Math 482: Linear Programming, Fall 2020

Due Wednesday, September 11, 6PM CST

Homework 2 Solutions

1. *Set up a linear program for the problem below. Do not solve. (Because there will be many similar constraints, it's fine if you just provide an example of each type of constraint, and say "do the same thing for every club" or "do the same thing for every pair of clubs".)*

   Every student at a certain school is a member of at least one (maybe more) of its five clubs: Athletics Club, Book Club, Chess Club, Drama Club, and Ethics Club. Moreover, each club is very large, and includes at least $\frac{1}{2}$ of the students at the school.

   The school gives a "friendship award" to the two clubs that have the largest overlap in students. Every student that's a member of both *clubs (not just one) will receive the award.*

   What is the smallest possible fraction of students that can receive the friendship award?

   (Hint: to minimize the largest overlap between two clubs, minimize an auxiliary variable z that's greater than or equal to the size of every overlap. You will need to think carefully about what your other variables are in this linear program. Think about a Venn diagram, splitting up the students into 32 possible categories, depending on the set of clubs in which they are members.)

   To be able to describe all possible ways to distribute students between clubs, we define 31 variables $x_a, x_b, x_c, x_d, x_e, x_{ab}, x_{ac}, \dots, x_{abcde}$. These are intended to describe each possible nonempty subset of clubs: for example, x_{acd} is the fraction of students that are in Athletics Club, Chess Club, and Drama Club, but not in the other two clubs.

   The linear program is:

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^{31}, z\in\mathbb{R}}{\text{minimize}} \quad & z \\
\text{subject to} \quad & x_a + x_{ab} + x_{ac} + x_{ad} + x_{ae} + x_{abc} + x_{abd} + x_{abe} + x_{acd} + x_{ace} + x_{ade} \\
& \quad + x_{abcd} + x_{abce} + x_{abde} + x_{acde} + x_{abcde} \ge \frac{1}{2} \\
& \text{(4 more constraints like the one above)} \\
& z \ge x_{ab} + x_{abc} + x_{abd} + x_{abe} + x_{abcd} + x_{abce} + x_{abde} + x_{abcde} \\
& \text{(9 more constraints like the one above)} \\
& x_a + x_b + x_c + x_d + x_e + x_{ab} + x_{ac} + x_{ad} + x_{ae} + x_{bc} + x_{bd} + x_{be} \\
& \quad + x_{cd} + x_{ce} + x_{de} + x_{abc} + x_{abd} + x_{abe} + x_{acd} + x_{ace} + x_{ade} + x_{bcd} \\
& \quad + x_{bce} + x_{bde} + x_{cde} + x_{abcd} + x_{abce} + x_{abde} + x_{acde} + x_{bcde} + x_{abcde} = 1 \\
& x_a, x_b, x_c, x_d, x_e, x_{ab}, x_{ac}, x_{ad}, x_{ae}, x_{bc}, x_{bd}, x_{be}, x_{cd}, x_{ce}, x_{de}, \\
& x_{abc}, x_{abd}, x_{abe}, x_{acd}, x_{ace}, x_{ade}, x_{bcd}, x_{bce}, x_{bde}, x_{cde}, \\
& x_{abcd}, x_{abce}, x_{abde}, x_{acde}, x_{bcde}, x_{abcde} \ge 0
\end{aligned}
$$

   Here is an explanation of all of the constraints.

   First, there are five constraints telling us that each club includes at least $\frac{1}{2}$ of the students. To do this, for every club, add up the 16 variables that represent students in that club, and ask for that sum to be at least $\frac{1}{2}$. (In the linear program above, an example of this constraint is given for a: Athletics Club.)

   Second, there are ten constraints telling us that the variable z is at least the overlap between every two clubs. This is equivalent to saying that z is at least the largest overlap between any two clubs, and since we're minimizing z, z will in fact be equal to the largest overlap in any optimal solution. Anyway, the way this works, for every pair of clubs, add up the 8 variables that represent students in both of those clubs, and ask for z to be at least that sum. (In the linear program above, an example of this constraint is given for the pair ab: Athletics Club and Book Club.)

Third, there is a constraint saying that all 31 variables add up to 1: that every student is in some nonempty subset of the clubs.

Finally, there is a constraint that all 31 variables are nonnegative.

2. *The linear program*

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{maximize}} \quad & 2x - 2y \\ \text{subject to} \quad & x - 3y \le 3 \\ & -4x + y \le 4 \\ & x - 2y \le 6 \\ & x, y \ge 0 \end{aligned}$$

is unbounded. Use the simplex method to find a ray (a starting point (x_0, y_0) and a direction (u, v)) along which every point is a feasible solution, and the objective value increases arbitrarily far.

Let's set up the simplex tableau:

	x	y	s_1	s_2	s_3	
s_1	1	−3	1	0	0	3
s_2	−4	1	0	1	0	4
s_3	1	−2	0	0	1	6
$-z$	1	−1	0	0	0	0

To maximize z, we should pivot on x, whose reduced cost is positive. Throwing out s_2 (with a -4 in x's column) we compare the ratios of s_1 and s_3. Since $\frac{3}{1} < \frac{6}{1}$, s_1 has the smallest ratio, so it leaves the basis and x replaces it. We get:

	x	y	s_1	s_2	s_3	
x	1	−3	1	0	0	3
s_2	0	−11	4	1	0	16
s_3	0	1	−1	0	1	3
$-z$	0	2	−1	0	0	−3

Now y enters the basis, since its reduced cost is positive. The only variable with a positive number in y's column is s_3, so it leaves the bases and y replaces it. We get:

	x	y	s_1	s_2	s_3	
x	1	0	−2	0	3	12
s_2	0	0	−7	1	11	49
y	0	1	−1	0	1	3
$-z$	0	0	2	0	−4	−18

Now s_1 should enter the basis, since its reduced cost is positive, but there is no negative entry in the s_1 column to pivot on. This is a sign that the LP is unbounded.

To find the ray along which we get solutions with arbitrarily large values of the objective function, take the equations for x and y in the tableau above:

$$\begin{cases} x - 2s_1 + 3s_3 = 12 \\ y - s_1 + s_3 = 3 \end{cases}$$

As we pivot on s_1, the nonbasic variable s_3 remains 0, so we can ignore it in these equations. We can solve for x and y to get $x = 12+2s_1$ and $y = 3+s_1$. This is our final answer; to express it as a starting point and a direction, we can write it as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 + 2s_1 \\ 3 + s_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} s_1$$

so the starting point is $(12, 3)$ and the direction is $(2, 1)$.

3. *Use the two-phase simplex method to solve*

$$\begin{aligned} \underset{x_1,x_2,x_3 \in \mathbb{R}}{\textit{minimize}} \quad & 3x_1 - x_2 \\ \textit{subject to} \quad & x_1 + x_2 + x_3 = 5 \\ & 2x_1 + x_2 - 2x_3 \geq 6 \\ & x_1 + x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

If we add slack variables, but make sure to keep the right-hand sides positive, we get $2x_1+x_2-2x_3-s_1 = 6$ and $x_1 + x_2 - x_3 + s_2 = 1$. Adding artificial variables x_1^a, x_2^a, x_3^a to the tableau gives us:

	x_1	x_2	x_3	s_1	s_2	x_1^a	x_2^a	x_3^a	
x_1^a	1	1	1	0	0	1	0	0	5
x_2^a	2	1	-2	-1	0	0	1	0	6
x_3^a	1	1	-1	0	1	0	0	1	1
$-z^a$	0	0	0	0	0	1	1	1	0
$-z$	3	-1	0	0	0	0	0	0	0

First, let's row-reduce to clean up $-z^a$'s row:

	x_1	x_2	x_3	s_1	s_2	x_1^a	x_2^a	x_3^a	
x_1^a	1	1	1	0	0	1	0	0	5
x_2^a	2	1	-2	-1	0	0	1	0	6
x_3^a	1	1	-1	0	1	0	0	1	1
$-z^a$	-4	-3	2	1	-1	0	0	0	-12
$-z$	3	-1	0	0	0	0	0	0	0

Let's pivot on x_1, whose reduced cost is -4, to decrease z^a. Then x_3^a leaves the basis and we get:

	x_1	x_2	x_3	s_1	s_2	x_1^a	x_2^a	x_3^a	
x_1^a	0	0	2	0	-1	1	0	-1	4
x_2^a	0	-1	0	-1	-2	0	1	-2	4
x_1	1	1	-1	0	1	0	0	1	1
$-z^a$	0	1	-2	1	3	0	0	4	-8
$-z$	3	-1	0	0	0	0	0	0	0

Let's pivot on x_3, whose reduced cost is -2, to decrease z^a. Then x_1^a leaves the basis and we get:

	x_1	x_2	x_3	s_1	s_2	x_1^a	x_2^a	x_3^a	
x_3	0	0	1	0	$-1/2$	$1/2$	0	$-1/2$	2
x_2^a	0	-1	0	-1	-2	0	1	-2	4
x_1	1	1	0	0	$1/2$	$1/2$	0	$1/2$	3
$-z^a$	0	1	0	1	2	1	0	3	-4
$-z$	3	-1	0	0	0	0	0	0	0

At this point, we see that z^a is still negative, but all the reduced costs are positive and so z^a has reached its minimum. We conclude that the linear program is infeasible.

(Note: we could have taken a shortcut and saved ourselves a little bit of work by using s_2 instead of x_3^a as the basic variable for the third row. If we did, we could have omitted x_3^a entirely, minimizing the artificial objective function $z^a = x_1^a + x_2^a$. This would have been a little bit faster.)

4. *Consider the following linear program:*

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{maximize}} & x_1 - 3x_2 - 2x_4 \\ \text{subject to} & \frac{1}{2}x_1 - \frac{7}{2}x_2 - \frac{3}{2}x_3 + \frac{7}{2}x_4 \le 0 \\ & \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \le 0 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{array}$$

(a) *Perform two iterations of the simplex method using the following pivoting rule: choose the entering variable with the highest reduced cost. When both rows are valid leaving variables (in which case they'll always be tied for the smallest ratio: both ratios will always be 0) choose the basic variable for the first row as the leaving variable.*

We go from

	x_1	x_2	x_3	x_4	s_1	s_2	
s_1	$1/2$	$-7/2$	$-3/2$	$7/2$	1	0	0
s_2	$1/2$	$-3/2$	$-1/2$	$1/2$	0	1	0
$-z$	1	-3	0	-2	0	0	0

to

	x_1	x_2	x_3	x_4	s_1	s_2	
x_1	1	-7	-3	7	2	0	0
s_2	0	2	1	-3	-1	1	0
$-z$	0	4	3	-9	-2	0	0

to

	x_1	x_2	x_3	x_4	s_1	s_2	
x_1	1	0	$1/2$	$-7/2$	$-3/2$	$7/2$	0
s_2	0	1	$1/2$	$-3/2$	$-1/2$	$1/2$	0
$-z$	0	0	1	-3	0	-2	0

(b) *Comparing the resulting tableau to the original tableau, argue that the simplex method with this pivoting rule will cycle forever, returning to the same tableau every six steps.*

Doing the pivot steps above, we've shifted every entry of the tableau two columns left. (For example, the new x_3 column is the same as the old x_1 column, the new x_4 column is the same as the old x_2 column, and so on.) If we do this two more times, we get the original tableau.

5. *(Only 4-credit students need to do this problem.)*

Consider a linear program of the form

$$\begin{array}{ll} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}. \end{array}$$

Suppose that points $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ are both feasible solutions of this linear program.

(a) *Show that if both* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions, then* $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ *is an optimal solution.*

First of all, $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ is also feasible, because $A(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}) = \frac{1}{2}(A\mathbf{x}) + \frac{1}{2}(A\mathbf{y}) \le \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{b} = \mathbf{b}$.

The point $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ has objective value $\mathbf{c}^\mathsf{T}(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}) = \frac{1}{2}\mathbf{c}^\mathsf{T}\mathbf{x} + \frac{1}{2}\mathbf{c}^\mathsf{T}\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are both optimal, they both have the same objective value z^*. But then $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ has objective value $\frac{1}{2}z^* + \frac{1}{2}z^* = z^*$ as well.

(b) *Show that if* $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ *is an optimal solution, then both* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions.*

If $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ has the optimal objective value z^*, then we have $\frac{1}{2}\mathbf{c}^\mathsf{T}\mathbf{x} + \frac{1}{2}\mathbf{c}^\mathsf{T}\mathbf{y} = z^*$, in the same way as in the first part, or $\mathbf{c}^\mathsf{T}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{y} = 2z^*$.

Since z^* is the maximum possible objective value, we have $\mathbf{c}^\mathsf{T}\mathbf{x} \le z^*$, so $\mathbf{c}^\mathsf{T}\mathbf{y} = 2z^* - \mathbf{c}^\mathsf{T}\mathbf{x} \ge 2z^* - z^* = z^*$. We can't have $\mathbf{c}^\mathsf{T}\mathbf{y} > z^*$, so we must have $\mathbf{c}^\mathsf{T}\mathbf{y} = z^*$. We prove $\mathbf{c}^\mathsf{T}\mathbf{x} = z^*$ in the same way, and conclude that both $\mathbf{x}$ and $\mathbf{y}$ are optimal (since they have the optimal objective value z^*).