1. For each of the following languages over the alphabet $\{0, 1\}$, give a regular expression that describes that language, and *briefly* argue why your regular expression is correct.

   (a) All strings except 010.

Solution (brute force):

$$\varepsilon + 0 + 1 + 00 + 01 + 10 + 11$$
$$+ 000 + 001 + 011 + 100 + 101 + 110 + 111$$
$$+ (0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)^*$$

The first line matches all strings of length that re shorter than 010; the second line matches all strings of length 3 except 010; the last line matches all strings that are longer than 010. ■

Solution (prefix case analysis):

$$\big(\varepsilon + 0 + 01\big) + \big(010(0 + 1)(0 + 1)^*\big) + (1 + 00 + 011)(0 + 1)^*$$

The first subexpression matches all proper prefixes of 010; the second subexpression matches all strings for which 010 is a proper prefix; the last subexpression matches all strings that have a prefix that is not a prefix of 010. ■

   (b) All strings that contain the substring 010.

Solution: $(0 + 1)^*\, 010\, (0 + 1)^*$ — Anything, then 010, then anything. ■

   (c) All strings that contain the subsequence 010.

Solution: $(0 + 1)^*\, 0\, (0 + 1)^*\, 1\, (0 + 1)^*\, 0\, (0 + 1)^*$
Any string can appear before, after, or inside the subsequence 010. ■

   (d) All strings that do not contain the substring 010.

Solution: $1^*(0111^*)^*(1 + \varepsilon)$ or $1^*(0 + 11 + 111)^*1^*$
Every run of 1s, except possibly at the start and end of the string, has length at least 2. (A *run* of 1s is a maximal nonempty substring consisting entirely of 1s.) Every run of length at least 2 is the concatenation of substrings of length 2 (11) and 3 (111). ■

   (e) All strings that do not contain the subsequence 010.

Solution: $1^*0^*1^*$ — Every string in this language has at most one run of 0s. ■

Rubric: 2 points each = ½ for syntactically correct regular expression + 1 for correctness + ½ for explanation. (This is the standard regular expression rubric scaled to 2 points.) As always, omitting the explanation entirely is a Deadly Sin! These are not the only correct solutions.

2. Let L be the set of all strings in $\{0, 1\}^*$ that contain *at least two* occurrences of the substring 010.

(a) Give a regular expression for L, and briefly argue why your expression is correct.

Solution: $(0+1)^*\,010\,(0+1)^*\,010\,(0+1)^* \;+\; (0+1)^*\,01010\,(0+1)^*$

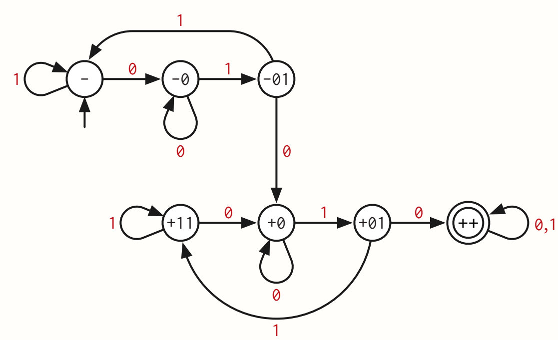
The first term describes all strings that contain at least two *disjoint* occurrences of the substring 010. The second term describes all strings that contain the substring 01010, and therefore contain at least two *overlapping* occurrences of the substring 010.

(Neither subexpression attempts the match the *first* two or *last* two occurrences of 010, and the first subexpression does not attempt to match *adjacent* occurrences of the substring 010.) ■

Rubric: 5 points: standard regular expression rubric (scaled). This is not the only correct solution.

(b) Describe a DFA over the alphabet $\Sigma = \{0, 1\}$ that accepts the language L.

Solution:



The seven states of the DFA have the following meanings:

- -: We have not seen the substring 010, and we have not just read a non-empty prefix of 010. This is the start state.
- -0: We have not seen the substring 010, and we just read 0.
- -01: We have not seen the substring 010, and we just read 0 followed by 1.
- +11: We have seen the substring 010 exactly once, and we just read 1 followed by 1.
- +0: We have seen the substring 010 exactly once, and we just read 0.
- +01: We have seen the substring 010 exactly once, and we just read 0 followed by 1.
- ++: We have seen the substring 010 at least twice. This is the only accepting state.

■

Rubric: 5 points: standard DFA rubric (scaled). This is not the only correct solution.

3. Let L denote the set of all strings $w \in \{0, 1\}^*$ that satisfy *at most two* of the following conditions:

   - The substring 01 appears in w an odd number of times.
   - $\#(1, w)$ is divisible by 3.
   - The binary value of w is *not* a multiple of 7.

   Formally describe a DFA with input alphabet $\Sigma = \{0, 1\}$ that accepts the language L, by explicitly describing the states Q, the start state s, the accepting states A, and the transition function δ.

Solution (formal description):

$$Q = \{0, 1\} \times \{0, 1\} \times \{0, 1, 2\} \times \{0, 1, 2, 3, 4, 5, 6\}$$
$$s = (1, 0, 0, 0)$$
$$A = \{(a, b, c, d) \mid b = 0 \text{ or } c \neq 0 \text{ or } d = 0\}$$

$$\delta((0, b, c, d), 0) = \big(0,\ b,\ c,\ (2d) \bmod 7\big)$$
$$\delta((0, b, c, d), 1) = \big(1,\ (b + 1) \bmod 2,\ (c + 1) \bmod 3,\ (2d + 1) \bmod 7\big)$$

$$\delta((1, b, c, d), 0) = \big(0,\ b,\ c,\ (2d) \bmod 7\big)$$
$$\delta((1, b, c, d), 1) = \big(1,\ b,\ (c + 1) \bmod 3,\ (2d + 1) \bmod 7\big)$$

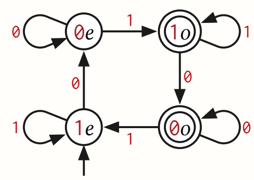
The state (a, b, c, d) indicates the following:

- a is the last symbol read by the DFA, or 1 if the DFA hasn't read anything yet.
- b is the number of times the DFA has read the substring 01, modulo 2.
- c is the number of times the DFA has read the symbol 1, modulo 3.
- d is the binary value of the string red so far, modulo 7.

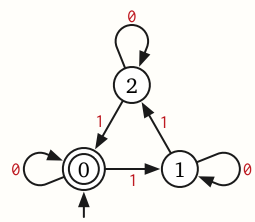
■

Solution (product construction): Our DFA M is the product of three smaller DFAs:

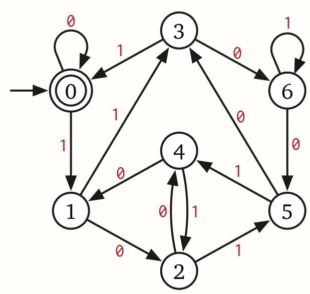
- The first DFA A accepts all strings in which the substring 01 appears an odd number of times. Each state records the last symbol read (of 1 if nothing has been read yet, and whether the DFA has read an even or odd number of 01s.



- The second DFA B accepts all strings where the number of 1s is divisible by 3. Each state records the number of 1s (mod 3) read so far).



- The third DFA C accepts all strings whose binary value is divisible by 3. Each state records the binary value (mod 7) of the string read so far).



Each state of the product DFA M is a triple (a, b, c), where a is a state of A, b is a state of B, and c is a state of C. For example, the start state of M is $(1e, 0, 0)$.

Finally, a state (a, b, c) of M is accepting if and only if $a \in \{0e, 1e\}$ (the number of 01s is even) or $b \neq 0$ (the number of 1s is not divisible by 3) or $c = 0$ (the binary value is divisible by 7). ■

Rubric: 10 points: standard DFA rubric. These are not the only correct solutions.