1 (100 PTS.) Review

1.A. (50 PTS.) Suppose S is a set of 111 integers. Prove that there is a subset $S' \subseteq S$ of at least 11 numbers such that the difference of any two numbers in S' is a multiple of 11.

Solution:

We will use the pigeonhole principle. Let the 111 numbers in S be $a_1, a_2, \ldots, a_{111}$. Note that since S is a set, the numbers are distinct, that is $a_i \neq a_j$ for $1 \leq i < j \leq 111$.

For $h \in \{0, 1, 2, \ldots, 10\}$ let $S_h = \{a_i \mid a_i \bmod 11 = h\}$ be the set of numbers from S whose remainder when divided by 11 is h. Every a_i is in exactly one S_h and therefore $S_0, S_1, \ldots, S_{10}$ is a partition of S. This implies that $|S| = \sum_{h=0}^{10} |S_h| = 111$.

We claim that there is an index $r \in \{0, 1, 2, \ldots, 10\}$ such that $|S_r| \geq 11$. This follows from the pigeon hole principle. A direct argument is the following. If $|S_h| \leq 10$ for each $h \in \{0, 1, \ldots, 10\}$ then,

$$\sum_{h=0}^{10} |S_h| \leq 11 \times 10 \leq 110$$

but $\sum_{h=0}^{10} |S_h| = 111$, a contradiction.

We claim that $S_r \subseteq S$, with $|S_r| \geq 11$ is our desired set. For any two distinct numbers $a_i, a_j \in S_r$ we have the property that $a_i \equiv a_j \bmod 11$ ($a_i \bmod 11$ and $a_j \bmod 11$ are equal to r) which implies that $a_i - a_j$ is divisible by 11.

Rubric: 50 points for a correct proof. Small issues with a proof (such as improperly using modular arithmetic, improperly stating the pigeonhole principle, not fully defining all variables, etc.) will lose anywhere from 5 to 20 points each. Large issues with a proof (such as not letting S be an arbitrary set, only proving that there is a pair of numbers in S' whose difference is a multiple of 11, not proving that the size of S' is at least 11, etc.) will lose anywhere from 25 to 40 points each.

1.B. (50 PTS.) The famous Basque computational arborist Gorka Oihanean has a favorite 26-node binary tree, in which each node is labeled with a letter of the alphabet. Inorder and postorder traversals of his tree visits the nodes in the following orders:

Inorder: *U C N O B I E L Z T F D S H W V Q A X G R J P K M Y*
Postorder: *U C O I E B L F S D H T Z N V A R G J X M K P Y Q W*

List the nodes in Professor Oihanean's tree according to a preorder traversal.

Solution:

Preorder: *W N C U Z L B O EI T H D F S Q V Y P X A J G R K M*

Rubric: 50 points for a correct answer. −10 points for each misplaced, missing, or repeated letter,but no negative scores. No proof is required. The correct solution is unique.

5points.

2 (100 PTS.) **A recurrence.**

Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + 4T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6. \end{cases}$$

Prove by induction that $T(n) = O(n \log n)$. (Recall that you need to show that $T(n) \leq c_1 n \log n + c_2$ for $n \geq 1$ where $c_1, c_2 \geq 0$ are some fixed but suitably chosen constants.

Solution:

We assume all logarithms are with respect to base 2. This is without loss of generality since only the choice of c_1, c_2 will change by a constant factor for other base values, and the rest of the proof remains the same.

We claim that $T(n) \leq n \log n + 1$ for all positive integers n (this is our inductive hypothesis). We prove this by induction on n. Let $g(n) = n \log n + 1$.

For the base of induction, consider the cases $n = 1, 2, 3, 4, 5$. By definition, $T(n) = 1$ for $n = 1, 2, 3, 4, 5$. For $n \geq 1, g(n) \geq 1$ since $g(n)$ is an increasing function of n and $g(1) = 1$. Therefore, $T(n) \leq g(n)$ for $n = 1, 2, 3, 4, 5$.

For the induction step, let $n \geq 6$ and suppose that the inductive hypotheis holds for all $k < n$. We will show that $T(n) \leq g(n)$ holds for n.

By definition, $T(n) = T(\lfloor n/3 \rfloor) + 4T(\lfloor n/6 \rfloor) + n$. Since $\lfloor n/3 \rfloor$ and $\lfloor n/6 \rfloor$ are smaller than n when $n \geq 6$, applying the inductive hypothesis, we have that

$$T(\lfloor n/3 \rfloor) \leq \lfloor n/3 \rfloor \log(\lfloor n/3 \rfloor) + 1$$

and

$$T(\lfloor n/6 \rfloor) \leq \lfloor n/6 \rfloor \log(\lfloor n/6 \rfloor) + 1.$$

Since the function $x \to \log(x)$ is an increasing function of x, and the $\lfloor x \rfloor \leq x$ for all x, we have that $\lfloor n/3 \rfloor \log(\lfloor n/3 \rfloor) \leq \frac{n}{3} \log(\frac{n}{3})$ and $\lfloor n/6 \rfloor \log(\lfloor n/6 \rfloor) \leq (n/6) \log(n/6)$.

Therefore,

$$\begin{aligned} T(n) &= T(\lfloor n/3 \rfloor) + 4T(\lfloor n/6 \rfloor) + n \\ &\leq (\lfloor n/3 \rfloor \log(\lfloor n/3 \rfloor) + 1) + 4 (\lfloor n/6 \rfloor \log(\lfloor n/6 \rfloor) + 1) + n \\ &\leq ((n/3) \log(n/3) + 1) + 4 ((n/6) \log(n/6) + 1) + n \\ &= (n/3) (\log(n/3) + 2 \log(n/6)) + 5 + n \\ &= (n/3) (3 \log(n/3) - 2) + 5 + n \\ &= n \log(n) + 5 - n \left(\frac{\log(3) - 1}{3} \right) \\ &\leq n \log(n) + 5 - \frac{n}{6} \\ &\leq n \log(n) + 1. \end{aligned}$$

The last inequality is valid because $n \geq 6$. This completes the inductive proof.

Rubric: Standard induction rubric. Additional problem specific rubric: -10 for removing floor signs for $T(.)$ instead of for $c_1 n \log n + c_2$ (as $T(.)$ is not known to be increasing). -10 each for not stating explicit, or stating incorrect c_1, c_2 values. -5 for only one base case, instead of 5 (as we require $n/6$ to exist for the $T(n/6)$ variable in the $T(n)$ definition).

3 (100 PTS.) Languages

Let $L \subseteq \{0, 1\}^*$ be a language defined recursively as follows:

(i) The string 0 is in L.
(ii) For any string x in L, the string $x1$ is also in L.
(iii) For any string x in L, the string $1x$ is also in L.
(iv) For any strings x and y in L, the string $x0y$ is also in L.
(v) These are the only strings in L.

Let $\#_0(w)$ denote the number of times 0 appears in string w and $\#_1(w)$ denote the number of times 1 appears in string w. You may assume without proof that $\#_0(xy) = \#_0(x) + \#_0(y)$, for any strings x, y.

3.A. (50 PTS.) Prove by induction that every string $w \in L$ contains an odd number of 0s.

Solution:

Let w be an *arbitrary* string in L.

Assume that $\#_0(x)$ is odd for every string $x \in L$ such that $\left|x\right| < \left|w\right|$.

There are four cases to consider (mirroring the four cases in the definition):

- If $w = 0$, then $\#_0(w) = 1$ which is odd.
- If $w = x1$ for some string $x \in L$, then

$$\#_0(w) = \#_0(x) + \#_0(1) = \#_0(x)$$

  $\#_0(x)$ is odd by the inductive hypothesis (since $|x| < |w|$), hence $\#_0(w)$ is odd.
- If $w = 1x$ for some string $x \in L$, then

$$\#_0(w) = \#_0(1) + \#_0(x) + = \#_0(x) +$$

  $\#_0(x)$ is odd by the inductive hypothesis (since $|x| < |w|$), hence $\#_0(w)$ is odd.
- Otherwise, $w = x0y$ for some strings $x, y \in L$. Then

$$\begin{aligned} \#_0(w) &= \#_0(x) + \#_0(0) + \#_0(y) \\ &= \#_0(x) + 1 + \#_0(y) \end{aligned}$$

  Both $\#_0(x), \#_0(y)$ are odd by the inductive hypothesis (since $|x| < |w|$ and $|y| < |w|$) and the sum of three odd numbers is always odd, so $\#_0(w)$ is also odd.

In all four cases, we conclude that $\#_0(w)$ is odd.

Rubric: Standard induction rubric.

3.B. (50 PTS.) Let $L' \subseteq \{0, 1\}^*$ be the language of strings with an odd number of 0s. Prove that $L = L'$.

Solution:

From the previous part, we have that if $w \in L \Rightarrow w \in L'$. Hence, $L \subseteq L'$.

Now, we will prove that if $w \in L' \Rightarrow w \in L$. We will prove by induction on the length of the string w that if $\#_0(w)$ is odd, then $w \in L$.

Proof: Let w be an *arbitrary* string such that $\#_0(w)$ is odd. Assume that every string x with $|x| < |w|$ and $\#_0(x)$ odd belongs to L. We consider four cases below, and every string w with $\#_0(w)$ odd falls into one of these cases.

Case 1: w starts with 1. That is $w = 1x$ for some string x. Then, we have $\#_0(x) = \#_0(w)$, therefore $\#_0(x)$ is odd. Since $|x| < |w|$, by induction hypothesis, we have that $x \in L$. By the second construction rule we have that $w = 1x$ is also in L.

Case 2: w ends with 1. That is $w = x1$ for some string x. Then, we have $\#_0(x) = \#_0(w)$, therefore $\#_0(x)$ is odd. Since $|x| < |w|$, by induction hypothesis, we have that $x \in L$. By the third construction rule we have that $w = x1$ is also in L.

Case 3: w starts and ends with 0 and $|w| = 1$. Then $w = 0$. By the first construction rule, $w \in L$.

Case 4: w starts and ends with 0 and $|w| > 1$. Then $\#_0(w) \geq 2$, but since $\#_0(w)$ is odd, we have that $\#_0(w) \geq 3$. Consider the second 0 in w. Let x be the prefix of w till the second 0 in w (not including it), let y be the suffix of of w after the second 0. Then $w = x0y$ where $\#_0(x) = 1$. Since $\#_0(w) \geq 3$ and odd, and $\#_0(x) = 1$, we have that $\#_0(y)$ is odd as well. Since $|x| < |w|$ and $|y| < |w|$, by induction hypothesis, we have that $x, y \in L$. By the fourth construction rule we have that $w = x0y \in L$. ■

Hence, $L' \subseteq L$ and $L \subseteq L'$. Thus, $L = L'$.

<u>*Rubric:*</u> Standard induction rubric. Additional problem specific rubric: -5 for not proving the if and only if.