Math 482: Linear Programming, Fall 2020

Due Monday, September 21, 6PM CST

Homework 3 Solutions

1. *Let $P \subseteq \mathbb{R}^3$ be the convex polyhedron with only the following four extreme points:* $(0,0,0)$*,* $(0,1,1)$*,* $(1,0,1)$*, and* $(1,1,0)$*.*

   (a) *Write down a set of four linear inequalities describing P.*

   The polyhedron P is a tetrahedron with four sides. Each side is defined by three of the four points. For example, we can solve for the equation of the plane containing $(0,0,0)$, $(0,1,1)$, and $(1,0,1)$ by finding a,b,c,d that satisfy $ax+by+cz=d$ for $(x,y,z)=(0,0,0)$, $(x,y,z)=(0,1,1)$ and $(x,y,z)=(1,0,1)$: this gives us

   $$\begin{cases} 0 = d \\ b + c = d \\ a + c = d \end{cases} \implies (a,b,c,d) = (a,a,-a,0)$$

   so one possibility is the equation $x+y-z=0$. (There are infinitely many equations for the same plane, since we can always multiply the equation by any nonzero real number.)

   If we do this for all four sides, we get the four equations $x+y-z=0$, $x-y+z=0$, $x-y-z=0$, and $x+y+z=2$. However, to define P, we need inequalities, which means figuring out which side of each plane contains P.

   For example, the plane $x+y-z=0$ passes through $(0,0,0)$, $(0,1,1)$, and $(1,0,1)$, and we want to turn it into an inequality that's true at the fourth point $(1,1,0)$ as well. At the point $(1,1,0)$, $x+y-z=2>0$. Therefore, we write down the inequality $x+y-z\geq 0$.

   Altogether, we get

   $$\begin{cases} x+y-z \geq 0 \\ x-y+z \geq 0 \\ -x+y+z \geq 0 \\ x+y+z \leq 2 \end{cases}$$

   as our final answer.

   (b) *Show directly from the definition that the point $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ is* not *an extreme point of P.*

   Here is one possible answer: because $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ lies on the line segment from $(1,1,0)$ to $(\frac{1}{2},\frac{1}{2},1)$, and both of these points are in P (we can check that they both satisfy all four inequalities from part (a)), it is not an extreme point.

   This is not the only possible line segment we could have picked. Here's an explanation of how we could have gotten the answer above, or many others.

   The definition of an extreme point is a point $\mathbf{x} \in P$ that does not lie on any line segment contained in P, except for line segments with an endpoint at $\mathbf{x}$. So to prove that $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ is *not* an extreme point, we want to find a line segment $[\mathbf{a},\mathbf{b}]$ containing $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ with $\mathbf{a},\mathbf{b}\in P$ and both $\mathbf{a},\mathbf{b}$ not equal to $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$.

   To help us along, notice that $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ lies on the plane $x+y+z=2$: one of the sides of P. If one of the endpoints $\mathbf{a}$ or $\mathbf{b}$ were below this plane, then for the segment to pass through $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$, the other endpoint would have to be above this plane. But then, it wouldn't be contained in P. So the only way to succeed is to pick both $\mathbf{a}$ and $\mathbf{b}$ on the plane $x+y+z=2$.

Otherwise, the problem is fairly unconstrained, and many choices of **a** and **b** work. For example, we could pick **a** to be one of the corners of P: say, $(1,1,0)$. Then **b** has to be on the line through $(1,1,0)$ and $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$, but on the other side of $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$, and still contained in P; one choice of such a **b** is $(\frac{1}{2},\frac{1}{2},1)$. This gives us the solution in the first paragraph.

2. *Use the two-phase simplex method to solve the following linear program:*

$$\begin{aligned} \underset{x_1,x_2,x_3\in\mathbb{R}}{\textit{maximize}} \quad & x_1 + x_2 + 3x_3 \\ \textit{subject to} \quad & 2x_1 \phantom{{}+x_2} + x_3 = 2 \\ & \phantom{2x_1+{}} x_2 + x_3 = 3 \\ & 4x_1 + x_2 + 3x_3 = 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

We add artificial variables x_1^a, x_2^a, x_3^a, and set up a tableau to minimize the artificial objective function $z^a = x_1^a + x_2^a + x_3^a$:

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1^a	2	0	1	1	0	0	2
x_2^a	0	1	1	0	1	0	3
x_3^a	4	1	3	0	0	1	7
$-z$	1	1	3	0	0	0	0
$-z^a$	0	0	0	1	1	1	0

$\rightsquigarrow$

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1^a	2	0	1	1	0	0	2
x_2^a	0	1	1	0	1	0	3
x_3^a	4	1	3	0	0	1	7
$-z$	1	1	3	0	0	0	0
$-z^a$	-6	-2	-5	0	0	0	-12

The clearest choice of entering variable is x_1: it has the first negative reduced cost, and also the highest negative reduced cost. The shortlist of leaving variables is x_1^a and x_3^a (because x_2^a's coefficient in x_1's column is 0, which is not positive), and their ratios are $\frac{2}{2}$ and $\frac{7}{4}$; x_1^a has the smallest ratio, so it is replaced by x_1. We get the tableau:

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1	1	0	$1/2$	$1/2$	0	0	1
x_2^a	0	1	1	0	1	0	3
x_3^a	0	1	1	-2	0	1	3
$-z$	0	1	$5/2$	$-1/2$	0	0	-1
$-z^a$	0	-2	-2	3	0	0	-6

Let's pick x_2 to enter the basis next. The shortlist of leaving variables is x_2^a and x_3^a, and they have equal ratios of $\frac{3}{1}$; we arbitrarily pick x_2^a to leave the basis. We get:

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1	1	0	$1/2$	$1/2$	0	0	1
x_2	0	1	1	0	1	0	3
x_3^a	0	0	0	-2	-1	1	0
$-z$	0	0	$3/2$	$-1/2$	-1	0	-4
$-z^a$	0	0	0	3	2	0	0

At this point, z^a has been minimized (all the reduced costs are positive) and it's reached an objective value of 0. This means that the linear program has a feasible solution, and we can find a tableau for one just by removing the artificial rows and columns from the tableau above.

Moreover (and this is the interesting feature of this problem), x_3^a has remained in the basis, so we have to do something about x_3^a's row. In this case, when we delete the artificial variables, that row corresponds to the equation "$0x_1 + 0x_2 + 0x_3 = 0$", so it is redundant. (This happened because, in our original program, the third equation could be derived from the other two.) In this case, we just delete the redundant row. Our new tableau is:

	x_1	x_2	x_3	
x_1	1	0	$1/2$	1
x_2	0	1	1	3
$-z$	0	0	$3/2$	-4

Since we are maximizing z, we should finish off the problem by pivoting on x_3. Both x_1 and x_2 are on the shortlist of valid leaving variables, and x_1's ratio of $\frac{1}{1/2}$ is smaller than x_2's ratio of $\frac{3}{1}$. Therefore x_1 leaves the basis, and we get the tableau:

	x_1	x_2	x_3	
x_3	2	0	1	2
x_2	-2	1	0	1
$-z$	-3	0	0	-7

Reading off the final answer from the tableau, we get $(x_1, x_2, x_3) = (0, 1, 2)$.

3. *Use lexicographic pivoting to solve the following linear program:*

$$\begin{array}{ll} \underset{x,y\in\mathbb{R}}{\textit{maximize}} & x - \ \ y \\ \textit{subject to} & x - 2y \le 0 \\ & x - 3y \le 0 \\ & \qquad y \le 3 \\ & x, y \ge 0 \end{array}$$

After slack variables are added, we add ϵ_1, ϵ_2, ϵ_3 to the right-hand sides of the three equations we get. This gives us the tableau

	x	y	s_1	s_2	s_3	
s_1	1	-2	1	0	0	ϵ_1
s_2	1	-3	0	1	0	ϵ_2
s_3	0	1	0	0	1	$3 + \epsilon_3$
$-z$	1	-1	0	0	0	0

To maximize, we pivot on x. The shortlist of leaving variables is s_1 and s_2, and s_2's ratio of $\frac{\epsilon_2}{1}$ is smaller than s_1's ratio of $\frac{\epsilon_1}{1}$. So x replaces s_2 in the basis, and we get the tableau

	x	y	s_1	s_2	s_3	
s_1	0	1	1	-1	0	$\epsilon_1 - \epsilon_2$
x	1	-3	0	1	0	ϵ_2
s_3	0	1	0	0	1	$3 + \epsilon_3$
$-z$	0	2	0	-1	0	$-\epsilon_2$

Next, we must pivot on y. The shortlist of leaving variables is s_1 and s_3, and s_1's ratio of $\frac{\epsilon_1 - \epsilon_2}{1}$ is smaller than s_3's ratio of $\frac{3+\epsilon_3}{1}$. So y replaces s_1 in the basis, and we get

	x	y	s_1	s_2	s_3	
y	0	1	1	-1	0	$\epsilon_1 - \epsilon_2$
x	1	0	3	-2	0	$3\epsilon_1 - 2\epsilon_2$
s_3	0	0	-1	1	1	$3 - \epsilon_1 + \epsilon_2 + \epsilon_3$
$-z$	0	0	-2	1	0	$-2\epsilon_1 + \epsilon_2$

Finally, we must pivot on s_2. Only s_3 can leave the basis, giving us the tableau

$$\begin{array}{c|ccccc|r}
 & x & y & s_1 & s_2 & s_3 & \\
\hline
y & 0 & 1 & 0 & 0 & 1 & 3+\epsilon_3 \\
x & 1 & 0 & 1 & 0 & 2 & 6+\epsilon_1+2\epsilon_3 \\
s_2 & 0 & 0 & -1 & 1 & 1 & 3-\epsilon_1+\epsilon_2+\epsilon_3 \\
\hline
-z & 0 & 0 & -1 & 0 & -1 & -3-\epsilon_1-\epsilon_2
\end{array}$$

The final answer, once we remove the ϵ's from the problem, is $(x, y) = (6, 3)$.

4. *Consider the following linear program:*

$$\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^{10}}{\textit{maximize}} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5 - x_6 + 4x_7 + 4x_8 + 2x_9 - x_{10} \\
\textit{subject to} \quad & x_1 + x_2 - x_3 + 2x_4 - x_5 + 3x_6 + 2x_7 - x_8 + x_9 + 2x_{10} = 3 \\
& 3x_1 + 4x_2 + 2x_3 + 7x_4 + 5x_5 + 6x_6 - 2x_7 + 9x_8 + 8x_9 + 9x_{10} = 10 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 0.
\end{aligned}$$

(a) *Starting with basic variables $\mathcal{B} = (x_1, x_2)$, compute the inverse matrix $A_{\mathcal{B}}^{-1}$ and the basic feasible solution corresponding to $\mathcal{B}$.*

The submatrix is $A_{\mathcal{B}}$ obtained by taking the columns for x_1 and x_2. Therefore

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \implies A_{\mathcal{B}}^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}.$$

We obtain the values of the basic variables at the current basic feasible solution by the formula $\mathbf{p} = A_{\mathcal{B}}^{-1}\mathbf{b}$, giving us

$$\mathbf{p} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which corresponds to the point

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (2, 1, 0, 0, 0, 0, 0, 0, 0, 0).$$

(b) *Perform one iteration of the revised simplex method from the basic feasible solution you found in part (a). Use Bland's rule for pivoting.*

Your answer should give the new basis $\mathcal{B}$, the new inverse matrix $A_{\mathcal{B}}^{-1}$, and the new basic feasible solution.

We begin by picking an entering variable, by looking at the reduced costs. The reduced cost of x_j is given by the formula $r_j = c_j - \mathbf{u}^\mathsf{T} A_j$, where $\mathbf{u}^\mathsf{T} = {\mathbf{c}_{\mathcal{B}}}^\mathsf{T} A_{\mathcal{B}}^{-1}$.

In this case,

$$\mathbf{u}^\mathsf{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

and we can move on to reduced costs. The first nonbasic variable is x_3, and its reduced cost is

$$r_3 = 3 - \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 3 - 2 - 2 = -1 < 0,$$

so we don't pivot on x_3. The second nonbasic variable is x_4, and its reduced cost is

$$r_4 = 4 - \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = 4 + 4 - 7 = 1 > 0,$$

so we pivot on x_4. This requires computing x_4's column in the mini-tableau:

$$Q_4 = A_{\mathcal{B}}^{-1} A_4 = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The mini-tableau is now

$$\begin{array}{c|c|c|c} & A_{\mathcal{B}}^{-1} & x_4 & \mathbf{p} \\ \hline \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{array}$$

The simplex method says that x_2 should leave the basis, because its ratio of $\frac{1}{1}$ is smaller than x_1's ratio of $\frac{2}{1}$. Doing the row-reduction gives us the mini-tableau

$$\begin{array}{c|c|c|c} & A_{\mathcal{B}}^{-1} & x_4 & \mathbf{p} \\ \hline \begin{matrix} x_1 \\ x_4 \end{matrix} & \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

from which we can read off the new basis $\mathcal{B} = (x_1, x_4)$, the new inverse matrix

$$A_{\mathcal{B}}^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix},$$

and the new basic feasible solution

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (1, 0, 0, 1, 0, 0, 0, 0, 0, 0).$$

5. *(Only 4-credit students need to do this problem.)*

Your friend was solving a linear program with two inequality constraints on the variables x and y, as well as the nonnegativity constraints $x, y \geq 0$. After adding slack variables s_1, s_2 to deal with the constraints, your friend used the simplex method to arrive at the following tableau:

	x	y	s_1	s_2	
x	1	2	0	0	3
s_2	0	1	-1	1	1
$-z$	0	-2	-1	0	-4

Show that your friend must have made a mistake: there is no linear program of the form described which can result in this final tableau.

Because the submatrix

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

given by the s_1 and s_2 columns is not invertible, it is impossible to put this tableau in the (s_1, s_2) basis.

But if your friend had started by adding slack variables s_1 and s_2 to two inequalities, then there would have been a basic solution in which (s_1, s_2) are the basic variables, corresponding to $x = y = 0$. (Even if a two-phase simplex method was involved because this solution wasn't feasible, this solution would still have been basic.)

This is a contradiction, so at some point something must have gone wrong.

Another possible argument: if our tableau begins with slack variables, then the matrix $A_{\mathcal{B}}^{-1}$ can always be found as the submatrix formed by the slack variable columns, so in our case,

$$A_{\mathcal{B}}^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

But $A_{\mathcal{B}}^{-1}$ must be invertible (its inverse is $A_{\mathcal{B}}$), and this matrix isn't invertible (it has determinant 0), so the tableau cannot be a valid tableau.