Line Equation

$$\overrightarrow{r} = t.\overrightarrow{v} + \overrightarrow{r_0}$$

Plane Equation

$$\overrightarrow{n}.\overrightarrow{P_0P} = 0$$

$$Arclength = \int_b^a \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\overrightarrow{v}(t)|\, dt$$

$$\text{Arc length Parameter} = s(t) = \int_a^t |\overrightarrow{v}(\tau)| d\tau$$

cartesian - polar coordinate

$$dxdy = \mathbf{r}\, dr d\theta$$

cartesian - Spherical coordinate:

$$dxdydz = \boldsymbol{\rho^2} \sin\phi\, d\rho d\phi d\theta$$

$$\text{Curvature} := \left|\frac{d\overrightarrow{T}}{ds}\right| = \kappa = \left|\frac{d\overrightarrow{T}}{dt}\frac{dt}{ds}\right|$$

$$\xrightarrow{\overrightarrow{v} = \frac{ds}{dt}} \kappa = \frac{1}{|\overrightarrow{v}|}\left|\frac{d\overrightarrow{T}}{dt}\right|$$

$$\text{Torsion} := -\frac{d\overrightarrow{B}}{ds}.\overrightarrow{N} = \tau$$

- Torsion τ shows how that plane twists.
- Curvature κ shows how points curve in the plane defined by $\overrightarrow{T}$ and $\overrightarrow{N}$

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt} \qquad \overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

$$a_{\overrightarrow{T}} = \frac{d|\overrightarrow{v}|}{dt} \qquad a_{\overrightarrow{N}} = \kappa|\overrightarrow{v}|^2$$

$$\text{Acceleration} := |\overrightarrow{a}| = \sqrt{|a_{\overrightarrow{T}}|^2 + |a_{\overrightarrow{N}}|^2}$$

Theorem: if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on an open region R, then f is differentiable on R

$$\text{Chain Rule} := \frac{dw}{dt} = \sum \frac{\partial w}{\partial x_i}\frac{dx_i}{dt}$$

$$\text{Directional Derivatives} := D_{\overrightarrow{u}} f(x_0, y_0) = \nabla f|_{(x_0,y_0)}.\overrightarrow{u}$$

$$\text{Tangent Plane} := \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0) + z_0 = z$$

Lagrange Multipliers:

$$\nabla f = \lambda \nabla g$$
$$g = 0$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$
$$g_1 = 0 \qquad g_2 = 0$$

tips to find the local Max/Min of a function:

1. Find critical points where $f_x(a,b) = f_y(a,b) = 0$
2. classify each critical point with 2nd derivative test:
   - f has a local **max** if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
   - f has a local **min** if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
   - f has a **saddle** if $f_{xx}f_{yy} - f_{xy}^2 < 0$
   - f is **inconclusive** if $f_{xx}f_{yy} - f_{xy}^2 = 0$

$$V = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} dzdydx$$

The Jacobian is useful when changing variables in multivariable integrals:

$$\iint_R f(x,y)\, dxdy = \iint_G f(g(u,v), h(u,v))J(u,v)\, dudv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\text{Line Integral} := \int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))\boxed{\sqrt{g'(t)^2 + h'(t)^2 + k'(t)^2}dt}$$

Arc Length

$$\text{Line Integral of Vector Field} \int_C \overrightarrow{F}.\overrightarrow{T}ds = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t))\frac{d\overrightarrow{r}}{dt}dt = \int_C \overrightarrow{F}d\overrightarrow{r}$$

conservative fields

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$

1. equivalent to $\overrightarrow{F} = \nabla f$
2. fundamental theorem of line integrals:

   $\int_C \overrightarrow{F}.d\overrightarrow{r} = f(B) - f(A)$
3. curl = 0, the vector field is conservative.

Flow (flow density = curl)

$$Curl\ \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & M & P \end{vmatrix}$$

- with parametrization:

$$Flow = \int_a^b M dx + \int_a^b N dy$$

- without parametrization: (Green's Theorem)

$$\oint_C \overrightarrow{F}.d\overrightarrow{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Flux (flux density = divergence)

$$Div\ \overrightarrow{F} = \nabla.\overrightarrow{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

- with parametrization:

$$Flux = \int_a^b M dy - \int_a^b N dx$$

- without parametrization: (Green's Theorem)

$$\oint_C \overrightarrow{F}.\overrightarrow{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

Surface Area

$$Parametric: \overrightarrow{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}$$

$$SA_p = \int_c^d \int_a^b |\overrightarrow{r}_u \times \overrightarrow{r}_v|\, dudv$$

$$(\overrightarrow{p} \in \hat{i}, \hat{j}, \hat{k})$$

$$Implicit: F(x,y,z) = c$$

$$SA_i = \iint_R \frac{|\nabla F|}{|\nabla F.\overrightarrow{p}|}\, dA$$

$$Explicit: z = f(x,y)$$

$$SA_e = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

Surface Integrals

$$SI = \iint_R G(f(u,v), g(u,v), h(u,v))\, \mathbf{SA_p}\, dudv$$

$$SI = \iint_R G(x,y,z)\, \mathbf{SA_i}\, dA$$

$$SI = \iint_R G(x,y,f(x,y))\, \mathbf{SA_e}\, dA$$

Flux of a vector field across a surface

$$Flux = \iint_S \overrightarrow{F}.\overrightarrow{n}\, d\sigma$$

as only boundary is important to us, we can choose a simpler surface instead of given one and apply the stokes or divergence theorem.

Parametric

$$Flux = \iint_S \overrightarrow{F}.(\overrightarrow{r}_u \times \overrightarrow{r}_v)\, dudv$$

Implicit

$$Flux = \iint_S \overrightarrow{F}.\frac{\nabla g}{|\nabla g.\overrightarrow{p}|}\, dudv$$

Stoke's Theorem

$$\oint_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S (\nabla \times \overrightarrow{F}).\overrightarrow{n}\, d\sigma$$

flow density over boundary

Divergence Theorem

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\, ds = \iiint_D \nabla.\overrightarrow{F}\, dV$$

flux density over boundary

Amin