---

1. Refer to Definition 1.10. Show that the divisibility relation — makes the set N of natural numbers a partially ordered set.
   Reflexivity:
   Because every number $x \in N$ can divides itself. Hence, the divisibility relation is reflexive.
   Transitivity:
   If $a|b$ and $b|c$ for $a, b, c \in N$. Then $b = a.k$ and $c = b.m$ and $c = a.k.m$. Therefore, c can divides a. Hence, the relation | is transitivity.
   Antisymmetry:
   If $a|b$ with $a, b \in N$, $a < b$. Hence, a cannot divide a. Therefore, the relation | is antisymmetric.
   Hence, the relation | is a partially order set.

2. Explain why the divisibility relation | does not define a partially ordering on the set Z of integers.
   For $x = -1$ and $y = 1$. $x|y$ and also $y|x$. Hence, the relation is not antisymmetric. Therefore, the relation is not a partially ordering set.

3. Consider the poset $(N, |)$. Are there any minimal elements? Are there any maximal elements? Explain.
   Because N = {1,2,3,4,...∞}. The minimal element is 1 and there is no maximal elements.

4. Let A = {a,b,c,...z}. In the poset(P(A), ⊂), find a pair of incomparable elements.
   A pair of incomparable elements is ({a,b,c},{d,e,f}).

5. Let W be the set of all web pages. For $x, y \in W$, let xRy if you can navigate from x to y by following links (Let's say it always possible to "navigate" from a page to itself; just do nothing.) Explain why R is not a partial ordering.
   Let $x, y \in W$, it is possible to navigate from x to y and from y to x. Hence, xRy and yRx. Therefore, R is not antisymmetric and not a partially ordering set.

6. Let a relation R be defined on the set of real numbers as follows:

$$xRy \Leftrightarrow 2x + y = 3$$

   Prove that this relation is antisymmetric.
   Let: $y = 3 - 2x$
   For yRx:

$$yRx \Leftrightarrow 2y + x = 3$$

$$2(3 - 2x) + x = 3$$

$$6 - 4x + x = 3$$

$$-3x = -3$$
$$x = 1$$
$$y = 3 - 2(1) = 1$$

Hence, x = y.
Therefore, the relation is antisymmetric.

7. Explain why the relation R on {0, 1, 2, 3} given by

$$R = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,2),(2,3),(0,2)\}$$

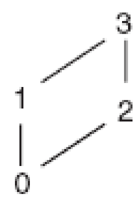
is not a partial ordering on {0, 1, 2, 3}. Be specific.
Because 1R2 and 2R3 but there is no relation between 1 and 3. Hence, the relation R is not transitive. Therefore, the relation is not a partially ordering set.

8. Explain why the relation R on {0, 1, 2, 3} given by

$$R = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,2),(0,2),(2,1)\}$$

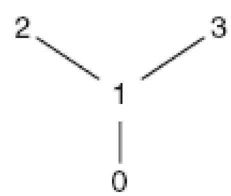
is not a partial ordering on {0, 1, 2, 3}. Be specific.
Because 1R2 and 2R1, the relation is not fully antisymmetric. Hence, the relation is not partial ordering.

9. The Hasse diagram below defines a partial ordering on the set {0, 1, 2, 3}. Give the set of ordered pairs corresponding to this relation.



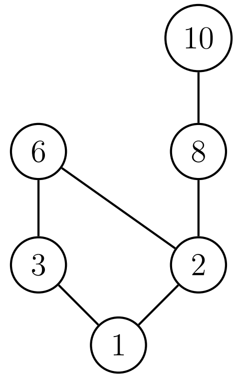
$$R = \{(0,1),(1,3),(0,2),(2,3),(0,3),(0,0),(1,1),(2,2),(3,3)\}$$

10. The Hasse diagram below defines a partial ordering on the set {0, 1, 2, 3}. Give the set of ordered pairs corresponding to this relation.



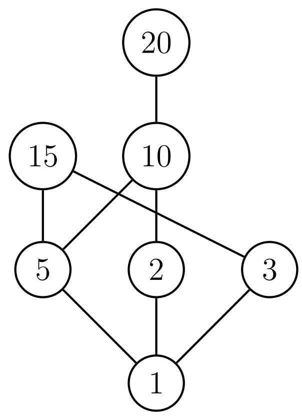
$$R = \{(0,1),(0,2),(0,3),(1,2),(1,3),(0,0),(1,1),(2,2),(3,3)\}$$

11. The divides relation "|" defines a partial ordering on the set $\{1, 2, 3, 6, 8, 10\}$. Draw the Hasse diagram for this poset. What are the maximal elements?



The maximal elements are 6 and 10.

12. Let S = $\{1, 2, 3, 5, 10, 15, 20\}$. It is a fact that (S, |) is a poset. Draw its Hasse diagram.



13. Let X be a set of different nonzero monetary values (in U.S. or Canadian cents). In other words, $X \subseteq N$. Define a relation $\vDash$ on x as follows. For $a, b \in X, a \vDash b$ if b can be obtained from a by adding a (possibly empty) collection of dimes (10 cents) and quarters (25 cents). So, for example, $25 \vDash 35$, but $25 \nvDash 30$. Prove that $\vDash$ is a partial ordering on X.

Reflexive:

For every $a \in X$, $a \vDash a$ because a can add a empty collection of dimes and quarters to become a.

Antisymmetric:

For every $a, b \in X$, if $a \vDash b$ and $a \neq b$, b must be larger than a. Hence, b cannot become a by adding a collection of dimes and quarters. Therefore, the relation is antisymmetric.

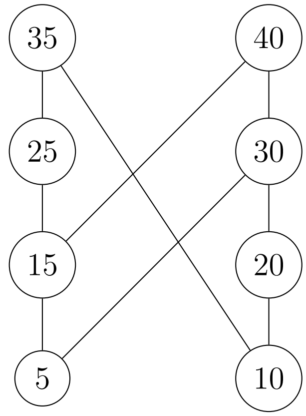
Transitivity:

For every $a, b, c \in X$, if $a \vDash b$ then $a+10k+25z = b$ and if $b \vDash c$ then $b+10m+25n = c$. Then a can become c by adding $10k + 25z + 10m + 25n$. Hence, $a \vDash c$. Therefore, the relation is transitive.

Therefore, the relation is partial ordering.

14. Let X = {5, 10, 15, 20, 25, 30, 35, 40}, and let $\models$ be as in Problem 13.

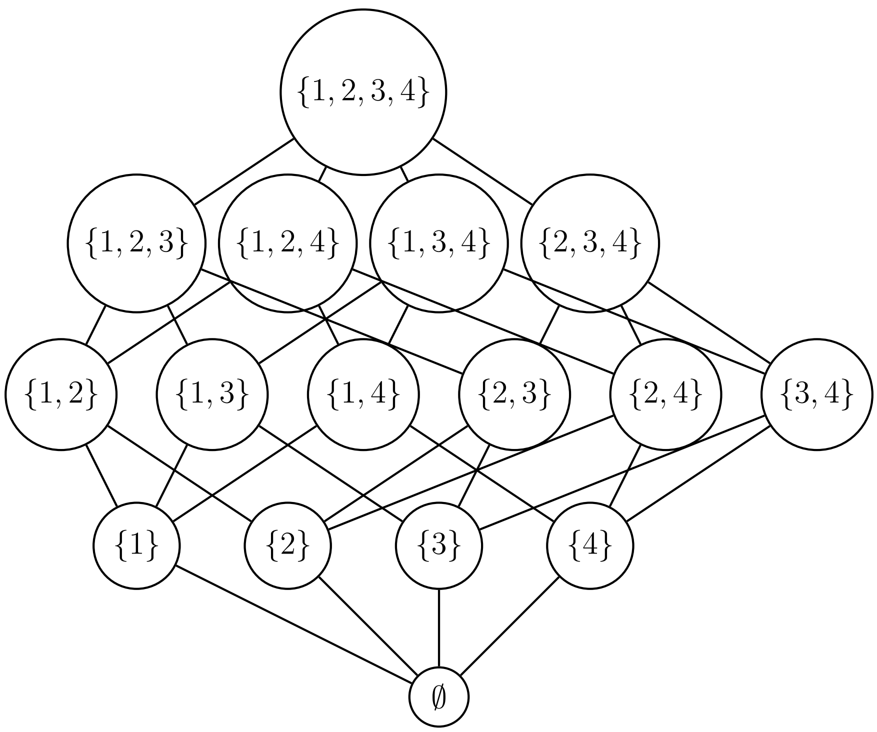
(a) Draw the Hasse diagram for the poset $(X, \models)$.



(b) List all the minimal elements of $(X, \models)$.
The minimal element of $(X, \models)$ is 5.

(c) Give a pair of incomparable elements in $(X, \models)$.
A pair of incomparable elements in $(X, \models)$ is (20,25).

19. Let $X = \{1, 2, 3, 4\}$. Draw the Hasse diagram for the poset $((X), \subset)$.



22. Let B be the set of all four-digit binary strings; that is,

$$B = \{0000, 0001, 0010, 0011, ...1111\}$$

Define a relation $\lhd$ on B as follows: Let $x, y \in B$, where $x = x_1x_2x_3x_4$ and $y = y_1y_2y_3y_4$. We say that $x \lhd y$ if $x_i \leq y_i$ for $i = 1, 2, 3, 4$. In other words, $x \lhd y$ if y has a 1 in every position where x does. So, for example, $0101 \lhd 0111$ and $0000 \lhd 0011$, but $1010 \not\lhd 0111$. The relation $\lhd$ is called the bitwise $\leq$. Show that $(B, \lhd)$ is a poset.
Reflexive:
For every $a = a_1a_2a_3a_4 \in B$, $a \lhd a$ because $a_i \leq a_i$ for $i = 1, 2, 3, 4$. Hence, the relation is reflexive.
Antisymmetric:
For every $a = a_1a_2a_3a_4 \in B$ and $b = b_1b_2b_3b_4 \in B$, if $a \lhd b$ and $b \lhd a$ then $a = b$ because $a_1a_2a_3a_4 \leq b_1b_2b_3b_4$ and $b_1b_2b_3b_4 \leq a_1a_2a_3a_4$. Hence, the relation is antisymmetric.
Transitivity:
For every $a, b, c \in B$ and $a = a_1a_2a_3a_4, b = b_1b_2b_3b_4, c = c_1c_2c_3c_4$. If $a \lhd b$ and $b \lhd c$, then $a_i \leq b_i$ and $b_i \leq c_i$ for $i = 1, 2, 3, 4$. Hence, $a_i \leq c_i$ and $a \lhd c$. Hence, the relation is transitive.
Therefore, the $(B, \lhd)$ is a poset.

23. Prove that $(B, \lhd) \cong (P(\{1, 2, 3, 4\}, \subseteq))$.
We can define a function $f : B \to P(\{1, 2, 3, 4\})$ with that maps each $x = x_1x_2x_3x_4 \in B$ if $x_i = 1$ for $i = 1, 2, 3, 4$ then $i \in f(x)$. Because every elements of B maps exactly to one element of $P(\{1, 2, 3, 4\})$, f is one-to-one correspondence. And because of $\lhd$ and $\subseteq$ behave exactly the same for the two sets. Hence, the edges in the Hasse diagram for $(B, \lhd)$ correspond exactly to the edges in the Hasse diagram for $(P(\{1, 2, 3, 4\}), \subseteq)$. Therefore, $(B, \lhd) \cong (P(\{1, 2, 3, 4\}), \subseteq)$.

24. In $(B, \lhd)$, give a counterexample to show that

0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1001, 1000, 1010, 0011, 1100, 1101, 1110, 1111

is not a valid topological sort of the elements of B.
Because 0011 is not the minimal element if 0100 is not deleted but 0011 is standing before 0100, this is not a valid topological sort of the elements of B.

25. Perform a topological sort on the elements of B.
A topological sort on the elements of B is:

0000, 0001, 0010, 0100, 1000, 0011, 0101, 0110, 1001, 1010, 1100, 0111, 1110, 1101, 1011, 1111

26. Let $F \subseteq N$ be the set of all factors of 210. In the poset $(F, |)$, find the following.

    (a) $30 \wedge 21$, the meet of 30 and 21.
    $30 \wedge 21 = gcd(30, 21) = 3$.

(b) $35 \vee 15$, join of 35 and 15.
$35 \vee 15 = lcm(35, 15) = 105$.

(c) $2 \wedge 7$.
$2 \wedge 7 = gcd(2, 7) = 1$.

(d) $2 \vee 7$.
$2 \vee 7 = lcm(2, 7) = 14$.

(e) $\neg 30$, the complement of 30.

$$\neg 30 = x \text{ with } (30 \wedge x) = 1 \text{ and } (30 \vee x) = 210$$

$$30 = 2 \times 3 \times 5$$

$$210 = 2 \times 3 \times 5 \times 7$$

Hence, x = 7 and $\neg 30 = 7$.