**Minh Anh Nguyen**
**Discrete Math** **Assignment-8**

---

Section 3.1:

1. Refer to the recurrence relation for the Fibonacci sequence in Definition 3.1

   (a) Answer Fibonacci's question by calculating F(12).

   $$F(12) = 144$$

   (b) Write F(1000) in terms of F(999) and F(998).

   $$F(1000) = F(999) + F(998)$$

   (c) Write F(1000) in terms of F(998) and F(997).

   $$F(1000) = 2F(998) + F(997)$$

2. In Fibonaccci's model, rabbits live forever. The following modification of Definition 3.1 accounts for dying rabbits:

   $$G(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } n = 1 \text{ or } n = 2, \\ G(n-1) + G(n-2) - G(n-8) & \text{if } n > 2. \end{cases}$$

   (a) Compute $G(n)$ for $n = 1, 2, \ldots, 12$:

   $$\begin{aligned} G(1) &= 1 \\ G(2) &= 1 \\ G(3) &= 2 \\ G(4) &= 3 \\ G(5) &= 5 \\ G(6) &= 8 \\ G(7) &= 13 \\ G(8) &= 21 \\ G(9) &= 33 \\ G(10) &= 53 \\ G(11) &= 86 \\ G(12) &= 136 \end{aligned}$$

   (b) In this modified model, how long do rabbits live?
   The rabbits live for 8 months.

3. Consider the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } n = 1 \text{ or } n = 2, \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

(a) Compute H(n) for n = 1,2,...10

$$\begin{aligned} H(1) &= 1 \\ H(2) &= 1 \\ H(3) &= 2 \\ H(4) &= 2 \\ H(5) &= 3 \\ H(6) &= 3 \\ H(7) &= 4 \\ H(8) &= 4 \\ H(9) &= 5 \\ H(10) &= 5 \end{aligned}$$

(b) Using the pattern from part(a), gues what H(100) is

$$H(100) = 50$$

4. The Lucas numbers L(n) have almost the same definition as the Fibonacci numbers:

$$L(n) = \begin{cases} 1 & \text{if } n = 1, \\ 3 & \text{if } n = 2 \\ L(n-1) + L(n-2) & \text{if } n > 2. \end{cases}$$

(a) How is the definition of L(n) different from the definition of F(n) in Definition 3.1?
The definition of L(n) is different from F(n) when n = 2.
L(2) = 3 and F(2) = 1.

(b) Compute the first 12 Lucas numbers.

$$\begin{aligned} L(1) &= 1 \\ L(2) &= 3 \\ L(3) &= 4 \\ L(4) &= 7 \\ L(5) &= 11 \\ L(6) &= 18 \\ L(7) &= 29 \\ L(8) &= 47 \\ L(9) &= 76 \\ L(10) &= 123 \end{aligned}$$

5. Compute the first seven terms of the following recurrence relations:

(a) C(n) of Example 3.5

$$C(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2.C(n-1) & \text{if } n > 0 \end{cases}$$

$$\begin{aligned} C(0) &= 1 \\ C(1) &= 2 \\ C(2) &= 4 \\ C(3) &= 8 \\ C(4) &= 16 \\ C(5) &= 32 \\ C(6) &= 64 \end{aligned}$$

(b) V(n) of Example 3.6

$$V(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2.V(n-1) + 1 & \text{if } n > 0 \end{cases}$$

$$\begin{aligned} V(0) &= 1 \\ V(1) &= 3 \\ V(2) &= 7 \\ V(3) &= 15 \\ V(4) &= 31 \\ V(5) &= 63 \\ V(6) &= 127 \end{aligned}$$

(c) E(n) of Example 3.7

$$E(n) = \begin{cases} 0 & \text{if } n = 1, \\ E(n-1) + n - 1 & \text{if } n > 1 \end{cases}$$

$$\begin{aligned} E(1) &= 0 \\ E(2) &= 1 \\ E(3) &= 3 \\ E(4) &= 6 \\ E(5) &= 10 \\ E(6) &= 15 \\ E(7) &= 21 \end{aligned}$$

6. Consider the following recurrence relation.

$$P(n) = \begin{cases} 0 & \text{if } n = 0 \\ [P(n-1)]^2 - n & \text{if } n > 0 \end{cases}$$

Use this recurrence relation to compute P(1), P(2), P(3), and P(4)

$$P(1) = -1$$
$$P(2) = -1$$
$$P(3) = -2$$
$$P(4) = 0$$

7. Given the following definition, compute P(4).

$$Q(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times [P(n-1)] & \text{if } n > 0 \end{cases}$$

$$P(4) = 4 \times P(3) = 4 \times 3 \times P(2) = 4 \times 3 \times 2 \times P(1) = 4 \times 3 \times 2 \times 1 \times P(0)$$
$$4 \times 3 \times 2 \times 1 \times 1 = 24$$

8. Given the following definition, compute Q(5).

$$Q(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ Q(n-1) + Q(n-2) + Q(n-3) & \text{if } n > 2 \end{cases}$$

$$Q(5) = Q(4) + Q(3) + Q(2) = Q(3) + Q(2) + Q(1) + Q(2) + Q(1) + Q(0) + Q(2)$$
$$= Q(2) + Q(1) + Q(0) + Q(2) + Q(1) + Q(2) + Q(1) + Q(0) + Q(2)$$
$$= 2 + 1 + 0 + 2 + 1 + 2 + 1 + 0 + 2 = 11$$

9. Consider the recurrence relation defined in Example 3.3. Suppose that, as in the example, you borrow \$500, but you pay her back \$100 each week. Each week, Ursula charges you 10% interest on the amount you still owe, after your \$100 payment is taken into account.

   (a) Write down a recurrence relation for M(n), the amount owed after n weeks.

$$Q(n) = \begin{cases} 500 & \text{if } n = 0 \\ 1.1 \times (M(n-1) - 100) & \text{if } n > 0 \end{cases}$$

   (b) How much will you owe after four weeks?

$$Q(4) = 1.1 \times (Q(3) - 100) = 1.1 \times (1.1 \times (Q(2) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (Q(1) - 100) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (1.1 \times (Q(0) - 100) - 100) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (1.1 \times (500 - 100) - 100) - 100) - 100) = 221.54$$

10. Every year, Alice gets a raise of \$3,000 plus 5% of her previous year's salary. Her starting salary is \$50,000. Give a recurrence relation for S(n), Alice's salary after n years, for $n \geq 0$.

$$S(n) = \begin{cases} 50000 & \text{if } n = 0 \\ 1.05 \times S(n-1) + 3000 & \text{if } n > 0 \end{cases}$$

11. Suppose that today (year 0) your car is worth \$10,000. Each year your car loses 10% of its value, but at the end of each year you add customizations to your car that increase its value by \$50. Write a recurrence relation to model this situation.
Let P(n) is the value of the car after n year(s).

$$P(n) = \begin{cases} 10000 & \text{if } n = 0 \\ 0.9 \times P(n-1) + 50 & \text{if } n > 0 \end{cases}$$

Section 3.4:

13. In Exercise 3 of Section 3.1, we defined the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \leq 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2 \end{cases}$$

Prove that $H(2n) = H(2n-1) = n$ for all $n \geq 1$.
**Step 1:**
For n = 1:

$$H(2 \times 1) = H(2) = 1$$

$$H(2 \times 1 - 1) = H(1) = 1$$

Hence, $H(2 \times 1) = H(2 \times 1 - 1) = 1$.
Then $H(2n) = H(2n-1) = n$ is true for n = 1.
For n = 2:

$$H(2 \times 2) = H(4) = H(3) + H(2) - H(1) = H(2) + 1 - 1 = 1 + 1 - 0 = 2$$

$$H(2 \times 2 - 1) = H(3) = 1 + 1 - 0 = 2$$

Hence, $H(2 \times 2) = H(2 \times 2 - 1) = 2$.
**Step 2:**
For n = k, assume $H(2k) = H(2k-1) = k$ is true.
**Step 3:**
For n = k + 1:

$$H(2(k+1)) = H(2(k+1) - 1)$$

$$H(2k+2) = H(2k+2-1)$$

$$H(2k+2) = H(2k+1)$$

$$H(2k+1) + H(2k) - H(2k-1) = H(2k) + H(2k-1) - H(2k-2)$$
$$H(2k+1) + k - k = k + k - H(2k-2)$$
$$H(2k) + H(2k-1) - H(2k-2) = 2k - H(2k-2)$$
$$k + k - H(2k-2) = 2k - H(2k-2)$$
$$0 = 0(true)$$

Hence, $H(2(k+1)) = H(2(k+1)-1)$ is true.

$$H(2(k+1)) = k+1$$

$$H(2k+2) = k+1$$
$$H(2k+1) + H(2k) - H(2k-1) = k+1$$
$$H(2k+1) + k - k = k+1$$
$$H(2k+1) = k+1$$