Math 482: Linear Programming, Fall 2020

Due Wednesday, September 2, 6PM CST

Homework 1 Solutions

1. *Write a linear program for the following problem. (Do not solve.)*

   A ship is transporting rice and wheat from California to Alaska. It has three cargo holds with the following capacities:

   - *The forward cargo hold can carry at most 10,000 tons, and at most 400,000 cubic feet.*
   - *The middle cargo hold can carry at most 5,000 tons, and at most 250,000 cubic feet.*
   - *The aft cargo hold can carry at most 12,000 tons, and at most 600,000 cubic feet.*

   In addition, for the ship to be balanced, each cargo hold must be filled to the same fraction of its total capacity, with respect to tonnage.

   A ton of wheat takes up 44.7 cubic feet and can be sold at a profit of \$20; a ton of rice takes up 40.9 cubic feet and can be sold at a profit of \$18.

   The goal is to maximize the profit from the ship's cargo.

   To keep track of all parameters of the cargo holds, we use six variables. Let x_f, x_m, x_a denote the number of tons of wheat in each of the cargo holds, and let y_f, y_m, y_a denote the number of tons of rice in each of the cargo holds.

   The limitations on tonnage tell us that:

$$\begin{cases} x_f + y_f \le 10000 \\ x_m + y_m \le 5000 \\ x_a + y_a \le 12000 \end{cases}$$

   The limitations on volume tell us that:

$$\begin{cases} 44.7x_f + 40.9y_f \le 400000 \\ 44.7x_m + 40.9y_m \le 250000 \\ 44.7x_a + 40.9y_a \le 600000 \end{cases}$$

   The requirement for the ship to be balanced tells us that

$$\frac{x_f + y_f}{10000} = \frac{x_m + y_m}{5000} = \frac{x_a + y_a}{12000}.$$

   (Note that in practice, to write the constraint $A = B = C$ in a linear program, we could write two equations such as $A - B = 0$ and $B - C = 0$, or three inequalities such as $A - B \le 0$, $B - C \le 0$, and $C - A \le 0$.)

   A final requirement to include is that all the variables are nonnegative:

$$x_f, x_m, x_a, y_f, y_m, y_a \ge 0.$$

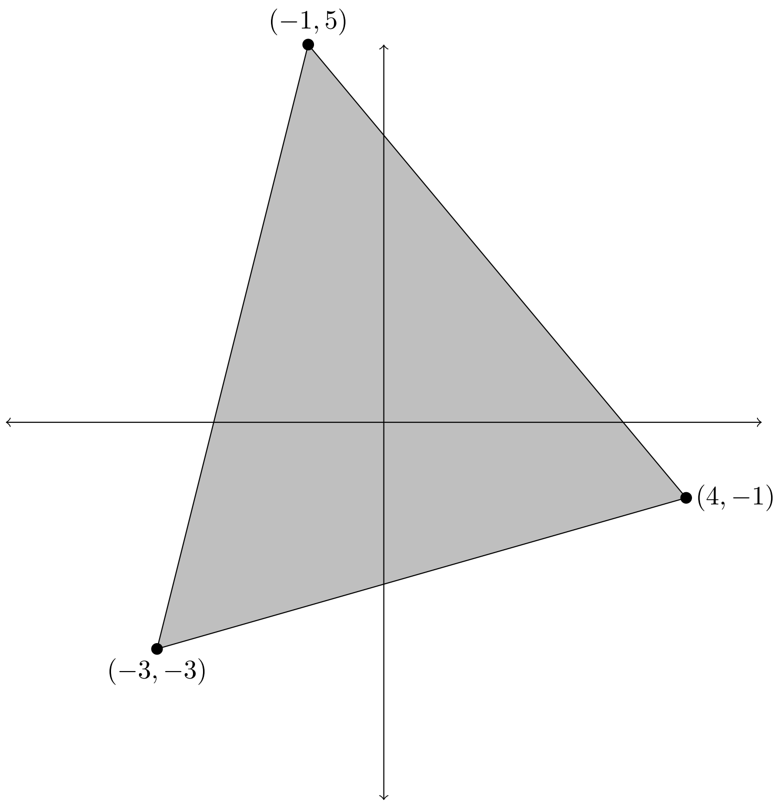
   We maximize, subject to the constraints above, the objective function

$$20x_f + 20x_m + 20x_a + 18y_f + 18y_m + 18y_a.$$

2. *Draw the feasible region of this linear program. Solve it using the naive approach from Lecture 1.*

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\textit{maximize}} \quad & x+y \\ \textit{subject to} \quad & 6x+5y \le 19, \\ & y \le 4x+9, \\ & 2x-7y \le 15. \end{aligned}$$

The feasible region is the triangle below:



The lines $6x+5y=19$ and $y=4x+9$ intersect at $(-1,5)$, where $x+y=4$. The lines $6x+5y=19$ and $2x-7y=15$ intersect at $(4,-1)$, where $x+y=3$. Finally, the lines $y=4x+9$ and $2x-7y=15$ intersect at $(-3,-3)$, where $x+y=-6$.

Since we are maximizing $x+y$ and the maximum is achieved at a corner point, the optimal solution is $(-1,5)$ with an objective value of 4.

3. *Convert the linear program below to equational form. Equational form requires that all variables are nonnegative, so you must substitute nonnegative variables for x as we did in Lecture 2.*

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\textit{minimize}} \quad & x-2y \\ \textit{subject to} \quad & x+y \le 10, \\ & y \ge 0. \end{aligned}$$

The variable y is already nonnegative, but x can be negative, so we substitute $x = x^+ - x^-$, where $x^+, x^- \geq 0$. This gives us the linear program

$$\begin{aligned} \underset{x^+,x^-,y\in\mathbb{R}}{\text{minimize}} \quad & x^+ - x^- - 2y \\ \text{subject to} \quad & x^+ - x^- + y \leq 10, \\ & x^+, x^-, y \geq 0. \end{aligned}$$

To put the constraint $x^+ - x^- + y \leq 10$ into equational form, we add a slack variable $s \geq 0$, writing it as $x^+ - x^- + y + s = 10$. This gives us the final answer

$$\begin{aligned} \underset{x^+,x^-,y,s\in\mathbb{R}}{\text{minimize}} \quad & x^+ - x^- - 2y \\ \text{subject to} \quad & x^+ - x^- + y + s = 10, \\ & x^+, x^-, y, s \geq 0. \end{aligned}$$

4. *Use the simplex method to solve the* minimization *problem given in the tableau below:*

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	2	−1	0	4
x_2	0	1	2	3	4	9
$-z$	2	0	3	−1	1	−7

The first step is to row-reduce the tableau:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	2	−1	0	4
x_2	0	1	2	3	4	9
$-z$	0	0	−1	1	1	−15

After this, to minimize we are forced to pivot on x_3. Both entries in x_3's column are positive, and of the ratios $\frac{4}{2}$ and $\frac{9}{2}$, the first is smaller, so x_1 leaves the basis and we get

	x_1	x_2	x_3	x_4	x_5	
x_3	$1/2$	0	1	$-1/2$	0	2
x_2	−1	1	0	4	4	5
$-z$	$1/2$	0	0	$1/2$	1	−13

Since all reduced costs are positive, this is the final and optimal solution: $(x_1, x_2, x_3, x_4, x_5) = (0, 5, 2, 0, 0)$ with objective value 13.

5. *(Only 4-credit students need to do this problem.)*

   (a) *Rewrite the constraint $|x| + |y| \leq 5$ as a combination of linear constraints.*

   This constraint is equivalent to the four linear constraints

$$\begin{cases} x + y \leq 5 \\ -x + y \leq 5 \\ x - y \leq 5 \\ -x - y \leq 5 \end{cases}$$

   We can find this by drawing the region and looking at the equations of each straight-line boundary. Alternatively, we can use the rule that $|x| \leq a \iff -a \leq x \leq a$ to first rewrite $|x| \leq 5 - |y|$ as the two constraints $x \leq 5 - |y|$ and $|y| - 5 \leq x$, and then similarly eliminate the absolute value on y.

(b) *Show that there is no way to rewrite the constraint* $|x| + |y| \geq 5$ *as a combination of linear constraints.*

The big problem with this constraint is that the set satisfying it is not convex. For example, the points $(10, 0)$ and $(-10, 0)$ both satisfy the constraint, but their midpoint $(0, 0)$ violates it.