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Calculus 1
Section: 04
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Section 2.1:

5. Find an equation of the tangent line to the curve at the given point.

$$y = 2x^2 - 5x + 1$$
$$(3, 4)$$

The slope of the tangent line at (3, 4) is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 5(x+h) + 1 - 2x^2 + 5x - 1}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 5x - 5h - 2x^2 + 5x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{4xh + 2h^2 - 5h}{h}$$

$$f'(x) = \lim_{h \to 0} (4x + 2h - 5)$$

$$m = f'(x) = 4x$$

$$m = f'(3) = 4 \times 3 = 12$$

The equation of the tangent line at (3, 4) is:

$$y - 4 = 4(x - 3)$$

$$\boxed{y = 4x - 8}$$

11. A cliff diver plunges from a height of $100ft$ above the water surface. The distance the diver falls in t seconds is given by the function $d(t) = 16t^2(ft)$.

(a) After how many seconds will the diver hit the water?
The diver hit the water when the distance equals the height above the water surface:

$$100 = 16t^2$$

$$6.25 = t^2$$

$$t = \pm 2.5 \ (second)$$

$$\boxed{\text{There are no negative time, so the diver will hit the water after 2.5 second.}}$$

(b) With what velocity does the diver hit the water?
The equation of the velocity is:

$$d'(t) = \lim_{h \to 0} \frac{d(t+h) - d(t)}{h}$$

$$d'(t) = \lim_{h \to 0} \frac{16(t+h)^2 - 16t^2}{h}$$

$$d'(t) = \lim_{h \to 0} \frac{16t^2 + 32th + 16h^2 - 16t^2}{h}$$

$$d'(t) = \lim_{h \to 0} \frac{32th + 16h^2}{h}$$

$$d'(t) = \lim_{h \to 0} (32t + 6h)$$

$$d'(t) = 32t$$

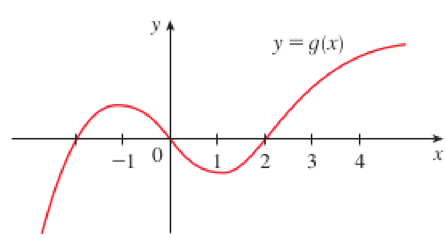
The diver hit the water after 2.5 second:

$$d'(t) = 32.5$$

$$\boxed{d'(t) = 80(ft/s)}$$

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



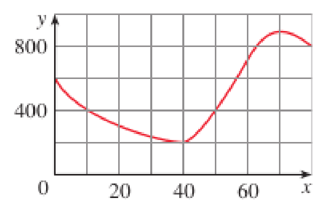
Slope of the tangent line are considered as following:

1. Slope at x = -2: The graph is increasing steeply, so the slope is positive and quite large.
2. Slope at x = 0: The graph is decreasing steeply, so the is negative.
3. Slope at x = 2: The graph is increasing steeply, so the slope is positive but it's not as large as x = -2.
4. Slope at x = 4: The graph is increasing gradually, so the slope is positive but it's smaller than x = 2.

Thus the increasing order is:

$$\boxed{g'(0) \quad 0 \quad g'(4) \quad g'(2) \quad g'(-2)}$$

19. The graph of a function f is shown.



(a) Find the average rate of change of f on the interval $[20, 60]$.
The average rate of change on the interval $[20, 60]$ is:

$$\frac{f(60) - f(20)}{60 - 20} = \frac{700 - 300}{60 - 20} = \boxed{100}$$

(b) Identify an interval on which the average rate of change of f is 0.
An interval on which the average rate of change of f is 0 is $\boxed{[10, 50]}$ because:

$$\frac{f(50) - f(10)}{50 - 10} = \frac{400 - 400}{50 - 210} = 0$$

(c) Compute and explain what does this value represent geometrically?

$$\frac{f(40) - f(10)}{40 - 10} = \boxed{\frac{-200 - 400}{40 - 10} = -200}$$

$\boxed{\text{This value represents the average rate of change of } f \text{ on the interval } [10, 40].}$

(d) Estimate the value of $f'(50)$.
The value of $f'(50)$ is approximately equal to the average of change on the interval $[40, 60]$:

$$f'(50) = \frac{f(60) - f(40)}{60 - 40} = \frac{700 - 200}{60 - 40} = \boxed{25}$$

(e) Is $f'(10) > f'(30)$?
The value of $f'(10)$ is approximately equal to the average of change on the interval $[0, 20]$:

$$f'(10) = \frac{f(20) - f(0)}{20 - 0} = \frac{300 - 600}{20 - 0} = -15$$

The value of $f'(30)$ is approximately equal to the average of change on the interval $[20, 40]$:

$$f'(30) = \frac{f(40) - f(20)}{40 - 20} = \frac{200 - 300}{40 - 20} = -5$$

$\boxed{\text{Hence, the statement is wrong because } f'(10) < f'(30).}$

(f) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.

The value of $f'(60)$ is approximately equal to the average of change on the interval $[50, 70]$:

$$f'(60) = \frac{f(70) - f(50)}{70 - 50} = \frac{900 - 400}{70 - 50} = 25$$

$$\frac{f(80) - f(40)}{80 - 40} = \frac{800 - 200}{80 - 40} = 15$$

$\boxed{\text{Hence, the statement is correct.}}$

21. Use Equation 5 to find $f'(a)$ at the given number a.

$$f(x) = \frac{x^2}{x+6}, a = 3$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{\frac{x^2}{x+6} - \frac{a^2}{a+6}}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{\frac{x^2(a+6) - a^2(x+6)}{(x+6)(a+6)}}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{x^2 a + 6x^2 - a^2 x - 6a^2}{(x - a)(x+6)(a+6)}$$

$$f'(a) = \lim_{x \to a} \frac{x^2 a - a^2 x + 6x^2 - 6a^2}{(x - a)(x+6)(a+6)}$$

$$f'(a) = \lim_{x \to a} \frac{xa(x - a) + 6(x^2 - a^2)}{(x - a)(x+6)(a+6)}$$

$$f'(a) = \lim_{x \to a} \frac{xa(x - a) + 6(x - a)(x + a)}{(x - a)(x+6)(a+6)}$$

$$f'(a) = \lim_{x \to a} \frac{(x-a)(xa+6x+6a)}{(x-a)(x+6)(a+6)}$$

$$f'(a) = \lim_{x \to a} \frac{(xa+6x+6a)}{(x+6)(a+6)}$$

$$f'(a) = \frac{(a^2+6a+6a)}{(a+6)(a+6)}$$

$$f'(a) = \frac{(a^2+12a)}{(a+6)^2}$$

$$f'(3) = \frac{(3^2+12\times 3)}{(3+6)^2} = \boxed{\frac{5}{9}}$$

23. Find $f'(a)$.

$$f(x) = 2x^2 - 5x + 3$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{2x^2 - 5x + 3 - 2a^2 + 5a - 3}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{2x^2 - 2a^2 - 5x + 5a}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{2(x^2 - a^2) - 5(x - a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{2(x-a)(x+a) - 5(x-a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{(x-a)(2x+2a-5)}{x - a}$$

$$f'(a) = \lim_{x \to a} (2x + 2a - 5)$$

$$f'(a) = 2a + 2a - 5$$

$$\boxed{f'(a) = 4a - 5}$$

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{3(a+h)^2 - (a+h)^3 - 3a^2 + a^3}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 - a^3 - 3a^2h - 3ah^2 - h^3 - 3a^2 + a^3}{h}$$

$$f'(a) = \lim_{h\to 0} \frac{6ah + 3h^2 - 3a^2h - 3ah^2 - h^3}{h}$$
$$f'(a) = \lim_{h\to 0}(6a + 3h - 3a^2 - 3ah - h^2)$$
$$f'(a) = \lim_{h\to 0}(6a + 3h - 3a^2 - 3ah - h^2)$$
$$f'(a) = 6a - 3a^2$$
$$f'(1) = 6 \times 1 - 3 \times 1^2 = 3$$

Equation of the tangent line is:

$$y - 2 = 3(x - 1)$$

$$\boxed{y = 3x - 1}$$

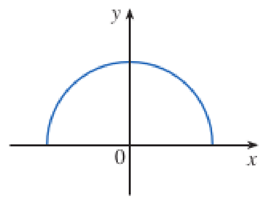
43. Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

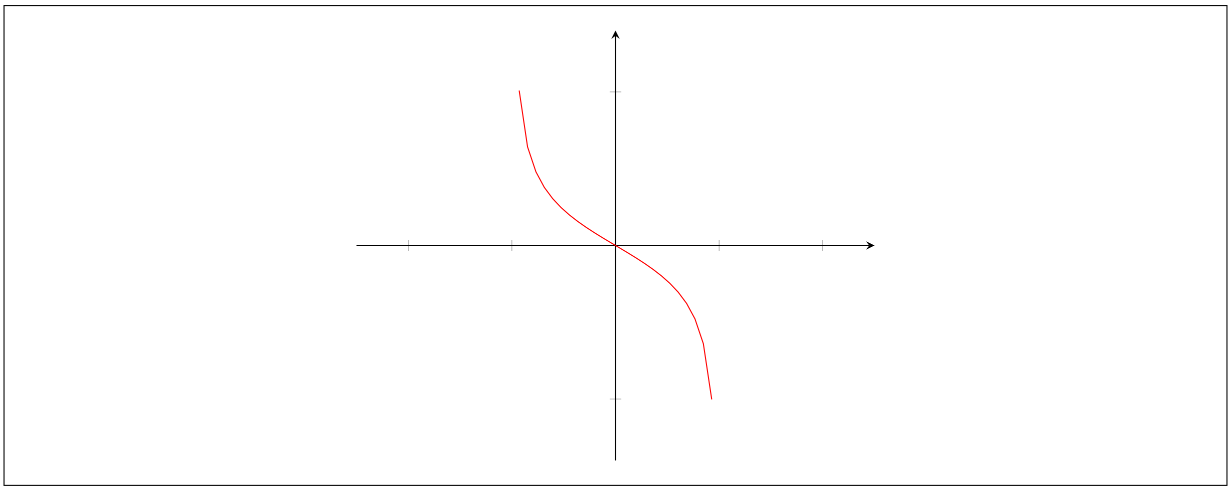
$$\lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h}$$

$$\boxed{f(x) = \sqrt{x}, \quad a = 9}$$

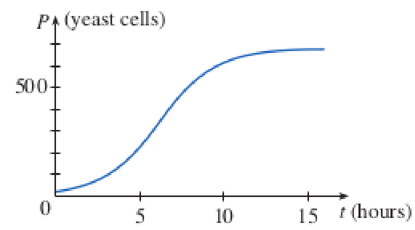
Section 2.2:

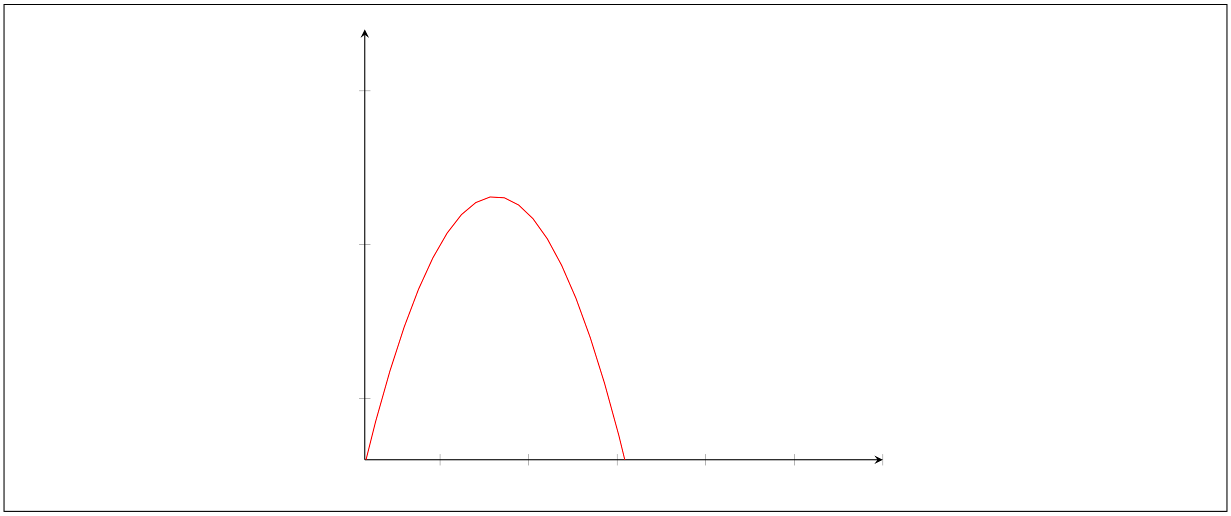
7. Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.





12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?





$\boxed{\text{The yeast population will grow rapidly at first, but will slow down over time.}}$

25. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \frac{1}{x^2 - 4}$$

The domain of $f(x)$ is:

$$x^2 - 4 \neq 0$$

$$x \neq \pm 2$$

Hence, the domain of $f(x)$ is: $\boxed{\mathbb{R} \setminus \{-2, 2\}}$

The derivative of $f(x)$ is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2 - 4} - \frac{1}{x^2 - 4}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x^2 + xh + h^2 - 4} - \frac{1}{x^2 - 4}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^2 - 4 - x^2 - xh - h^2 + 4}{h(x^2 + xh + h^2 - 4)(x^2 - 4)}$$

$$f'(x) = \lim_{h \to 0} \frac{-xh - h^2}{h(x^2 + xh + h^2 - 4)(x^2 - 4)}$$

$$f'(x) = \lim_{h \to 0} \frac{-x - h}{(x^2 + xh + h^2 - 4)(x^2 - 4)}$$

$$f'(x) = \frac{-x}{(x^2 - 4)(x^2 - 4)}$$

$$\boxed{f'(x) = \frac{-x}{(x^2 - 4)^2}}$$

The domain of $f'(x)$ is:

$$(x^2 - 4)^2 \neq 0$$

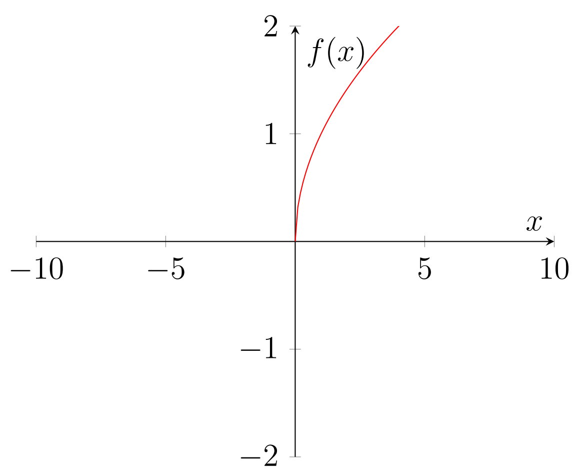
$$x^2 - 4 \neq 0$$

$$x \neq \pm 2$$

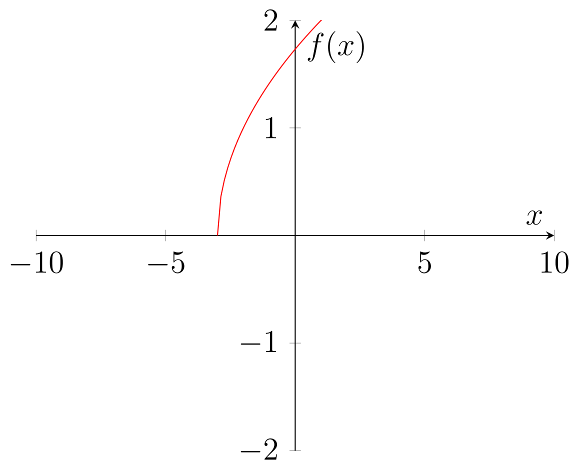
Hence, the domain of $f'(x)$ is: $\boxed{\mathbb{R} \setminus \{-2, 2\}}$

31. Sketch the graph of $f(x) = 1 + \sqrt{x + 3}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

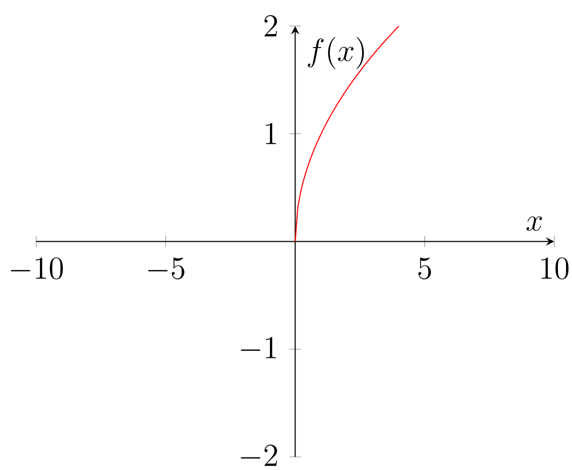
First, we draw $f(x) = \sqrt{x}$.



And then, we draw $f(x+3)$:



Finally, we draw $f(x+3)+1$.



37. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2020.

(a) What does dP/dt represent in this context?

dP/dt is the rates of change of the percentage P of a city's electrical power with respect to t years.

(b) Interpret the statement

$$\frac{dP}{dt}\Big|_{t-2} = 3.5$$

On January 2018, the rates of increasing of the percentage of a city's electrical power is 3.5(%).

55. Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.1.5 to find $f'(a)$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{(\sqrt[3]{x} - \sqrt[3]{a})(\sqrt[3]{x}^2 + \sqrt[3]{a}^2 + \sqrt[3]{ax})}{(x - a)(\sqrt[3]{x}^2 + \sqrt[3]{a} + \sqrt[3]{ax})}$$

$$f'(a) = \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt[3]{x}^2 + \sqrt[3]{a}^2 + \sqrt[3]{ax})}$$

$$f'(a) = \lim_{x \to a} \frac{1}{(\sqrt[3]{x}^2 + \sqrt[3]{a}^2 + \sqrt[3]{ax})}$$

$$f'(a) = \frac{1}{(\sqrt[3]{a}^2 + \sqrt[3]{a}^2 + \sqrt[3]{a^2})}$$

$$\boxed{f'(a) = \frac{1}{3\sqrt[3]{a}^2}}$$

(b) Show that $f'(0)$ does not exist.

$$f'(0) = \frac{1}{3\sqrt[3]{0}^2}$$

$$f'(0) = \frac{1}{0}$$

$\boxed{\text{Because we cannot divide by 0, } f'(0) \text{ does not exist.}}$

(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at (0,0). (Recall the shape of the graph f. See Figure 1.2.13.)
The slope of the tangent line at (0,0) is:

$$f'(0) = \frac{1}{0}$$

Because the slope of the tangent line approachs infinity as x approachs 0, we can conclude that there is a vertical tangent line at (0, 0).