Math 482: Linear Programming, Fall 2020

Due Friday, October 30, 6PM CST

Homework 7 Solutions

1. *Show that any $n \times n$ matrix following the pattern*

$$\begin{bmatrix} 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

is totally unimodular: any submatrix obtained by taking any k rows and any k columns has determinant -1, 0, or 1.

There are only two 1×1 submatrices: $[\,0\,]$ and $[\,1\,]$, which have determinant 0 and 1, respectively.

There are many possible 2×2 submatrices, but since their entries are all 0 or 1, the formula for a determinant,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

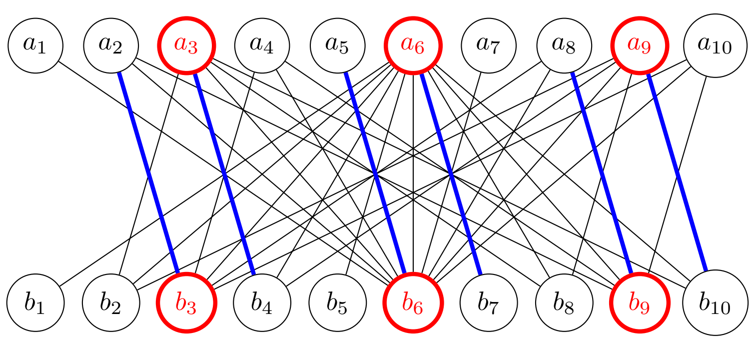
can only give us -1, 0, or 1: ad and bc are both 0 or 1, so the difference can be $1-0$, $1-1$, $0-0$, or $0-1$.

Finally, the $n \times n$ matrix has only two distinct types of rows: rows with 1's in the odd positions, and rows with 1's in the even positiions. So any $k \times k$ submatrix with $k \geq 3$ will have to include two rows of the same type. These will contribute identical entries to the $k \times k$ submatrix, and any matrix with two identical rows has determinant 0.

2. *Consider the bipartite graph with vertices $\{a_1, a_2, \dots, a_{10}\}$ on one side, vertices $\{b_1, b_2, \dots, b_{10}\}$ on the other side, and an edge between a_i and b_j if the product ij is a multiple of 6.*

Find a largest matching in this graph, and show that it cannot be any larger by finding a vertex cover of the same size.

A vertex cover of 6 vertices (in red) and one of many possible matchings of 6 edges (in blue):



To convince yourself that $S = \{a_3, a_6, a_9, b_3, b_6, b_9\}$ is a vertex cover without having to look at each of the edges in the diagram carefully, note that if we pick vertices a_i and b_j such that neither of them is in S, then neither i nor j is divisible by 3. In that case, ij is not divisible by 3, so in particular, it's not a multiple of 6: and that means there's no edge (a_i, b_j). In other words, all edges have to have at least one endpoint in S.

3. *A bipartite graph (X, Y, E) has $|X| = |Y| = n$ and is r-*regular*: every vertex (in X or in Y) is the endpoint of exactly r edges.*

   (a) *Determine $|E|$, the number of edges in the graph.*

   There are n vertices in X, and each is the endpoint of r edges, for $r \cdot n$ edges. This counts every edge exactly once, because every edge has exactly one endpoint in X.

   (b) *Show that any vertex cover must contain at least n vertices.*

   Because every vertex is the endpoint of exactly r edges, each vertex in the vertex cover can only cover r edges. Therefore there must be at least $\frac{|E|}{r} = \frac{r \cdot n}{r} = n$ vertices in the vertex cover.

4. *Find examples of networks with the following properties. For each example, describe the maximum flow(s) and the minimum cut(s).*

   (a) *A network with a unique maximum flow, but multiple minimum cuts.*

   One example here is a network where many arcs of equal capacity are connected in series, such as:

$$\begin{tikzcd} s \arrow[r, "0/1"] & a \arrow[r, "0/1"] & b \arrow[r, "0/1"] & c \arrow[r, "0/1"] & t \end{tikzcd}$$

   Then taking S to be any of $\{s\}$, $\{s, a\}$, $\{s, a, b\}$, or $\{s, a, b, c\}$ (and T to be the complement $\{a, b, c, t\}$, $\{b, c, t\}$, $\{c, t\}$, or $\{t\}$ respectively) gives a minimum cut: there is only one crossing edge, which has capacity 1.

   On the other hand, there is only one flow with value 1: setting $x_{sa} = x_{ab} = x_{bc} = x_{ct} = 1$.

   (b) *A network with multiple maximum flows, but a unique minimum cut.*

   Here is one possible example:

$$\begin{tikzcd} & a \arrow[dr, "0/1"] & & \\ s \arrow[ur, "0/1"] \arrow[dr, "0/1"'] & & c \arrow[r, "0/1"] & t \\ & b \arrow[ur, "0/1"'] & & \end{tikzcd}$$

   Here, the minimum cut (which has capacity 1) is unique: take $S = \{s, a, b, c\}$ and $T = \{t\}$. However, there are multiple flows with value 1: for any real number $\alpha \in [0, 1]$, we could set $x_{sa} = x_{ac} = \alpha$, set $x_{sb} = x_{bc} = 1 - \alpha$, and set $x_{ct} = 1$.

   (c) *A network with multiple maximum flows and multiple minimum cuts.*

   One way to get here is to combine our two previous examples:

$$\begin{tikzcd} & a \arrow[dr, "0/1"] & & & & \\ s \arrow[ur, "0/1"] \arrow[dr, "0/1"'] & & c \arrow[r, "0/1"] & d \arrow[r, "0/1"] & e \arrow[r, "0/1"] & t \\ & b \arrow[ur, "0/1"'] & & & & \end{tikzcd}$$

   The maximum flows are similar to the ones in (b): for any real number $\alpha \in [0, 1]$, we could set $x_{sa} = x_{ac} = \alpha$, set $x_{sb} = x_{bc} = 1 - \alpha$, and set $x_{cd} = x_{de} = x_{et} = 1$.

The minimum cuts are similar to the ones in (c): we can take S to be any of $\{s,a,b,c\}$, $\{s,a,b,c,d\}$, or $\{s,a,b,c,d,e\}$, taking T to be the complement $\{d,e,t\}$, $\{e,t\}$, or $\{t\}$ respectively.

5. *(Only 4-credit students need to do this problem.)*

   Consider a bipartite graph (X,Y,E) with $X=\{x_1,x_2,\ldots,x_m\}$ and $Y=\{y_1,y_2,\ldots,y_n\}$. The biadjacency matrix *of this graph is the $m\times n$ matrix A where $A_{ij}=1$ if there is an edge $(x_i,y_j)\in E$, and $A_{ij}=0$ otherwise.*

   If $m=n$ (so that the matrix A is square) and $\det(A)=-3$*, show that the graph contains a matching of size n.*

   Recall the Leibniz formula for a determinant:

   $$\det(A)=\sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^{n}A_{i,\sigma(i)}\right)$$

   where the sum is over all permutations of $\{1,2,\ldots,n\}$. For example, if A is a 3×3 matrix,

   $$\det(A)=A_{11}A_{22}A_{33}+A_{12}A_{23}A_{31}+A_{13}A_{21}A_{32}-A_{11}A_{23}A_{32}-A_{13}A_{22}A_{31}-A_{12}A_{21}A_{33}.$$

   When A is the biadjacency matrix, each of the products $\prod_{i=1}^{n}A_{i,\sigma(i)}$ is 1 if and only if each one of the edges $(x_i,y_{\sigma(i)})$ exists in the graph, and when that happens, these n edges form a matching of size n. For example, if a graph with $|X|=|Y|=3$ contained the matching $\{(x_1,y_2),(x_2,y_1),(x_3,y_3)\}$, then the product $A_{12}A_{21}A_{33}$ would equal 1.

   If the graph had *no* matching of size n, then all of these products would be 0, and so $\det(A)$ would be 0. We are given that $\det(A)=-3\neq 0$; therefore some matchings (at least 3 matchings, in fact) must exist.

   Note that this implication does not go the other way. It's possible for the graph to contain some matchings and for $\det(A)$ to still be 0, because the terms have different signs (depending on the sign of the permutation σ) and some of them might cancel.