

1. Refer to Definition 1.10. Show that the divisibility relation $|$ makes the set N of natural numbers a partially ordered set.

Reflexivity:

Because every number $x \in N$ can divide itself. Hence, the divisibility relation is reflexive.

Transitivity:

If $a|b$ and $b|c$ for $a, b, c \in N$. Then $b = a.k$ and $c = b.m$ and $c = a.k.m$. Therefore, c can divide a . Hence, the relation $|$ is transitivity.

Antisymmetry:

If $a|b$ with $a, b \in N$, $a < b$. Hence, a cannot divide b . Therefore, the relation $|$ is antisymmetric.

Hence, the relation $|$ is a partially ordered set.

2. Explain why the divisibility relation $|$ does not define a partial ordering on the set Z of integers.

For $x = -1$ and $y = 1$. $x|y$ and also $y|x$. Hence, the relation is not antisymmetric. Therefore, the relation is not a partial ordering set.

3. Consider the poset $(N, |)$. Are there any minimal elements? Are there any maximal elements? Explain.

Because $N = \{1, 2, 3, 4, \dots, \infty\}$. The minimal element is 1 and there is no maximal elements.

4. Let $A = \{a, b, c, \dots, z\}$. In the poset $(P(A), \subset)$, find a pair of incomparable elements. A pair of incomparable elements is $(\{a, b, c\}, \{d, e, f\})$.

5. Let W be the set of all web pages. For $x, y \in W$, let xRy if you can navigate from x to y by following links (Let's say it's always possible to "navigate" from a page to itself; just do nothing.) Explain why R is not a partial ordering.

Let $x, y \in W$, it is possible to navigate from x to y and from y to x . Hence, xRy and yRx . Therefore, R is not antisymmetric and not a partial ordering set.

6. Let a relation R be defined on the set of real numbers as follows:

$$xRy \Leftrightarrow 2x + y = 3$$

Prove that this relation is antisymmetric.

Let: $y = 3 - 2x$

For yRx :

$$yRx \Leftrightarrow 2y + x = 3$$

$$2(3 - 2x) + x = 3$$

$$6 - 4x + x = 3$$

$$-3x = -3$$

$$x = 1$$

$$y = 3 - 2(1) = 1$$

Hence, $x = y$.

Therefore, the relation is antisymmetric.

7. Explain why the relation R on $\{0, 1, 2, 3\}$ given by

$$R = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 2), (2, 3), (0, 2)\}$$

is not a partial ordering on $\{0, 1, 2, 3\}$. Be specific.

Because $1R2$ and $2R3$ but there is no relation between 1 and 3. Hence, the relation R is not transitive. Therefore, the relation is not a partially ordering set.

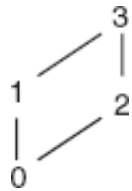
8. Explain why the relation R on $\{0, 1, 2, 3\}$ given by

$$R = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 2), (0, 2), (2, 1)\}$$

is not a partial ordering on $\{0, 1, 2, 3\}$. Be specific.

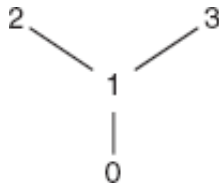
Because $1R2$ and $2R1$, the relation is not fully antisymmetric. Hence, the relation is not partial ordering.

9. The Hasse diagram below defines a partial ordering on the set $\{0, 1, 2, 3\}$. Give the set of ordered pairs corresponding to this relation.



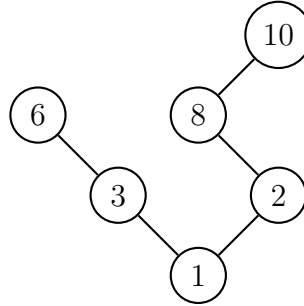
$$R = \{(0, 1), (1, 3), (0, 2), (2, 3), (0, 3), (0, 0), (1, 1), (2, 2), (3, 3)\}$$

10. The Hasse diagram below defines a partial ordering on the set $\{0, 1, 2, 3\}$. Give the set of ordered pairs corresponding to this relation.



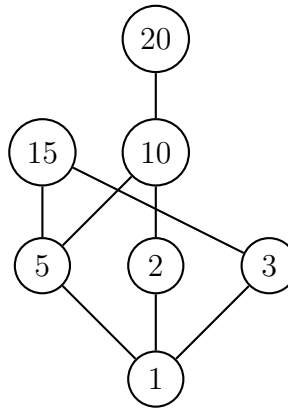
$$R = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (0, 0), (1, 1), (2, 2), (3, 3)\}$$

11. The divides relation “ \mid ” defines a partial ordering on the set $\{1, 2, 3, 6, 8, 10\}$. Draw the Hasse diagram for this poset. What are the maximal elements?



The maximal elements are 6 and 10.

12. Let $S = \{1, 2, 3, 5, 10, 15, 20\}$. It is a fact that (S, \mid) is a poset. Draw its Hasse diagram.



13. Let X be a set of different nonzero monetary values (in U.S. or Canadian cents). In other words, $X \subseteq \mathbb{N}$. Define a relation \models on X as follows. For $a, b \in X$, $a \models b$ if b can be obtained from a by adding a (possibly empty) collection of dimes (10 cents) and quarters (25 cents). So, for example, $25 \models 35$, but $25 \not\models 30$. Prove that \models is a partial ordering on X .

Reflexive:

For every $a \in X$, $a \models a$ because a can add a empty collection of dimes and quarters to become a .

Antisymmetric:

For every $a, b \in X$, if $a \models b$ and $a \neq b$, b must be larger than a . Hence, b cannot become a by adding a collection of dimes and quarters. Therefore, the relation is antisymmetric.

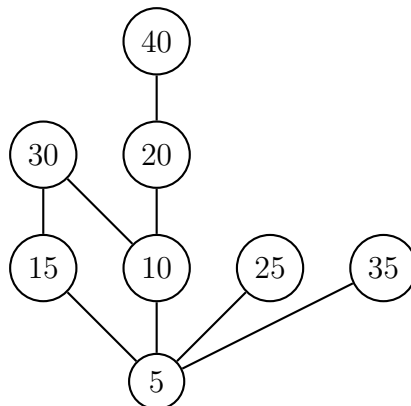
Transitivity:

For every $a, b, c \in X$, if $a \models b$ then $a + 10k + 25z = b$ and if $b \models c$ then $b + 10m + 25n = c$. Then a can become c by adding $10k + 25z + 10m + 25n$. Hence, $a \models c$. Therefore, the relation is transitive.

Therefore, the relation is partial ordering.

14. Let $X = \{5, 10, 15, 20, 25, 30, 35, 40\}$, and let \models be as in Problem 13.

(a) Draw the Hasse diagram for the poset (X, \models) .



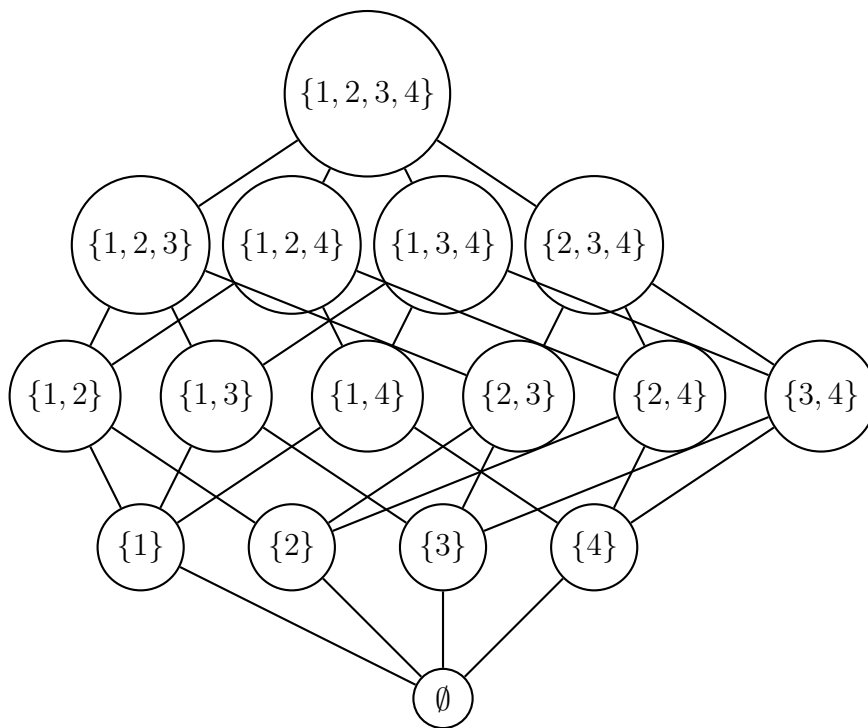
(b) List all the minimal elements of (X, \models) .

The minimal element of (X, \models) is 5.

(c) Give a pair of incomparable elements in (X, \models) .

A pair of incomparable elements in (X, \models) is $(20, 25)$.

19. Let $X = \{1, 2, 3, 4\}$. Draw the Hasse diagram for the poset $((X), \subset)$.



22. Let B be the set of all four-digit binary strings; that is,

$$B = \{0000, 0001, 0010, 0011, \dots, 1111\}$$

Define a relation \triangleleft on B as follows: Let $x, y \in B$, where $x = x_1x_2x_3x_4$ and $y = y_1y_2y_3y_4$. We say that $x \triangleleft y$ if $x_i \leq y_i$ for $i = 1, 2, 3, 4$. In other words, $x \triangleleft y$ if y has a 1 in every position where x does. So, for example, $0101 \triangleleft 0111$ and $0000 \triangleleft 0011$, but $1010 \not\triangleleft 0111$. The relation \triangleleft is called the bitwise \leq . Show that (B, \triangleleft) is a poset.

Reflexive:

For every $a = a_1a_2a_3a_4 \in B$, $a \triangleleft a$ because $a_i \leq a_i$ for $i = 1, 2, 3, 4$. Hence, the relation is reflexive.

Antisymmetric:

For every $a = a_1a_2a_3a_4 \in B$ and $b = b_1b_2b_3b_4 \in B$, if $a \triangleleft b$ and $b \triangleleft a$ then $a = b$ because $a_1a_2a_3a_4 \leq b_1b_2b_3b_4$ and $b_1b_2b_3b_4 \leq a_1a_2a_3a_4$. Hence, the relation is antisymmetric.

Transitivity:

For every $a, b, c \in B$ and $a = a_1a_2a_3a_4, b = b_1b_2b_3b_4, c = c_1c_2c_3c_4$. If $a \triangleleft b$ and $b \triangleleft c$, then $a_i \leq b_i$ and $b_i \leq c_i$ for $i = 1, 2, 3, 4$. Hence, $a_i \leq c_i$ and $a \triangleleft c$. Hence, the relation is transitive.

Therefore, the (B, \triangleleft) is a poset.

23. Prove that $(B, \triangleleft) \cong (P(\{1, 2, 3, 4\}), \subseteq)$.

We can define a function $f : B \rightarrow P(\{1, 2, 3, 4\})$ with that maps each $x = x_1x_2x_3x_4 \in B$ if $x_i = 1$ for $i = 1, 2, 3, 4$ then $i \in f(x)$. Because every elements of B maps exactly to one element of $P(\{1, 2, 3, 4\})$, f is one-to-one correspondence. And because of \triangleleft and \subseteq behave exactly the same for the two sets. Hence, the edges in the Hasse diagram for (B, \triangleleft) correspond exactly to the edges in the Hasse diagram for $(P(\{1, 2, 3, 4\}), \subseteq)$. Therefore, $(B, \triangleleft) \cong (P(\{1, 2, 3, 4\}), \subseteq)$.

24. In (B, \triangleleft) , give a counterexample to show that

$$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1001, 1000, 1010, 0011, 1100, 1101, 1110, 1111$$

is not a valid topological sort of the elements of B .

Because 0011 is not the minimal element if 0100 is not deleted but 0011 is standing before 0100, this is not a valid topological sort of the elements of B .

25. Perform a topological sort on the elements of B .

A topological sort on the elements of B is:

$$0000, 0001, 0010, 0100, 1000, 0011, 0101, 0110, 1001, 1010, 1100, 0111, 1110, 1101, 1110, 1111$$

26. Let $F \subseteq N$ be the set of all factors of 210. In the poset $(F, |)$, find the following.

- (a) $30 \wedge 21$, the meet of 30 and 21.
 $30 \wedge 21 = \gcd(30, 21) = 3$.

- (b) $35 \vee 15$, join of 35 and 15.
 $35 \vee 15 = lcm(35, 15) = 105.$
- (c) $2 \wedge 7$.
 $2 \wedge 7 = gcd(2, 7) = 1.$
- (d) $2 \vee 7$.
 $2 \vee 7 = lcm(2, 7) = 14.$
- (e) $\neg 30$, the complement of 30.

$$\neg 30 = x \text{ with } (30 \wedge x) = 1 \text{ and } (30 \vee x) = 210$$

$$30 = 2 \times 3 \times 5$$

$$210 = 2 \times 3 \times 5 \times 7$$

Hence, $x = 7$ and $\neg 30 = 7$.