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Calculus 1 Assignment-6
Section: 04
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Section 3.1:

35. Find the critical numbers of the function.

$$g(y) = \frac{y-1}{y^2-y+1}$$
$$g'(y) = \frac{(y-1)'(y^2-y+1)-(y^2-y+1)'(y-1)}{(y^2-y+1)^2}$$
$$g'(y) = \frac{y^2-y+1-(2y-1)(y-1)}{(y^2-y+1)^2}$$
$$g'(y) = \frac{y^2-y+1-2y^2+2y+y-1}{(y^2-y+1)^2}$$
$$g'(y) = \frac{-y^2+2y}{(y^2-y+1)^2}$$
$$-y^2+2y=0$$
$$-y(y-2)=0$$
$$y=0 \text{ or } y=2$$

The critical numbers of the function are 0 and 2.

41. Find the critical numbers of the function.

$$F(x) = x^{4/5}(x-2)^2$$
$$F(x) = x^{4/5}(x^2-4x+4)$$
$$F(x) = x^{14/5}-4x^{9/5}+4x^{4/5}$$
$$F'(x) = \frac{14}{5}x^{9/5}-\frac{36}{5}x^{4/5}+\frac{16}{5}x^{-1/5}$$
$$\frac{14}{5}x^{9/5}-\frac{36}{5}x^{4/5}+\frac{16}{5}x^{-1/5}=0$$
$$x=\frac{4}{7} \text{ or } x=2$$

The critical numbers of the function are $\frac{4}{7}$ and 2.

45. Find the critical numbers of the function.

$$f(\theta) = 2\cos\theta + \sin^2\theta$$
$$f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$$
$$0 = -2\sin\theta + 2\sin\theta\cos\theta$$
$$2\sin\theta = 2\sin\theta\cos\theta$$
$$\theta = n\pi \text{ with n is a integer.}$$

The critical numbers of the function are $n\pi$ with n is a integer.

53. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \text{ [-2,3]}$$
$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$12x^3 - 12x^2 - 24x = 0$$
$$12x(x^2 - x - 2) = 0$$
$$12x(x-2)(x+1) = 0$$
$$x = 0 \text{ or } x = 2 \text{ or } x = -1$$
$$f(-2) = 33$$
$$f(-1) = -4$$
$$f(0) = 1$$
$$f(2) = -31$$
$$f(3) = 28$$

The absolute maximum value of f is 33 and the absolute minimum value of f is -31.

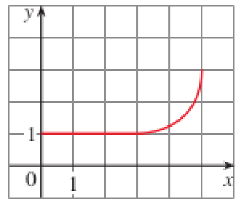
59. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(t) = 2\cos t + \sin 2t, [0, \frac{\pi}{2}]$$
$$f'(t) = -2\sin t + 2\cos 2t$$
$$0 = -2\sin t + 2\cos 2t$$
$$2\cos 2t = 2\sin t$$
$$\cos 2t = \cos(\frac{\pi}{2} - t)$$
$$2t = \frac{\pi}{2} - t$$
$$3t = \frac{\pi}{2}$$
$$t = \frac{\pi}{6}$$
$$f(0) = 2$$
$$f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2}$$
$$f(\frac{\pi}{2}) = 0$$

The absolute maximum value of f is $\frac{3\sqrt{3}}{2}$ and the absolute minimum value of f is 0.

Section 3.2:

7. The graph of a function f is shown. Does f satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$? If so, find a value c that satisfies the conclusion of the Mean Value Theorem on that interval.



Based on the graph, f is both continuous on the interval $[0, 5]$ and differentiable on the interval $(0, 5)$.

$$f'(c) = \frac{f(5) - f(0)}{5 - 0}$$

$$f'(c) = \frac{3 - 1}{5}$$

$$f'(c) = \frac{2}{5}$$

Based on the graph:

$$c \approx 4$$

11. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = \sin(x/2),\ [\frac{\pi}{2}, \frac{3\pi}{2}]$$

Because $f(x)$ is a trigonometric function, $f(x)$ is both continuous on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and differentiable on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$.

$$f(\frac{\pi}{2}) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

$$f(\frac{3\pi}{2}) = \sin(3\pi/4) = \frac{\sqrt{2}}{2}$$

Hence, $f(\frac{\pi}{2}) = f(\frac{3\pi}{2}) = \frac{\sqrt{2}}{2}$. Therefore, there exists c in $(\frac{\pi}{2}, \frac{3\pi}{2})$ such as:

$$f'(c) = 0$$

$$\frac{1}{2}\cos(c/2) = 0$$

$$\cos(c/2) = 0$$

$$c/2 = \pi/2 + k\pi \text{ with k is a integer.}$$

$$c = \pi + 2k\pi \text{ with k is a integer.}$$

Therefore, in the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, the only c satisfies the conclusion of Rolle's Theorem is π.

13. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?
Firstly, $f(x)$ is continuous on the interval $[-1, 1]$.

$$f'(x) = -\frac{2}{3}x^{-1/3}$$

$$f'(x) = -\frac{2}{3\sqrt[3]{x}}$$

But $f(x)$ is not differentiable with x = 0.
Therefore, this function cannot contradict with Rolle's Theorem.

17. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = \sqrt[3]{x},\ [0, 1]$$
$$f(x) = x^{1/3}$$
$$f'(x) = \frac{1}{3}x^{-2/3}$$

Firstly, f(x) is continuous on the interval [0,1].
Secondly, f(x) is differentiable on the integer (0,1). Then there is a number c in (0,1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$
$$\frac{1}{3}c^{-2/3} = \frac{1 - 0}{1 - 0}$$
$$\frac{1}{3}c^{-2/3} = 1$$
$$c = \frac{\sqrt{3}}{9}$$

21. Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in (1,4) such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

$$f(x) = (x-3)^{-2}$$

$$f(x) = \frac{1}{(x-3)^2}$$

The function $f(x)$ is not defined at x =3. Therefore, it's not continuous on x = 3 in the inteval (1,4). Hence, it does not contradict the Mean Value Theorem.

23. Show that the equation has exactly one real solution.

$$2x + \cos x = 0$$

Let say:

$$f(x) = 2x + \cos x$$

$$f'(x) = 2 - \sin x$$

Because:

$$-1 < \sin x < 1$$

$$1 > -\sin x > -1$$

$$3 > 2 - \sin x > 1$$

$$3 > f'(x) > 1$$

Because f'(x) is always positive, the function is always increasing.
For x = 0:

$$f(0) = 1 > 0$$

For x = -1:

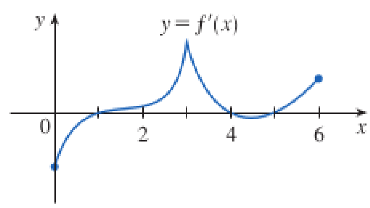
$$f(-1) \approx -1.46 < 0$$

Therefore, f(x) has a root between (-1,0) and because f(x) is always increasing. f(x) only has one real solution.

25. Show that the equation $x^3 - 15x + c = 0$ has at most one solution in the interval [-2,2].

Section 3.3:

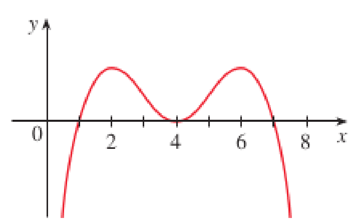
6. The graph of the derivative f' of a function f is shown.



   (a) On what intervals is f increasing? Decreasing?
   f is increasing on the intervals (1,4),(5,6).
   f is decreasing on the intervals (0,1),(4,5).

   (b) At what values of x does f have a local maximum? Local minimum?
   f has local maximum at x = 4.
   f has local minimum at x = 1, and x = 5.

7. In each part use the given graph to state the x-coordinates of the inflection points of f. Give reasons for your answers.



(a) The curve is the graph of f.
The inflection points are 3, 5.

(b) The curve is the graph of f'.
The inflection points are 2, 4, 6.

(c) The curve is the graph of f''.
The inflection points are 1, 7.

9. Find the intervals on which f is increasing or decreasing, and find the local maximum and minimum values of f.

$$f(x) = 2x^3 - 15x^2 + 24x - 5$$

$$f'(x) = 6x^2 - 30x + 24$$

$$6x^2 - 30x + 24 = 0$$

$$x = 1 \text{ or } x = 4$$

Function f(x) is increasing in the intervals $(-\infty, 1)$ and $(4, \infty)$
Function f(x) is decreasing in the intervals $(1, 4)$.
The local maximum of the function is f(x) with x = 1.
The local minimum of the function is f(x) with x = 4.

13. Find the intervals on which f is increasing or decreasing, and find the local maximum and minimum values of f.

$$f(x) = \frac{x^2 - 24}{x - 5}$$

$$f'(x) = \frac{(x^2 - 24)'(x - 5) - (x - 5)'(x^2 - 24)}{(x - 5)^2}$$

$$f'(x) = \frac{(2x)(x - 5) - x^2 + 24}{(x - 5)^2}$$

$$f'(x) = \frac{2x^2 - 10x - x^2 + 24}{(x - 5)^2}$$

$$f'(x) = \frac{x^2 - 10x + 24}{(x - 5)^2}$$

$$x^2 - 10x + 24 = 0$$

$$x = 4 \text{ or } x = 6$$

Function is increasing on the intervals $(-\infty, 4)$ and $(6, \infty)$.
Function is decreasing on the interval $(4, 6)$.
The local maximum of the function is f(x) with x = 4.
The local minimum of the function is f(x) with x = 6.

17. Find the intervals on which f is concave upward or concave downward, and find the infection points of f.

$$f(x) = \sin^2(x) - \cos 2x,\ 0 \leq x \leq \pi$$

$$f'(x) = 2\sin(x)\cos(x) + 2\sin 2x$$
$$f''(x) = 2[(\sin(x))'\cos(x) + \sin(x)(\cos(x))'] + 4\cos 2x$$
$$f''(x) = 2[\cos^2(x) - \sin^2(x)] + 4\cos 2x$$
$$f''(x) = 2\cos^2(x) - 2\sin^2(x) + 4(2\cos^2 x - 1)$$
$$f''(x) = 2\cos^2(x) - 2\sin^2(x) + 8\cos^2 x - 4$$
$$f''(x) = 2\cos^2(x) - 2\sin^2(x) + 8\cos^2 x - 4\cos^2 x - 4\sin^2 x$$
$$f''(x) = 6\cos^2(x) - 6\sin^2(x)$$
$$6\cos^2(x) - 6\sin^2(x) = 0$$
$$6\cos^2(x) = 6\sin^2(x)$$
$$\tan^2 x = 1$$
$$\tan x = \pm 1$$

Since x is in the intervals $[0,\ \pi]$.

$$x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}$$

The inflection points of f(x) are $(\frac{\pi}{4}, \frac{1}{2})$ and $(\frac{3\pi}{4}, \frac{1}{2})$.
The function f(x) is concave up in the intervals $(0, \frac{\pi}{4})$ and $(\frac{3\pi}{4}, \pi)$.
The function f(x) is concave down in the interval $(\frac{\pi}{4}, \frac{3\pi}{4})$

22.

$$f(x) = \cos^2 x - 2\sin x,\ 0 \leq x \leq 2\pi$$

(a) Find the intervals on which f is increasing or decreasing.

$$f'(x) = -2\cos x \sin x - 2\cos x$$

$$-2\cos x \sin x - 2\cos x = 0$$

$$-2\cos x(\sin x + 1) = 0$$

$$\cos x = 0 \text{ or } \sin x + 1 = 0$$

$$x = \pi/2 + \pi n \text{ with n is a integer or } \sin x = -1$$

$$x = \pi/2 + \pi n \text{ or } x = 3\pi/2 + 2\pi n \text{ with n is a integer.}$$

Since x is in the inteval $[0, 2\pi]$.

$$x \in \{\pi/2, 3\pi/2\}$$

The function $f(x)$ is increasing on the intevals $(\pi/2, 3\pi/2)$.
The function $f(x)$ is decreasing on the intevals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$.

(b) Find the local maximum and minimum values of f.
The local maximum value of f is: $f(3\pi/2) = 2$.
The local minimum value of f is: $f(\pi/2) = -2$

(c) Find the intervals of concavity and the inflection points.

$$f''(x) = -2(-\sin^2 x + \cos^2 x) - 2\cos x$$

$$f''(x) = 2\sin^2 x - 2\cos^2 x - 2\cos x$$

$$2\sin^2 x - 2\cos^2 x - 2\cos x = 0$$

$$x \in \{\pi/3, \pi, 5\pi/3\}$$

The inflection points are $(\pi/3, \frac{1-4\sqrt{3}}{4}), (\pi, 1), (5\pi/3, \frac{1+4\sqrt{3}}{4})$.
The function is concave up in the interval $(\pi/3, \pi)$.
The function is concave down in the interval $(0, \pi/3)$ and $(5\pi/3, 2\pi)$

23. Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

$$f(x) = 1 + 3x^2 - 2x^3$$

Let:

$$f'(x) = 6x - 6x^2$$

$$6x - 6x^2 = 0$$

$$x = 0 \text{ or } x = 1$$

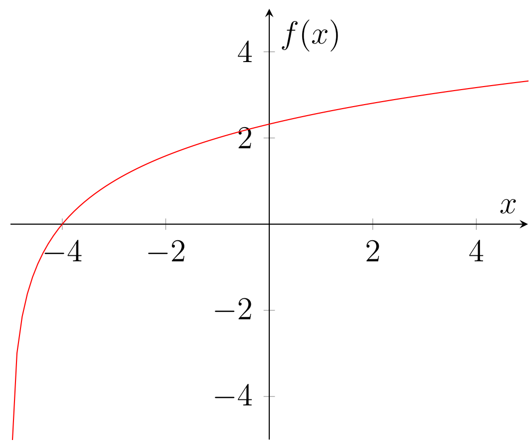
Let:

$$f''(x) = 6 - 12x$$

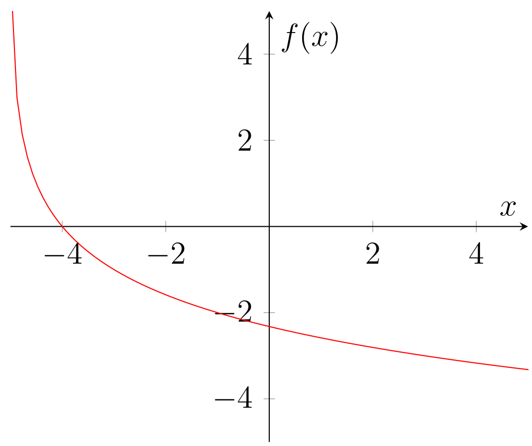
The First Derivative Test:
f'(x) change from negative to positive at x = 0. Therefore, we have a local minimum at x = 0.
f'(x) change from positive to negative at x = 1. Therefore, we have a local maximum at x = 1.
The Second Derivative Test:
$f'(x) = 0$ at x = 0 and $f''(0) > 0$. Therefore, we have a local minimum at x = 0.
$f'(x) = 0$ at x = 1 and $f''(1) < 0$. Therefore, we have a local maximum at x = 1.
I personally prefer the first one.

29. Sketch the graph of a function that satisfies all of the given conditions.

(a) $f'(x) > 0$ and $f''(x) < 0$ for all x.



(b) $f'(x) < 0$ and $f''(x) > 0$ for all x.



31. Sketch the graph of a function that satisfies all of the given conditions.
$f'(0) = f'(2) = f'(4) = 0$, $f'(x) > 0$ if $x < 0$ or $2 < x < 4$, $f'(x) < 0$ if $0 < x < 2$ or $x > 4$, $f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$.