1. For each statement below, check "True" if the statement is ***always*** true and "False" otherwise. Each correct answer is worth +1 point; each incorrect answer is worth −½ point; checking "I don't know" is worth +¼ point; and flipping a coin is (on average) worth +¼ point. You do ***not*** need to prove your answer is correct.

   Read each statement *very* carefully. Some of these are deliberately subtle.

~~Yes~~ No — If $2+2=5$, then zero is odd.
The hypothesis is false, so the implication is true.

Yes ~~No~~ — Language L is regular if and only if there is a DFA that accepts every string in L.
L is regular iff there is a DFA that accepts every string in L *and rejects every string not in* L.

Yes ~~No~~ — Two languages L and L' are regular if and only if $L \cup L'$ is regular.
Let $L = \{0^n 1^n \mid n \geq 0\}$ and $L' = (0+1)^*$.

~~Yes~~ No — For every language L, if L^* is empty, then L is empty.
L^* always contains the empty string ε, so the hypothesis is false, which means the implication is true. It is also true that if L^* is empty, then L is an elephant.

Yes ~~No~~ — Every regular language is recognized by a DFA with exactly one accepting state.
Consider the language $\varepsilon + 0$.

Yes ~~No~~ — If L has a fooling set of size 374, then L is regular.
Every non-regular language has an infinite fooling set, and every subset of a fooling set is a fooling set.

~~Yes~~ No — The language $\{0^{374n} \mid n \geq 374\}$ is regular.
This is $(0^{374})^{374}(0^{374})^*$

Yes ~~No~~ — The language $\{0^{37n} 1^{4n} \mid n \geq 374\}$ is regular.
Intuitively, any algorithm to detect strings in this language would need to count the number of 0s. More directly, $(0^{37})^*$ is a fooling set for this language.

~~Yes~~ No — The language $\{0^{3n} 1^{74n} \mid n \leq 374\}$ is regular.
This language is finite.

Yes ~~No~~ — Every language is either regular or context-free.
Every regular language is context-free, and most languages are not context-free.

Rubric: Max 10 points = +1 for each correct answer — ½ for each incorrect answer + ¼ for each IDK. Round negative scores up to 0.

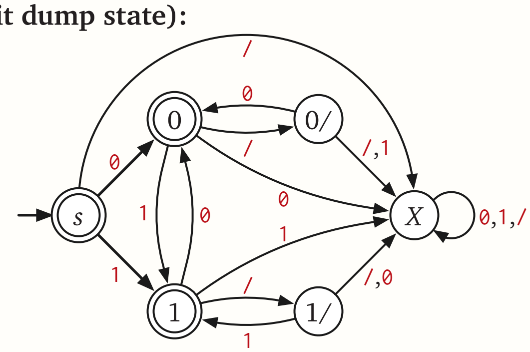
2. For any string $w \in \{0,1\}^*$, let $slash(w)$ be the string in $\{0,1,/\}^*$ obtained from w by inserting a new symbol / between any two consecutive appearances of the same symbol. For example:

$$slash(\varepsilon) = \varepsilon$$
$$slash(10101) = 10101$$
$$slash(001010111) = 0/010101/1/1$$

For any language $L \subseteq \{0,1\}^*$, let $slash(L) = \{slash(w) \mid w \in L\}$.

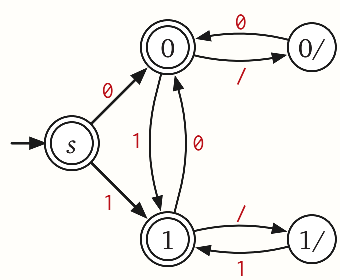
(a) Draw or describe a DFA that accepts the language $slash(\{0,1\}^*)$.

Solution (explicit dump state):



■

Solution:



(All missing transitions go to a hidden dump state.) ■

Rubric: 2½ points, standard DFA rubric (scaled). These are not the only correct solutions. The state names are not necessary for credit.

(b) Give a regular expression for the language $slash(\{0,1\}^*)$.

Solution: $(0(/0)^* + \varepsilon)(1(/1)^*0(/0)^*)^*(1(/1)^* + \varepsilon)$ ■

Rubric: 2½ points, standard regular expression rubric (scaled). This is not the only correct solution.

(c) ***Prove*** that for any regular language L, the language $slash(L)$ is also regular.

Solution (direct): Given a DFA $M = (Q, s, A, \delta)$ for L, we construct an NFA $N = (Q, s, A, \delta)$ for $slash(L)$ as follows. The second component of each state indicates the last one or two symbols read.

$$Q' = Q \times \{\mathtt{0}, \mathtt{0/}, \mathtt{1}, \mathtt{1/}\} \cup \{s'\}$$
$$s' \text{ is a new explicit state}$$
$$A' = Q \times \{\mathtt{0}, \mathtt{1}\} \cup \{s'\}$$

$$\delta'(s', \varepsilon) = \{(s, \mathtt{0/}),\ (s, \mathtt{1/})\}$$

$$\delta'((q,\mathtt{0}),\mathtt{1}) = \{(\delta(q,\mathtt{1}),\mathtt{1})\} \qquad \delta'((q,\mathtt{1}),\mathtt{0}) = \{(\delta(q,\mathtt{0}),\mathtt{0})\}$$
$$\delta'((q,\mathtt{0}),\mathtt{/}) = \{(q,\mathtt{0/})\} \qquad \delta'((q,\mathtt{1}),\mathtt{/}) = \{(q,\mathtt{1/})\}$$
$$\delta'((q,\mathtt{0/}),\mathtt{0}) = \{(\delta(q,\mathtt{0}),\mathtt{0})\} \qquad \delta'((q,\mathtt{1/}),\mathtt{1}) = \{(\delta(q,\mathtt{1}),\mathtt{1})\}$$

(All unspecified transitions go to the empty set $\varnothing$.) ■

Solution (almost-product): Let $M_1 = (Q_1, s_1, A_1, \delta_1)$ be an arbitrary DFA (over the alphabet $\{\mathtt{0}, \mathtt{1}, \mathtt{/}\}$) for the language $slash(\{\mathtt{0}+\mathtt{1}\}^*)$, and let $M_2 = (Q_2, s_2, A_2, \delta_2)$ be an arbitrary DFA (over the alphabet $\{\mathtt{0}, \mathtt{1}\}$) for L. We construct a ***DFA*** $M = (Q, s, A, \delta)$ (over the alphabet $\{\mathtt{0}, \mathtt{1}, \mathtt{/}\}$) for $slash(L)$ as follows:

$$Q = Q_1 \times Q_2$$
$$s = (s_1, s_2)$$
$$A = A_1 \times A_2$$
$$\delta((p,q),a) = \begin{cases} (\delta_1(p,a),\ q) & \text{if } a = \mathtt{/} \\ (\delta_1(p,a),\ \delta_2(q,a)) & \text{otherwise} \end{cases}$$

■

Rubric: 5 points, standard language transformation rubric (scaled). These are not the only correct solutions.

3. Let L be the language $\{\mathtt{0}^a\mathtt{1}^b\mathtt{0}^c \mid 2a = b + c\}$.

   (a) ***Prove*** that L is not a regular language.

Solution: Let $F = \mathtt{0}^*$.
Let x and y be arbitrary distinct strings in F.
Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some integers $i \neq j$.
Let $z = \mathtt{1}^i\mathtt{0}^i$.
Then $xz = \mathtt{0}^i\mathtt{1}^i\mathtt{0}^i \in L$ because $2i = i + i$.
But $yz = \mathtt{0}^j\mathtt{1}^i\mathtt{0}^i \notin L$ because $2j \neq i + i$.
We conclude that F is an infinite fooling set for L. ■

Rubric: 5 points, standard fooling set rubric (scaled). This is not the only correct solution.

   (b) Describe a context-free grammar for L.

Solution:

$$\begin{aligned} S &\to \mathtt{0}S\mathtt{00} \mid \mathtt{0}A\mathtt{10} \mid A && \{\mathtt{0}^a\mathtt{1}^b\mathtt{0}^c \mid 2a = b + c\} \\ A &\to \mathtt{0}A\mathtt{11} \mid \varepsilon && \{\mathtt{0}^a\mathtt{1}^b \mid 2a = b\} \end{aligned}$$

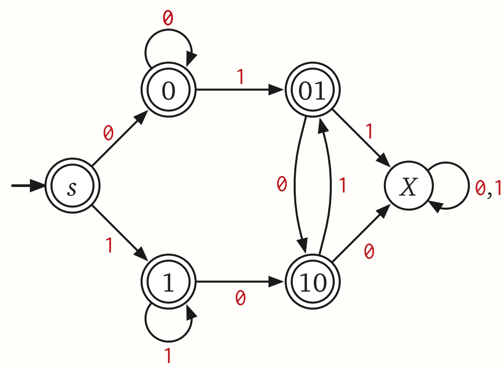
■

Rubric: 5 points, standard CFG rubric (scaled). This is not the only correct solution. The language specifications for the non-terminals are not necessary for credit.

4. For each of the following languages L, give a regular expression that represents L ***and*** draw or describe a DFA that recognizes L.

(a) All strings in $\{0, 1\}^*$ that do not contain either 100 or 011 as a substring

Solution: $(0^* + 1^*)(10)^*(1 + \varepsilon)$



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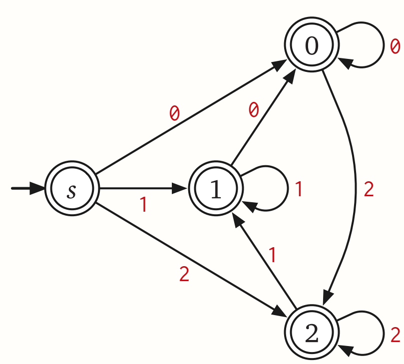
Rubric: 5 points = 2½ for regular expression (standard rubric, scaled) + 2½ for DFA (standard rubric, scaled). These are not the only correct solutions.

(b) All strings in $\{0, 1, 2\}^*$ that do not contain either 01 or 12 or 20 as a substring

Solution: $(2^+1^+0^+)^*(2^+1^* + \varepsilon) + (1^+0^+2^+)^*(1^+0^* + \varepsilon) + (0^+2^+1^+)^*(0^+2^* + \varepsilon)$

[Or more concisely: $1^* + (0^* + 1^*0^+)(2^+1^+0^+)^*(2^* + 2^+1^*)$]



(All missing transitions go to a hidden dump state.) ■

Rubric: 5 points = 2½for regular expression (standard rubric, scaled) + 2½for DFA (standard rubric, scaled). These are not the only correct solutions.

5. For any string $w \in \{0,1\}^*$, let $stupefy(w)$ denote the string obtained from w by deleting the first 1 (if any) and replacing each remaining 1 with a 0. Let L be an arbitrary regular language.

   (a) ***Prove*** that the language $\{stupefy(w) \mid w \in L\}$ is regular.

Solution: Let L' denote the language $\{stupefy(w) \mid w \in L\}$.

Let $M = (Q, s, A, \delta)$ be a DFA for L. We construct an NFA $M' = (Q', s', A', \delta')$ for L' as follows. Intuitively, we need to nondeterministically guess (1) where the first 1 (if any) was deleted, and (2) which later 0s (if any) were originally 1s.

$$
\begin{aligned}
Q' &= Q \times \{before, after\}\\
s' &= (s, before)\\
A' &= A \times \{before, after\}
\end{aligned}
$$

$$
\begin{aligned}
\delta'((q, before), \varepsilon) &= \{(\delta(q,1), after)\}\\
\delta'((q, after), \varepsilon) &= \varnothing\\
\delta'((q, before), 0) &= \{(\delta(q,0), before)\}\\
\delta'((q, after), 0) &= \{(\delta(q,0), after),\ (\delta(q,1), after)\}\\
\delta'((q, before), 1) &= \varnothing\\
\delta'((q, after), 1) &= \varnothing
\end{aligned}
$$

Alternatively, we can assume that the input alphabet for M' is just $\{0\}$. ■

Rubric: 5 points: standard transformation rubric (scaled). This is not the only correct solution. No penalty for implicitly assuming the alphabet of M' is just $\{0\}$.

   (b) ***Prove*** that the language $\{w \in \{0,1\}^* \mid stupefy(w) \in L\}$ is regular.

Solution: Let L' denote the language $\{w \in \{0,1\}^* \mid stupefy(w) \in L\}$. Given any DFA $M = (Q, s, A, \delta)$ for L, we construct a **DFA** $M' = (Q', s', A', \delta')$ for L' as follows. Intuitively, we need to remember whether we've deleted the first 1.

$$
\begin{aligned}
Q' &= Q \times \{before, after\}\\
s' &= (s, before)\\
A' &= A \times \{before, after\}
\end{aligned}
$$

$$
\begin{aligned}
\delta'((q, before), 0) &= (\delta(q,0), before)\\
\delta'((q, after), 0) &= (\delta(q,0), after)\\
\delta'((q, before), 1) &= (q, after)\\
\delta'((q, after), 1) &= (\delta(q,0), after)
\end{aligned}
$$

■

Rubric: 5 points: standard transformation rubric (scaled). This is not the only correct solution.