Assignment 2

1. Prove with induction that $|w^n| = n \cdot |w|$.

   Proof. Base case: $|w^0| = |\epsilon| = 0 = 0 \cdot |w|$
   Inductive Hypothesis: $\forall k < n$
   $|w^k| = k \cdot |w|$
   Inductive Step:
   $|w^(k+1)| = |w^k| + |w| = k \cdot |w| + |w|$ (according to Inductive Hypothesis)
   $k \cdot |w| + |w| = (k+1) \cdot |w|$
   Therefore, $|w^n| = n \cdot |w|, \forall n$ $\square$

2. Prove $w^i = w$ if and only if $w = \epsilon$.

   Proof. We need to prove 2 parts
   $\star$ If $w^i = w$ then $w = \epsilon$
   We prove this by contradiction. Suppose w is a word and $i \in N$
   Suppose $w^i = w$ and suppose that $w \neq \epsilon$.
   Since $w \neq \epsilon$, $\exists x \in w$
   If $x \in w$, $x \in w^0$ since $w = w^i$. We also have: $w^0 = \epsilon$
   Therefore, $x = \epsilon$.
   This contradicts our assumption. Therefore, $w = \epsilon$

   $\star$ If $w = \epsilon$ then $w^i = w$
   Since $w = \epsilon$, then $w^1 = \epsilon$. Because we concatenate ϵ to w.
   Then $w^2 = \epsilon$ since we concatenate ϵ to w^1
   ...
   Therefore, $w^i = \epsilon \ \forall i$

   Since we prove: If $w^i = w$ then $w = \epsilon$ and If $w = \epsilon$ then $w^i = w$. It follows that $w^i = w$ if and only if $w = \epsilon$ $\square$

3. Suppose you have two languages L_1 and L_2 over the alphabet $\{a, b\}$. Give an example of L_1 and L_2 such that $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$. For all possible choices of L_1, L_2, what is the smallest value of $|L_1| + |L_2|$ such that $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$? Show why this is true.

   Example: Let $L_1 = \{a, \epsilon\}, L_2 = \{a, \epsilon\}$
   Then: $L_1 \cdot L_2 = \{\epsilon, a, aa\}$
   $|L_1 \cdot L_2| = 3$
   $|L_1| \cdot |L_2| = 2 \cdot 2 = 4$
   Therefore, $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$
   The smallest value of $|L_1| + |L_2|$ such that $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ is 4.

Proof. We prove this by Case Analysis Technique.
Case 1: $|L_1| + |L_2| = 0$
This implies that $|L_1| = 0$ and $|L_2| = 0$.
We have $|L_1 \cdot L_2| = 0$ and $|L_1| \cdot |L_2| = 0 \cdot 0 = 0$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $0 = 0$

Case 2: $|L_1| + |L_2| = 1$
This implies that ($|L_1| = 1$ and $L_2 = 0$) or ($L_1 = 0$ and $|L_2| = 1$).
The only word we concatenate is word from L_1 or L_2.
We have only 1 word concatenated from $L_1 \cdot L_2$, then $|L_1 \cdot L_2| = 1$ and $|L_1| \cdot |L_2| = 1 \cdot 0 = 0$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $1 > 0$

Case 3: $|L_1| + |L_2| = 2$
3a) Let $|L_1| = 1$ and $|L_2| = 1$
We have 1 word concatenated from $L_1 \cdot L_2$, then $|L_1 \cdot L_2| = 1$ and $|L_1| \cdot |L_2| = 1 \cdot 1 = 1$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $1 = 1$
3b) Let ($|L_1| = 0$ and $|L_2| = 2$) or ($|L_1| = 2$ and $|L_2| = 0$)
Since $|L_1| = 0$ and $|L_2| = 2$, $|L_2|$ has 2 distinct words, we still have 2 words after concatenating.
Therefore, we have 2 words concatenated from $L_1 \cdot L_2$, then $|L_1 \cdot L_2| = 2$ and $|L_1| \cdot |L_2| = 2 \cdot 0 = 0$ (similar to the case$|L_1| = 2$ and $|L_2| = 0$)
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $2 > 0$

Case 4: $|L_1| + |L_2| = 3$
4a) Let ($|L_1| = 0$ and $|L_2| = 3$) or ($|L_1| = 3$ and $|L_2| = 0$)
We will have 3 words from L_1 or L_2 that does not change. Then $|L_1 \cdot L_2| = 3$ and $|L_1| \cdot |L_2| = 3 \cdot 0 = 0$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $0 = 0$
4b) Let ($|L_1| = 1$ and $|L_2| = 2$) or ($|L_1| = 2$ and $|L_2| = 1$)
Since $|L_1| = 1$ and $|L_2| = 2$, $|L_2|$ has 2 distinct words, we concatenate 2 words with 1 word from L_1, we will still have 2 different words.
Therefore, we have 2 words concatenated from $L_1 \cdot L_2$, then $|L_1 \cdot L_2| = 2$ and $|L_1| \cdot |L_2| = 2 \cdot 1 = 2$. (similar to the case$|L_1| = 2$ and $|L_2| = 1$)
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $2 = 2$

Case 5: $|L_1| + |L_2| = 4$
5a) Let ($|L_1| = 0$ and $|L_2| = 4$) or ($|L_1| = 4$ and $|L_2| = 0$)
Since $|L_1| = 0$ and $|L_2| = 4$, $|L_2|$ has 4 distinct words, we concatenate 4 words from L_1 or L_2.
Therefore, we have $|L_1 \cdot L_2| = 4$ and $|L_1| \cdot |L_2| = 4 \cdot 0 = 0$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $4 > 0$
5b) Let ($|L_1| = 1$ and $|L_2| = 3$) or ($|L_1| = 3$ and $|L_2| = 1$)
Since $|L_1| = 1$ and $|L_2| = 3$, $|L_2|$ has 3 distinct words, we concatenate 3

words with 1 word from L_1, we will still have 3 different words.
Therefore, we have 3 words concatenated from from $L_1 \cdot L_2$, then $|L_1 \cdot L_2| = 3$ and $|L_1| \cdot |L_2| = 3 \cdot 1 = 3$. (similar to the case$|L_1| = 3$ and $|L_2| = 1$)
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ does not hold since $3 = 3$
5c) Let $|L_1| = 2$ and $|L_2| = 2$
Since we could have repetition from this case, we can have 3 words concatenated from $L_1 and L_2$
Therefore, $|L_1 \cdot L_2| = 3$ and $|L_1| \cdot |L_2| = 2 \cdot 2 = 4$
Then the condition $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$ holds since $3 < 4$

Therefore, the smallest value of $|L_1| + |L_2|$ is 4 such that $|L_1 \cdot L_2| < |L_1| \cdot |L_2|$.

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