Math 482: Linear Programming, Fall 2020

Due Friday, October 9, 6PM CST

Homework 5 Solutions

1. *Use the dual simplex method with an artificial objective function to find a solution to the system of equations*

$$\begin{cases} x_1 - x_2 + 4x_3 = 4 \\ x_1 + x_2 = 2 \\ x_1 + 2x_2 - 2x_3 + x_4 = 3 \end{cases}$$

in which $x_1, x_2, x_3, x_4 \geq 0$.

Row-reducing the system produces the following tableau:

x_1	x_2	x_3	x_4	
1	−1	4	0	4
1	1	0	0	2
1	2	−2	1	3

$\rightsquigarrow$

	x_1	x_2	x_3	x_4	
x_1	1	0	2	0	3
x_2	0	1	−2	0	−1
x_4	0	0	0	1	2

This is still not feasible; it represents the basic solution $(3, -1, 0, 2)$. To make it feasible using the dual simplex method (which is admittedly overkill in this problem) we add an artificial objective function: to minimize the sum of nonbasic variables, or in this case to minimize x_2. (There is no special point to this artificial objective function, except that we want to end up with a dual feasible tableau.)

We get the tableau on the left. The dual simplex method tells us to pick x_2 as the leaving variable (it is the only one with a negative value) and x_3 as the entering variable (it is the only one with a negative entry in x_2's row). Pivoting, we get the tableau on the right, which is optimal.

	x_1	x_2	x_3	x_4	
x_1	1	0	2	0	3
x_2	0	1	−2	0	−1
x_4	0	0	0	1	2
$-z$	0	0	1	0	0

$\rightsquigarrow$

	x_1	x_2	x_3	x_4	
x_1	1	1	0	0	2
x_3	0	−1/2	1	0	1/2
x_4	0	0	0	1	2
$-z$	0	1/2	0	0	−1/2

This gives us the solution $(x_1, x_2, x_3, x_4) = (2, 0, \frac{1}{2}, 2)$, which is nonnegative and satisfies all three equations.

For problems 2 and 3, suppose that you have solved the linear program below on the left, and gotten the simplex tableau below on the right.

$$\begin{array}{ll} \underset{x,y\in\mathbb{R}}{\textit{minimize}} & x + 2y \\ \textit{subject to} & x + y \geq 3 \\ & x + 4y \geq 10 \\ & x \geq 2 \\ & x, y \geq 0 \end{array}$$

	x	y	s_1	s_2	s_3	
x	1	0	0	0	−1	2
y	0	1	0	−1/4	1/4	2
s_1	0	0	1	−1/4	−3/4	1
$-z$	0	0	0	1/2	1/2	−6

2. *Describe how the objective value will change, for sufficiently small values of δ, in each of the following cases. State whether your prediction will be a lower bound or an upper bound in general (when δ is large).*

(a) *The objective function changes from $x+2y$ to $x+(2+\delta)y$.*

Since $y=2$ in the optimal solution, changing the coefficient of y by δ should change the objective value from 6 to $6+2\delta$.

This is an upper bound in general: the point $(x,y)=(2,2)$ will always remain feasible, but it's possible that another point will become optimal when the objective value changes significantly. In that case, that other point could have a smaller objective value than $6+2\delta$.

(b) *The constraint $x \ge 2$ changes to $x \ge 2+\delta$.*

(Be careful! In equational form, $x \ge 2+\delta$ is represented as $-x+s_3=-2-\delta$.)

In equational form, the dual program has objective function $-3u-10v-2w$ (because all the constraints are negated), and the optimal dual solution (based on the tableau) is $(u,v,w)=(0,-\frac{1}{2},-\frac{1}{2})$. So changing the coefficient of w from -2 to $-2-\delta$ will change the objective value by $-\delta \cdot -\frac{1}{2}=\frac{1}{2}\delta$: from 6 to $6+\frac{1}{2}\delta$.

This is a lower bound in general. This dual solution with objective value $6+\frac{1}{2}\delta$ will remain dual feasible, so the dual program (which is a maximization problem) will achieve an objective value of at least $6+\frac{1}{2}\delta$, possibly higher.

(c) *The constraint $x+y \ge 3$ changes to $x+y \ge 3+\delta$.*

The constraint $x+y \ge 3$ is slack, so when the right-hand side changes by a small amount, the objective value will not change.

As for part (b), this is a lower bound in general.

3. *Use the dual simplex method to add the constraint $x+5y \le 11$ to the linear program and find the new optimal solution.*

When we add the constraint (in the form $x+5y+s_4=11$), we get the tableau on the left below, which we immediately row-reduce to get the tableau below on the right:

	x	y	s_1	s_2	s_3	s_4	
x	1	0	0	0	-1	0	2
y	0	1	0	$-1/4$	$1/4$	0	2
s_1	0	0	1	$-1/4$	$-3/4$	0	1
s_4	1	5	0	0	0	1	11
$-z$	0	0	0	$1/2$	$1/2$	0	-6

$\rightsquigarrow$

	x	y	s_1	s_2	s_3	s_4	
x	1	0	0	0	-1	0	2
y	0	1	0	$-1/4$	$1/4$	0	2
s_1	0	0	1	$-1/4$	$-3/4$	0	1
s_4	0	0	0	$5/4$	$-1/4$	1	-1
$-z$	0	0	0	$1/2$	$1/2$	0	-6

Now we use the dual simplex method. Our only choice of leaving variable is s_4 (with a value of -1) and our only choice of entering variable is s_3 (which has a negative entry in s_4's row), giving us the following tableau:

	x	y	s_1	s_2	s_3	s_4	
x	1	0	0	-5	0	-4	6
y	0	1	0	1	0	1	1
s_1	0	0	1	-4	0	-3	4
s_3	0	0	0	-5	1	-4	4
$-z$	0	0	0	3	0	2	-8

This tableau is optimal and so $(x,y)=(6,1)$ is the new optimal solution.

4. *Illinois Instruments (II) is a company that makes calculators. Their three models are:*

- *The II-91, which can do basic arithmetic operations.*
  (5 ounces of plastic, 3 hours to produce, sells for \$65)

- *The II-92, which can solve linear equations.*
  (8 ounces of plastic, 5 hours to produce, sells for \$100)
- *The II-93, which can perform the simplex method.*
  (12 ounces of plastic, 8 hours to produce, sells for \$160)

Their factory in Champaign receives a shipment of 320 ounces of plastic every week. They have 5 employees making calculators, each of which works for 40 hours every week.

(a) *How much of each calculator model should II produce each week to maximize profit?*

We set up a linear program, with variables x_1, x_2, x_3 representing the number of II-91, II-92, II-93 calculators respectively. We get:

$$\begin{array}{ll} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{maximize}} & 65x_1 + 100x_2 + 160x_3 \\ \text{subject to} & 5x_1 + 8x_2 + 12x_3 \le 320 \\ & 3x_1 + 5x_2 + 8x_3 \le 200 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

After solving, we obtain the optimal tableau

	x_1	x_2	x_3	s_1	s_2	
x_1	1	1	0	2	-3	40
x_3	0	$1/4$	1	$-3/4$	$5/4$	10
$-z$	0	-5	0	-10	-5	-4200

So the factory should produce 40 II-91's and 10 II-93's per week, which sell for \$4200 total.

(b) *The factory manager is considering purchasing more plastic, at a price of \$$D$ per ounce. For which range of D is this profitable?*

The optimal dual solution is $(u, v) = (10, 5)$ from the tableau, where u corresponds to the plastic constraint and v corresponds to the work hours constraint.

Since $u = 10$, this means that if the RHS of the plastic constraint is increased by δ (corresponding to purchasing δ ounces of plastic), the objective value will increase by 10δ. So purchasing more plastic is profitable if it is obtained at a cost of less than \$10 per ounce.

(c) *At most how many extra ounces of plastic per week could be purchased before your answer to (b) might stop being valid?*

Following the rule for sensitivity analysis for a change in the RHS of a tight constraint, we compute the negative of the ratios of the RHS column in the optimal tableau over the s_1 column. This gives us $-\frac{40}{2} = -20$ and $-\frac{10}{3/4} = \frac{40}{3}$.

So the answer in part (b) is valid if $\delta \in [-20, \frac{40}{3}]$. This means that, up to $\frac{40}{3} = 13.\overline{3}$ ounces of plastic can be purchased before the answer in (b) might stop being valid.

(Note: in general, what this means is that the "shadow cost" of \$10 per ounce might become smaller at that point. In this particular case, if we look more closely, we can see what happens when more than $\frac{40}{3}$ extra ounces of plastic are bought per week. At that point, no matter what the employees work on, they won't be able to use up all the plastic, so the shadow cost will drop to \$0 per ounce: it won't make sense to buy any more plastic for any price.)

5. *(Only 4-credit students need to do this problem.)*

All linear programs are either unbounded, infeasible, or have an optimal solution.

Is it possible to have a linear program with constraints $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ such that, just by changing the value of $\mathbf{b}$, we can get a linear program of all three types?

No, this is impossible.

When we change the value of $\mathbf{b}$, we are changing only the objective value of the dual linear program. So the dual program either is infeasible (and remains infeasible) or is feasible (and remains feasible, though it can be sometimes bounded and sometimes unbounded).

In the first case, the dual program is infeasible, so the primal program is either infeasible or unbounded. We only achieve two of the three types.

In the second case, the dual program has a feasible solution, which puts a bound on the primal program, so the primal program is either infeasible or has an optimal solution. (That is, the primal program cannot be unbounded, because we've obtained a bound on it.) Again, we only achieve two of the three types.