Assignment 3

1. Suppose that $M = (Q, \Sigma, \delta, q_0, A)$ and let $\delta^\star : Q \times \Sigma^\star \to Q$ be its extended transition function. Prove that for any $q \in Q$ and $x, y \in \Sigma^\star$ that $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$. (Hint: use induction on the length of x).

   Proof. Prove by induction on the length of x
   Base case: $\mid x \mid = 0$
   We have $x = \epsilon$, then $\delta^\star(q, xy) = \delta^\star(q, \epsilon y)$
   Then $\delta^\star(q, \epsilon y) = \delta^\star(\delta^\star(q, \epsilon), y) = \delta^\star(\delta^\star(q, x), y)$ (Since x = ϵ)

   Inductive Hypothesis: $\mid x \mid = n$
   Let $q \in Q$ and $x, y \in \Sigma^\star$ and $n \in N$.
   Let x = a, we have $\delta^\star(q, ay) = \delta^\star(\delta^\star(q, a), y)$

   Inductive Step: we need to prove for $\mid x \mid = n + 1$
   We can write $x = ai$ where $|a| = n$ and i
   We have: $\delta^\star(q, xy) = \delta^\star(q, aiy)$
   Then $\delta^\star(\delta^\star(q, a), iy) = \delta^\star(\delta^\star(\delta^\star(q, a), i), y) = \delta^\star(\delta^\star(q, ai), y)$ (according to IH)
   Then $\delta^\star(\delta^\star(q, ai), y) = \delta^\star(\delta^\star(q, x), y)$ (Since $x = ai$) Therefore, $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$ for any $q \in Q$ and $x, y \in \Sigma^\star$

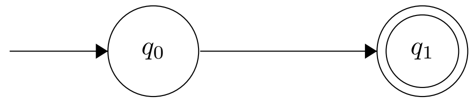
   □

2. Suppose L is a regular language, and $M = (Q, \Sigma, \delta, q_0, A)$ is a deterministic finite state machine such that $L(M) = L$. Prove that if $|Q| = 2$ then one of the following hold: (i) $L = \emptyset$ (ii) $\epsilon \in L$, or (iii) $\exists a \in \Sigma$ such that $a \in L$. (Hint: prove one of the three hold for each possible configuration of A and q_0).

   Proof. Suppose if $|Q| = 2$ then none of the following hold: (i) $L = \emptyset$ (ii) $\epsilon \in L$, or (iii) $\exists a \in \Sigma$ such that $a \in L$.

   Case 1: There is no accepting states or no paths from state q_0 to q_1. In this case, $L = \emptyset$ holds since there is no language that machine accepts or accepting states.

   Case 2: There is one accepting state. Since M is a deterministic finite machine, then $\exists \delta(q_0, a) = q_1$ that if the automaton is in state q_0 and reads input a then it goes to state q_1. Therefore, $\exists a \in \Sigma$ holds such that $a \in L$



Case 3: There are 2 accepting states. If the automaton is in state q_0, it already in accepting state and does not need to receive any inputs.Therefore, $\epsilon \in L$ holds.

There is a contradiction with what we suppose. Therefore, when $|Q| = 2$ then one of the following hold: (i) $L = \emptyset$ (ii) $\epsilon \in L$, or (iii) $\exists a \in \Sigma$ such that $a \in L$. □

3. If L is a finite-size language then L is a regular language, meaning that it can be accepted by a finite state machine. Prove this by defining how to build—for any finite-size language L—a finite state machine M that accepts L and prove that $L(M) = L$. (Hint: a finite-size language has the notion of the "longest" word and a finite state machine can have *a lot* of states as long as its a finite number.)
   1) Since L is a finite size language, L is a regular language, we can build a machine that accepts L.
   If a finite-size language has the notion of the longest word and the finite state machine can have a lot of states - finite number, we can start from q_0 and add states after each inputs, until we reach the accepting states.

   Define the construction for a machine:
   Q is a finitenon empty sets of states
   Σ is a finite non empty alphabet of symbols
   q_0 is an initial state
   A is a set of accepting states
   $\delta^\star(q, x) = p$ where $x \in \Sigma^\star$ and $q \in Q$ is a transition function from each states.
   We know that if the automaton is in state q and reads input x then it goes to state p. The machine takes input as a state and a word, output a state such that the result of applying the single instance of the transition function to each letter keeping track of the state along the way.

   2) Prove that $L(M) = L$
   Assume $x \in L(M)$.
   Since $L(M) = \{x \in \Sigma^\star : \delta^\star(q_0, x) \in A\}$, there exists a transition function $\delta^\star(q_0, x) = A$.
   Since L is a finite size regular language, there is a machine that will accept it.

We know x is arbitrary number and x has to be in some L. Then $x \in L$. Therefore, $L(M) \in L$ for L is some regular language.

Assume $x \in L$. Let x and L are arbitrary.
Since L is a finite size language and regular language, which means that there exists a Deterministic Finite Automata that accepts the language. Then there exist a transition function $\delta^{\star}(q_0, x) = A$ that transition to accepting states with some input x. Therefore, $x \in L(M)$, then $L \in L(M)$.

Therefore, $L(M) = L$