Assignment 3

** All the proofs in this assignment are screenshots from proof checker **

1. For each item below, provide a Fitch-style proof of the claim.

(a) ⊢ ∀xA(x) ↔ ∀x(A(x) ∨ A(x))

Due to syntax error, replaced A with P in proof checker

1	∀xPx	
2	Pa	∀E 1
3	Pa ∨ Pa	∨I 2
4	∀x (Px ∨ Px)	∀I 3
5	∀x (Px ∨ Px)	
6	Pa ∨ Pa	∀E 5
7	Pa	
8	Pa	R 7
9	Pa	∨E 6, 7–8, 7–8
10	∀x Px	∀I 9
11	∀x Px ↔ ∀x (Px ∨ Px)	↔I 1–4, 5–10

(b) ⊢ ∀x∀y(x = y → (G(x,y) ↔ G(y, x)))

1	a = b	
2	Gab	
3	Gbb	=E 1, 2
4	Gba	=E 1, 3
5	Gba	
6	Gbb	=E 1, 5
7	Gab	=E 1, 6
8	Gab ↔ Gba	↔I 2–4, 5–7
9	a = b → (Gab ↔ Gba)	→I 1–8
10	∀y(a = y → (Gay ↔ Gya))	∀I 9
11	∀x∀y(x = y → (Gxy ↔ Gyx))	∀I 10

(c) ∀x(F(x) ∨ G(x)) ⊢ ∃x ¬F(x) → ∃xG(x)

1	$\forall x(Fx \lor Gx)$	
2	$\mid\ \exists x \neg Fx$	
3	$\mid\ \mid\ \neg Fa$	
4	$\mid\ \mid\ Fa \lor Ga$	∀E 1
5	$\mid\ \mid\ \mid\ Fa$	
6	$\mid\ \mid\ \mid\ \bot$	¬E 3, 5
7	$\mid\ \mid\ \mid\ Ga$	X 6
8	$\mid\ \mid\ \mid\ Ga$	
9	$\mid\ \mid\ \mid\ Ga$	R 8
10	$\mid\ \mid\ Ga$	∨E 4, 5–7, 8–9
11	$\mid\ \mid\ \exists x Gx$	∃I 10
12	$\mid\ \exists x Gx$	∃E 2, 3–11
13	$\exists x \neg Fx \to \exists x Gx$	→I 2–12

(d) ∀z[G(z) → ∀y(K(y) → H (z,y))], (K(i) ∧G(j)) ∧ i = j ⊢ H (i,i)

1	$\forall z[Gz \to \forall y(Ky \to Hzy)]$	
2	$(Ki \land Gj) \land i = j$	
3	$Gj \to \forall y(Ky \to Hjy)$	∀E 1
4	$i = j$	∧E 2
5	$Ki \land Gj$	∧E 2
6	Ki	∧E 5
7	Gj	∧E 5
8	$\forall y(Ky \to Hjy)$	→E 3, 7
9	$Ki \to Hji$	∀E 8
10	Hji	→E 9, 6
11	Hii	=E 4, 10

2. For each of the following claims:

i) If it is true, provide a Fitch-style proof of the claim

ii) If it is not true, provide an interpretation that is a counterexample to the claim.

(a) $\exists xF(x, a), \forall y(y = a \rightarrow y = b) \vdash \exists yF(y,y)$

The claim is false. Consider the following counterexample:

Domain: 0,1

a: 1 b: 1

F(x,y) is true of ⟨0,1⟩

We have:

- 1st premise: There exists x in domain (x = 0) that F(x,a) is true because F(x,y) is true of ⟨0,1⟩. ∃xF(x,a) is true.
- 2nd premise: For all y in domain, if y = a = 1 then y = b = 1, which is also true.
- Conclusion: There exists y in domain (y=1) that F(y,y) which is F(1,1) false.

⇨ Therefore, we have premises are true and conclusion is false. The claim is false.

(b) $L(a) \leftrightarrow \forall xL(x) \dashv\vdash \exists xL(x)$

The claim is false. Consider the following counterexample:

Domain: 0,1

a: 0

L(x) is true of 0

We have:

- L(x) is true of 0, then L(a) is true.
- For all x in domain, let x = 1. L(1) is false since L(x) is true of 0. ∀xL(x) is false.
  - ⇨ The biconditional is false since L(a) is true and ∀xL(x) is false.
- Conclusion: There exists y in domain (x=0) then L(x) is true. The conclusion is true.

⇨ The premise is true and the conclusion is false. The claim is false.

(c) $\forall x(F(x) \rightarrow \exists y(G(y, x) \land \neg G(x,y))), \exists xF(x) \vdash \exists x\exists y\neg x = y$

The claim is true.

Proof:

1	$\forall x[Fx \rightarrow \exists y(Gyx \wedge \neg Gxy)]$	
2	$\exists x Fx$	
3	$Fa \rightarrow \exists y(Gya \wedge \neg Gay)$	∀E 1
4	$\quad Fa$	
5	$\quad \exists y(Gya \wedge \neg Gay)$	→E 3, 4
6	$\quad\quad Gba \wedge \neg Gab$	
7	$\quad\quad\quad a = b$	
8	$\quad\quad\quad Gba$	∧E 6
9	$\quad\quad\quad \neg Gab$	∧E 6
10	$\quad\quad\quad Gaa$	=E 7, 8
11	$\quad\quad\quad \neg Gaa$	=E 7, 9
12	$\quad\quad\quad \perp$	¬E 10, 11
13	$\quad\quad \neg a = b$	¬I 7–12
14	$\quad\quad \exists y \neg a = y$	∃I 13
15	$\quad\quad \exists x \exists y \neg x = y$	∃I 14
16	$\quad \exists x \exists y \neg x = y$	∃E 5, 6–15
17	$\exists x \exists y \neg x = y$	∃E 2, 4–16

(d) ∀xF(x, a) ↔ ¬∀x¬G(x, b), ∃x(F(x, a) ∧ ¬G(x, b)) ⊢ ⊥

The claim is false. Consider the following counterexample:

Domain: 0,1

a: 0 b: 1

F(x,y) is true of ⟨0,0⟩, ⟨1,0⟩

G(x,y) is true of ⟨0,1⟩

We have:

- 1st premise: For all x in domain (x = 0 or x = 1), F(x,a) is true because F(x,a) is true of ⟨0,0⟩, ⟨1,0⟩.
  Not all x in domain such that ¬G(x, b), let x = 1. Then G(1, 1) is false, ¬G(1, 1) is true. ¬∀x¬G(x, b) is true. Therefore, the 1st premise are true.

- 2nd premise: There exists x in domain (x=1) such that $F(x, a) \land \neg G(x, b)$ is true. Since $F(1,0) \land \neg G(0,1)$ is true, $F(x, a) \land \neg G(x, b)$ is true.

⇨ We have both premises are true, the conclusion is false. The claim is false.

(e) $\forall y(H(y) \land (J(y,y) \land M(y))) \vdash \exists x\, J(x, b) \land \forall x M(x)$

The claim is true. Proof:

1	$\forall y[Hy \land (Jyy \land My)]$	
2	$Hb \land (Jbb \land Mb)$	∀E 1
3	Hb	∧E 2
4	$Jbb \land Mb$	∧E 2
5	Jbb	∧E 4
6	Mb	∧E 4
7	$\forall x\, Mx$	∀I 6
8	$\exists x\, Jxb$	∃I 5
9	$\exists x Jxb \land \forall x Mx$	∧I 7, 8

3. Find an example of an invalid argument where only ∀I is used improperly because the name that is quantified away still remains in the formula

Example of invalid argument:

Everyone loves themselves; therefore everyone loves Alice.

$\forall x\, L(x,x) \therefore \forall x\, L(x, a)$

1	$\forall x Lxx$	
2	Laa	∀E 1
3	$\forall x\, Lxa$	∀I 2

It is true that everyone loves themselves, but not everyone loves Alice. The argument is invalid. **∀I** is used improperly because the name that is quantified is still remains in the formula at line 3.

Bonus: Provide an informal proof of the following claim:

$\forall x \forall y(R(x,y) \rightarrow R(y, x)), \forall x \forall y \forall z((R(x,y) \land R(y,z)) \rightarrow R(x,z)), \forall x \exists y R(x,y) \models \forall x R(x, x)$

Informal Proof:

Let I be an interpretation such that:

1. $I \models \forall x \forall y(R(x,y) \rightarrow R(y, x))$
2. $I \models \forall x \forall y \forall z((R(x,y) \land R(y,z)) \rightarrow R(x,z))$
3. $I \models \forall x \exists y R(x,y)$

For every objects in D, the domain of the interpretation, there is an object that it relates to by I(R) according to 3. Assume D is finite. Since they are only finitely many objects and each object must relate to some object, there is an object a relates to an object b.

Since a relates to b and I(R) is symmetric (according to 1), b relates to a. That means both (a,b) and (b,a) are in I(R).

By transitivity for (a,b) and (b,a), (a,a) is also in I(R) (according to 2). Therefore, every object relates to itself by I(R).