Math 482: Linear Programming, Fall 2020

Due Wednesday, September 30, 6PM CST

Homework 4 Solutions

1. *Write down the dual of the linear program below. (Do not solve).*

$$\begin{array}{rl} \underset{x,y,z\in\mathbb{R}}{\textit{maximize}} & x + y + z \\ \textit{subject to} & 2x + y + 2z \leq 14 \\ & x + z \leq 8 \\ & 2x + 2y - z \leq 18 \\ & x, y, z \geq 0. \end{array}$$

The dual is:

$$(\mathbf{P}) \begin{cases} \underset{x,y,z\in\mathbb{R}}{\text{maximize}} & x + y + z \\ \text{subject to} & 2x + y + 2z \leq 14 \quad (u) \\ & x + z \leq 8 \quad (v) \\ & 2x + 2y - z \leq 18 \quad (w) \\ & x, y, z \geq 0. \end{cases} \rightsquigarrow (\mathbf{D}) \begin{cases} \underset{u,v,w\in\mathbb{R}}{\text{minimize}} & 14u + 8v + 18w \\ \text{subject to} & 2u + v + 2w \geq 1 \quad (x) \\ & u + 2w \geq 1 \quad (y) \\ & 2u + v - w \geq 1 \quad (z) \\ & u, v, w \geq 0. \end{cases}$$

2. *Determine whether $(x, y, z) = (5, 4, 0)$ is the optimal solution to the linear program from problem 1, using complementary slackness.*

If we assume that $(x, y, z) = (5, 4, 0)$ is optimal, then complementary slackness tells us the following about the optimal dual solution (u, v, w):

- Either $2x + y + 2z = 14$ or $u = 0$. However, since $2x + y + 2z = 2(5) + (4) + 2(0) = 14$, we learn nothing about u.
- Either $x + z = 8$ or $v = 0$. Since $x + z = 5 + 0 < 8$, we must have $v = 0$.
- Either $2x + 2y - z = 18$ or $w = 0$. However, since $2x + 2y - z = 2(5) + 2(4) - (0) = 18$, we learn nothing about w.
- Either $x = 0$ or $2u + v + 2w = 1$. Since $x = 5 > 0$, $2u + v + 2w = 1$.
- Either $y = 0$ or $u + 2w = 1$. Since $y = 4 > 0$, $u + 2w = 1$.
- Either $z = 0$ or $2u+v-w = 1$. However, since $z = 0$, we can't conclude anything about $2u+v-w$.

(Note that in the process, we also check that $(5, 4, 0)$ is feasible.)

Altogether, we have the following three equations:

$$\begin{cases} v = 0 \\ 2u + v + 2w = 1 \\ u + 2w = 1 \end{cases}$$

There is a unique solution: $(u, v, w) = (0, 0, \frac{1}{2})$.

As a final check, we want to know if this solution is dual feasible. We have $u, v, w \geq 0$; however, $2u + v - w = -\frac{1}{2} < 1$. Therefore we did not get a dual feasible solution. This means that our assumption was incorrect: $(x, y, z) = (5, 4, 0)$ is **not** an optimal solution.

3. *Consider the problem below:*

$$\begin{aligned} \mathop{\textit{maximize}}_{\mathbf{x}\in\mathbb{R}^n} \quad & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \textit{subject to} \quad & a_1x_1 + a_2x_2 + \dots + a_nx_n \le 1, \\ & x_1, x_2, \dots, x_n \ge 0. \end{aligned}$$

Assume that $a_1, \dots, a_n, c_1, \dots, c_n > 0$.

(a) *Write down the dual program.*

Since the primal program has a single $\le$ constraint, the dual will have a single nonnegative variable; call it u; the objective function will be to minimize u (since the right-hand side of the constraint is 1). For each variable x_i, we get the constraint $a_iu \ge c_i$. This gives us:

$$\begin{aligned} \mathop{\text{minimize}}_{u\in\mathbb{R}} \quad & u \\ \text{subject to} \quad & a_1u \ge c_1 \\ & a_2u \ge c_2 \\ & \dots \\ & a_nu \ge c_n \\ & u \ge 0. \end{aligned}$$

(b) *Determine the optimal dual solution. (This will of course depend on* $a_1, \dots, a_n$ *and* $c_1, \dots, c_n$*, but you should describe how.)*

The n dual constraints give us the lower bounds $u \ge \frac{c_1}{a_1}, u \ge \frac{c_2}{a_2}, \dots, u \ge \frac{c_n}{a_n}$. Since we are trying to minimize u, we want to pick the smallest value that satisfies all of these lower bounds. That's just the largest of all the bounds:

$$u = \max\left\{\frac{c_1}{a_1}, \frac{c_2}{a_2}, \dots, \frac{c_n}{a_n}\right\}.$$

(c) *Find a primal solution with the same objective value.*

Suppose that the maximum in part (b) is achieved by $\frac{c_j}{a_j}$. Then one primal solution that achieves the same objective value (and therefore the optimal solution) sets $x_j = \frac{1}{a_j}$ and $x_i = 0$ for all $i \ne j$.

Note that if we just guess this answer, that's fine, and we automatically know it's optimal. However, one approach that can lead us to finding this solution is complementary slackness. If $\frac{c_j}{a_j} > \frac{c_i}{a_i}$ for all $i \ne j$ (which is not guaranteed to happen, but is likely), then most of the dual constraints are slack: each constraint $a_iu \le c_i$ with $i \ne j$ is slack when we set $u = \frac{c_j}{a_j}$. Therefore complementary slackness says that $x_i = 0$ for all $i \ne j$.

Meanwhile, because $u \ne 0$, we must have $a_1x_1 + a_2x_2 + \dots + a_nx_n = 1$, which simplifies (if most of the x_i are zero) to $a_jx_j = 1$. So we should set $x_j = \frac{1}{a_j}$.

4. *Use the simplex method to solve the linear program below. Then, use your final simplex tableau to find the optimal dual solution.*

$$\begin{aligned} \mathop{\textit{maximize}}_{x,y,z\in\mathbb{R}} \quad & x - \ \ y + z \\ \textit{subject to} \quad & x + 2y + z \le 5 \\ & 2x + \ \ y + z \le 6 \\ & x, y, z \ge 0. \end{aligned}$$

We start with the simplex tableau below (I renamed the objective function to w because we're using z for one of the variables):

	x	y	z	s_1	s_2	
s_1	1	2	1	1	0	5
s_2	2	1	1	0	1	6
$-w$	1	-1	1	0	0	0

We pivot on x. Both s_1 and s_2 have positive entries in x's column, but s_1's ratio of $\frac{5}{1}$ is bigger than s_2's ratio of $\frac{6}{2}$, so s_2 leaves the basis and we get:

	x	y	z	s_1	s_2	
s_1	0	$3/2$	$1/2$	1	$-1/2$	2
x	1	$1/2$	$1/2$	0	$1/2$	3
$-w$	0	$-3/2$	$1/2$	0	$-1/2$	-3

We pivot on z. Both s_1 and x have positive entries in z's column; s_1's ratio is $\frac{2}{1/2} = 4$ and x's ratio is $\frac{3}{1/2} = 6$, so s_1 leaves the basis and we get:

	x	y	z	s_1	s_2	
z	0	3	1	2	-1	4
x	1	-1	0	-1	1	1
$-w$	0	-3	0	-1	0	-5

Since all reduced costs are negative, this is the optimal solution. By taking the negatives of the reduced costs in the slack variables, we get $(u, v) = (1, 0)$ for the optimal dual solution, where u is the dual variable for the $x + 2y + z \leq 5$ constraint and v is the dual variable for the $2x + y + z \leq 6$ constraint.

5. *(Only 4-credit students need to do this problem.)*

Consider the following linear program discussed in class:

$$\begin{aligned}
\underset{\mathbf{x} \in \mathbb{R}^d}{\textit{maximize}} \quad & x_d \\
\textit{subject to} \quad & 0.1 \leq x_1 \leq 1 - 0.1, \\
& 0.1x_1 \leq x_2 \leq 1 - 0.1x_1, \\
& \dots \\
& 0.1x_{d-1} \leq x_d \leq 1 - 0.1x_{d-1}, \\
& x_1, x_2, \dots, x_d \geq 0.
\end{aligned}$$

Let $\mathcal{P}_d$ be the "terrible trajectory"—the path between adjacent basic feasible solutions defined recursively as follows:

- *$\mathcal{P}_1$ starts at $(0, 0, 0)$ and increases x_1 from its lower bound to its upper bound;*
- *$\mathcal{P}_k$ follows $\mathcal{P}_{k-1}$, then increases x_k from its lower bound to its upper bound, then undoes the steps of $\mathcal{P}_{k-1}$ in reverse order.*

Show that the objective value increases with every step along $\mathcal{P}_d$. (Induct on d.)

First, we check the base case: that the objective value is increasing with every step along $\mathcal{P}_1$. For $\mathcal{P}_1$, there is only one pivot step: we increase x_1 from 0.1 to 0.9. The objective function in the 1-dimensional case is x_1, so this is increasing.

Assume that for the $(d-1)$-dimensional linear program, the objective value is always increasing along $\mathcal{P}_{d-1}$. Then:

- For the first $2^{d-1} - 1$ steps of $\mathcal{P}_d$, we are just following $\mathcal{P}_{d-1}$, so the $(d-1)$-dimensional objective function x_{d-1} is increasing at every step. Meanwhile, x_d is at its lower bound of $0.1x_{d-1}$. So it is always increasing at every step.
- On the 2^{d-1}-th step, x_d switches from the lower to the upper bound. The lower bound is $0.1x_{d-1}$, which is at most 0.1. The upper bound is $1 - 0.1x_{d-1}$, which is at least 0.9. So this is also an improvement in x_d.
- For the last $2^{d-1} - 1$ steps of $\mathcal{P}_d$, we are following $\mathcal{P}_{d-1}$ in reverse. This means, by the inductive hypothesis, that the $(d-1)$-dimensional objective function x_{d-1} is *decreasing* at every step. But x_d is now at its upper bound of $1 - 0.1x_{d-1}$. So any increase in x_{d-1} is an increase in x_d, and therefore the objective value is increasing.

This completes the inductive step; therefore the value of x_d is always increasing along $\mathcal{P}_d$, for each $d \geq 1$.