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Discrete Math

Section 3.1:

1. Refer to the recurrence relation for the Fibonacci sequence in Definition 3.1

    (a) Answer Fibonacci's question by calculating F(12).

    $$F(12) = 144$$

    (b) Write F(1000) in terms of F(999) and F(998).

    $$F(1000) = F(999) + F(998)$$

    (c) Write F(1000) in terms of F(998) and F(997).

    $$F(1000) = 2F(998) + F(997)$$

2. In Fibonaccci's model, rabbits live forever. The following modification of Definition 3.1 accounts for dying rabbits:

$$G(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } n = 1 \text{ or } n = 2, \\ G(n-1) + G(n-2) - G(n-8) & \text{if } n > 2. \end{cases}$$

    (a) Compute $G(n)$ for $n = 1, 2, \ldots, 12$:

    $$G(1) = 1$$
    $$G(2) = 1$$
    $$G(3) = 2$$
    $$G(4) = 3$$
    $$G(5) = 5$$
    $$G(6) = 8$$
    $$G(7) = 13$$
    $$G(8) = 21$$
    $$G(9) = 33$$
    $$G(10) = 53$$
    $$G(11) = 86$$
    $$G(12) = 136$$

    (b) In this modified model, how long do rabbits live?
        The rabbits live for 8 months.

3. Consider the following recurrence relation:
$$H(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } n = 1 \text{ or } n = 2, \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

(a) Compute H(n) for n = 1,2,...10
$$H(1) = 1$$
$$H(2) = 1$$
$$H(3) = 2$$
$$H(4) = 2$$
$$H(5) = 3$$
$$H(6) = 3$$
$$H(7) = 4$$
$$H(8) = 4$$
$$H(9) = 5$$
$$H(10) = 5$$

(b) Using the pattern from part(a), gues what H(100) is
$$H(100) = 50$$

4. The Lucas numbers L(n) have almost the same definition as the Fibonacci numbers:
$$L(n) = \begin{cases} 1 & \text{if } n = 1, \\ 3 & \text{if } n = 2 \\ L(n-1) + L(n-2) & \text{if } n > 2. \end{cases}$$

(a) How is the definition of L(n) different from the definition of F(n) in Definition 3.1?
The definition of L(n) is different from F(n) when n = 2.
L(2) = 3 and F(2) = 1.

(b) Compute the first 12 Lucas numbers.
$$L(1) = 1$$
$$L(2) = 3$$
$$L(3) = 4$$
$$L(4) = 7$$
$$L(5) = 11$$
$$L(6) = 18$$
$$L(7) = 29$$
$$L(8) = 47$$
$$L(9) = 76$$
$$L(10) = 123$$
$$L(11) = 199$$
$$L(12) = 322$$

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5. Compute the first seven terms of the following recurrence relations:

(a) C(n) of Example 3.5

$$C(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2.C(n-1) & \text{if } n > 0 \end{cases}$$

$$C(0) = 1$$
$$C(1) = 2$$
$$C(2) = 4$$
$$C(3) = 8$$
$$C(4) = 16$$
$$C(5) = 32$$
$$C(6) = 64$$

(b) V(n) of Example 3.6

$$V(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2.V(n-1) + 1 & \text{if } n > 0 \end{cases}$$

$$V(0) = 1$$
$$V(1) = 3$$
$$V(2) = 7$$
$$V(3) = 15$$
$$V(4) = 31$$
$$V(5) = 63$$
$$V(6) = 127$$

(c) E(n) of Example 3.7

$$E(n) = \begin{cases} 0 & \text{if } n = 1, \\ E(n-1) + n - 1 & \text{if } n > 1 \end{cases}$$

$$E(1) = 0$$
$$E(2) = 1$$
$$E(3) = 3$$
$$E(4) = 6$$
$$E(5) = 10$$
$$E(6) = 15$$
$$E(7) = 21$$

6. Consider the following recurrence relation.

$$P(n) = \begin{cases} 0 & \text{if } n = 0 \\ [P(n-1)]^2 - n & \text{if } n > 0 \end{cases}$$

Use this recurrence relation to compute P(1), P(2), P(3), and P(4)

$$P(1) = -1$$
$$P(2) = -1$$
$$P(3) = -2$$
$$P(4) = 0$$

7. Given the following definition, compute P(4).

$$Q(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times [P(n-1)] & \text{if } n > 0 \end{cases}$$

$$P(4) = 4 \times P(3) = 4 \times 3 \times P(2) = 4 \times 3 \times 2 \times P(1) = 4 \times 3 \times 2 \times 1 \times P(0)$$
$$4 \times 3 \times 2 \times 1 \times 1 = 24$$

8. Given the following definition, compute Q(5).

$$Q(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ Q(n-1) + Q(n-2) + Q(n-3) & \text{if } n > 2 \end{cases}$$

$$Q(5) = Q(4) + Q(3) + Q(2) = Q(3) + Q(2) + Q(1) + Q(2) + Q(1) + Q(0) + Q(2)$$
$$= Q(2) + Q(1) + Q(0) + Q(2) + Q(1) + Q(2) + Q(1) + Q(0) + Q(2)$$
$$= 2 + 1 + 0 + 2 + 1 + 2 + 1 + 0 + 2 = 11$$

9. Consider the recurrence relation defined in Example 3.3. Suppose that, as in the example, you borrow $500, but you pay her back $100 each week. Each week, Ursula charges you 10% interest on the amount you still owe, after your $100 payment is taken into account.

(a) Write down a recurrence relation for M(n), the amount owed after n weeks.

$$M(n) = \begin{cases} 500 & \text{if } n = 0 \\ 1.1 \times (M(n-1) - 100) & \text{if } n > 0 \end{cases}$$

(b) How much will you owe after four weeks?

$$M(4) = 1.1 \times (M(3) - 100) = 1.1 \times (1.1 \times (M(2) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (M(1) - 100) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (1.1 \times (M(0) - 100) - 100) - 100) - 100)$$
$$= 1.1 \times (1.1 \times (1.1 \times (1.1 \times (500 - 100) - 100) - 100) - 100) = 221.54$$

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10. Every year, Alice gets a raise of $3,000 plus 5% of her previous year's salary. Her starting salary is $50,000. Give a recurrence relation for S(n), Alice's salary after n years, for $n \geq 0$.

$$S(n) = \begin{cases} 50000 & \text{if } n = 0 \\ 1.05 \times S(n-1) + 3000 & \text{if } n > 0 \end{cases}$$

11. Suppose that today (year 0) your car is worth $10,000. Each year your car loses 10% of its value, but at the end of each year you add customizations to your car that increase its value by $50. Write a recurrence relation to model this situation.
Let P(n) is the value of the car after n year(s).

$$P(n) = \begin{cases} 10000 & \text{if } n = 0 \\ 0.9 \times P(n-1) + 50 & \text{if } n > 0 \end{cases}$$

Section 3.4:

13. In Exercise 3 of Section 3.1, we defined the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \leq 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2 \end{cases}$$

Prove that $H(2n) = H(2n-1) = n$ for all $n \geq 1$.
Step 1:
For n = 1:
$$H(2 \times 1) = H(2) = 1$$
$$H(2 \times 1 - 1) = H(1) = 1$$

Hence, $H(2 \times 1) = H(2 \times 1 - 1) = 1$.
Then $H(2n) = H(2n-1) = n$ is true for n = 1.
For n = 2:

$$H(2 \times 2) = H(4) = H(3) + H(2) - H(1) = H(2) + 1 - 1 = 1 + 1 - 0 = 2$$

$$H(2 \times 2 - 1) = H(3) = 1 + 1 - 0 = 2$$

Hence, $H(2 \times 2) = H(2 \times 2 - 1) = 2$.
Step 2:
For n = k, assume $H(2k) = H(2k-1) = k$ is true.
Step 3:
For n = k + 1:
$$H(2(k+1)) = H(2(k+1) - 1)$$
$$H(2k+2) = H(2k+2-1)$$
$$H(2k+2) = H(2k+1)$$

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$$H(2k+1) + H(2k) - H(2k-1) = H(2k) + H(2k-1) - H(2k-2)$$

$$H(2k+1) + k - k = k + k - H(2k-2)$$

$$H(2k) + H(2k-1) - H(2k-2) = 2k - H(2k-2)$$

$$k + k - H(2k-2) = 2k - H(2k-2)$$

$$0 = 0 (true)$$

Hence, $H(2(k+1)) = H(2(k+1) - 1)$ is true.

$$H(2(k+1)) = k+1$$

$$H(2k+2) = k+1$$

$$H(2k+1) + H(2k) - H(2k-1) = k+1$$

$$H(2k+1) + k - k = k+1$$

$$H(2k+1) = k+1$$