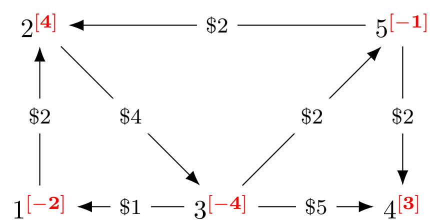
Math 482: Linear Programming, Fall 2020

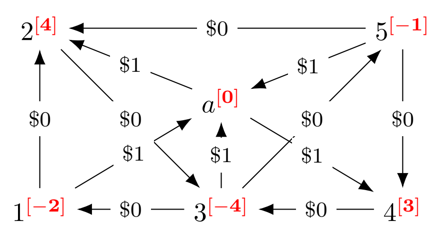
Due Wednesday, November 18, 6PM CST

Homework 9 solutions

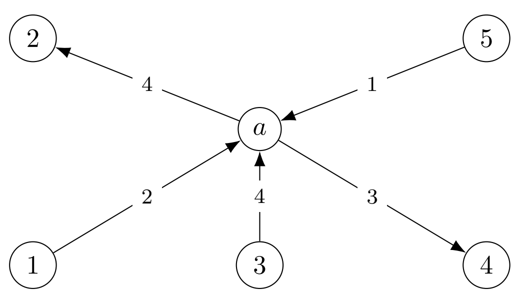
1. Consider the minimum-cost flow problem given in the diagram below. Solve this by adding an artificial node, and performing diagramatic two-step simplex method as in Lecture 29. Make sure you draw the diagrams at each step.



In the first phase, we modify the network by adding an artificial node a with demand $d_a = 0$. For each node with positive demand, we add an arc from a; for each node with negative demand, we add an arc to a. The costs of the artificial arcs are all 1 for this phase; the costs of the original arcs are all 0.

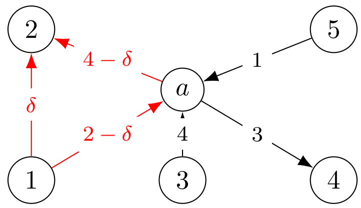


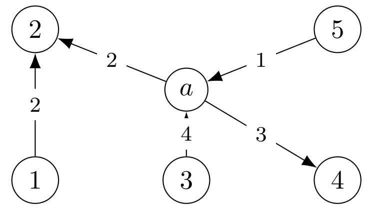
Start with the spanning tree solution below.



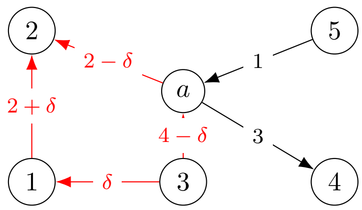
To make sure that I do not run into cycling, I will order the arcs: $(1, 2)(2, 3)(3, 1)(3, 4)(3, 5)(5, 2)(5, 4)$. I will always compute reduced costs for arcs not in the spanning tree in this order and always take the first one that is the correct sign.

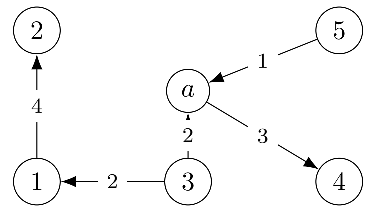
Arc $(1, 2)$ forms a cycle with arcs $(1, a)$ and $(a, 2)$ in the spanning tree. Both of these arcs go in the opposite direction of $(1, 2)$ around the cycle, so the reduced cost is $c_{12} - c_{a1} - c_{2a} = 0 - 1 - 1 = 2$. This is a valid pivot.



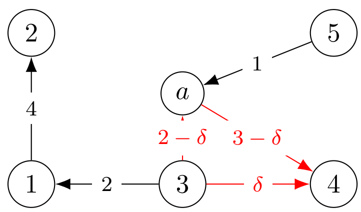


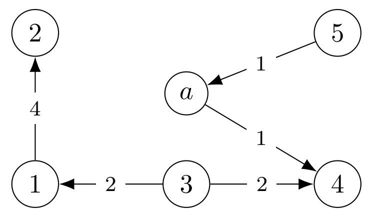
Arc $(1,2)$ is in the spanning tree. Arc $(2,3)$ is not a valid pivot. Arc $(3,1)$ forms a cycle with arcs $(1,2),(3,a)$ and $(a,2)$ in the spanning tree. Two of these arcs go in the opposite direction of $(3,1)$ around the cycle, so the reduced cost is $c_{31}+c_{12}-c_{3a}-c_{a2}=0+0-1-1=-2$. This is a valid pivot.



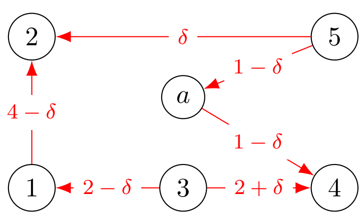


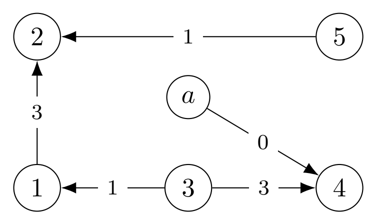
Arcs $(1,2),(3,1)$ are in the spanning tree. Arc $(2,3)$ is not a valid pivot (reduced cost is positive). Arc $(3,4)$ forms a cycle with arcs $(3,a)$ and $(a,4)$. These arcs go in the opposite direction around the cycle from $(3,4)$, so the reduced costs is $c_{34}-c_{3a}-c_{a4}=0-1-1=-2$. This is a valid pivot.



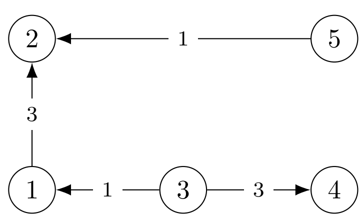


Arcs $(1,2),(3,1),(3,4)$ are in the spannign tree. Arc $(2,3),(3,5)$ are not a valid pivots (reduced costs are positive). Arc $(5,2)$ forms a cycle with arcs $(1,2)$, $(3,1)$,$(4,3)$,$(a,4)$ and $(5,a)$ in the spanning tree. The reduced cost is $c_{52}+c_{34}-c_{12}-c_{31}-c_{a4}-c_{5a}=0+0-0-0-1-1=-2$. This is a valid pivot.

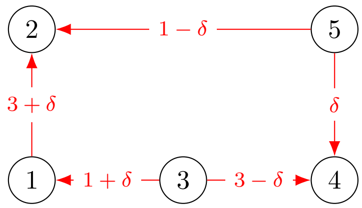


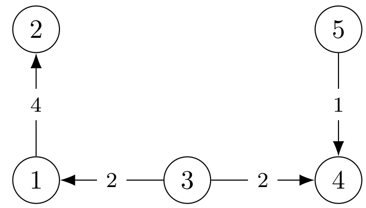


We are done with phase one. We are left with a spanning tree solution to the original problem:

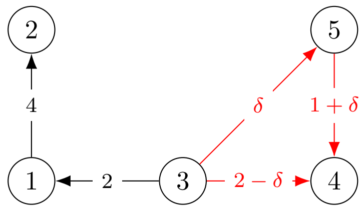


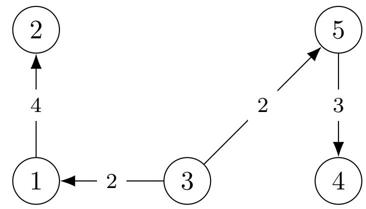
Arcs $(2,3)$, $(3,5)$ are not valid pivots. Arc $(5,4)$ forms a cycle with arcs $(1,2)$, $(3,1)$,$(4,3)$, and $(5,2)$ in the spanning tree. The reduced cost is $c_{54}+c_{12}+c_{31}-c_{43}-c_{52}=2+2+1-5-2=-2$. This is a valid pivot.





Arcs $(2,3)$ is not a valid pivot. Arc $(3,5)$ forms a cycle with arcs $(5,4)$ and $(3,4)$ in the spanning tree. The reduced cost is $c_{35}+c_{54}-c_{34}=-1$. This is a valid pivot.





All reduced costs are positive and we are done.

2. *Suppose that we are using the primal-dual method to solve the linear program*

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\textit{minimize}} & 3x_1+x_2+x_4 \\ \textit{subject to} & x_1+2x_2-2x_3-\ \ x_4=2 \\ & x_1+\ \ x_2 \qquad\quad -2x_4=3 \\ & x_1-2x_2-\ \ x_3+2x_4=4 \\ & x_1,x_2,x_3,x_4\geq 0 \end{array}$$

and that we are currently at the dual solution $\mathbf{u}=(1,1,1)$.

(a) *Write down and solve* (**RP**)*: the restricted primal program.*

The dual program is

$$\begin{array}{ll} \underset{u_1,u_2,u_3\in\mathbb{R}}{\text{maximize}} & 2u_1+3u_2+4u_3 \\ \text{subject to} & u_1+u_2+u_3\leq 3 \\ & 2u_1+u_2-2u_3\leq 1 \\ & -2u_1-u_3\leq 0 \\ & -u_1-2u_2+2u_3\leq 1 \end{array}$$

Of these constraints,the first two are tight at $(1,1,1)$. So (**RP**) only includes x_1,x_2 from the primal program: it is

$$\begin{array}{ll} \underset{x_1,y_1,y_2,y_3\in\mathbb{R}}{\text{minimize}} & y_1+y_2+y_3 \\ \text{subject to} & x_1+2x_2+y_1=2 \\ & x_1+x_2+y_2=3 \\ & x_1+-2x_2+y_3=4 \\ & x_1,y_1,y_2,y_3\geq 0 \end{array}$$

We solve this program in one pivot step:

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & y_1 & y_2 & y_3 & \\ \hline y_1 & 1 & 2 & 1 & 0 & 0 & 2 \\ y_2 & 1 & 1 & 0 & 1 & 0 & 3 \\ y_3 & 1 & -2 & 0 & 0 & 1 & 4 \\ \hline -w & -3 & -1 & 0 & 0 & 0 & -9 \end{array} \rightsquigarrow \begin{array}{c|ccccc|c} & x_1 & x_2 & y_1 & y_2 & y_3 & \\ \hline x_1 & 1 & 2 & 1 & 0 & 0 & 2 \\ y_2 & 0 & -1 & -1 & 1 & 0 & 1 \\ y_3 & 0 & -4 & -1 & 0 & 1 & 2 \\ \hline -w & 0 & 5 & 3 & 0 & 0 & -3 \end{array}$$

The optimal solution is $(x_1, x_2, y_1, y_2, y_3) = (2, 0, 0, 1, 2)$.

(b) *Find the optimal solution* $\mathbf{v}$ *to* (**DRP**)*: the dual of the restricted primal.*

Since v_i is 1 minus the reduced cost of y_i, we have $\mathbf{v} = (1, 1, 1) - (3, 0, 0) = (-2, 1, 1)$.

(c) *Use* $\mathbf{v}$ *to find an improved dual solution to the original linear program.*

We go from $\mathbf{u}$ to $\mathbf{u} + t\mathbf{v} = (1 - 2t, 1 + t, 1 + t)$ for some $t > 0$.

The first two constraints are automatic. The third constraint requires $-2(1 - 2t) - (1 + t) \le 0$, or $t \le 1$. The final constraint requires $-(1 - 2t) - 2(1 + t) + 2(1 + t) \le 1$, or $t \le 1$ again.

So we set $t = 1$ and end up at the new dual solution $(-1, 2, 2)$.

(d) *Show that the new dual solution is optimal by finding a corresponding optimal primal solution.*

At this dual solution, the first, third, and fourth dual constraints are tight, while the last is slack. We conclude that $x_2 = 0$ in the primal.

We solve the system of equations that implies: $x_1 - 2x_3 - x_4 = 2$, $x_1 - 2x_4 = 3$, and $x_1 - x_3 + 2x_4 = 4$. This gives $x_1 = \frac{27}{7}$, $x_3 = \frac{5}{7}$, and $x_4 = \frac{3}{7}$. This does satisfy $\mathbf{x} \ge \mathbf{0}$, so complementary slackness holds, and

$$\mathbf{x} = \left(\frac{27}{7}, 0, \frac{5}{7}, \frac{3}{7}\right)$$

is the final optimal primal solution.

3. *Consider the linear program*

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4 \in \mathbb{R}}{\textit{minimize}} & x_1 + 3x_2 - 2x_3 - x_4 \\ \textit{subject to} & x_1 + 2x_2 - 2x_3 - x_4 = 2 \\ & x_1 + x_2 \phantom{{}-2x_3} - 2x_4 = 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

(a) *Use the assumption that* $x_1 + x_2 + x_3 + x_4 \le 100$ *in all feasible solutions to the linear program above, to write down an equivalent linear program and a feasible dual solution for it.*

If $x_1 + x_2 + x_3 + x_4 \le 100$ in all feasible solutions, then there is always an $x_5 \ge 0$ such that $x_1 + x_2 + x_3 + x_4 + x_5 = 100$, leading to

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4,x_5 \in \mathbb{R}}{\text{minimize}} & x_1 + 3x_2 - 2x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 - 2x_3 - x_4 \phantom{{}+x_5} = 2 \\ & x_1 + x_2 \phantom{{}-2x_3} - 2x_4 \phantom{{}+x_5} = 3 \\ & x_1 + x_2 + x_3 + x_4 + x_5 = 100 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{array}$$

The dual of this linear program is

$$
\begin{aligned}
\underset{u_1,u_2,u_3\in\mathbb{R}}{\text{maximize}} \quad & 2u_1 + 3u_2 + 100u_3 \\
\text{subject to} \quad & u_1 + u_2 + u_3 \le 1 \\
& 2u_1 + u_2 + u_3 \le 3 \\
& -2u_1 + u_3 \le -2 \\
& -u_1 - 2u_2 + u_3 \le -1 \\
& u_3 \le 0
\end{aligned}
$$

and we can get a feasible dual solution of $(0, 0, -2)$ by setting $u_1 = u_2 = 0$ and setting u_3 to the minimum of the numbers on the right-hand sides of the inequalities.

(b) *Perform two steps of the primal-dual method, starting from the feasible dual solution you found.*

Initially, only the third constraint $-2u_1 + u_3 \le -2$ is going to be tight, so only x_3 is going to appear in the restricted primal. I will use the "frozen variable" approach and keep all variables in the primal, some of them frozen, but you don't have to use this method.

If we do, though, then we start with the tableau

	$[x_1]$	$[x_2]$	x_3	$[x_4]$	$[x_5]$	y_1	y_2	y_3	
y_1	$[1]$	$[2]$	-2	$[-1]$	$[0]$	1	0	0	2
y_2	$[1]$	$[1]$	0	$[-2]$	$[0]$	0	1	0	3
y_3	$[1]$	$[1]$	1	$[1]$	$[1]$	0	0	1	100
$-w$	$[-3]$	$[-4]$	1	$[2]$	$[-1]$	0	0	0	-105

and this tableau is actually already optimal. So we take $\mathbf{v} = (1, 1, 1)$ and try to go from $(0, 0, -2)$ to $(t, t, t - 2)$ in the dual.

The constraints on t are that $t + t + (t - 2) \le 1$ or $t \le 1$; that $2t + t + (t - 2) \le 3$ or $t \le \frac{5}{4}$; we can skip the third constraint; that $-t - 2t + (t - 2) \le -1$ or $t \ge -\frac{1}{2}$; that $t - 2 \le -1$ or $t \le 1$.

So we can set $t = 1$, at which point we go to the dual solution $(1, 1, -1)$. The first constraint becomes tight; the third constraint becomes slack. So x_1 unfreezes while x_3 becomes frozen, and we arrive at

	x_1	$[x_2]$	$[x_3]$	$[x_4]$	$[x_5]$	y_1	y_2	y_3	
y_1	1	$[2]$	$[-2]$	$[-1]$	$[0]$	1	0	0	2
y_2	1	$[1]$	$[0]$	$[-2]$	$[0]$	0	1	0	3
y_3	1	$[1]$	$[1]$	$[1]$	$[1]$	0	0	1	100
$-w$	-3	$[-4]$	$[1]$	$[2]$	$[-1]$	0	0	0	-105

Pivoting on x_1 yields

	x_1	$[x_2]$	$[x_3]$	$[x_4]$	$[x_5]$	y_1	y_2	y_3	
x_1	1	$[2]$	$[-2]$	$[-1]$	$[0]$	1	0	0	2
y_2	0	$[-1]$	$[2]$	$[-1]$	$[0]$	-1	1	0	1
y_3	0	$[-1]$	$[3]$	$[2]$	$[1]$	-1	0	1	98
$-w$	0	$[2]$	$[-5]$	$[-1]$	$[-1]$	3	0	0	-99

At this point, the tableau is optimal.

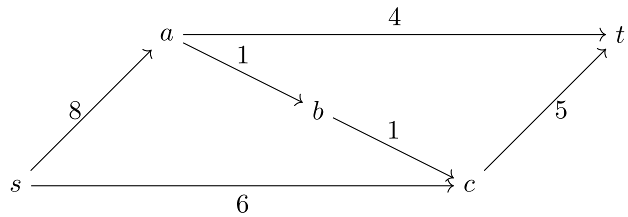
We compute $\mathbf{v} = (1, 1, 1) - (3, 0, 0) = (-2, 1, 1)$ and so we are going to move from $(1, 1, -1)$ to $(1 - 2t, 1 + t, -1 + t)$ in the dual.

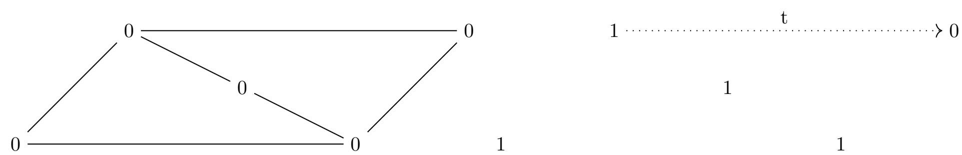
Skipping the first constraint, the requirements on t are that $2(1 - 2t) + (1 + t) + (-1 + t) \le 3$ or $t \ge -\frac{1}{2}$; that $-2(1 - 2t) + (-1 + t) \le -2$ or $t \le \frac{1}{5}$; that $-(1 - 2t) - 2(1 + t) + (-1 + t) \le -1$ or $t \le 3$; that $(-1 + t) \le 0$ or $t \le 1$.

So we set $t = \frac{1}{5}$ and end up at the dual solution $(\frac{3}{5}, \frac{6}{5}, -\frac{4}{5})$.

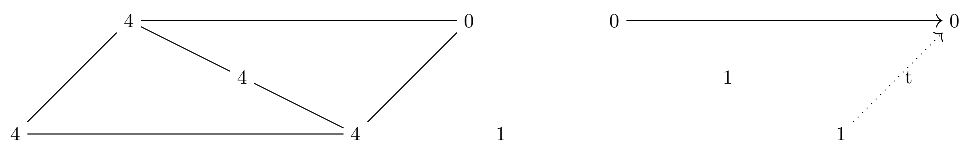
We could keep going, but that's the two steps we were going to do.

4. Use the primal-dual version of Dijkstra's algorithm from Lecture 33 to find the shortest path from s to t in the network below. Make sure you draw the diagram associated to the dual and the dual of the restricted primal at each step.

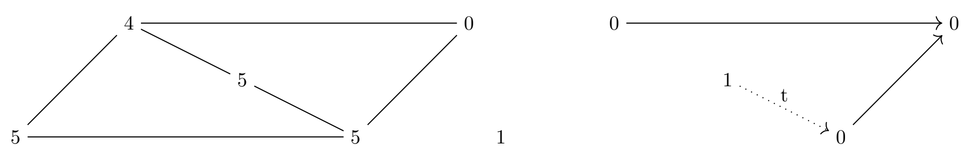




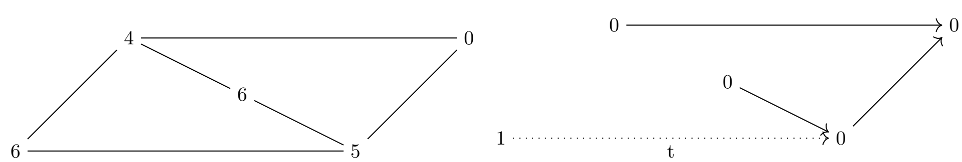
Above: $u = (0, 0, 0, 0, 0)$, $J = \emptyset$, $v = (1, 1, 1, 1, 0)$. Thus $t = 4$, and our new $u = (4, 4, 4, 4, 0)$.



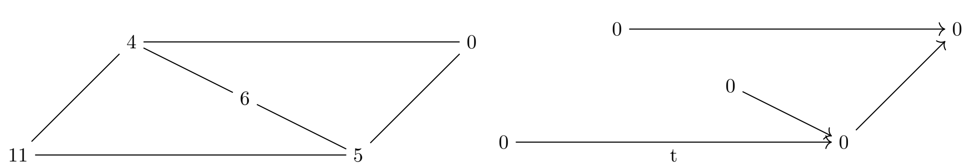
Above: $u = (4, 4, 4, 4, 0)$, $J = \{(a, t)\}$, $v = (1, 0, 1, 1, 0)$. Thus $t = 1$, and our new $u = (5, 4, 5, 5, 0)$.



Above: $u = (5, 4, 5, 5, 0)$, $J = \{(a, t), (c, t)\}$, $v = (1, 0, 1, 0, 0)$. Thus $t = 1$, and our new $u = (6, 4, 6, 5, 0)$.



Above: $u = (6, 4, 6, 5, 0)$, $J = \{(a, t), (c, t), (b, c)\}$, $v = (1, 0, 0, 0, 0)$. Thus $t = 5$, and our new $u = (11, 4, 6, 5, 0)$.



Above: $u = (11, 4, 6, 5, 0)$, $J = \{(a, t), (c, t), (b, c), (s, c)\}$, $v = (0, 0, 0, 0, 0)$. Our final $J = \{(a, t), (c, t), (b, c), (s, c)\}$. Thus the shortest path from s to t is $s \to c \to t$.

5. *(Only 4-credit students need to do this problem.)*

Suppose that you have a network in which, instead of every arc having a capacity, there is a capacity associated through every node (other than the source s or the sink t). The flow along an arc can be arbitrary, but the total flow going into a node (equivalently, the total flow going out of a node) can be at most the capacity of that node.

Explain how to convert a maximum-flow problem for such a network into a standard maximum-flow problem.

First, give all existing arcs capacity ∞.

Second, subdivde each interior node v into two nodes: v^- and v^+. For every arc that pointed into v, direct it toward v^- instead; for every arc that pointed out of v, direct it out of v^+. Then, add an arc (v^-, v^+) whose capacity is the "capacity" of the node v.

If previously we directed flow through v we can still do that in the same way: it goes into v^-, flows along the new arc (v^-, v^+), and then goes out of v^+. However, all such flow has to use the arc (v^-, v^+), so it is limited by the capacity of that arc, enforcing the constraint that there can be at most that much flow through node v.