Prove that each of the following problems is NP-hard.

1. Given an undirected graph G, does G contain a simple path that visits all but 374 vertices?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained from G by adding 374 isolated vertices. Call a path in H ***almost-Hamiltonian*** if it visits all but 374 vertices. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian path.

$\Longrightarrow$ Suppose G has a Hamiltonian path P. Then P is an almost-Hamiltonian path in H, because it misses only the 374 isolated vertices.

$\Longleftarrow$ Suppose H has an almost-Hamiltonian path P. This path must miss all 374 isolated vertices in H, and therefore must visit every vertex in G. Every edge in H, and therefore every edge in P, is also an edge in G. We conclude that P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

2. Given an undirected graph G, does G have a spanning tree in which every node has degree at most 374?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained by attaching a fan of 372 edges to every vertex of G. Call a spanning tree of H ***almost-Hamiltonian*** if it has maximum degree 374. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

$\Longrightarrow$ Suppose G has a Hamiltonian path P. Let T be the spanning tree of H obtained by adding every fan edge in H to P. Every vertex v of H is either a leaf of T or a vertex of P. If $v \in P$, then $\deg_P(v) \leq 2$, and therefore $\deg_H(v) = \deg_P(v) + 372 \leq 374$. We conclude that H is an almost-Hamiltonian spanning tree.

$\Longleftarrow$ Suppose H has an almost-Hamiltonian spanning tree T. The leaves of T are precisely the vertices of H with degree 1; these are also precisely the vertices of H that are not vertices of G. Let P be the subtree of T obtained by deleting every leaf of T. Observe that P is a spanning tree of G, and for every vertex $v \in P$, we have $\deg_P(v) = \deg_T(v) - 372 \leq 2$. We conclude that P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

3. Given an undirected graph G, does G have a spanning tree with at most 374 leaves?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.[a] Given an arbitrary graph G, let H be the graph obtained from G by adding the following vertices and edges:

- First we add a vertex z with edges to every other vertex in z.
- Then we add 373 vertices $\ell_1, \ldots, \ell_{373}$, each with edges to z and nothing else.

Call a spanning tree of H ***almost-Hamiltonian*** if it has at most 374 leaves. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

$\Longrightarrow$ Suppose G has a Hamiltonian path P. Suppose P starts at vertex s and ends at vertex t. Let T be subgraph of H obtained by adding the edge tz and all possible edges $z\ell_i$. Then T is a spanning tree of H with exactly 374 leaves, namely s and all 373 new vertices ℓ_i.

$\Longleftarrow$ Suppose H has an almost-Hamiltonian spanning tree T. Every node ℓ_i is a leaf of T, so T must consist of the 373 edges $z\ell_i$ and a simple path from z to some vertex s of G. Let t be the only neighbor of z in T that is not a leaf ℓ_i, and let P be the unique path in T from s to t. This path visits every vertex of G; in other words, P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

[a] Are you noticing a pattern here?