) = 0$, $u_2(R, P) = 1$, $u_2(R, S) = -1$,...

Finite games

Definition 1 (Finite games)

A ***finite game in normal (or strategic) form*** is a tuple $(\mathcal{N}, \{\mathcal{A}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$, where:

- $\mathcal{N} = \{1, 2, \dots, N\}$ is a finite set of ***players***.
- For each player $i \in \mathcal{N}$, $\mathcal{A}_i$ is a finite set of ***actions*** (or ***pure strategies***) for player i.
- For each player $i \in \mathcal{N}$, $u_i : \Pi_{j\in\mathcal{N}}\mathcal{A}_j \to \mathbb{R}$ is a ***payoff function*** mapping each combination (or *profile*) of actions $(a_1, ..., a_N)$ to a real number $u_i(a_1, ..., a_N)$ that is the payoff to player i when players $1, 2, \dots, N$ do actions $a_1, a_2, \dots, a_N$, respectively.

Notation: $\mathcal{A} := \Pi_j\mathcal{A}_j$ the set of all profiles of actions, with a generic element $a := (a_1, ..., a_N)$.

Example 1: The Prisoner’s Dilemma

Imagine the following situation:

- Two bank robbers, Thomas and Julian, have been arrested by the authorities and are being interrogated in separate rooms.
- The authorities need one robber to testify to convict them.
- Each robber faces a choice:
  - *cooperate* with his accomplice and remain silent
  - *defect* from the gang and testify for the prosecution
- There are a number of different scenarios that could occur:
  - If Thomas and Julian cooperate, they will each get 1 year in jail.
  - If one testifies against the other, the one who testifies goes free and the other gets 5 years.
  - If they both testify against each other, they will each get 3 years.

How to model this game in normal form?

Example 1: The Prisoner's Dilemma

Normal form representation

- **Players**: $\mathcal{N} = \{1 \text{ (Thomas)}, 2 \text{ (Julian)}\}$
- **Actions** associated to each player: $\mathcal{A}_i = \{C\text{(ooperate)}, D\text{(efect)}\}, i \in \mathcal{N}$
- **Payoff matrix**:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{Julian}} \\ & & C & D \\ \hline \text{Thomas} & C & (-1,-1) & (-5,0) \\ & D & (0,-5) & (-3,-3) \end{array}$$

- **Payoff functions**: for each player,
  Thomas: $u_1(C,C) = -1$, $u_1(C,D) = -5$, $u_1(D,C) = 0$, $u_1(D,D) = -3$.
  Julian: $u_2(C,C) = -1$, $u_2(C,D) = 0$, $u_2(D,C) = -5$, $u_2(D,D) = -3$.

Example 2: Split or Steal?

- The game **Split or Steal** is a two-player game.
- Each player must decide whether to split the pot of money in front of them or steal from the other player.
- One of these situations will occur.
  - If both players choose split, they each get half of the money.
  - If both players choose steal, neither player gets any money.
  - If only one player steals, the player who chooses steal gets all of the money and the other player gets nothing.

Watch a playthrough of the £66885 “Split or Steal” game and then model this game in normal form.

Example 2: Split or Steal?

Normal form representation

- **Players**: $\mathcal{N} = \{1\text{ (Tony)}, 2\text{ (Lucy)}\}$
- **Actions** associated to each player: $\mathcal{A}_i = \{Sp\text{(lit)}, St\text{(eal)}\}, i \in \mathcal{N}$
- **Payoff matrix**:

		Lucy	
		Sp	*St*
Tony	*Sp*	(£33442.5, £33442.5)	(£0, £66885)
	St	(£66885, £0)	(£0, £0)

- **Payoff functions**: for each player,
  Tony: $u_1(Sp, Sp) = £33442.5$, $u_1(Sp, St) = £0$, $u_1(St, Sp) = £66885$, $u_1(St, St) = £0$.
  Lucy: $u_2(Sp, Sp) = £33442.5$, $u_2(Sp, St) = £66885$, $u_2(St, Sp) = £0$, $u_2(St, St) = £0$.

Example 3: Chicken game

- The **Chicken Game** is a game in which two players drive towards each other on a collision course.
- Each driver has two choices: swerve or go straight.
  - If **both drivers swerve**, they will both have a minor accident and get 0 points.
  - If **one driver swerves and the other does not**, the driver who swerves will be hit and get -1 point, while the driver who goes straight will be unharmed and get 1 point.
  - If **neither driver swerves**, they will both crash head-on and be seriously injured, so they will get -1000 points.

How to model this game in normal form?

Example 3: Chicken game

Normal form representation

- **Players**: $\mathcal{N} = \{1 \text{ (Driver 1)}, 2 \text{ (Driver 2)}\}$
- **Actions** associated to each player: $\mathcal{A}_i = \{Sw\text{(erve)}, St\text{(raight)}\}, i \in \mathcal{N}$
- **Payoff matrix**:

		Driver 2	
		Sw	St
Driver 1	Sw	$(0, 0)$	$(-1, 1)$
	St	$(1, -1)$	$(-1000, -1000)$

- **Payoff functions**: for each driver,
  Driver 1: $u_1(Sw, Sw) = 0$, $u_1(Sw, St) = -1$, $u_1(St, Sw) = 1$, $u_1(St, St) = -1000$.
  Driver 2: $u_2(Sw, Sw) = 0$, $u_2(Sw, St) = 1$, $u_2(St, Sw) = -1$, $u_2(St, St) = -1000$.

Dominant Strategies

In some games, there is a ***dominant strategy*** that always produces the best outcome for a player, regardless of what the other players do.
Before definitions, note that for $i \in \mathcal{N}$, $(a', a_{-i}) := (a_1, \dots, a_{i-1}, a', a_{i+1}, \dots, a_N)$, $\mathcal{A}_{-i} := \Pi_{j \neq i} \mathcal{A}_j$.

Definition 2 (Dominant strategies)

1. A strategy $a_i \in \mathcal{A}_i$ ***weakly dominates*** $a_i' \in \mathcal{A}_i$ (noted by $a_i \succcurlyeq a_i'$) if

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in \mathcal{A}_{-i} \quad \text{and} \quad u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \in \mathcal{A}_{-i}.$$

2. A strategy $a_i \in \mathcal{A}_i$ ***strictly dominates*** $a_i' \in \mathcal{A}_i$ (noted by $a_i \succ a_i'$) if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in \mathcal{A}_{-i}.$$

3. A strategy $a_i \in \mathcal{A}_i$ is ***weakly dominant*** if $a_i \succcurlyeq a_i'$ for all $a_i' \in \mathcal{A}_i$.
4. A strategy $a_i \in \mathcal{A}_i$ is ***strictly dominant*** if $a_i \succ a_i'$ for all $a_i' \in \mathcal{A}_i$, $a_i' \neq a_i$.

In other words, $a_i \succcurlyeq a_i'$ (resp. $a_i \succ a_i'$) means that a_i' is **weakly** (resp. **strictly**) **dominated** by a_i.

Return to Examples 1-3

Weakly / Strictly dominant strategies for players for each example?

Example 1: The Prisoner's Dilemma

		Julian	
		C	D
Thomas	C	$(-1,-1)$	$(-5,0)$
	D	$(0,-5)$	$(-3,-3)$

Example 2: Split or Steal?

		Lucy	
		Sp	St
Tony	Sp	$(£33442.5, £33442.5)$	$(£0, £66885)$
	St	$(£66885, £0)$	$(£0, £0)$

Example 3: Chicken game

		Driver 2	
		Sw	St
Driver 1	Sw	$(0,0)$	$(-1,1)$
	St	$(1,-1)$	$(-1000,-1000)$

Iteratively Dominated Strategies

Given a 2-player game with the following payoff matrix:

$$\begin{array}{cc|ccc} & & \multicolumn{3}{c}{\text{Player 2}} \\ & & A & B & C \\ \hline & X & (4,10) & (1,0) & (1,2) \\ \text{Player 1} & Y & (3,5) & (6,4) & (5,3) \\ & Z & (2,3) & (0,6) & (8,2) \end{array}$$

Questions: In this game,

1. What are the strictly dominated strategies (by which strategies) for players?
2. What happens to this game if eliminating one of these strategies? Does new strictly dominated strategies appear?
3. If repeating elimination iteratively, what does the game become?

Dominance-Solvable Game

After repeating elimination iteratively, the game becomes as follows:

		Player 2		
		A	*B*	*C*
Player 1	*X*	(4,10)	~~(1,0)~~	~~(1,2)~~
	Y	~~(3,5)~~	~~(6,4)~~	~~(5,3)~~
	Z	~~(2,3)~~	~~(0,6)~~	~~(8,2)~~

Definition 3 (Dominance-solvable game)

A game is called ***dominance-solvable*** if the iterated elimination of dominated strategies leads to **a singleton**.

Best Responses

How to make a good decision if we only know the strategy of the opposing players?

Definition 4 (Best responses)

- A strategy $a_i^* \in \mathcal{A}_i$ is a ***best response*** (or ***best reply***) of player i to a profile of strategies $a_{-i} \in \mathcal{A}_{-i}$ for the other players if

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \text{ for all } a_i \in \mathcal{A}_i, \text{ equivalently, } a_i^* \in \operatorname{argmax}_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}).$$

- A ***best-response correspondence*** of player i is a set-valued function $\mathrm{BR}_i : \mathcal{A}_{-i} \to \mathcal{A}_i$ defined by

$$\mathrm{BR}_i(a_{-i}) := \operatorname{argmax}_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i})$$

Return to Examples 1-3

Best-response correspondence for players for each example?

Example 1: The Prisoner's Dilemma

		Julian	
		C	D
Thomas	C	$(-1, -1)$	$(-5, 0)$
	D	$(0, -5)$	$(-3, -3)$

Example 2: Split or Steal?

		Lucy	
		Sp	St
Tony	Sp	$(£33442.5, £33442.5)$	$(£0, £66885)$
	St	$(£66885, £0)$	$(£0, £0)$

Example 3: Chicken game

		Driver 2	
		Sw	St
Driver 1	Sw	$(0, 0)$	$(-1, 1)$
	St	$(1, -1)$	$(-1000, -1000)$

Nash equilibrium

Definition 5 (Nash equilibrium)

A profile of strategies $a^* = (a_1^*, \dots, a_i^*, \dots, a_N^*) \in \mathcal{A}$ is a ***Nash equilibrium*** (NE) if

$$a_i^* \in \mathrm{BR}_i(a_{-i}^*) \text{ for all } i \in \mathcal{N}$$

or, equivalently, if

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } i \in \mathcal{N} \text{ and for all } a_i \in \mathcal{A}_i.$$

Properties

- **Stability**: No player has any incentive to change their current strategy. In other words, no player regrets their decision in an NE.
- **Unilateral resilience**: An NE is resilient to **uni**lateral deviations, but it may not be stable for **multi**-lateral deviations.
- **Non-uniqueness**: In many games, there are multiple Nash equilibria.

Return to Examples 1-3

Nash equilibrium for each example?

Example 1: The Prisoner's Dilemma

		Julian	
		C	D
Thomas	C	$(-1,-1)$	$(-5,0)$
	D	$(0,-5)$	$(-3,-3)$

Example 2: Split or Steal?

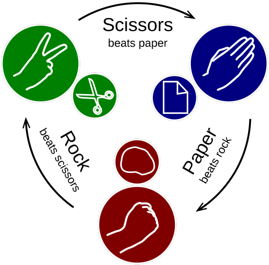
		Lucy	
		Sp	St
Tony	Sp	$(£33442.5, £33442.5)$	$(£0, £66885)$
	St	$(£66885, £0)$	$(£0, £0)$

Example 3: Chicken game

		Driver 2	
		Sw	St
Driver 1	Sw	$(0,0)$	$(-1,1)$
	St	$(1,-1)$	$(-1000,-1000)$

Return to the rock-paper-scissors game

		Player 2		
		R	P	S
Player 1	R	$(0,0)$	$(-1,1)$	$(1,-1)$
	P	$(1,-1)$	$(0,0)$	$(-1,1)$
	S	$(-1,1)$	$(1,-1)$	$(0,0)$



Nash equilibrium?

Summary

This lesson

- Finite games: definition + examples
- Strategic dominance: strict, weak, iterated
- Best responses and Nash equilibrium

Next lesson

- Mixed strategies
- Nash's existence theorem
- ...