

Section 3.1:

35. Find the critical numbers of the function.

$$\begin{aligned}g(y) &= \frac{y-1}{y^2-y+1} \\g'(y) &= \frac{(y-1)'(y^2-y+1) - (y^2-y+1)'(y-1)}{(y^2-y+1)^2} \\g'(y) &= \frac{y^2-y+1 - (2y-1)(y-1)}{(y^2-y+1)^2} \\g'(y) &= \frac{y^2-y+1 - 2y^2+2y+y-1}{(y^2-y+1)^2} \\g'(y) &= \frac{-y^2+2y}{(y^2-y+1)^2} \\-y^2+2y &= 0 \\-y(y-2) &= 0 \\y &= 0 \text{ or } y = 2\end{aligned}$$

The critical numbers of the function are 0 and 2.

41. Find the critical numbers of the function.

$$\begin{aligned}F(x) &= x^{4/5}(x-2)^2 \\F(x) &= x^{4/5}(x^2-4x+4) \\F(x) &= x^{14/5} - 4x^{9/5} + 4x^{4/5} \\F'(x) &= \frac{14}{5}x^{9/5} - \frac{36}{5}x^{4/5} + \frac{16}{5}x^{-1/5} \\\frac{14}{5}x^{9/5} - \frac{36}{5}x^{4/5} + \frac{16}{5}x^{-1/5} &= 0 \\x &= \frac{4}{7} \text{ or } x = 2\end{aligned}$$

The critical numbers of the function are $\frac{4}{7}$ and 2.

45. Find the critical numbers of the function.

$$f(\theta) = 2 \cos \theta + \sin^2 \theta$$

$$f'(\theta) = -2 \sin \theta + 2 \sin \theta \cos \theta$$

$$0 = -2 \sin \theta + 2 \sin \theta \cos \theta$$

$$2 \sin \theta = 2 \sin \theta \cos \theta$$

$$\theta = n\pi \text{ with } n \text{ is a integer.}$$

The critical numbers of the function are $n\pi$ with n is a integer.

53. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1, [-2, 3]$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$12x^3 - 12x^2 - 24x = 0$$

$$12x(x^2 - x - 2) = 0$$

$$12x(x - 2)(x + 1) = 0$$

$$x = 0 \text{ or } x = 2 \text{ or } x = -1$$

$$f(-2) = 33$$

$$f(-1) = -4$$

$$f(0) = 1$$

$$f(2) = -31$$

$$f(3) = 28$$

The absolute maximum value of f is 33 and the absolute minimum value of f is -31.

59. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(t) = 2 \cos t + \sin 2t, [0, \frac{\pi}{2}]$$

$$f'(t) = -2 \sin t + 2 \cos 2t$$

$$0 = -2 \sin t + 2 \cos 2t$$

$$2 \cos 2t = 2 \sin t$$

$$\cos 2t = \cos(\frac{\pi}{2} - t)$$

$$2t = \frac{\pi}{2} - t$$

$$3t = \frac{\pi}{2}$$

$$t = \frac{\pi}{6}$$

$$f(0) = 2$$

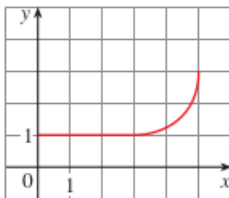
$$f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2}$$

$$f(\frac{\pi}{2}) = 0$$

The absolute maximum value of f is $\frac{3\sqrt{3}}{2}$ and the absolute minimum value of f is 0.

Section 3.2:

7. The graph of a function f is shown. Does f satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$? If so, find a value c that satisfies the conclusion of the Mean Value Theorem on that interval.



Based on the graph, f is both continuous on the interval $[0, 5]$ and differentiable on the interval $(0, 5)$.

$$\begin{aligned} f'(c) &= \frac{f(5) - f(0)}{5 - 0} \\ f'(c) &= \frac{3 - (-1)}{5} \\ f'(c) &= \frac{4}{5} \end{aligned}$$

Based on the graph:

$$c \approx 4$$

11. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = \sin(x/2), \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Because $f(x)$ is a trigonometric function, $f(x)$ is both continuous on the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and differentiable on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

$$f\left(\frac{\pi}{2}\right) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

$$f\left(\frac{3\pi}{2}\right) = \sin(3\pi/4) = \frac{\sqrt{2}}{2}$$

Hence, $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}$. Therefore, there exists c in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such as:

$$f'(c) = 0$$

$$\frac{1}{2} \cos(c/2) = 0$$

$$\cos(c/2) = 0$$

$$c/2 = \pi/2 + k\pi \text{ with } k \text{ is a integer.}$$

$$c = \pi + 2k\pi \text{ with } k \text{ is a integer.}$$

Therefore, in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the only c satisfies the conclusion of Rolle's Theorem is π .

13. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?
 Firstly, $f(x)$ is continuous on the interval $[-1, 1]$.

$$f'(x) = -\frac{2}{3}x^{-1/3}$$

$$f'(x) = -\frac{2}{3\sqrt[3]{x}}$$

But $f(x)$ is not differentiable with $x = 0$.

Therefore, this function cannot contradict with Rolle's Theorem.

17. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = \sqrt[3]{x}, [0, 1]$$

$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

Firstly, $f(x)$ is continuous on the interval $[0, 1]$.

Secondly, $f(x)$ is differentiable on the interior $(0, 1)$. Then there is a number c in $(0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\frac{1}{3}c^{-2/3} = \frac{1 - 0}{1 - 0}$$

$$\frac{1}{3}c^{-2/3} = 1$$

$$c = \frac{\sqrt{3}}{9}$$

21. Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

$$f(x) = (x - 3)^{-2}$$

$$f(x) = \frac{1}{(x - 3)^2}$$

The function $f(x)$ is not defined at $x = 3$. Therefore, it's not continuous on $x = 3$ in the interval $(1, 4)$. Hence, it does not contradict the Mean Value Theorem.

23. Show that the equation has exactly one real solution.

$$2x + \cos x = 0$$

Let say:

$$f(x) = 2x + \cos x$$

$$f'(x) = 2 - \sin x$$

Because:

$$-1 < \sin x < 1$$

$$1 > -\sin x > -1$$

$$3 > 2 - \sin x > 1$$

$$3 > f'(x) > 1$$

Because $f'(x)$ is always positive, the function is always increasing.

For $x = 0$:

$$f(0) = 1 > 0$$

For $x = -1$:

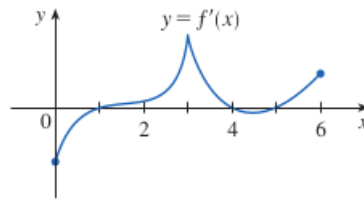
$$f(-1) \approx -1.46 < 0$$

Therefore, $f(x)$ has a root between $(-1, 0)$ and because $f(x)$ is always increasing. $f(x)$ only has one real solution.

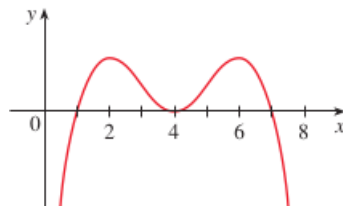
25. Show that the equation $x^3 - 15x + c = 0$ has at most one solution in the interval $[-2, 2]$.

Section 3.3:

6. The graph of the derivative f' of a function f is shown.



- (a) On what intervals is f increasing? Decreasing?
 f is increasing on the intervals $(1, 4), (5, 6)$.
 f is decreasing on the intervals $(0, 1), (4, 5)$.
- (b) At what values of x does f have a local maximum? Local minimum?
 f has local maximum at $x = 4$.
 f has local minimum at $x = 1$, and $x = 5$.
7. In each part use the given graph to state the x -coordinates of the inflection points of f . Give reasons for your answers.



- (a) The curve is the graph of f .
The inflection points are 3, 5.
- (b) The curve is the graph of f' .
The inflection points are 2, 4, 6.
- (c) The curve is the graph of f'' .
The inflection points are 1, 7.

9. Find the intervals on which f is increasing or decreasing, and find the local maximum and minimum values of f .

$$f(x) = 2x^3 - 15x^2 + 24x - 5$$

$$f'(x) = 6x^2 - 30x + 24$$

$$6x^2 - 30x + 24 = 0$$

$$x = 1 \text{ or } x = 4$$

Function $f(x)$ is increasing in the intervals $(-\infty, 1)$ and $(4, \infty)$

Function $f(x)$ is decreasing in the intervals $(1, 4)$.

The local maximum of the function is $f(x)$ with $x = 1$.

The local minimum of the function is $f(x)$ with $x = 4$.

13. Find the intervals on which f is increasing or decreasing, and find the local maximum and minimum values of f .

$$f(x) = \frac{x^2 - 24}{x - 5}$$

$$f'(x) = \frac{(x^2 - 24)'(x - 5) - (x - 5)'(x^2 - 24)}{(x - 5)^2}$$

$$f'(x) = \frac{(2x)(x - 5) - x^2 + 24}{(x - 5)^2}$$

$$f'(x) = \frac{2x^2 - 10x - x^2 + 24}{(x - 5)^2}$$

$$f'(x) = \frac{x^2 - 10x + 24}{(x - 5)^2}$$

$$x^2 - 10x + 24 = 0$$

$$x = 4 \text{ or } x = 6$$

Function is increasing on the intervals $(-\infty, 4)$ and $(6, \infty)$.

Function is decreasing on the interval $(4, 6)$.

The local maximum of the function is $f(x)$ with $x = 4$.

The local minimum of the function is $f(x)$ with $x = 6$.

17. Find the intervals on which f is concave upward or concave downward, and find the inflection points of f .

$$f(x) = \sin^2(x) - \cos 2x, 0 \leq x \leq \pi$$

$$f'(x) = 2 \sin(x) \cos(x) + 2 \sin 2x$$

$$f''(x) = 2[(\sin(x))' \cos(x) + \sin(x)(\cos(x))'] + 4 \cos 2x$$

$$f''(x) = 2[\cos^2(x) - \sin^2(x)] + 4 \cos 2x$$

$$f''(x) = 2 \cos^2(x) - 2 \sin^2(x) + 4(2 \cos^2 x - 1)$$

$$f''(x) = 2 \cos^2(x) - 2 \sin^2(x) + 8 \cos^2 x - 4$$

$$f''(x) = 2 \cos^2(x) - 2 \sin^2(x) + 8 \cos^2 x - 4 \cos^2 x - 4 \sin^2 x$$

$$f''(x) = 6 \cos^2(x) - 6 \sin^2(x)$$

$$6 \cos^2(x) - 6 \sin^2(x) = 0$$

$$6 \cos^2(x) = 6 \sin^2(x)$$

$$\tan^2 x = 1$$

$$\tan x = \pm 1$$

Since x is in the intervals $[0, \pi]$.

$$x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}$$

The inflection points of $f(x)$ are $(\frac{\pi}{4}, \frac{1}{2})$ and $(\frac{3\pi}{4}, \frac{1}{2})$.

The function $f(x)$ is concave up in the intervals $(0, \frac{\pi}{4})$ and $(\frac{3\pi}{4}, \pi)$.

The function $f(x)$ is concave down in the interval $(\frac{\pi}{4}, \frac{3\pi}{4})$

22.

$$f(x) = \cos^2 x - 2 \sin x, 0 \leq x \leq 2\pi$$

- (a) Find the intervals on which f is increasing or decreasing.

$$f'(x) = -2 \cos x \sin x - 2 \cos x$$

$$-2 \cos x \sin x - 2 \cos x = 0$$

$$-2 \cos x(\sin x + 1) = 0$$

$$\cos x = 0 \text{ or } \sin x + 1 = 0$$

$$x = \pi/2 + \pi n \text{ with } n \text{ is a integer or } \sin x = -1$$

$$x = \pi/2 + \pi n \text{ or } x = 3\pi/2 + 2\pi n \text{ with } n \text{ is a integer.}$$

Since x is in the interval $[0, 2\pi]$.

$$x \in \{\pi/2, 3\pi/2\}$$

The function $f(x)$ is increasing on the intervals $(\pi/2, 3\pi/2)$.

The function $f(x)$ is decreasing on the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$.

(b) Find the local maximum and minimum values of f .

The local maximum value of f is: $f(3\pi/2) = 2$.

The local minimum value of f is: $f(\pi/2) = -2$

(c) Find the intervals of concavity and the inflection points.

$$f''(x) = -2(-\sin^2 x + \cos^2 x) - 2\cos x$$

$$f''(x) = 2\sin^2 x - 2\cos^2 x - 2\cos x$$

$$2\sin^2 x - 2\cos^2 x - 2\cos x = 0$$

$$x \in \{\pi/3, \pi, 5\pi/3\}$$

The inflection points are $(\pi/3, \frac{1-4\sqrt{3}}{4})$, $(\pi, 1)$, $(5\pi/3, \frac{1+4\sqrt{3}}{4})$.

The function is concave up in the interval $(\pi/3, \pi)$.

The function is concave down in the interval $(0, \pi/3)$ and $(5\pi/3, 2\pi)$

23. Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

$$f(x) = 1 + 3x^2 - 2x^3$$

$$f'(x) = 6x - 6x^2$$

$$f''(x) = 6 - 12x$$

The First Derivative Test: