Math 482: Linear Programming, Fall 2020

Due Friday, October 16, 6PM CST

Homework 6

1. *You are given a set of points on a line at locations $a_1, a_2, \dots, a_n$. Write down linear programs to find a point x on that line that minimizes*

   (a) *The maximum distance from x to any of the points.*

   If we write the objective function as

   $$\max\{|x - a_1|, |x - a_2|, \dots, |x - a_n|\}$$

   then we are trying to minimize it with no constraints. But this objective function isn't linear, so we're not done yet.

   We can think of $|x - a_i|$ as $\max\{x - a_i, a_i - x\}$, so the objective function is really the maximum of $2n$ terms:

   $$\max\{x - a_1, x - a_2, \dots, x - a_n, a_1 - x, a_2 - x, \dots, a_n - x\}.$$

   To minimize the maximum of $2n$ things, minimize an auxiliary variable z, subject to the condition that z is at least each of those things (so that it is at least the maximum). This gives us the linear program

   $$\begin{array}{ll} \underset{x,z \in \mathbb{R}}{\text{minimize}} & z \\ \text{subject to} & z \geq x - a_1 \\ & z \geq x - a_2 \\ & \dots \\ & z \geq x - a_n \\ & z \geq a_1 - x \\ & z \geq a_2 - x \\ & \dots \\ & z \geq a_n - x \end{array}$$

   (We can make z nonnegative if we want to, but it's not required. However, x should not be nonnegative; if some values a_i are negative, x might need to be negative as well.)

   (b) *The sum of the distances from x to all of the points.*

   Here, we are trying to minimize the objective function

   $$\sum_{i=1}^{n} |x - a_i|.$$

   Once again, we have no constraints, but we'll get some from trying to make this objective function linear.

   As before, $|x - a_i| = \max\{x - a_i, a_i - x\}$, but now these maximums are independent from each other. We can model them by adding n artificial variables $z_1, \dots, z_n$ and minimize the sum of these variables. We'll require that z_i is at least $\max\{x - a_i, a_i - x\}$ by asking for it to be bigger than both of those. We get the following linear program as a result:

$$\begin{array}{ll}
\underset{x\in\mathbb{R},\mathbf{z}\in\mathbb{R}^n}{\text{minimize}} & z_1 + z_2 + \cdots + z_n \\
\text{subject to} & z_1 \ge x - a_1 \\
& z_2 \ge x - a_2 \\
& \dots \\
& z_n \ge x - a_n \\
& z_1 \ge a_1 - x \\
& z_2 \ge a_2 - x \\
& \dots \\
& z_n \ge a_n - x
\end{array}$$

(As in the previous part, we can add the condition $\mathbf{z} \ge \mathbf{0}$ if we want to, but it's not required. However, x should not be nonnegative; if some values a_i are negative, x might need to be negative as well.)

2. *Alice and Bob each have two coins: a nickel (5 cents) and a dime (10 cents). They simultaneously put a coin down on the table. If the coins are equal in value, Alice wins Bob's coin; if Alice's coin is more valuable, Bob wins Alice's coin; if Bob's coin is more valuable, nothing happens.*

   Determine optimal strategies for Alice and Bob.

   The matrix of Alice's payoffs in this game is given below:

	Bob : 5	Bob : 10
Alice : 5	5	0
Alice : 10	−10	10

   Alice and Bob's linear programs are

$$\begin{array}{ll}
\underset{x_1,x_2,u\in\mathbb{R}}{\text{maximize}} & u \\
\text{subject to} & u \le 5x_1 - 10x_2 \\
& u \le 10x_2 \\
& x_1 + x_2 = 1 \\
& x_1, x_2 \ge 0
\end{array}
\qquad
\begin{array}{ll}
\underset{y_1,y_2,v\in\mathbb{R}}{\text{minimize}} & v \\
\text{subject to} & v \ge 5y_1 \\
& v \ge -10y_1 + 10y_2 \\
& y_1 + y_2 = 1 \\
& y_1, y_2 \ge 0
\end{array}$$

   We can check that no pure strategy (that is, no row or column) for Alice or Bob dominates the other pure strategy, so both players will use a mixed strategy with positive probability on both options. In other words, the optimal solutions have $x_1, x_2 > 0$ and $y_1, y_2 > 0$.

   By complementary slackness, Alice's optimal solution must have $u = 5x_1 - 10x_2$ and $u = 10x_2$, so $5x_1 - 10x_2 = 10x_2$. Solving this together with $x_1 + x_2 = 1$, we get $(x_1, x_2) = (\frac{4}{5}, \frac{1}{5})$.

   By complementary slackness, Bob's optimal solution must have $v = 5y_1$ and $v = -10y_1 + 10y_2$, so $5y_1 = -10y_1 + 10y_2$. Solving this together with $y_1 + y_2 = 1$, we get $(y_1, y_2) = (\frac{2}{5}, \frac{3}{5})$.

3. *Use Fourier–Motzkin elimination to find a point (x, y, z) satisfying*

$$\begin{aligned}
x \phantom{{}+y} - z &\le 1 \\
x + y + z &\le 1 \\
-x + y + 3z &\le 1 \\
-3x - 5y - z &\le -1
\end{aligned}$$

We begin by solving for z. The four inequalities give us two lower bounds and two upper bounds:

$$\begin{aligned}
z &\geq x - 1 \\
z &\leq 1 - x - y \\
z &\leq \frac{1}{3}(1 + x - y) \\
z &\geq 1 - 3x - 5y
\end{aligned}$$

To eliminate z, we compare every lower bound to every upper bound:

$$\begin{aligned}
x - 1 &\leq 1 - x - y \\
x - 1 &\leq \frac{1}{3}(1 + x - y) \\
1 - 3x - 5y &\leq 1 - x - y \\
1 - 3x - 5y &\leq \frac{1}{3}(1 + x - y)
\end{aligned}$$

Next, we solve each of these inequalities for y:

$$\begin{aligned}
y &\leq 2 - 2x \\
y &\leq 4 - 2x \\
y &\geq -\frac{1}{2}x \\
y &\geq \frac{1}{7}(1 - 5x)
\end{aligned}$$

To eliminate y, we compare every lower bound to every upper bound:

$$\begin{aligned}
-\frac{1}{2}x &\leq 2 - 2x \\
-\frac{1}{2}x &\leq 4 - 2x \\
\frac{1}{7}(1 - 5x) &\leq 2 - 2x \\
\frac{1}{7}(1 - 5x) &\leq 4 - 2x
\end{aligned}$$

Finally, we solve each of these inequalities for x, getting $x \leq \frac{4}{3}$, $x \leq \frac{8}{3}$, $x \leq \frac{13}{9}$, and $x \leq 3$. All of these are upper bounds, so any sufficiently small value of x will work; for example, we can take $x = 0$.

Substituting $x = 0$ into the x, y system, the inequalities on y become $y \leq 2$, $y \leq 4$, $y \geq 0$, and $y \geq \frac{1}{7}$. One choice of y that's in the range $[\frac{1}{7}, 2]$ is $y = 2$.

Substituting $x = 0, y = 2$ into the original system, the inequalities on z become $z \geq -1$, $z \leq -1$, $z \leq -\frac{1}{3}$, and $z \geq -9$. The only valid choice of z for this x and this y is $z = -1$.

So we have found the feasible solution $(x, y, z) = (0, 2, -1)$. (There are many others.)

4. *(Only 4-credit students need to do this problem.)*

   Use Farkas's lemma to prove LP duality in the following form: if the linear program (**P**) *below* cannot achieve *an objective value of at least z^*, and the dual program* (**D**) *is feasible, then the dual linear* *program* $(\mathbf{D})$ *has a feasible solution* $\mathbf{u}$ *with objective value less than* z^**.*

$$(\mathbf{P}) \begin{cases} \underset{\mathbf{x}\in\mathbb{R}^n}{\textit{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \textit{subject to} & A\mathbf{x} \le \mathbf{b} \end{cases} \qquad (\mathbf{D}) \begin{cases} \underset{\mathbf{u}\in\mathbb{R}^m}{\textit{minimize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \textit{subject to} & \mathbf{u}^\mathsf{T}A = \mathbf{c}^\mathsf{T} \\ & \mathbf{u} \ge \mathbf{0} \end{cases}$$

Saying that $(\mathbf{P})$ cannot achieve an objective value of z^* means that the system of inequalities

$$\begin{aligned} A\mathbf{x} &\le \mathbf{b} \\ -\mathbf{c}^\mathsf{T}\mathbf{x} &\le -z^* \end{aligned}$$

has no solution. (We negate the second inequality to put it in $\le$ form.) We can also write this as

$$\begin{bmatrix} A \\ -\mathbf{c}^\mathsf{T} \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ -z^* \end{bmatrix}.$$

By Farkas's lemma, this means that there is an $(m+1)$-dimensional vector $(\mathbf{v}, w)$ with $\mathbf{v} \ge \mathbf{0}$ and $w \ge 0$ such that

$$\begin{bmatrix} \mathbf{v}^\mathsf{T} & w \end{bmatrix} \begin{bmatrix} A \\ -\mathbf{c}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \mathbf{v}^\mathsf{T} & w \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ -z^* \end{bmatrix} < 0$$

or in other words

$$\mathbf{v}^\mathsf{T}A - w\mathbf{c}^\mathsf{T} = \mathbf{0}^\mathsf{T} \qquad \text{and} \qquad \mathbf{v}^\mathsf{T}\mathbf{b} - wz^* < 0.$$

Define $\mathbf{u} = \frac{1}{w}\mathbf{v}$, which still has $\mathbf{u} \ge \mathbf{0}$. Then the equation on the left says that $\mathbf{v}^\mathsf{T}A = w\mathbf{c}^\mathsf{T}$ or $\mathbf{u}^\mathsf{T}A = \mathbf{c}^\mathsf{T}$. Meanwhile, the inequality on the right says that $\mathbf{v}^\mathsf{T}\mathbf{b} < wz^*$ or $\mathbf{u}^\mathsf{T}\mathbf{b} < z^*$. This is precisely a solution to $(\mathbf{D})$ which achieves an objective value less than z^*.

The exceptional case when this doesn't work is when $w = 0$. In this case, we just have $\mathbf{v}^\mathsf{T}A = \mathbf{0}^\mathsf{T}$ and $\mathbf{v}^\mathsf{T}\mathbf{b} < 0$. This is going to be the case where the dual is unbounded.

Here, we need to use the fact that the dual program is feasible. In this case, let $\mathbf{u}$ be any feasible dual solution.

For any $t \ge 0$, $\mathbf{u} + t\mathbf{v}$ is another feasible dual solution: we have $\mathbf{u} + t\mathbf{v} \ge 0$ and

$$(\mathbf{u} + t\mathbf{v})^\mathsf{T}A = \mathbf{u}^\mathsf{T}A + t\mathbf{v}^\mathsf{T}A = \mathbf{c}^\mathsf{T} + t\mathbf{0}^\mathsf{T} = \mathbf{c}^\mathsf{T}.$$

Meanwhile, the objective value is $(\mathbf{u} + t\mathbf{v})^\mathsf{T}\mathbf{b} = \mathbf{u}^\mathsf{T}\mathbf{b} + t(\mathbf{v}^\mathsf{T}\mathbf{b})$. Since $\mathbf{v}^\mathsf{T}\mathbf{b} < 0$, we can make this objective value arbitrarily small by increasing t; in particular, less than z^*.