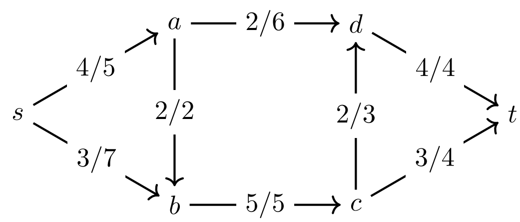
Math 482: Linear Programming, Fall 2020

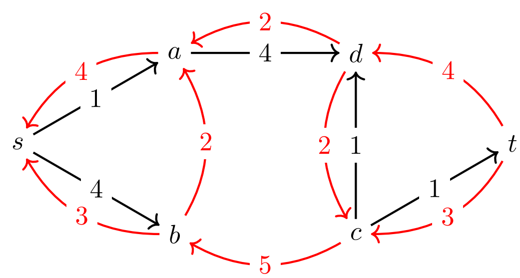
Due Wednesday, November 11, 6PM CST

Homework 8 solutions

1. *Given the network below, with label x/y denoting a flow of x and a total capacity of y along an edge, draw the residual graph, and use it to list all possible augmenting paths.*



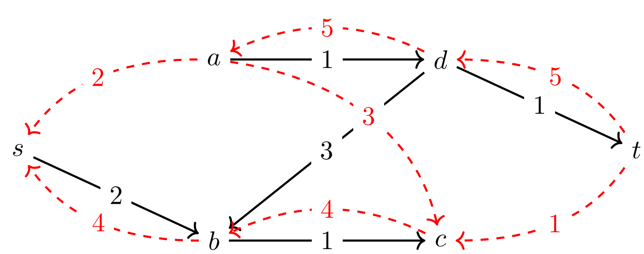
We remove the arcs $a \to b, b \to c, d \to t$, which are at capacity. We add a reverse arc practically everywhere, since all the other arcs have positive capacity, getting the following residual graph:



Augmenting paths are paths from s to t in the residual graph. These have to eventually get from the left half of the picture to the right half, which means they have to use the edge $a \to d$. To get from s to a, we can either go directly, or via b; to get from d to t, we must go via c. This gives us two paths:

- $s \to a \to d \leftarrow c \to t$.
- $s \to b \leftarrow a \to d \leftarrow c \to t$.

2. *The diagram below gives a residual graph for a network. (Black edges are "forward" edges, red dashed edges are "backward" edges.)*

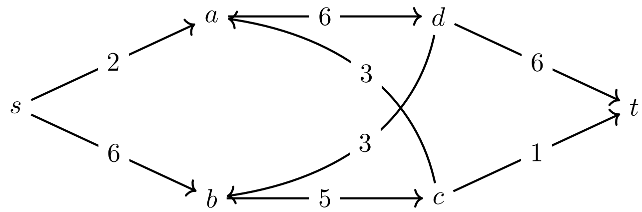


(a) *Determine the edges of the original network, and their capacities.*

We get an arc (i, j) in the original network whenever there is a "forward" residual arc (i, j) in the same direction or a "backward" residual arc (j, i) in the opposite direction. (Or both.)

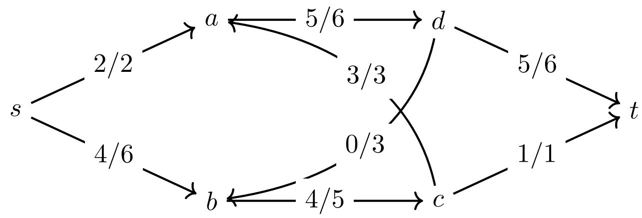
The residual capacities of these arcs are $c_{ij} - x_{ij}$ and x_{ij}, respectively, so the total capacity c_{ij} of the arc is the sum of the residual capacities on these arcs.

This gives us the following network:

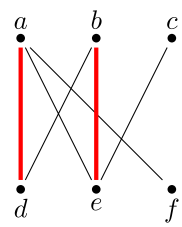


(b) *Find the flow which produces this residual graph.*

The flow x_{ij} is exactly the residual capacity of the backward residual arc (j, i), if there is one. So we get the following flows:

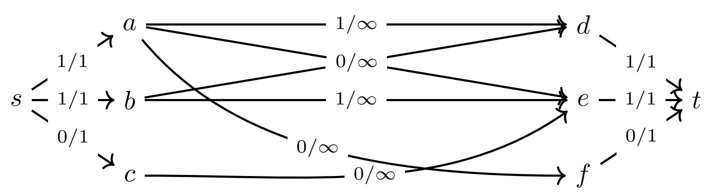


3. *Consider the following matching (that is, $M = \{(a, d), (b, e)\}$) in a bipartite graph:*



First, convert this matching into a feasible flow in a network. Then, find an augmenting path in that network, and use it to improve the matching to a larger one.

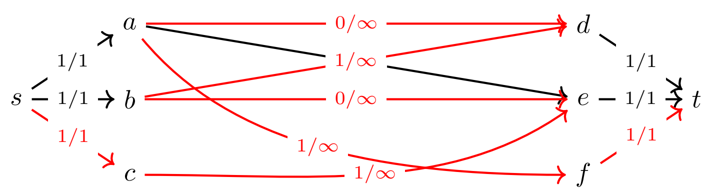
We start with the network below:



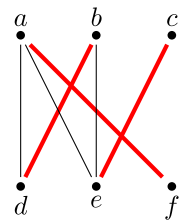
An augmenting path must get from the $s \to c$ arc at the beginning (which is the only arc out of s below capacity) to the $f \to t$ arc at the end (which is the only arc into t below capacity). We can do this by going

$$s \xrightarrow{0/1} c \xrightarrow{0/\infty} e \xleftarrow{1/\infty} b \xrightarrow{0/\infty} d \xleftarrow{1/\infty} a \xrightarrow{0/\infty} f \xrightarrow{0/1} t.$$

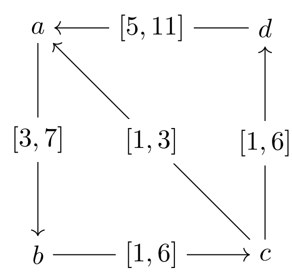
We can increase the forward arcs by 1 and decrease the backward arcs by 1, giving us the flow below (with the augmenting path marked in red):



The corresponding matching in the bipartite graph is

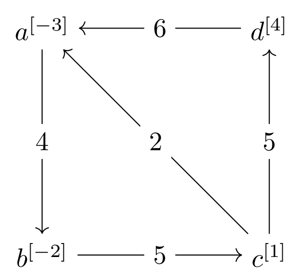


4. *Suppose that we want to find a feasible circulation in the network below with flows on each edge in the specified lower and upper bounds.*

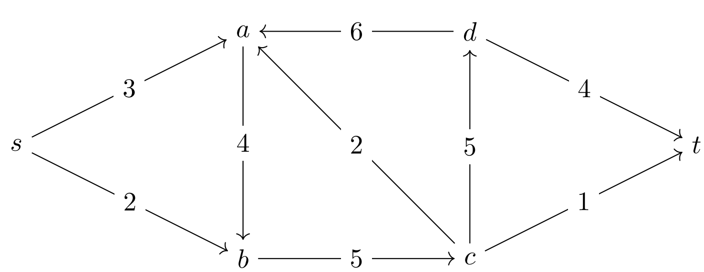


Find a feasible circulation in this network by writing down an equivalent maximum-flow problem, finding the maximum flow in that network, and converting it back to an equivalent feasible circulation.

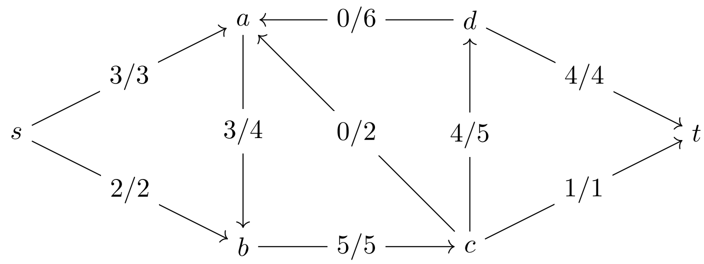
We first convert this problem to a supply/demand problem (labels on arcs are capacities):



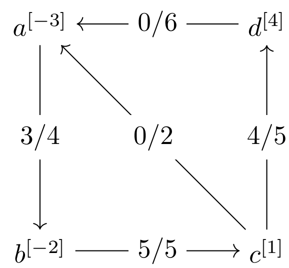
Then, we convert the supply/demand problem into a network flow problem:



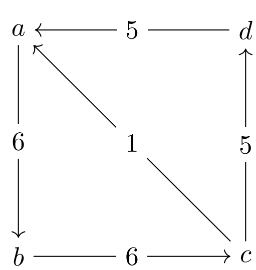
Solving the maximum flow problem (e.g., by augmenting along $s \to b \to c \to t$, then $s \to b \to c \to d \to t$, then $s \to a \to b \to c \to d \to t$), we get the following flow:

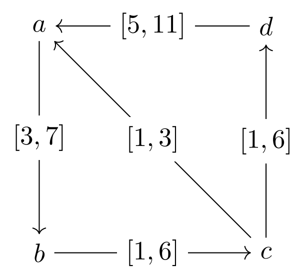


Erasing s and t, we get a solution to the supply/demand problem:



Finally, to the flow along each arc, we add the lower bound in the original problem, getting the following circulation (with the intervals in the diagram on the right for comparison):





5. *Write down a linear program for a general feasible circulation problem. (There is no objective function, so make the objective function just "maximize 0".)*

   Then, take the dual of this linear program.

   The linear program will be:

$$\begin{array}{lll}
\underset{\mathbf{x}\in\mathbb{R}^{|A|}}{\text{maximize}} & 0 & \\
\text{subject to} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = 0 & (k \in N) \\
& a_{ij} \le x_{ij} \le b_{ij} & (i,j) \in A
\end{array}$$

   Here, we take the same flow conservation constraints as for the maximum-flow problem, but at every node. We replace the capacity constraints by the constraints $a_{ij} \le x_{ij} \le b_{ij}$; for taking the dual, we should split these up $x_{ij} \ge a_{ij}$ and $x_{ij} \le b_{ij}$. The variables are now unconstrained, since it's possible for a_{ij} to be negative sometimes.

We assign a dual variable u_k to every node for the flow conservation constraint, and a dual variable v_{ij} for each lower bound and w_{ij} for each upper bound.

In the dual, we'll get:

$$
\begin{array}{lll}
\underset{\mathbf{u},\mathbf{v},\mathbf{w}}{\text{minimize}} & \displaystyle\sum_{(i,j)\in A} (a_{ij}v_{ij} + b_{ij}w_{ij}) & \\
\text{subject to} & v_{ij} + w_{ij} + u_j - u_i = 0 & ((i,j) \in A) \\
& v_{ij} \leq 0, w_{ij} \geq 0, u_i \text{ unrestricted.} &
\end{array}
$$