1. Recall that a 5-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0,1,2,3,4\}$, such that for any edge uv, vertices u and v are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

Solution: We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph G, we construct a new graph H by replacing each edge in G with a path of length three. I claim that H has a careful 5-coloring if and only if G has a (not necessarily careful) 5-coloring.

$\Longleftarrow$ Suppose G has a 5-coloring. Consider a single edge uv in G, and suppose $color(u) = a$ and $color(v) = b$. We color the path from u to v in H as follows:

- If $b = (a+1) \bmod 5$, use colors $(a,\ (a+2) \bmod 5,\ (a-1) \pmod 5,\ b)$.
- If $b = (a-1) \bmod 5$, use colors $(a,\ (a-2) \bmod 5,\ (a+1) \pmod 5,\ b)$.
- Otherwise, use colors $(a,\ b,\ a,\ b)$.

In particular, every vertex in G retains its color in H. The resulting 5-coloring of H is careful.

$\Longrightarrow$ On the other hand, suppose H has a careful 5-coloring. Consider a path (u,x,y,v) in H corresponding to an arbitrary edge uv in G. There are exactly eight careful colorings of this path with $color(u) = 0$, namely: $(0,2,0,2)$, $(0,2,0,3)$, $(0,2,4,1)$, $(0,2,4,2)$, $(0,3,0,3)$, $(0,3,0,2)$, $(0,3,1,3)$, $(0,3,1,4)$. It follows immediately that $color(u) \neq color(v)$. Thus, if we color each vertex of G with its color in H, we obtain a valid 5-coloring of G.

Given G, we can clearly construct H in polynomial time. ■

2. Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors.

Solution: Let G' be the union of 374 copies of G, with additional edges between *every* vertex of each copy and *every* vertex in *every* other copy. Given G, we can easily build G' in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of G, and define $\chi(G')$ similarly.

$\Longrightarrow$ Fix any coloring of G with $\chi(G)$ colors. We can obtain a proper coloring of G' with $374 \cdot \chi(G)$ colors, by using a distinct set of $\chi(G)$ colors in each copy of G. Thus, $\chi(G') \leq 374 \cdot \chi(G)$.

$\Longleftarrow$ Now fix any coloring of G' with $\chi(G')$ colors. Each copy of G in G' must use its own distinct set of colors, so at least one copy of G uses at most $\lfloor \chi(G')/374 \rfloor$ colors. Thus, $\chi(G) \leq \lfloor \chi(G')/374 \rfloor$.

These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if k is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G')/374 = \lceil k/374 \rceil$. Thus, if we could compute such an integer k in polyomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard! ■

3. A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

   Solution: It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3Color problem.

   Let G be an arbitrary undirected graph. I claim that G has a proper 3-coloring if and only if G has a weak bicoloring with 3 colors.

   - Suppose G has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of G using only the colors cyan, magenta, and yellow by recoloring each red vertex with $\{\text{magenta}, \text{yellow}\}$, recoloring each blue vertex with $\{\text{magenta}, \text{cyan}\}$, and recoloring each green vertex with $\{\text{yellow}, \text{cyan}\}$.
   - Suppose G has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of G by defining red $= \{\text{magenta}, \text{yellow}\}$, defining blue $= \{\text{magenta}, \text{cyan}\}$, and defining green $= \{\text{yellow}, \text{cyan}\}$.

   More generally, for any integer k and any graph G, every weak k-bicoloring of G is also a proper $\binom{k}{2}$-coloring of G, and vice versa. ■

(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Solution: It suffices to prove that deciding whether a graph has a strong bicoloring with five colors is NP-hard, using the following reduction from the standard 3COLOR problem.

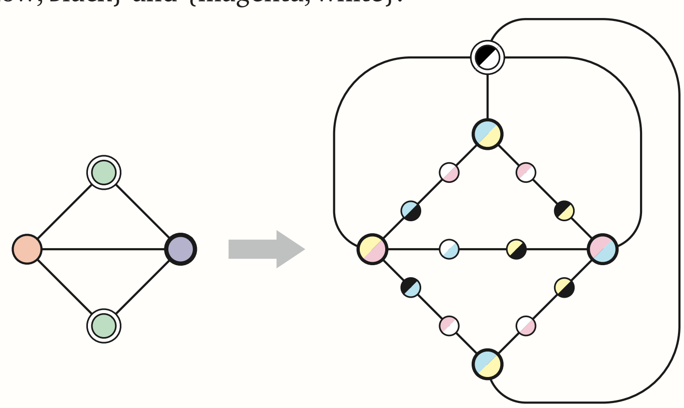
Let G be an arbitrary undirected graph. We build a new graph from G as follows:

- Add a new vertex z and edges zv to every vertex v of G.
- Subdivide every edge of G into a path of length 3. (But don't subdivide the new edges incident to z.)

I claim that G has a proper 3-coloring if and only if H has a strong bicoloring with five colors.

$\Longrightarrow$ Suppose G has a proper coloring with colors red, green, and blue. We obtain a strong bicoloring of H with colors cyan, magenta, yellow, white, and black as follows:

- Color vertex z $\{\text{white}, \text{black}\}$.
- Recolor the vertices of G by defining red $= \{\text{magenta}, \text{yellow}\}$ and green $= \{\text{cyan}, \text{yellow}\}$ and blue $= \{\text{magenta}, \text{cyan}\}$
- Color the new vertices on each red-green edge $\{\text{cyan}, \text{black}\}$ and $\{\text{magenta}, \text{white}\}$, the new vertices on each red-blue edge $\{\text{cyan}, \text{white}\}$ and $\{\text{yellow}, \text{black}\}$, and the new vertices on each blue-green edge $\{\text{yellow}, \text{black}\}$ and $\{\text{magenta}, \text{white}\}$.



$\Longleftarrow$ On the other hand, suppose H has a strong bicoloring with colors cyan, magenta, yellow, white, and black. Without loss of generality, vertex z is colored $\{\text{white}, \text{black}\}$, and therefore each vertex of G is colored either $\{\text{magenta}, \text{yellow}\}$ or $\{\text{cyan}, \text{yellow}\}$ or $\{\text{magenta}, \text{cyan}\}$.

Consider an arbitrary edge uv of G. Suppose for the sake of argument that u and v are assigned the same pair of colors, without loss of generality $\{\text{magenta}, \text{yellow}\}$. Then the intermediate vertices on the corresponding path in H only use the colors cyan, white, and black. But this is impossible, because two adjacent vertices of H must use four distinct colors. Thus, u and v must be assigned distinct (but not disjoint!) pairs of colors.

We conclude that defining red = {magenta, yellow} and blue = {magenta, cyan} and green = {yellow, cyan} gives us a proper 3-coloring of G.

We can easily construct H from G in polynomial time by brute force. ■

Five is the smallest number of colors for which strong bicoloring is NP-hard. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.