HW1 Solutions

CS/ECE 374 B: Algorithms & Models of Computation, Spring 2020

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1 Problem 1

Solution 1.1

By the definition of remainders, there are 11 equivalent classes of different remainders for a number divided by 11, i.e. 0,1,2,3,4,...10. That is, $\forall x, y \in \mathbb{N}, \exists$ a,b $\in \mathbb{Z}$ such that x=11a+n, y=11b+n, where n is the remainder divided by 11.

Therefore, by Pigeonhole Principle, with a set of 111 natural numbers, there are at least 11 numbers in this set with the same remainder divided by 11, which means, in the same equivalent classes. In this case, as long as we choose 11 numbers from this equivalent class, we will be able to find a set S' of 11 numbers such that the difference between any two numbers is a multiple of 11.

Solution 1.2

The nodes of Professor Oihanean's tree according to preorder traversal is:
W N C U Z L B O E I T H D F S Q V Y P X A J G R K M

2 Problem 2

Solution:

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + 4T(\lfloor n/6 \rfloor) + n & \text{for} \quad n >= 6 \\ 1 & \text{for} \quad n < 6 \end{cases}$$

Claim: $T(n) = O(n \log n)$ for any $n \in \mathbb{N}$. That is, $\forall n \in \mathbb{N}$, $T(n) \leqq c_1 n \log n + c_2$, for some constants c_1 and c_2

Base Case: When $n = 1, 2, 3, 4, 5$, $T(n) = 1 \leqq c_1 n \log(n) + c_2$ provided that $c_2 \geq 1$. Therefore, the claim is true for $n < 6$.

Inductive Hypothesis: Suppose for any $k \in \mathbb{N}$ such that $k \geq 6$, for all $n < k$, the claim is true for $T(n)$. We need to show that the claim holds for $n = k$. And suppose $c_2 = 1$.

Inductive Step:

When $n = k$, $T(k) = T(\lfloor k/3 \rfloor) + 4T(\lfloor k/6 \rfloor) + k$

$\leqq c_1 \lfloor k/3 \rfloor \log \lfloor k/3 \rfloor + c_2 + 4(c_1/6\rfloor \log \lfloor k/6 \rfloor + c_2) + k$ (by inductive hypothesis)

$\leqq c_1(k/3)\log(k/3) + 4c_1(k/6)\log(k/6) + 5c_2 + k$

$\leqq c_1 k \log k - c_1(k/3)\log 3 - 4c_1(k/6)\log 6 + 5c_2 + k$

$\leqq c_1 k \log k - 2c_1(k/3) - 8c_1(k/6) + 5 + k$ (since $\log 3$ and $\log 6$ are smaller than 2)

$= c_1 k \log k - 2c_1 k + k + 5$

$\leqq c_1 k \log k + c_2$ (provided that $-2c_1 k + k + 5 \leqq 1 \Longleftrightarrow c_1 \geq 5/6$)

Conclusion: Therefore, for any given $c_1 \geq 5/6$ and $c_2 \geq 1$, we have $T(n) \leqq c_1 n \log n + c_2$ for any $n \in \mathbb{N}$. In other words, $T(n) = O(n \log n)$. QED.