1. Prove that the closed-form solution in Equation 3.2.2 matches the recurrence relation in Equation 3.2.1. (See Example 3.8.)
   Step 1: When n = 0:
   From the recursive definition:
   $$M(0) = 500$$
   From the closed-form expression:
   $$M(0) = 500 \times (1.10)^0 = 500$$
   Thus, the base case is true.
   Step 2: Assume $M(k-1) = 500 \times (1.10)^{k-1}$ is true with some $k > 0$.
   Step 3: Prove that $M(k) = 500 \times (1.10)^k$.
   $$M(k) = 1.10.M(k-1) = 1.10.500 \times (1.10)^{k-1} = 500 \times (1.10)^k$$
   Hence, this is true.
   According to the induction principle, the closed-form solution in Equation 3.2.2 matches the recurrence relation in Equation 3.2.1.

2. Prove that the closed-form solution in Equation 3.1.2 matches the recurrence relation in Equation 3.1.3. (See page 108.)
   Step 1: When n = 1
   From the recursive definition:
   $$P(1) = 1$$
   From the closed-form expression:
   $$P(1) = \frac{1 \times (1+1)}{2} = \frac{2}{2} = 1$$
   Thus, the base case is true.
   Step 2: Assume $P(k-1) = \frac{(k-1)(k-1+1)}{2} = \frac{(k-1)(k)}{2}$ with some $k > 1$.
   Step 3: Prove that $P(k) = \frac{k(k+1)}{2}$.
   $$P(k) = k + P(k-1) = k + \frac{(k-1)(k)}{2} = \frac{2k + (k-1)k}{2}$$
   $$= \frac{k(2+k-1)}{2} = \frac{k(k+1)}{2}$$
   Hence, this is true.
   According to the induction principle, the closed-form solution in Equation 3.1.2 matches the recurrence relation in Equation 3.1.3.

3. Consider the following recurrence relation:

$$B(n) = \begin{cases} 2 & \text{if } n = 1 \\ 3 \cdot B(n-1) + 2 & \text{if } n > 1 \end{cases}$$

Use induction to prove that $B(n) = 3^n - 1$ for all $n \geq 1$.
Step 1: When n = 1.
From the recursive expression:

$$B(1) = 2$$

From the closed-form expression:

$$B(1) = 3^1 - 1 = 2$$

Thus, the base case is true.
Step 2: Assume $B(k-1) = 3^{k-1} - 1$ with some $k > 1$.
Step 3: Prove that $B(k) = 3^k - 1$.

$$B(k) = 3.B(k-1) + 2 = 3.(3^{k-1} - 1) + 2 = 3^k - 3 + 2 = 3^k - 1$$

Hence, this is true.
According to the induction principle, $B(n) = 3^n - 1$ with all $n \geq 1$.

4. Consider the following recurrence relation:

$$P(n) = \begin{cases} 0 & \text{if } n = 0 \\ 5 \cdot P(n-1) + 1 & \text{if } n > 0 \end{cases}$$

Prove by induction that $P(n) = \frac{5^n - 1}{4}$ for all $n \geq 0$.
Step 1: When n = 0
From the recursive expression:

$$P(0) = 0$$

From the closed-form expression:

$$P(0) = \frac{5^0 - 1}{4} = \frac{1 - 1}{4} = 0$$

Hence, the base case is true.
Step 2: Assume

$$P(k-1) = \frac{5^{k-1} - 1}{4}$$

is true with some $k > 0$.
Step 3: Prove that $P(k) = \frac{5^k - 1}{4}$.

$$P(k) = 5.P(k-1) + 1 = 5(\frac{5^{k-1} - 1}{4}) + 1 = \frac{5^k - 5}{4} + \frac{4}{4} = \frac{5^k - 1}{4}$$

Hence, this is true.
According to the induction principle, $P(n) = \frac{5^n - 1}{4}$ for all $n \geq 0$.

5. Consider the following recurrence relation:

$$C(n) = \begin{cases} 0 & \text{if } n = 0 \\ n + 3 \cdot C(n-1) & \text{if } n > 0 \end{cases}$$

Prove by induction that $C(n) = \frac{3^{n+1}-2n-3}{4}$ for all $n \geq 0$.
Step 1: For n = 0
From the recursive expression:

$$C(0) = 0$$

From the closed-form expression:

$$C(0) = \frac{3^{0+1} - 2(0) - 3}{4} = \frac{3-3}{4} = 0$$

Hence, the base case is true.
Step 2: Assume that

$$C(k-1) = \frac{3^{k-1+1} - 2(k-1) - 3}{4} = \frac{3^k - 2k + 2 - 3}{4} = \frac{3^k - 2k - 1}{4}$$

with some $k > 0$.
Step 3: Prove that $C(k) = \frac{3^{k+1}-2k-3}{4}$ for all $k \geq 0$

$$C(k) = k + 3 \cdot C(k-1) = k + 3 \cdot \frac{3^k - 2k - 1}{4} = \frac{4k}{4} + \frac{3^{k+1} - 6k - 3}{4} = \frac{3^{k+1} - 2k - 3}{4}$$

Hence, this is true.
According to the induction principle, $C(n) = \frac{3^{n+1}-2n-3}{4}$ for all $n \geq 0$.

6. Consider the following recurrence relation:

$$Q(n) = \begin{cases} 4 & \text{if } n = 0 \\ 2 \cdot Q(n-1) - 3 & \text{if } n > 0 \end{cases}$$

Prove by induction that $Q(n) = 2^n + 3$, for all $n \geq 0$.
Step 1: For n = 0
From the recursive expression:

$$Q(0) = 4$$

From the closed-from expression:

$$Q(0) = 2^0 + 3 = 1 + 3 = 4$$

Hence, the base case is true.
Step 2: Assume that

$$Q(k-1) = 2^{k-1} + 3$$

with some $k > 0$.
Step 3: Prove that $Q(k) = 2^k + 3$

$$Q(k) = 2Q(k-1) - 3 = 2 \times (2^{k-1} + 3) - 3 = 2^k + 6 - 3 = 2^k + 3$$

Hence, this is true.
According to the induction principle, $Q(n) = 2^n + 3$, for all $n \geq 0$.

7. Consider the following recurrence relation:

$$R(n) = \begin{cases} 1 & \text{if } n = 0 \\ R(n-1) + 2n & \text{if } n > 0 \end{cases}$$

Prove by induction that $R(n) = n^2 + n + 1$ for all $n \geq 0$.
Step 1: For n = 0
From the recursive expression:

$$R(0) = 1$$

From the closed-form expression:

$$R(0) = 0^2 + 0 + 1 = 1$$

Hence, the base case is true.
Step 2: Assume that

$$R(k-1) = (k-1)^2 + k - 1 + 1 = (k-1)^2 + k$$

with some $k > 0$. Step 3: Prove that $R(k) = k^2 + k - 1$.

$$R(k) = R(k-1) + 2k = (k-1)^2 + k + 2k = k^2 - 2k - 1 + 3k = k^2 + k - 1$$

Hence, this is true.
According to the induction principle, $R(n) = n^2 + n + 1$ for all $n \geq 0$.

8. Guess a closed-form solution for the following recurrence relation:

$$K(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 \cdot K(n-1) - n + 1 & \text{if } n > 0 \end{cases}$$

Prove that your guess is correct.
The closed-form expression of the function is

$$K(n) = n + 1$$

Step 1: For n = 0
From the recursive expression:

$$K(0) = 1$$

From the closed-form expression:

$$K(0) = 0 + 1 = 1$$

Hence, the base case is true.
Step 2: Assume that

$$K(k-1) = k - 1 + 1 = k$$

with some $k > 0$.
Step 3: Prove that $K(k) = k + 1$

$$K(k) = 2K(k-1) - k + 1 = 2k - k + 1 = k + 1$$

Hence, this is true.
According to the induction principle, $K(n) = n + 1$ with all $n \geq 0$.

9. Guess a closed-form solution for the following recurrence relation:

$$P(n) = \begin{cases} 5 & \text{if } n = 0 \\ P(n-1) + 3 & \text{if } n > 0 \end{cases}$$

Prove that your guess is corerct.
The closed-form of this equation is:

$$P(n) = 5 + 3n$$

Step 1: For n = 0
From the recursive expression:

$$P(0) = 5$$

From the closed-form expression:

$$P(0) = 5 + 3(0) = 5$$

Step 2: Assume that

$$P(k-1) = 5 + 3(k-1)$$

is true with some $k > 0$.
Step 3: Prove that $P(k) = 5 + 3k$

$$P(k) = P(n-1) + 3 = 5 + 3(k-1) + 3 = 5 + 3k$$

Hence, this is true.
According to the induction principle, $P(n) = 5 + 3n$ with all $n \geq 0$.

10. Consider the following recurrence relation:

$$P(n) = \begin{cases} 1 & \text{if } n = 0 \\ P(n-1) + n^2 & \text{if } n > 0 \end{cases}$$

(a) Compute the first eight values of P(n).

$$P(0) = 1$$

$$P(1) = 2$$

$$P(2) = 6$$

$$P(3) = 15$$

$$P(4) = 31$$

$$P(5) = 56$$

$$P(6) = 92$$

$$P(7) = 141$$

(b) Analyze the sequences of differences. What does this suggest about the closed-form solution?
To find a pattern, let's look at the differences between consecutive values:

$$\begin{aligned} P(1) - P(0) &= 2 - 1 = 1 \\ P(2) - P(1) &= 6 - 2 = 4 \\ P(3) - P(2) &= 15 - 6 = 9 \\ P(4) - P(3) &= 31 - 15 = 16 \\ P(5) - P(4) &= 56 - 31 = 25 \\ P(6) - P(5) &= 92 - 56 = 36 \\ P(7) - P(6) &= 141 - 92 = 49 \end{aligned}$$

These differences $1, 4, 9, 16, 25, 36, 49$ are perfect squares: $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.
The sequence of differences suggests that $P(n)$ is related to the sum of squares formula.

(c) Find a good candidate for a closed-form solution.

$$P(n) = 1 + \frac{n(n+1)(2n+1)}{6}$$

(d) Prove that your candidate solution is the correct closed-form solution.
Step 1: For n = 0:
From the recursive expression:

$$P(0) = 1$$

From the closed-form expression:

$$P(0) = 1 + \frac{0(0+1)(2(0)+1)}{6} = 1$$

Hence, the base case is true
Step 2: Assume that

$$P(k-1) = 1 + \frac{(k-1)(k-1+1)(2(k-1)+1)}{6}$$

$$P(k-1) = 1 + \frac{(k-1)(k)(2k-1)}{6}$$

with some $k > 0$.
Step 3: Prove that $P(k) = 1 + \frac{k(k+1)(2k+1)}{6}$

$$P(k) = P(k-1) + k^2 = 1 + \frac{(k-1)(k)(2k-1)}{6} + k^2$$

$$P(k) = 1 + \frac{(k-1)(k)(2k-1) + 6k^2}{6} = 1 + \frac{k[(k-1)(2k-1) + 6k]}{6}$$

$$P(k) = 1 + \frac{k[2k^2 - 3k + 1 + 6k]}{6} = 1 + \frac{k[2k^2 + 3k + 1]}{6} = 1 + \frac{k(k+1)(2k+1)}{6}$$

Hence, this is true.
According to the induction principle, $P(n) = 1 + \frac{n(n+1)(2n+1)}{6}$ with all $n \geq 0$.

11. Consider the following recurrence relation:

$$G(n) = \begin{cases} 1 & \text{if } n = 0 \\ G(n-1) + 2n - 1 & \text{if } n > 0 \end{cases}$$

(a) Calculate G(0), G(1), G(2), G(3), G(4), and G(5).

$$G(0) = 1$$

$$G(1) = 2$$
$$G(2) = 5$$
$$G(3) = 10$$
$$G(4) = 17$$
$$G(5) = 26$$

(b) Use sequences of differences to guess at a closed-form solution for G(n).
To find a pattern, let's look at the differences between consecutive values:

$$G(1) - G(0) = 2 - 1 = 1$$
$$G(2) - G(1) = 5 - 2 = 3$$
$$G(3) - G(2) = 10 - 5 = 5$$
$$G(4) - G(3) = 17 - 10 = 7$$
$$G(5) - G(4) = 26 - 17 = 9$$

These differences are an sequence of increasing odd numbers.
Hence, the closed-form solution for G(n) is:

$$G(n) = 1 + n^2$$

(c) Prove that your guess is correct.
Step 1: For n = 0
From the recursive expression:

$$G(0) = 1$$

From the closed-form expression:

$$G(0) = 1 + 0^2 = 1$$

Hence, the base case is true.
Step 2: Assume that

$$G(k-1) = 1 + (k-1)^2 = 1 + k^2 - 2k + 1 = k^2 - 2k + 2$$

with some $k > 0$.
Step 3: Prove that $G(k) = 1 + k^2$

$$G(k) = G(n-1) + 2k - 1 = k^2 - 2k + 2 + 2k - 1 = 1 + k^2$$

Hence, this is true.
According to the induction principle, $G(n) = 1 + n^2$ with all $n \geq 0$.

12. Find a polynomial function f(n) such that f(1), f(2), . . . , f(8) is the following sequence:

$$2, 7, 12, 17, 22, 27, 32, 37.$$

A polynomial function f(n) can produce this sequence is:

$$f(n) = 5n - 3$$