~~PHA~~ Physics Department

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Subject : ~~[illegible]~~ (PHA-203) MATHEMATICAL PHYSICS (SI)

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1. Tensors : ||

টেন্সর বস্তুর রূপের জ্যামিতিক উপস্থাপনাই হলো tensor ||

importance of tensor in physics

e tensor এর অনেক উপকারিতা আছে পদার্থবিজ্ঞানে, কারন এর কারনে আমরা (Stress, স্থিতিস্থাপকতা, জড়তা ইত্যাদি) জ্যামিতিক ভাবে উপস্থাপনা করতে সাহায্য করে।

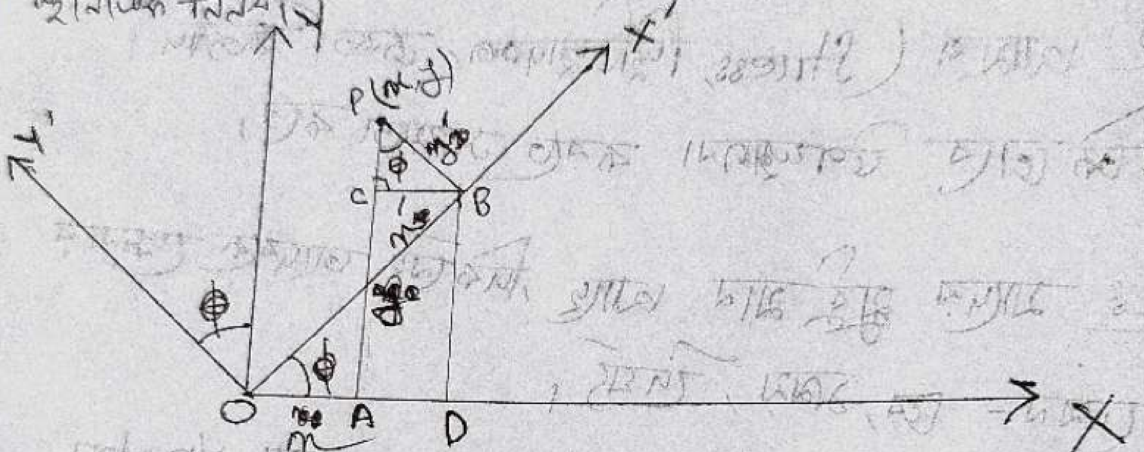
Scalar : যাদের শুধু মান আছে, দিক নেই তাদেরকে স্কেলার বলে। যেমন- ভর, সময়, দৈর্ঘ্য।

vector : যাদের মান ও দিক উভয়ই আছে তাদেরকে vector quantity বলে। যেমন : বেগ, সরন, বল ও কৌনিক বেগ,

কোনো একটি নির্দিষ্ট অক্ষের দিক পরিবর্তনের ফলে বা ঘূর্ণনের ফলে যদি একটি বস্তুর স্থানাংকের পরিবর্তন হয় তাহলে এটি vector কিন্তু যদি স্থানাংক অথবা মানের কোনো পরিবর্তন না হয় তাহলে এটি Scalar.

Rotation of the co-ordinate axis |

কোনো বিন্দুর অবস্থান অপরিবর্তিত রাখিয়া অক্ষদ্বয়কে θ কোণে আবর্তন করিলে, অক্ষের নতুন অক্ষের সাপেক্ষে রূপান্তরিত স্থানাঙ্ক নির্ণয়।



চিত্র হতে, $AP \perp OX$ এবং $PB \perp OX'$ আবার, $DB \perp OX$ এবং $BC \perp AP$, এখানে OX এবং OY অক্ষের স্থানাঙ্ক OX' এবং OY' রূপান্তরিত হলে $P(x, y)$ এর স্থানাঙ্ক নতুন অক্ষের সাপেক্ষে স্থানাঙ্ক হবে $P(x', y')$

$\therefore x = OA = OD - AD$

$\Rightarrow x = OD - BC$ ——— (i)

$y = AP = AC + CP = BD + CP$

$\Rightarrow y = BD + CP$ ——— (ii)

∴ $\triangle PBC$ হতে পাই,

$\sin\phi = \dfrac{BC}{PB}$, $\cos\theta = \dfrac{PC}{PB}$

$\Rightarrow BC = PB.\sin\phi$, $PC = PB\cos\phi$

∴ $BC = y'\sin\phi$, $PC = y'\cos\phi$

∴ $\triangle BOD$ হতে পাই,

$\sin\phi = \dfrac{BD}{OB}$, $\cos\phi = \dfrac{OD}{OB}$

$\Rightarrow BD = OB\sin\phi$, $\Rightarrow OD = OB\cos\phi$

∴ $BD = x'\sin\phi$, ∴ $OD = x'\cos\phi$

(i) নং সমীকরণ হতে

∴ $x = OD - BC = x'\cos\phi - y'\sin\phi$ —— (iii)

(ii) নং সমীকরণ হতে

$y = BD + CP = x'\sin\phi + y'\cos\phi$ —— (iv)

(iii × ~~sin~~ $\cos\phi$) + (iv × $\sin\phi$)

$x\cos\phi = x'\cos^2\phi - y'\sin\phi\cos\phi$

$y\sin\phi = x'\sin^2\phi + y'\sin\phi\cos\phi$

$x\cos\phi + y\sin\phi = x'(\sin^2\phi + \cos^2\phi)$

$\Rightarrow x' = x\cos\phi + y\sin\phi$

(iii $\times \sin\theta$) $-$ (iv $\times \cos\theta$)

$x \sin\phi = x_1 \sin\phi \cos\phi - y_1 \sin^2\phi$

$y \cos\phi = x_1 \sin\phi \cos\phi + y_1 \cos^2\phi$

$$\Rightarrow x \sin\phi - y\cos\phi = -y_1(\sin^2\phi + \cos^2\phi)$$

$$\Rightarrow y' = -x\sin\phi + y\cos\phi$$

∴ অক্ষের দিক θ কোনে আবর্তন করিলে নতুন অক্ষের সাপেক্ষে P বিন্দুর স্থানাঙ্ক রূপান্তরিত হবে,

$$(x', y') = (x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi)$$

∴ Coordinate System এর সাধারণ (x, y) হলো unprime system

এবং ~~(x_1, y_1) হলো বা~~ (x', y') এবং ϕ কোনো কোণে ঘূর্ণনের ফলে যেই নতুন অক্ষ হবে তার জন্য Prime System টি হবে, (x', y')

∴ (x', y') এর স্থানাঙ্ক হলো,

$$(x', y') = (x\cos\phi + y\sin\phi,\ x\sin\phi + y\cos\phi) \quad \text{①}$$

যদি (x', y') কে vector আকারে লেখা হয় তাহলে

$$\left.\begin{aligned} A_{x'} &= A_x\cos\phi + A_y\sin\phi \\ A_{y'} &= -A_x\sin\phi + A_y\cos\phi \end{aligned}\right\} \quad \text{②}$$

আমরা এখন হতে,

$x \to x_1,\ y \to x_2,\ z \to x_3$ হিসাবে লিখবো

কারণ 2 dimention math এ x, y কে x_1, x_2 লেখা বলে ও n তম dimention এ তা লেখা কঠিন তাই x_1, x_2 আকারে লেখবো. ~~ধরি~~,

$$\therefore \left.\begin{aligned} x &\to x_1 \\ y &\to x_2 \end{aligned}\right\} \quad \text{③}$$

আবার,

~~angle~~ angular, ~~a_{11}~~ এর জন্য a ব্যবহার করবো

② এর simplified মান

যেমন,

$$\left.\begin{aligned} a_{11} &= \cos\phi, & a_{12} &= \sin\phi \\ a_{21} &= -\sin\phi, & a_{22} &= \cos\phi \end{aligned}\right\} \quad \text{④}$$

(1) নং সমীকরণে (3,4) এর মান গুলো বসিয়ে পাই →

$$\left.\begin{aligned} x_1' &= x_1 a_{11} + x_2 a_{12} \\ x_2' &= x_1 a_{21} + x_2 a_{22} \end{aligned}\right\} \quad (5)$$

The coefficient a_{ij} may be interpreted as a direction the cosine of the angle between (x_i') & (x_j) that is,

অর্থাৎ a_{ij} এখানে i ও j এর মান এর cosine হবে, অক্ষ সমূহের মধ্যবর্তী কোণ, যেমন

যদি $i=1$, $j=2$ হলে, x_1' ও $x_2 = X'$ ও Y এর মধ্যবর্তী কোণ $(\pi/2 - \phi)$ এর cosine হবে অর্থাৎ $\cos(\pi/2 - \phi) = \sin\phi$ হবে অর্থাৎ $(a_{12} = \sin\phi)$

আবার, $i = j$ হলে, $i=1$, $j=1$ হলে, $i=2$, $j=2$

$a_{ij} = a_{11}$	$a_{ij} = a_{22}$
x_i' ও $x_j = x_1'$ ও x_1	x_2' ও x_2
⇓	⇓
X' ও X	Y' ও Y এর মধ্যবর্তী কোণ
এর মধ্যবর্তী কোণ,	
$a_{11} = \cos\phi$	$a_{22} = \cos\phi$

∴ যদি, $i=2$, $j=1$ হয় তাহলে - $a_{ij} = a_{21}$ হবে এবং ~~m_2~~ x'_i ও x_j এর মধ্যবর্তী কোণ x'_2 ও x_1 অর্থাৎ Y' ও X ~~এর মধ্যবর্তী~~ ~~অক্ষের মধ্যবর্তী~~ কোণ $(\pi/2+\phi)$ এর cosine, অর্থাৎ, $a_{ij} = a_{21} = \cos(\pi/2+\phi)$

$$\Rightarrow a_{21} = -\sin\phi$$

∴ দেখা গেলো যে,

a_{ij} এর জন্য x'_i ও x_j এর মধ্যবর্তী কোণের cosine ই হবে a_{ij} এর মান। ——— (6)

\# The advantage of new rotation is that it permits us to use the summation symbol & to rewrite equation (5) i.e.,

$$x'_i = \sum_{j=1}^{2} a_{ij} x_j \, ; \quad i = 1, 2 \qquad (7)$$

অর্থাৎ (5) নং equation কে সংক্ষেপে আমরা উক্ত (7) নং equation আকারে লিখতে পারি।

$i=1$, ~~$j=1$~~ হলে,

$$x'_1 = a_{11}x_1 + a_{12}x_2$$

$i=2$, হলে,

$$x'_2 = a_{21}x_1 + a_{22}x_2$$

(7a)

\# The set of (N) quantities V_j is said to be the components of (N) dimentional vector $\vec{V}$ is & if their values relative to the rotate co-ordinate axis are given by,

$$V_i' = \sum_{j=1}^{N} a_{ij} V_j \quad i = 1, 2, \ldots N \quad \text{(8)}$$

N তম dimention এর জন্য co-ordinate এর vector equation formula

~~From equation (7),~~

\# Equation 7 কে (∂ / ডেল) দ্বারা অন্তরীকরণ করে পাই

$$\partial x_i' = a_{ij}\, \partial x_j$$

$$\Rightarrow a_{ij} = \frac{\partial x_i'}{\partial x_j} \quad \text{(9a)}$$

\# Using the inverse rotation অর্থাৎ ϕ কোণের দিকে না গিয়ে যদি ($-\phi$) হয় তাহলে x_j এর জায়গায় x_i' এবং x_i' এর জায়গায় x_j হবে

$$\therefore x_j = \sum_{i=1}^{2} a_{ij} x_i' \quad \text{(7a)}$$

(7a) equation কে প্রতি Persial অন্তরীকরণ করে পাই

$$\partial x_j = a_{ij}\, \partial x'_i$$

$$\Rightarrow a_{ij} = \frac{\partial x_j}{\partial x'_i} \quad \text{--- (9b)}$$

∴ equation (8, 9a, 9b) তে a_{ij} এর মান বসিয়ে,

$$\cancel{V'_i = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j}} \qquad V'_i = \sum_{j=1}^{N} a_{ij} V_j \quad ; \; i = 1, 2 \ldots N$$

$$\therefore V'_i = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x'_i} V_j$$

The direction cosine (a_{ij}) satisfy as orthogonal condition,

$$\sum_i a_{ij} \cdot a_{ik} = \delta_{jk} \quad \text{--- (11)}$$

& equivalently

$$\sum_i a_{ji}\, a_{ki} = \delta_{jk} \quad \text{--- (12)}$$

Here the Symbol δ_{jk} is the Kronecker delta deffined by

$$\left.\begin{array}{ll} \delta_{jk} = 1 & j = k \\ \delta_{jk} = 0 & j \neq k \end{array}\right\} \rightarrow (18)$$

To verify equation (17) we can use (9a) & (9b)

$$\sum_i \cancel{a_{ij}\, a_{ik}}$$

$$= \sum \cancel{\frac{\partial x_i'}{\partial x_j} \cdot \frac{\partial x_k}{\partial x_i'}}$$

$$\sum_i \frac{\partial x_j}{\partial x_i'} \cdot \frac{\partial x_k}{\partial x_i'} = \sum_i \frac{\partial x_j}{\partial x_i'} \cdot \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k}$$

$$= \delta_{jk}$$

Orthogonal condition; হলো ~~একটি~~ 3-dimentional ২৩ যেমন দু অক্ষগুলো 90° কোণে অবস্থান করবে। Cartesian (x,y,z), Spherical polar (r, θ, ϕ), cylenderical (ρ, ϕ, z) এরা orthogal condition মেনে চলে।

Defination // for tensor.

Scalar // A quantity that did not change under rotation of the co-ordinate system in 3-dimentional space & in variant is labelled is a scalar.

* অর্থাৎ অক্ষকে ϕ কোণে পরিবর্তন করার পরও যদি কোনো রাশির স্থানাঙ্কের পরিবর্তন না হয়, তাহলে সেটি স্কেলার রাশি

A scalar is spacific by one real number & is a tensor rank 0.

* অর্থাৎ স্কেলার রাশির শুধু মাত্র একটি বাস্তব মান আছে যার কারনে এর টেন্সর র‍্যাঙ্ক শূন্য।

vector : A quantity whose ~~compony~~ component transformed under rotation of the co-ordinate system is called A vector.

In 3 dimentional space, A vector is specified by $\underbrace{3}_{\text{number of component or Real number}} = \underbrace{3}_{\text{Dimention}}^{1 \rightarrow \text{tensor rank}}$

অর্থাৎ যদি কোনো অক্ষের ঘূর্ণনের ফলে কোনো রাশির স্থানাঙ্কের পরিবর্তন হয় তাহলে সেই রাশিটি একটি vector রাশি।

ত্রি-মাত্রিক অক্ষের জন্য ভেক্টরের ভেক্টরের রাশির সংখ্যা

$3 = \underbrace{3}_{\text{dimention}}^{1 \rightarrow \text{power, হলো Tensor Rank}}$

Ex// & question//

If Cartesian Components has 9 Real number then what is the Rank of tensor is?

Ans// orthogonal condition, অর্থাৎ Cartesian, spherical polar, cylendarical এর জন্য dimention 3

$\therefore \underbrace{9}_{\text{Real number}} = \underbrace{3}_{\text{dimention}}^{2}$ অর্থাৎ power (2) হলো tensor Rank.

∴ A tensor of rank n has 3^n components or real number for 3-dimentional space.

☆ ∴ In N dimentional space a tensor of rank has $= N^n$ components or real number.

most important

∴ যদি $x' = x\cos\theta + y\sin\theta$ হয়, একে এভাবে $x'_1 = x_1 a_{11} + x_2 a_{12}$ লেখা যায়. এবং এভাবেও, $x'_i = \sum_{j=1}^{2} a_{ij} x_j$; $i = 1, 2$

∴ ij এর যেকোনো মানের জন্য, $\left. x'_i = a_{ij} x_j \right]$ লেখা যায় (1)

~~একটি vector এর ক্ষেত্রে,~~ x_i, x_j অক্ষ সাপেক্ষে A_i, A_j ~~রাশি~~ এবং x'_i, x'_j অক্ষ সাপেক্ষে A'_i, A'_j রাশি, তাহলে আমরা, 1 নং সমীকরণকে আংশিক অন্তরীকরণ করলে, $a_{ij} = \dfrac{\partial x'_i}{\partial x_j}$ লেখা যায়

∴ রূপান্তর সমীকরণটি হবে, ~~যদি দুই~~ A'_{ij} হতো তাহলে tensor rank 2 হতো,

$$A'_i = \frac{\partial x'_i}{\partial x_j} A_j$$

একটি component তাই এর rank (1)

☆ most most important for exam

☆ টেনসর কাকে বলে?

⇒ ভেক্টর বিশ্লেষণ (vector analysis) কে সাধারণীকরণই হলো, tensor বা tensor analysis ~~বলে~~।

Defination of tensor Rank 1

☆ প্রতিচল (ভেক্টর/টেনসর) বা Contravarient vector

Any set of quantities A^j transforming according to $\boxed{A'^i = \frac{\partial x'^i}{\partial x^j} A^j}$ is define as contravariant (vector/tensor).

Here, A^j is called a component of ~~contra~~ contravarient (vector/tensor).

এখানে $(x_1, x_2, \ldots x_n)$ স্থানাঙ্ক ব্যবস্থায় n সংখ্যক রাশি $(A^1, A^2 \ldots A^n)$ এবং $(x'_1, x'_2, \ldots x'_n)$ স্থানাঙ্ক ব্যবস্থায় n সংখ্যক রাশি $(A'^1, A'^2 \ldots A'^n)$ এর তাহলে রূপান্তরিত সমীকরণ হবে,

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

সহচল (ভেক্টর/টেনসর) or Covariant (vector/tensor)

Any set of quantities A_j transformed according to

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j$$

is define as covarient (vector/tensor).

এখানে $(x_1, x_2 \ldots x_n)$ স্থানাঙ্ক ব্যবস্থায় n সংখ্যক রাশি A_j অর্থাৎ $(A_1, A_2 \ldots A_n)$ এবং $(x'_1, x'_2 \ldots x'_n)$ স্থানাঙ্ক ব্যবস্থার জন্য n সংখ্যক রাশি A'_i $(A'_1, A'_2 \ldots A'_n)$ হলে রূপান্তরিত সমীকরণ, $A'_i = \frac{\partial x^j}{\partial x'^i} A_j$.

এখানে A_j হলো covarient (vector/tensor) এর উপাংশ or component..

Contravarient vector	Co varient vector
$\vec{r}$ হলো একটি vector রাশি, $\vec{dr}$ অর্থাৎ position vector এর change কে contravarient vector দ্বারা প্রকাশ করা হয়।	ϕ হলো একটি scalar component। $\vec{\nabla}\phi$ হলো scalar component এর গ্রেডিয়েন্ট ব্যবহার না হলো vector component যাকে co varient vector দ্বারা প্রকাশ করা হয়

☆ The gradient of scalar field is covariant vector

Defination of Tensor rank ~~2~~ 2

Tensor rank 2 Orthogonal condition মানলেও tensor rank 2 এর ক্ষেত্রে তা মানে না। যদি মানে ~~২~~ তাহলে contravariant ও covariant সমান হয়ে যাবে।

∴ Contravariant Tensor rank 2

~~Any set of quantities quantities A^{kl} transformed~~

Any set of quantities A^{kl} transformed according to

$$\boxed{A'^{ij} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{l}} A^{kl}}$$

~~is~~ is define as contravariant tensor rank ^{2}

Here, A^{kl} is called ~~a~~ component of contravariant tensor rank ^{2}.

এখানে A^{kl} হল তাই tensor ~~rank~~ rank 2

Covariant Tensor rank 2 :

Any set of quantities A_{kl} transformed according to $A'_{ij} = \frac{\partial x}{\partial x'_i} \frac{\partial x}{\partial x'_j} A_{kl}$

$$A'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}$$

is define as covariant Tensor rank 2,

Here A_{kl} is called a component of covariant tensor rank 2.

TAB: যদি Real tensor rank 2 এর orthogonal condition মেনে চলতে থাকে তাহলে, contravariant এর

$$\frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} = \text{covariant এর } \frac{\partial x^k}{\partial x'^i} \cdot \frac{\partial x^l}{\partial x'^j} \text{ হবে}$$

তবে covariant = contravariant হবে
Same

Mixed Tensor rank 2 :

Any set of quantities A^k_l transformed according to, $\boxed{A'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} A^k_l}$ is

define as mixed tensor rank 2.

Here, A^k_l is called a component of mixed tensor rank 2.

Addition and Subtraction of tensor

The addition and subtraction of tensores is defind in terms of individual elements, if

$\vec{A} + \vec{B} = \vec{C}$ then $A^{ij} + B^{ij} = C^{ij}$ | $A_{ij} + B_{ij} = C_{ij}$ |

of course A and B must be tensor of the same rank and both expresed in space of the same number of the dimention.

বাংলা। টেনসরের যোগ বিয়োগ পৃথক উপাদানে শর্তাবলী দ্বারা সংজ্ঞায়িত করা হয়। যদি $\vec{A}+\vec{B}=\vec{C}$ হয় তাহলে টেনসরের জন্য A যদি contravariant হয় তাহলে $A^{ij}+B^{ij}=C^{ij}$ এবং covariant হলে $A_{ij}+B_{ij}=C_{ij}$ হবে।

* শর্ত :- A এবং B উভয়ের এর টেনসর rank k সমান হতে হবে এদের যোগফলের ~~টেনসর~~ ভেক্টরের টেনসর পৃথক সমান হবে। এবং এদের স্থানাঙ্ক ব্যবস্থা একই হতে হবে।

Einstein Summation convention

This convention is simply that where the same index appears in any term as a subscript and superscript. This term stands for the sum of all terms obtaind by giving that index all the values it may take

বাংলা। আইনস্টাইনের যোজন রীতি অনুসারে কোনো রাশিমালায় একই (সূচক/ index) যদি subscript এবং superscript আকারে লেখা থাকে তাহলে তাকে সংক্ষেপে আকারে ($a_i x^i$) আকারে লেখা যায়।

যেমন: $a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots a_n x^n = \sum_{i=1}^{n} a_i x^i$

এই ~~সাধারণ~~ সমষ্টির প্রতীক $\sum_{i=1}^{n} a_i x^i$ কে আইনস্টাইনের মতে শুধু $a_i x^i$ আকারে লেখা যায় যেমন $i = 1, 2, 3, \dots, n$ পর্যন্ত মান গ্রহণ করে।

mixed tensor rank 2 তে einstein summation convention

$$B'^i_j = \sum_{\underbrace{k,l}_{\text{Unprime system}}} \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial x'^j} B^k_l$$ এটি

Summation equation কে einstein এর যোজনরীতি অনুসারে, $B'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} B^k_l$ লেখা যায়

Dummy index/ কোনো একটা (index/সূচক) যদি একটি term(পদে) এই দুইবার repeate হয় একবার subscript এবং আরেক বার superscript হিসাবে তা সেই (index/সূচক) কে dummy index বলে।

Ex// $a^i_j x^j$ এখানে (j index) একবার sub এবং আরেক বার sup আকারে আছে তাই j হলো dummy index। dummy index কে অন্য index দ্বারা replace করা যায়। যেমন $a^i_j x^j$ কে $a^i_k x^k$ আকারে লেখা যায়।

~~এখানে j = k হলো।~~

কারণ, $a^j_1 x^1 + a^j_2 x^2 + a^j_3 x^3$ এখানে dummy index repeat হয় তাই একে আলাদা index আকারেও লেখা যায়।

dummy index এ

$$\boxed{a^j_k x^k} = a^j_l x^l$$ হবে

$\boxed{\text{index} = \text{Suffix}}$

Free index: কোনো একটা (index / সূচক) যদি কোনো term এ repeat না হয় অর্থাৎ একটির অধিক তাকে Free index বলে এবং এটি replaceable না। Ex: $a^i_j x^j$, এখানে i হলো Free index

Free index: $\left[a^i_j x^j \neq a^k_j x^j\right]$

Important:

Kronecker Delta (δ^i_j)

In mathematics, the kronecker delta (named after Leopold Kronecker) is a function of two variable.

(i) independent variable

(ii) Dependent variable.

আবার যদি,

$i = j$, হয় তাহলে $\delta^i_j = 1$ হবে

তবে $i \neq j$, হয় তাহলে $\delta^i_j = 0$ হবে।

যেমন $\frac{dx}{dy} = 1$, $x = y$ হয়

$\frac{dx}{dy} = 0$, $x \neq y$ হয়

$\frac{\partial x^i}{\partial x^j} = \delta^i_j$ লেখা যায়। আবার $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$

যেমন, $\delta^1_1 = 1$, $\delta^2_2 = 1$ but $\delta^2_1 = 0$, $\delta^1_2 = 0$

আবার, Proved that, $\delta^i_i = n$

$L.H.S = \delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots + \delta^n_n$

$= 1 + 1 + 1 + \cdots + 1$

এখন n তম বার যোগফল

$= n = R.H.S$

Question

Prove, $\delta^k{}_j$ is really a mixed tensor of rank 2. (Index 2টা থাকলে tensor rank 2 হবে)

আমরা জানি formula of mixed tensor rank 2 হল

$$\underbrace{A'^{i}{}_{j}}_{\text{prime system}} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} \underbrace{A^{k}{}_{l}}_{\text{unprime system}}$$

∴ আমাদেরকে ডান পক্ষ হতে বাম পক্ষ প্রমাণ করতে হবে

Ans://

$$= \delta^{k}{}_{l} \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}}$$

$$= \delta^{k}{}_{k} \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{k}}{\partial x'^{j}}$$

[ধরি, unprime system এ k= index, k=l]

$$= 1 \cdot \frac{\partial x^{k}}{\partial x^{k}} \cdot \frac{\partial x'^{i}}{\partial x'^{j}} = \frac{\partial x'^{i}}{\partial x'^{j}} = \delta'^{i}{}_{j}$$

However, x'^i and x'^j are independent co-ordinates and therefore the variation of one with respect to other must be zero $\left(\frac{\partial x'^i}{\partial x'^j} = \delta'^i_j = 0 \;;\; i \neq j\right)$ if they are different, unity if the coincide that is $\left(\frac{\partial x'^i}{\partial x'^j} = \delta'^i_j = 1 \;;\; i = j\right)$

বাংলা: যদি x'^i এবং x'^j দুটি স্বাধীন স্থানাঙ্ক হয় তাহলে একের ~~পরিবর্তনের~~ প্রতি অপরের সাপেক্ষে ~~সাপেক্ষ~~ পরিবর্তনের হার $\left(\frac{\partial x'^i}{\partial x'^j}\right.$ = শূন্য হবে $\left. i \neq j\right)$
কিন্তু যদি অধীন স্থানাঙ্ক হয় তাহলে, $\left[\frac{\partial x'^i}{\partial x'^j} = 1\right.$ হবে $\left.;\; i = j\right]$

$$\therefore \frac{\partial x'^i}{\partial x'^j} = \delta^i_j$$

তাহলে, $\delta'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} \delta^{k}_{l}$

So, it's proved that, δ^{k}_{j} is a mixed tensor of rank 2.

Symmetric tensor / প্রতিসম টেন্সর // কোনো টেন্সর এর উপাংশ গুলোতে দুটি উর্ধ্ব সূচক বা দুটি নিম্ন সূচক এর পরস্পর স্থান বিনিময় করলে যদি উপাংশ গুলোর মান অপরিবর্তিত থাকে, তবে ঐ সূচকদ্বয়ের সাপেক্ষে উক্ত টেন্সর কে প্রতিসম টেন্সর বলে।

$A^{ij} = A^{ji}$ বা $A_{ij} = A_{ji}$

(Skew / anty) Symmetric / অপ্রতিসম টেন্সর // কোনো ~~টেন্সরের~~ ~~টেন্সোর~~ টেন্সরের উপাংশ গুলোতে উর্ধ্ব সূচক বা নিম্ন সূচক পরস্পর স্থান বিনিময় করলে যদি উপাংশ গুলোর মান অপরিবর্তিত এবং ঋণাত্মক হয় তাহলে উক্ত টেন্সরকে ঐ সূচকদ্বয়ের সাপেক্ষে উক্ত টেন্সরকে ~~টেন্সরকে~~ অপ্রতিসম টেন্সর বলে। $A^{ij} = -A^{ji} / A_{ij} = A_{ji}$

Spinors :

α- Particle → spin 0 ;
deuterons particle → spin 1
grabiton particle → spin 2

electron → ~~spin~~ spin 1/2

* spin = 0 হলে তার tensor rank = 0 হবে, scalar
* spin = 1 " " " " = 1 হবে ~~এবং~~ ~~vector হবে~~

* ও যুক্তরাং বলা যেতে পারে,
spin = 0 হলে, তাকে scalar দ্বারা define করা হয়
spin = 1 " " vector "

$ii_A = \dot{i}{}^{i}A = {}^{ii}{}_A$

But, spin = 1/2 হলে তাকে spinor বলা হয়

সুতরাং, "electron এর spin যেহেতু 1/2
~~যেহেতু তাকে spinor দ্বারা প্রকাশ করা হয়~~ "

যার spin 1/2 তাকে ~~sp~~ spinors দ্বারা define করা হয়।

contractions of tensors | টেন্সরের সংকোচন

কোনো টেন্সরের নিম্ন সূচক ও অন্তত একটি ঊর্ধ্ব সূচক সমান করে যোজন রীতি রীতি অনুসারে যোজন করে দেখা যায়, প্রাপ্ত ফলাফল একটি টেন্সর যার ~~র‍্যাঙ্ক~~ র‍্যাঙ্ক র‍্যাঙ্ক প্রদত্ত টেন্সরের র‍্যাঙ্ক অপেক্ষা 2 কম, এই প্রক্রিয়াটিকে টেন্সরের সংকোচন বলে।

contraction of tensor এ মূলত tensor এর rank (2) করে কমিয়ে দেয়।

Ex: A^{ijk}_{lm} এখানে উপরের ও নিচে (3, 2) টা index আছে তাই একে (3, 2) type বলা হয়, $(3+2) = 5$ তাই এই tensor এর rank 5 বলা হয়।

1 বার contract করি তাহলে tensor এর rank $(5-2) = 3$ হবে।

2 বার contract করলে tensor এর rank $\{5-(2\times 2)\} = 1$ হবে।

* মূলত contravariant ও covariant এর (1, 1) টি index সমান ~~করলেই~~ contraction ~~করা যায়~~।

$\left(\frac{m}{a}\right)^2 - 2\frac{m}{a}\frac{n}{b}\cos\delta + \left(\frac{n}{b}\right)^2$

$\left(\frac{m}{a}\right)^2 - 2\frac{m}{a}\frac{y}{b}\cos\delta + \left(\frac{n}{b}\right)^2 = \sin^2\delta$

Low

A'

definition// If we set in a tensor one covariant & one contravariant index equal, the process is called contraction.

প্রক্রিয়ার tensor এর (contraction / সংকোচন) rank (2) করে কমাতে থাকে, করলে tensor এর rank

ঃ যদি tensor, A^{ijk}_{lm} কে contraction (l) করি তাহলে new tensor হবে, B^{jk}_{m} [if $i = l$]

Question// Show that, in the process of contraction the rank of tensor is reduced by two.

Solv// Let, A^{ijk}_{lm} & A'^{ijk}_{lm} are components (উপাদান) of a tensor rank 5 & type (3,2). In x^i ~~...~~ & x'^i ~~co-ordi~~ co-ordinate system respectively.

Law of transformation | contraction's of tensor rank 5

$$A'^{ijk}_{lm} = \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^s}{\partial x'^l} \frac{\partial x^t}{\partial x'^m} A^{pqr}_{st}$$

Prime system — contravariant index — unprime system

Let, $i = l$ → covariant index

$$\Rightarrow A'^{ijk}_{im} = \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^s}{\partial x'^i} \frac{\partial x^t}{\partial x'^m} A^{pqr}_{st}$$

$$\Rightarrow A'^{ijk}_{im} = A^{pqr}_{st} \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^t}{\partial x'^m} \cdot \delta^s_p$$

$\delta^s_p = \frac{\partial x^s}{\partial x^p}$

$s = p$ হলে $\delta^s_p = 1$

kronecker delta

$$\Rightarrow A'^{ijk}_{im} = A^{pqr}_{pt} \cdot \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^t}{\partial x'^m} \cdot \delta^s_p$$

$\delta^s_p = 1$ when $s = p$

$$\Rightarrow B'^{jk}_{m} = B^{qr}_{t} \cdot \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^t}{\partial x'^m}$$

$$\therefore \boxed{B'^{jk}_{m} = \frac{\partial x'^j}{\partial x^q} \frac{\partial x'^k}{\partial x^r} \frac{\partial x^t}{\partial x'^m} B^{qr}_{t}} \quad (i)$$

$\therefore$ (i) নং equation এর type(2, 1)

এবং rank (2+1) = 3

যেহেতু contraction করার সময় tensor

rank 5 এবং contraction করার পর

tensor rank 3 হয়েছে তাই এ

প্রমানিত হলো যে, contraction করলে tensor rank 2 করে

~~কমে~~ reduced হয়।

Outer product | tensor গুনন

~~contractions of tensor~~

The outer product of two tensors is a tensor whose order (rank) is the sum of the ranks (orders) of the two tensors.

অর্থাৎ দুটি tensor এর গুনন করলে, নতুন tensor এর rank পূর্বের দুই tensor rank এর সমষ্টি হবে।

Ex// $(A^{ij} \times B_k)$ করলে, নতুন tensor এর rank হবে, $(2+1) = 3$

Question / If A^{ij} is a contravariant tensor and B_k is a covariant (vector | tensor) (rank-1 হলেই tensor কে vector বলা হয়)

Show that, $A^{ij}B_k$ is a tensor rank 3.

Soln

$$A'^{ij} = \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial x'^{j}}{\partial x^{n}} A^{mn} \quad \text{---(i)}$$

$$B'_{k} = \frac{\partial x^{p}}{\partial x'^{k}} B_{p} \quad \text{---(ii)}$$

(i)×(ii) or order product করলে,

$$A'^{ij} \cdot B'_{k} = \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial x'^{j}}{\partial x^{n}} \frac{\partial x^{p}}{\partial x'^{k}} A^{mn} B_{p} \quad \text{---(iii)}$$

∴ উপরের (iii) নং equation এ দেখা যাচ্ছে যে ~~টি~~ persial derivative এর index সংখ্যা ৩ টি অর্থাৎ ~~rank~~ tensor rank 3

∴ প্রমানিত হলো যে, 2 ~~tem~~ tensor এর order product করলে বা (multiple) করলে, tensor এর rank ~~কমতে থাকে~~ হয়, এবং এদের rank এর সমষ্টি হলো New tensor এর rank contraction of tensor rank 2—।

$$B^{i}_{j} \to B'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} B^{k}_{l}$$

∴ ~~contra~~ super ~~sub~~ উপাদানের সূচক সংকোচন বিধায় সমূহ সমান হলে, ~~i=j~~ $i=j$

$$B'^{i}_{i} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{i}} B^{k}_{l}$$

$$\Rightarrow B'^{i}_{i} = \frac{\partial x^{l}}{\partial x^{k}} B^{k}_{l}$$

$$\left[\delta^{l}_{k} = \frac{\partial x^{l}}{\partial x^{k}} \right.$$

$$\Rightarrow B'^{i}_{i} = \delta^{l}_{k} B^{k}_{l}$$

$\delta^{l}_{k} = 1$, when $l=k$

$\delta^{l}_{k} = 0$, when $l \neq k$

$$\Rightarrow B'^{i}_{i} = B^{k}_{k} \quad [l=k]$$

এখানে, tensor দুবার এলো partial derivative নাই অর্থাৎ partial derivative = zero

তাই tensor rank ও zero

∴ ২টিকে rank-2 এর tensor কে contraction করলে নতুন rank $(2-2)=0$ হয় অর্থাৎ একটি scalar হয়।

~~The Transformation of co~~

Covariant tensor transformation Law

MATH

(Q) Show that, the law of transformation of a covariant tensor of second order (A_{ij}) possess the "group property".

অর্থাৎ প্রমাণ করতে হবে (A_{ij}) বা Covariant tensor rank 2 কে দুই বার ধাপে ধাপে ~~transformed~~ স্থানাঙ্কের উপাদানের পরিবর্তন করা directly পরিবর্তন করার সমান।

~~ধরি,~~ $x^i \to x'^i \to x''^i$ তিনটি transformed co-ordinate

এবং এদের টেনসরের উপাদান হলো $A_{ij} \to A'_{ij} \to A''_{ij}$

$\therefore$ ~~যদি x^i coordinate $\to x'^i$ cordinat g tensor transformed~~ করলে,

$$\cancel{A'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} A_{km}}$$

অনুরূপ, $x'^i \to x''^i$ coordinate এ transformed হলে

$$A''_{ij} = \frac{\partial x'^k}{\partial x''^i} \frac{\partial x'^l}{\partial x''^j} A'_{kl} \quad \text{--- (ii)}$$

(i) ও (ii) নং equation কে combined ~~কে~~ করলে,

~~$A''_{ij} \cdot A'_{ij} =$~~

$$A_{ij} \to A'_{ij} \to A''_{ij}$$

In case of $x^i \to x'^i$

$$A'_{lk} = \frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k} A_{mn} \quad \text{--- (i)}$$

In case of $x'^i \to x''^i$

$$A''_{ij} = \frac{\partial x'^l}{\partial x''^i} \frac{\partial x'^k}{\partial x''^j} A'_{lk} \quad \text{--- (ii)}$$

(ii) নং সমীকরণে A'_{lk} এর মান বসিয়ে,

$$A''_{ij} = \frac{\partial x'^l}{\partial x''^i} \frac{\partial x'^k}{\partial x''^j} \frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k} A_{mn}$$

$$\Rightarrow \boxed{A''_{ij} = \frac{\partial x^m}{\partial x''^i} \frac{\partial x^n}{\partial x''^j} A_{mn}}$$

সুতরাং দেখা যাচ্ছে যে ~~এর~~ transformation of covariant tensor rank 2 তে unprime to prime to double prime co-ordinate system এ গেলে ~~একই~~ যা হতে unprime to double prime co-ordinate system এ গেলেও একই equation পাওয়া যাবে।

type of tensor rank 2

contravariant tensor

$$A'^{ij} = \frac{\partial x'^{i}}{\partial x^{a}} \frac{\partial x'^{j}}{\partial x^{b}} A^{ab}$$ → second order type $(2,0)$

উপরের index সংখ্যা | নিচের index সংখ্যা

covariant tensor

$$A'_{ij} = \frac{\partial x^{a}}{\partial x'^{i}} \frac{\partial x^{b}}{\partial x'^{j}} A_{ab}$$ → type $(0,2)$

rank $(0+2) = 2$

mixed tensor

$$A'^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{a}} \frac{\partial x^{b}}{\partial x'^{j}} A^{a}{}_{b}$$ → type $(1,1)$

tensor rank $(1+1) = 2$

$$\boxed{\text{index সংখ্যা} = \text{tensor rank}}$$

Transformation law of mixed tensor

MATH

(2) Show that, a mixed tensor of second order "passess group property."

Solve// Let, A^{i}_{j}, A'^{i}_{j} & A''^{i}_{j} are be the component's of a mixed tensor of second order in $\{x^{i}, x'^{i}, x''^{i}\}$ co-ordinates system respectively

Cosider the co-ordinat transformation.

$x^{i} \to x'^{i} \to x''^{i}$ & their respectively component

$A^{i}_{j} \to A'^{i}_{j} \to A''^{i}_{j}$

∴ In case of,

$x^{i} \to x'^{i}$

$$A'^{l}_{k} = \frac{\partial x'^{l}}{\partial x^{m}} \frac{\partial x^{n}}{\partial x'^{k}} A^{m}_{n} \quad \text{—— (i)}$$

In case of,

$x'^i \to x''^i$

$$A''^i_j = \frac{\partial x''^i}{\partial x'^k} \frac{\partial x'^l}{\partial x''^j} A'^k_l \quad \text{—(ii)}$$

(ii) নং ও (i) এর মান বসিয়ে পাই,

$$A''^i_j = \frac{\partial x''^i}{\cancel{\partial x'^l}} \frac{\partial x^k}{\partial x''^j} \frac{\cancel{\partial x'^l}}{\partial x^m} \frac{\cancel{\partial x^k}}{\cancel{\partial x'^k}} A^m_n$$

$$\Rightarrow \boxed{A''^i_j = \frac{\partial x''^i}{\partial x^m} \frac{\partial x^n}{\partial x''^j} A^m_n}$$

∴ এটি দ্বারা প্রমাণিত হয় যে,

$x^i \to x''^i$ co-ordinate system এ একসাথে directly mixed tensor লেখা যায়,

এই process কে, (1 to 2 to 3) বা or (1 to 3)

গ্রুপ কে প্রমাণ করাই হলো (group property)

Direct Product | The components of a covariant vector a_i & those of a contravariant vector b^j maybe multiplied component by component to give the general term $a_i b^j$. This is accordingly a second rank Tensor.

$$Q'^{ij} = a'_i b'^j = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} a_k b^l$$

Contraction করলে rank 2 এর tensor এর rank 0 হবে।

কারন, index $i=j$ হলে ∂x^i ও $\partial x'^j$ একই হবে এবং $\frac{\partial x^k}{\partial x^l}$ = ~~1 হবে যদি~~

$\frac{\partial x^k}{\partial x^l} = \delta^k_l = 1$ হবে যদি $k=l$ হয়।

$\therefore$ দেখা যায়, $a'_i b'^j = a_k b^l$।

The operation of ~~adding~~ adjoining two vector a_i & b^j is known as forming the direct product. For the case of two vectors, the direct product is a tensor of second rank

direct product এর tensor rank product ~~গুলির~~ index এর
সমষ্টি -এ tensor ~~rank~~ rank এর সমষ্টি হয়ে থাকে,

∴ In general, the direct product of two tensor is a tensor of rank equal to the sum of the two ~~fi~~ initial ranks.

$$A^i_j \, B^{kl} = C^{ikl}_j \text{, where } C^{ikl}_j \text{ is a tensor of forth rank.}$$

$$\therefore \cancel{C'^{ikl}_j = \frac{\partial x'^i}{\partial x^m} \ldots C^{mno}_p}$$

$$\cancel{C'^{ikl}_j = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \ldots C^{mno}_p}$$

$$C'^{ikl}_{j} = \frac{\partial x'^i}{\partial x^p} \cdot \frac{\partial x^q}{\partial x'^j} \cdot \frac{\partial x'^k}{\partial x^r} \frac{\partial x'^l}{\partial x^s} C^{prs}_{q}$$

The direct product is a ~~teenical~~ new, higher tensors.

Quotient Law of Tensors

It states that if the inner product of a set of function with an arbitrary ~~arbitrary~~ tensor which is a tensor, then the set of functions is itself a tensor.

অর্থাৎ, যদি,
A^{ik}_{j} একটি Set of function এবং B^{j}_{k} হলো ~~arbitrary tensor~~ arbitrary tensor.
এবং ~~যদি~~ উহাদের গুনফল $A^{ik}_{j} \cdot B^{j}_{k} = C^{i}$ হলে,
A^{ik}_{j} ~~হবে~~ নিজেই একটি tensor হবে।

এটি হলো quotient law.

$k_i A_i = B$ (Rank 0)

$k_i A_j = B_{ij}$ (Rank 2)

$k_{ij} A_{jk} = B_{ik}$ (Rank 2)

$k_{ijkl} A_{ij} = B_{kl}$ (Rank 2)

$k_{ij} A_k = B_{ijk}$ (Rank 3)

যদি A vector হলো arbitrary tensor হয় এবং K যদি হয় set of function এবং যেকোনো B হয় tensor হয় তাহলে quotient Rule অনুসারে K হবে একটি tensor.

Proof // Let A^{ik}_j be a set of function in x^i co-ordinate & B^j_k is a ~~arbitrary~~ arbitrary tensor. প্রমাণ করতে হবে A^{ik}_j একটি tensor.

$\therefore$ Their multipication will be,

$$A^{ik}_j \cdot B^j_k = C^i$$

Now, in different co-ordinate system $(\bar{x}^i)$

$$\bar{A}^{ik}_j \cdot \bar{B}^j_k = \bar{C}^i \quad \text{——— (i)}$$

$\Rightarrow \left(\frac{x}{a}\right)^2 - 2\frac{x}{a}\frac{y}{b}\cos\delta + \left(\frac{y}{b}\right)^2 = \sin^2\delta$

∴ Law of tensor transformation for equation (i)

$$A'^{ik}_{j} \left\{ B^{l}_{m} \frac{\partial x'^{j}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{k}} \right\} = C^{q} \frac{\partial x'^{i}}{\partial x^{q}}$$

$$\Rightarrow A'^{ik}_{j} \frac{\partial x'^{j}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{k}} \left(\frac{\partial x^{q}}{\partial x'^{i}} \right) B^{l}_{m} = C^{q}$$

$$\Rightarrow \left\{ A'^{ik}_{j} \frac{\partial x'^{j}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{k}} \frac{\partial x^{q}}{\partial x'^{i}} \right\} B^{l}_{m} = C^{q} \quad \text{—— (ii)}$$

আমরা জানি, exam এর সময় মাত্রাযুক্ত লেখন নিয়ম

$$A'^{ik}_{j} = A^{qm}_{l} \cdot \frac{\partial x'^{i}}{\partial x^{q}} \cdot \frac{\partial x'^{k}}{\partial x^{m}} \cdot \frac{\partial x^{l}}{\partial x'^{j}} \quad \text{হবে}$$

$$\Rightarrow A^{qm}_{l} = A'^{ik}_{j} \frac{\partial x'^{j}}{\partial x^{l}} \cdot \frac{\partial x^{m}}{\partial x'^{k}} \cdot \frac{\partial x^{q}}{\partial x'^{i}}$$

∴ A^{qm}_{l} এর মান (ii) নং এ বসিয়ে পাই,

$$\Rightarrow A^{qm}_{l} \cdot B^{l}_{m} = C^{q} \quad \text{—— (iv)}$$

(iv − ii) করে পাই,

$$\Rightarrow \left\{A^{qm}_{l} - A'^{ik}_{j}\frac{\partial x'^{j}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{k}}\frac{\partial x^{q}}{\partial x'^{i}}\right\} B^{l}_{m} = C^{q} - C^{q}$$

$$\Rightarrow \left\{A^{qm}_{l} - A'^{ik}_{j}\frac{\partial x'^{j}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{k}}\frac{\partial x^{q}}{\partial x'^{i}}\right\} B^{l}_{m} = 0$$

যেহেতু B^{l}_{m} একটি arbitrary tensor তাই একে সরানো যায়,

$$\Rightarrow A^{qm}_{l} - A'^{ik}_{j}\frac{\partial x'^{j}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{k}}\frac{\partial x^{q}}{\partial x'^{i}} = \frac{0}{B^{l}_{m}} = 0$$

$$\Rightarrow A'^{ik}_{j}\frac{\partial x'^{j}}{\partial x^{l}}\frac{\partial x^{m}}{\partial x'^{k}}\frac{\partial x^{q}}{\partial x'^{i}} = A^{qm}_{l}$$

$$\cancel{\Rightarrow A'^{ik}_{j} = \frac{\partial x'^{i}}{\partial x^{q}}\frac{\partial x'^{k}}{\partial x^{m}}\frac{\partial x^{l}}{\partial x'^{j}} A^{qm}_{l}}$$

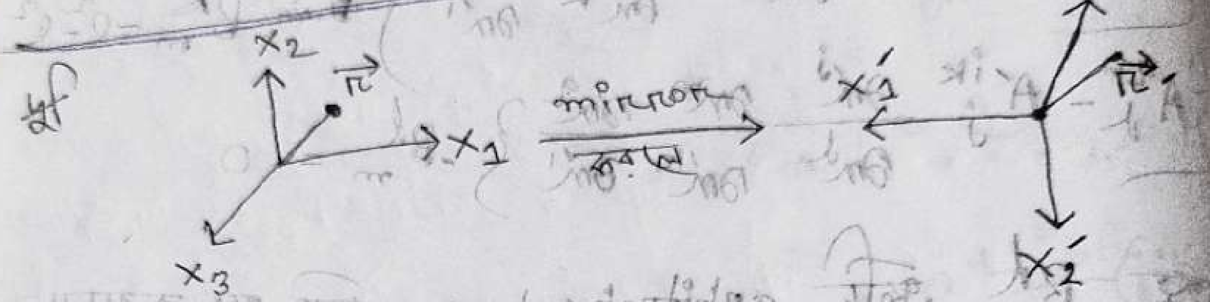
উভয় দিকে Prime চিহ্নগুলো replace করলে পাই

$$\Rightarrow \boxed{A'^{qm}_{l} = A^{ik}_{j}\frac{\partial x^{j}}{\partial x'^{l}}\frac{\partial x'^{m}}{\partial x^{k}}\frac{\partial x'^{q}}{\partial x^{i}}}$$

∴ দেখা যাচ্ছে A^{ik}_{j} tensor এর একটি component। ∴ A^{ik}_{j} হচ্ছে যেকোন arbitrary set of function যদি হিসাব যদি Law of transformation মেনে চলে তবে A^{ik}_{j} একটি tensor এর component.

Pseudo tensor | Mirror tensor

ধরি



∴ অর্থাৎ কোনো অক্ষের সাপেক্ষে একটি অবস্থান ভেক্টর যদি একটি unprime system হতে অক্ষের inverse করে prime system এ নেয়া হয় তাহলে অবস্থান vector ও prime system এ হয় $\vec{r}'$.

এখানে দেখা যাচ্ছে যে, co-ordinat system revers করেছিলে,

x^i ~~coor~~ co-ordinat $= -x'^i$ হয়ে যায়

unprime থেকে prime system এ যায়, but negative হয়ে যায়।

এক্ষেত্রে $a_{ij} = \frac{\partial x'^i}{\partial x_j} = -\delta_{ij}$ হয়ে যায়।

অর্থাৎ,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r}(x, y, z)$

$\Rightarrow \vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$, $\vec{r}(x_1, x_2, x_3)$

$\Rightarrow$ ~~$\vec{r} = x^i$~~, ~~হয়~~ যদি এরগুলো unprime system to prime system এ যায়, এবং tensor ~~যদি~~ ~~contravar~~ contravariant ~~হয়~~ হয় তাহলে, অবস্থান vector হবে

$\vec{r}' = (x'^1, x'^2, x'^3)$] $\rightarrow$ Prime system

$= (-x^1, -x^2, -x^3)$] $\rightarrow$ unprime system

অর্থাৎ Prime to unprime system এর ~~মধ্যে~~ থেকে বা ~~lett~~ ~~of~~ নতুন অক্ষে ভেলে দেখা যায় যে, অবস্থান ভেক্টরের উপাদানগুলো negative হয়ে যায় এমন গুলো কে Polar vector বলে।

অর্থাৎ,

$\vec{r}' = (x'^1, x'^2, x'^3) = (-x^1, -x^2, -x^3)$

This new vector, $\vec{r}'$, has negative components, relative to the new transformed set of axis. The position vector $\vec{r}$ ~~and~~ & all other vector whose components behave this way are called polar vectors and have odd parity.

[Proved]

Let, $\vec{C} = \vec{A} \times \vec{B}$, where both

$\vec{A}$ & $\vec{B}$ are polar vectors

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$= A_{x^1}\hat{i} + A_{x^2}\hat{j} + A_{x^3}\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$= B_{x^1}\hat{i} + B_{x^2}\hat{j} + B_{x^3}\hat{k}$

$$\therefore \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ Ax^1 & Ax^2 & Ax^3 \\ Bx^1 & Bx^2 & Bx^3 \end{vmatrix}$$

$$\Rightarrow \vec{C} = \cancel{\hat{i}(Ax^2Bx^3) - \hat{j}(Bx^3 - Ax^1) + \hat{k}}$$

$$\Rightarrow \vec{C} = \hat{i}(Ax^2Bx^3 - Ax^3Bx^2) - \hat{j}(Bx^3Ax^1 - Bx^1Ax^3) + \hat{k}(Ax^1Bx^2 - Ax^2Bx^1)$$

$$\Rightarrow \vec{C} = C^1\hat{i} + C^2\hat{j} + C^3\hat{k}$$

আমরা ~~ও লিখতে পারি~~ $C^1 = A^2B^3 - A^3B^2$

$C^2 = A^1B^3 - A^3B^1$

$C^3 = A^1B^2 - A^2B^1$

now, when co-ordinate system are ~~& &~~ inverted: $A^1 \to -A'^1$, $B_2 \to -B'_2$

কিন্তু আমাদের C^1 এর component গুলো ~~Prime~~ unprime থেকে আমার invert ~~হওয়ার~~ এর prime system এ

$\therefore C^1 = \{A^2B^3 - A^3B^2\}$

$= +\{(-A'^2 \times -B'^3) - (-A'^3 \times -B'^2)\}$

$= +\{A'^2B'^3 - A'^3B'^2\}$

$= + C'^1$

$\Rightarrow C^1 = + C'^1$

আমরা জানি,

~~Co-ordinate system inverted~~ Amis inverse করে দিলে,

$C^k = -C'^k$ হয় এবং একে Polar vector বলে।

কিন্তু এখানে, Amis inverse করার পরেও co-ordinate এর form change হয় নি.

$\therefore$ বলা যেতে পারে, দুটি Polar vector এর গুণের জন্য সৃষ্ট $(\vec{A} \times \vec{B}) = \vec{C}$, Polar হয়নি বরং তা হলো Psudo vector.

if, ~~[illegible]~~ $+C^k = -C'^k$ হয়, Axis inverse করার পর তাহলে সেটি, C^k হলো Polar vector

if, $+C^k = +C'^k$ হয়, Axis inverse করার পর তাহলে সেটি $+C^k$ হলো Psudo vector.

~~Psudo~~ Pseudo tensorের মধ্যে, axis inverse করা হলেও "Pseudo vector = axial vector"

Levi-Civita or Permutation symbol

→ maximally anti-symmetric tensor in 3-dimentional of value unity.

যদি, ϵ_{ijk} হলে Levi-civita symbol. 3-dimentional

so, $\epsilon_{ijk} = +1$, if (ijk) is an even peremutation of $(1,2,3)$

$= -1$, if (ijk) is an odd peremutation of $(1,2,3)$

$= 0$, other wise or any two index are same.

অর্থাৎ, যদি Levi-civita symbol এর মধ্যে, ϵ_{ijk} ijk এর যে কয়বার স্থান পরিবর্তন করবে, স্থান পরিবর্তন করার সংখ্যা জোড় হলে, ϵ_{ijk} এর value 1 হবে, বিজোড় হলে -1 হবে, আর index যদি দুটো same হয় তাহলে value zero হবে [0 = জোড় সংখ্যা]

$\epsilon_{123} = 1$; $\epsilon_{132} = -1$; $\epsilon_{231} = +1$

একবার স্থান পরিবর্তন — দুই বার স্থান পরিবর্তন হয়েছে।

$\epsilon_{312} = +1$; $\epsilon_{321} = -1$

অর্থাৎ,

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ → জোড় সংখ্যক permutation.

$\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$ → বিজোড় সংখ্যক permutation.

$\epsilon_{113} = 0$ → index দুইটা same তাই Levi-civita symbol value = zero

Line elements / matric tensor

If (S) is define as length. Then the differential of arc length (ds) in rectangular co-ordinates (x, y, z) in obtained form,

$\cancel{ds = dx\hat{i} + dy\hat{j} + dz\hat{k}}$

$$ds^2 = dx^2 + dy^2 + dz^2$$ $\Rightarrow$ दूरी (ds×ds) करने पर आता

By transforming to general curvilinear co-ordinates, this (ds^2) will become

$$ds^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du^p\, du^q$$

Such spaces are called 3-D Euclidean space.

<u>Line element / Matric</u> | A formula which express the distance between adjacent points is called a line elements.

A generalization to (n-dimensional) space with co-ordinate $(x^1, x^2, \dots x^n)$ is immediate. we define the line elements (ds) in this space is given by the quadratic form called the metric form.

$$~~ds~~ \; ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} g_{pq} \, dx^p \, dx^q$$

or using the Einstein summation convention

$$ds^2 = \underbrace{g_{pq}}_{\text{metric tensor}} dx^p dx^q$$

the ~~quat~~ quantities g_{pq} ~~are the~~ are the components of co-variant tensor of ~~rea~~ rank 2 called the Metric tensor.

important basic start

আমরা জানি,

দুটি বিন্দুর মধ্যবর্তী দূরত্ব, (x_1, y_1, z_1) (x_2, y_2, z_2)

$\therefore D = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ কে differential form এ $ds^2 = dx^2+dy^2+dz^2$ → for 3-dimention

$\therefore$ N তম dimention এর জন্য,

~~$ds^2 = g_{ij}\, dx^i\, dx^j$~~ $\boxed{ds^2 = g_{ij}\, dx^i\, dx^j}$ $(i = 1, 2, \ldots n)$ → (1)

eq-1 কে ~~রিমানিয়ান মেট্রিক~~ মেট্রিক বলে। এবং যে space এ একে ব্যক্ত করা হয় তাকে Riemannian space বলে।

$r = r(x^1, x^2, x^3 \ldots x^n)$ → formula

$\Rightarrow \boxed{dr = \frac{\partial r}{\partial x^i} dx^i}$ $i = 1, 2, \ldots n$

$\therefore ds^2 = dr \cdot dr = \frac{\partial r}{\partial x^i} dx^i \cdot \frac{\partial r}{\partial x^j} \cdot dx^j$

$\Rightarrow ds^2 = \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} dx^i dx^j$

$\Rightarrow \boxed{ds^2 = g_{ij}\, dx^i\, dx^j}$

g_{ij} = matric tensor rank 2 $= \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j}$

$\left(\frac{m}{a}\right)^2 - 2\frac{m}{a}\frac{y}{b}\cos\theta + \left(\frac{y}{b}\right)^2$

$\left(\frac{m}{a}\right)^2 - 2\frac{m}{a}\frac{y}{b}\cos\theta + \left(\frac{y}{b}\right)^2 = \sin^2\theta$

Question: The metric tensor (g_{ij}) is co-variant Symmetric tensor rank two. (Prove it)

Solve// By changing the co-ordinate from x^i to x'^i. Let the metric $(g_{ij}\, dx^i\, dx^j)$ be ~~transformed~~ to $(g'_{ij}\, dx'^i\, dx'^j)$

$$\therefore \boxed{ds^2 = g_{ij}\, dx^i\, dx^j = g'_{ij}\, dx'^i\, dx'^j}$$

→ ~~Axis rotation~~ co-ordinate change হলেও distance same থাকে,

$$\therefore g_{ij}\, dx^i\, dx^j = g'_{ij}\, dx'^i\, dx'^j$$

Prime to unprime system এ যাওয়ার formula

$$\Rightarrow g_{ij}\, dx^i\, dx^j = g'_{ij}\, \frac{\partial x'^i}{\partial x^k}\, dx^k\, \frac{\partial x'^j}{\partial x^l}\, dx^l \quad \text{—(2)}$$

$$g_{ij}\, dx^i dx^j = g_{kl}\, dx^k dx^l \quad \text{——} (3)$$

এখানে দুইটিরই same, এক যায়গায় index i ও j এবং অন্যটির ক্ষেত্রে index ~~kl~~ k ও l, distance same

∴ ~~g_{ij}~~ eq (3) তে eq (2) এর মান বসিয়ে,

$$g_{kl}\, dx^k dx^l = g_{ij} \cdot \frac{\partial x'^i}{\partial x^k} dx^k \cdot \frac{\partial x'^j}{\partial x^l} dx^l$$

$$\Rightarrow \left\{ g_{kl} - g'_{ij} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} \right\} dx^k dx^l = 0$$

dx^k ও dx^l arbitrary vector, contravariant ~~তাই এদের মধ্যে যাই~~

$$\Rightarrow g_{kl} - g'_{ij} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} = 0$$

$$\Rightarrow \boxed{g'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}}$$

→ এটি একটি covariant tensor rank-2 এর Law of transformation

এখানে, $g_{ij} = g_{ji}$ হলে এটি symmetric tensor

Chapter - 3

Fourier series

Periodic function / A function $f(x)$ is periodic, if there is a constant (P) such that,

$$f(x+P) = f(x)$$

ex// if, $f(x) = \sin x$ $\therefore f(x+2\pi) = \sin(x+2\pi) = \sin x$

$\therefore f(x)$ is a periodic function.

$\therefore y = a \sin \frac{2\pi}{\lambda} x = f(x)$

$$\Rightarrow f(x+\lambda) = a \sin \frac{2\pi}{\lambda}(x+\lambda) = a \sin\left(\frac{2\pi}{\lambda}x + 2\pi\right)$$

$$\boxed{= a \sin\left(\frac{2\pi}{\lambda}x\right) = f(x)}$$

$\therefore y = a \sin \omega t$

$$f(t) = a \sin \frac{2\pi}{T} t$$

$$\therefore f(t+T) = a\sin\frac{2\pi}{T}\cdot(t+T) = a\sin\left(\frac{2\pi}{T}\cdot t + 2\pi\right)$$

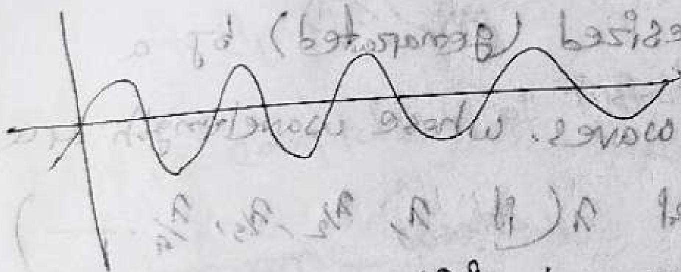
$$= a\sin\frac{2\pi}{T}\cdot t$$

$$= f(t)$$

$\therefore f(t+T) = f(t)$ এখানে T হলো Temporal constant,

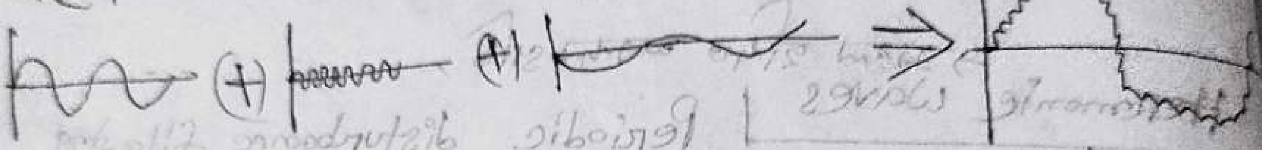
→ সরল ছন্দিত ~~তরঙ্গ~~ গতি,

Hermonic waves | Periodic disturbance fllowing the equation, $F = -kx$



Anhermonic waves | creative by the superposition of many hermonic waves with different Amplitude & different (wave-length | frequency). This is also periodic disturbance.

বাংলা// অসংখ্য সরল ছন্দিত স্পন্দন গতি মিলে যাদের বিস্তার, তরঙ্গ দৈর্ঘ্য, কম্পাঙ্ক ভিন্ন তারা মিলে শুধু মাত্র একটি তরঙ্গ তৈরী করে যাকে Anharmonic wave বলে।



Anharmonic wave

A function $f(x)$ having a specially period of λ can be synthesized (generated) by a number of harmonic waves. whose wavelength are integral submultiples of λ (if $\lambda, \lambda/2, \lambda/3, \lambda/4 \dots$)

অর্থাৎ, $f(x)$ হলো এমন একটি function যার তরঙ্গ দৈর্ঘ্য (λ) অনেক গুলো বিদ্যমান, অর্থাৎ সরল ছন্দিত স্পন্দন গতির অনেক গুলো তরঙ্গ মিলে একটি তরঙ্গ তৈরী করে। সকল তরঙ্গের λ হলো, ($\lambda, \lambda/2, \lambda/3, \lambda/4 \dots$)

আমরা জানি,

∴ এর একটি সরল ছন্দিত তরঙ্গের গতির সমীকরণ, $\phi_p = (kx \pm \omega t + \delta)$ দশাকোণ

$$y = A\cos(kx + \epsilon)$$

$$= A\cos\left(\frac{2\pi}{\lambda}\cdot x + \epsilon\right)$$

$A \rightarrow$ Amplitude

$\frac{2\pi}{\lambda} \rightarrow$ Propagation no. $k = \frac{2\pi}{\lambda}$

$\epsilon \rightarrow$ phase constant

∴ দুই রকম অসংখ্য সরল ছন্দিত তরঙ্গের গতির সমীকরণকে মিলি হলে। fourier series.

$$f(x) = C_0 + C_1\cos\left(\frac{2\pi}{\lambda}\cdot x + \epsilon_1\right) + C_2\cos\left(\frac{2\pi}{\lambda/2}\cdot x + \epsilon_2\right)$$

$$+ C_3\cos\left(\frac{2\pi}{\lambda/3}\cdot x + \epsilon_3\right) + \cdots\cdots$$

$f(x)$ is comprised of infinite number of terms

→ anharmonic waves

∴ ধরি, m তম wave এর equation.

$$C_m\cos\left(\frac{2\pi}{\lambda/m}\cdot x + \epsilon_m\right) \qquad \left[\frac{2\pi}{\lambda} = k\right]$$

$$= C_m\cos(mkx + \epsilon_m)$$

$$= C_m\cos(mkx)\cdot\cos(\epsilon_m) - C_m\sin(mkx)\sin(\epsilon_m)$$

এই fourier series এ এই রকম অসীম terms আছে

যাকে সংক্ষেপে,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mkx) + \sum_{m=1}^{\infty} B_m \sin(mkx) \quad (1)$$

{fourrier Siries}

যেখানে,

$A_m = C_m \cos \epsilon_m$; $B_m = -C_m \sin \epsilon_m$

A_m & B_m are called fourrier coefficients.

even & odd function/

যে function (±) দুটি একই value input ~~করার~~ পর যদি output same আসে তাহলে তাকে even function বলে।

$f(x) = x^2$; $\therefore f(+x) = (+x)^2 = x^2$ } same

$f(-x) = (-x)^2 = x^2$

$\therefore$ even function.

তাই ~~$f(x) = x$ হলো~~ $f(x) = x^2$ হলো even function.

$\therefore f(x) = \cos x$

$\therefore f(+x) = \cos(+x) = \cos x$ } even function.

$f(-x) = \cos(-x) = \cos x$

Odd function

$f(x) = x^3$; $f(x) = \sin x$ কোনো value এসেছে যদি x এর value (±) হলে f(x) এর উল্টো value আসে,
তবে f(x) হলো odd function.

Some Integrals /

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad [\text{even হলে}]$$

$$\int_{-a}^{a} f(x)\,dx = 0 \quad [\text{Odd হলে}]$$

Chain Rule formula

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{d}{dx}u \int v\,dx\right]dx$$

Chose, u in this order LIATE

Logs, Inverse, Algebric, Trigono, Exponential.

Fourier series হলো,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left[A_m \cos mkx + B_m \sin mkx \right]$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty} \left[A_m \cos \frac{mx 2\pi}{\lambda} + B_m \sin \frac{2\pi m x}{\lambda} \right]$$

This periodic function $F(x)$, having a period (λ). in the interval (a, b) may be represented by a series of sine and cosine in the form

here, $A_0 = \frac{2}{\lambda} \int_a^b f(x)\, dx$ $\quad \lambda = b - a$

$A_m = C_m \cos \epsilon_m$

$= \frac{2}{\lambda} \int_a^b f(x) \cos \left(\frac{2m\pi x}{\lambda} \right) . dx$

$B_m = - C_m \sin \epsilon_m$

$= \frac{2}{\lambda} \int_a^b f(x) \cdot \sin \left(\frac{2m\pi x}{\lambda} \right) . dx$

calculate A_0, A_m, B_m & put the equation (1) to get fourier series.

Dirichlet's condition for a fourier series

If the function $f(x)$ for the interval $(-\pi, \pi)$

(i) is single valued

(ii) is bounded

(iii) has at most a finite number of maxima & minima

(iv) has only a finite number of discontinuous

(v) if is $f(x+2\pi) = f(x)$ for values of x out side $[-\pi, \pi)$ then

$$f(x) = \frac{A_0}{2} + \left[\sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx\right]$$

এখানে, K এর মান যাতে বের করা যায় $(-\pi, \pi)$ হলে (a, b)

$\lambda = b - a = \pi - (-\pi) = 2\pi$

$\therefore \lambda = \frac{2\pi}{K} \Rightarrow K = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$

তাই, $K = 1$ বলে furier equation এ K নাই।

আমরা জানি, important

fourier Series,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx \quad \text{——(i)}$$

fourier Series হতে একটি নির্দিষ্ট পরিসরে A_0, A_m, B_m নির্ণয় করা যায়

(i) নং সমীকরণকে $(0 \to \lambda)$ ব্যবধিতে যোগজীকরণ করে পাই,

$$\int_0^\lambda f(x)\,dx = \frac{A_0}{2}\int_0^\lambda dx + \sum_{m=1}^{\infty} A_m \int_0^\lambda \cos mkx\,dx + \sum_{m=1}^{\infty} B_m \int_0^\lambda \sin mkx\,dx \quad \text{——(ii)}$$

এখন, $\int_0^\lambda \cos mkx\,dx = \left[\sin mkx\right]_0^\lambda = \left[\sin mk\lambda - \sin mk0\right]$

$$= \left[\sin m\frac{2\pi}{\lambda}\cdot\lambda - \sin 0\right] = \sin m2\pi - 0$$

$$= \sin m2\pi \quad [m = 0, 1, 2 \ldots \text{ যাই হোক না কেন}]$$

$$\therefore \boxed{\int_0^\lambda \cos mkx\,dx = 0}$$

$$\int_0^\lambda \sin mkx\,dx = \left[-\cos mkx\right]_0^\lambda = -\left[\cos m\frac{2\pi}{\lambda}\cdot\lambda - \cos mk0\right]$$

$$= -\cos m2\pi + \cos 0 \quad [\cos\ m = 0, 1, 2, 3 \text{ যাই হোক না কেন}]$$

$$= -1 + 1$$

$$\therefore \boxed{\int_0^\lambda \sin mkx\,dx = 0}$$

(ii) নং সমীকরণে, $\int_0^\lambda \cos mkx\, dx = 0$ ও $\int_0^\lambda \sin mkx\, dx = 0$ মান বসিয়ে পাই,

$$\int_0^\lambda f(x)\,dx = \frac{A_0}{2}\int_0^\lambda dx = \frac{A_0}{2}[x]_0^\lambda = \frac{A_0}{2}\lambda$$

$$\Rightarrow \boxed{A_0 = \frac{2}{\lambda}\int_0^\lambda f(x)\,dx} \quad \text{(iii)}$$

equation (i) কে $\cos lkx$ দ্বারা গুণ করে integral করে পাই, $(0 \to \lambda)$ ব্যবধিতে,

$$\int_0^\lambda f(x).\cos lkx\, dx = \frac{A_0}{2}\int_0^\lambda \cos lkx\, dx + \sum_{m=1}^{\infty} A_m \int_0^\lambda \cos mkx \cos lkx$$
$$+ \sum_{m=1}^{\infty} B_m \int_0^\lambda \sin mkx \cos lkx\, dx \quad \text{(iv)}$$

Using the condition of orthogonality,

(i) $\int_0^\lambda \cos lkx \cos mkx\, dx = \frac{\lambda}{2}\delta_{ml}$; $m = l$, $\delta_{ml} = 1$

$= 0$; $m \neq l$, $\delta_{ml} = 0$

(ii) $\int_0^\lambda \sin mkx \cdot \cos lkx\, dx = 0$, for any value of (m & l)

(iii) $\int_0^\lambda \sin lkx \sin mkx = \frac{\lambda}{2}\cdot\delta_{ml}$; $m = l$, $\delta_{ml} = 1$

$= 0$; $m \neq l$, $\delta_{ml} = 0$

(iv) নং সমীকরণে orthogonal condition এর মান বসিয়ে পাই,

$$\int_0^\lambda f(x)\cos lkx\,dx = 0+0+\sum_{m=1}^{\infty} A_m \int_0^\lambda \cos mkx \cos lkx\,dx$$

$$\int_0^\lambda f(x)\cos lkx\,dx = 0+0+\sum_{m=1}^{\infty} A_m \cdot \frac{\lambda}{2}\cdot \delta_{ml}$$

$$\Rightarrow \int_0^\lambda f(x)\cos lkx\,dx \qquad [m=l \text{ হলে } \delta_{ml}=1$$

$$\Rightarrow A_m = \frac{2}{\lambda}\int_0^\lambda f(x)\cdot \cos lkx\,dx$$

$$\therefore \boxed{A_m = \frac{2}{\lambda}\int_0^\lambda f(x)\cdot\cos lkx\,dx} \text{ — (v)}$$

(i) নং সমীকরণকে $\sin lkx$ দ্বারা গুণ করে পাই,

$$\int_0^\lambda f(x)\cdot\sin lkx\,dx = \frac{A_0}{2}\int_0^\lambda \sin lkx\,dx + \sum_{m=1}^{\infty} A_m\int_0^\lambda \cos mkx \sin lkx$$

$$+\sum_{m=1}^{\infty} B_m \int_0^\lambda \sin mkx \sin lkx\,dx$$

$$\Rightarrow \int_0^\lambda f(x)\cdot\sin lkx\,dx = 0+0+\sum_{m=1}^{\infty} B_m\cdot\frac{\lambda}{2}\cdot\delta_{ml}$$

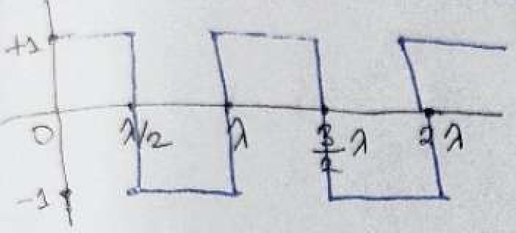
$$\Rightarrow \boxed{B_m = \frac{2}{\lambda}\cdot\int_0^\lambda f(x)\cdot\sin lkx\,dx} \text{ — (vi)}$$

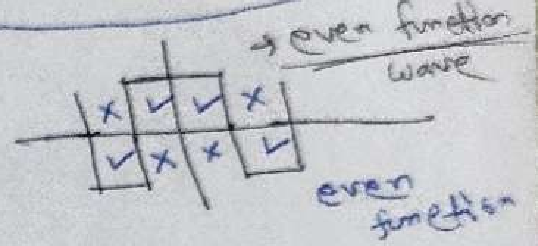
When,
$\delta_{ml} = 1$, $m=l$

$f(x) = f(-x)$; even function or Symmetric function.

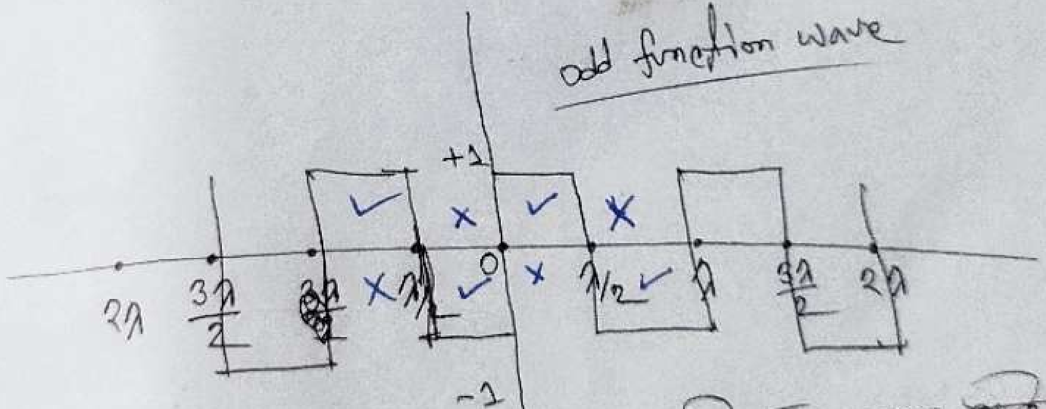
$f(x) \neq f(-x)$; odd function or Scew-Symmetric function.

Compute the ~~four~~ furier Siries crosponding to a square wave, $f(x) = \begin{cases} +1, & 0 < x < \lambda/2 \\ -1, & \frac{\lambda}{2} < x < \lambda \end{cases}$





odd function এ শুধু Sin function থাকে cosine term বাদ যায়।



X অক্ষের (+ও-) কাছের প্রতিবিম্ব দৈর্ঘ্য পরস্পর একই অবস্থান থাকে তাহলে তাকে odd function wave বলে,

Physics Department

Md. Emon

Roll: 2X0091

session: 2020-2021

Subject: (PHA-203) Mathematicle Physics (SI)

2

Math solve

$(0 \to \lambda)$ ব্যবধি মধ্যে $f(x)$ ফাংশনের furier Siries নির্ণয়,

আমরা জানি, furier Siries,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx \quad \text{---(i)}$$

আবার,

$$\cancel{A_0 = \frac{2}{\lambda} \int_0^{\lambda/2} f(x) \cos mkx \, dx}$$

$$A_0 = \frac{2}{\lambda} \int_0^{\lambda} f(x)\,dx = \frac{2}{\lambda} \int_0^{\lambda/2} f(x)\,dx + \frac{2}{\lambda} \int_{\lambda/2}^{\lambda} f(x)\,dx$$

$$= \frac{2}{\lambda} \int_0^{\lambda/2} (+1)\,dx + \frac{2}{\lambda} \int_{\lambda/2}^{\lambda} (+1)\,dx$$

আমরা ~~যা~~ জানি কোনো ফাংশনের $f(x) = \begin{cases} +1, & 0 \le x \le \lambda/2 \\ -1, & \lambda/2 \le x \le \lambda \end{cases}$

$$= \frac{2}{\lambda} [x]_0^{\lambda/2} - \frac{2}{\lambda} [x]_{\lambda/2}^{\lambda}$$

$$= \frac{2}{\lambda} \times \frac{\lambda}{2} - \frac{2}{\lambda} \left(\lambda - \frac{\lambda}{2}\right) = 1 - \left(\frac{2}{\lambda} \times \frac{\lambda}{2}\right) = 1 - 1$$

$$= 0$$

$\therefore \boxed{A_0 = 0}$

$$A_m = \frac{2}{\lambda}\int_0^{\lambda} \cos mkx \cdot f(x)\, dx$$

$$= \frac{2}{\lambda}\int_0^{\lambda/2} f(x)\cos mkx\, dx + \frac{2}{\lambda}\int_{\lambda/2}^{\lambda} f(x)\cos mkx\, dx$$

$$= \frac{2}{\lambda}\int_0^{\lambda/2} (+1)\cos mkx\, dx + \frac{2}{\lambda}\int_{\lambda/2}^{\lambda} (-1)\cos mkx\, dx$$

$$= \frac{2}{\lambda}\left[\frac{\sin mkx}{mk}\right]_0^{\lambda/2} - \frac{2}{\lambda}\left[\frac{\sin mkx}{mk}\right]_{\lambda/2}^{\lambda}$$

$$= \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{\sin m\cdot\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2} - \sin 0^\circ\right\} - \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{\sin m\frac{2\pi}{\lambda}\cdot\lambda - \sin m\frac{2\pi}{\lambda}\cdot\lambda/2\right\}$$

$$= \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{\sin m\pi - 0\right\} - \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{\sin m2\pi - \sin m\pi\right\}$$

$$= \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{\sin m\pi - \sin m2\pi + \sin m\pi\right\}$$

$$= \frac{2}{\lambda}\cdot\frac{1}{mk}\left\{2\sin m\pi - \sin m2\pi\right\}$$

$$= \frac{2}{\lambda}\cdot\frac{\lambda}{m2\pi}\left\{2\sin m\pi - \sin m2\pi\right)$$

$\sin m\pi$ ও $(-\sin m2\pi)$ এর m এর মান যেই কোনো মানের জন্যই (0) আসবে $\therefore$

$$\Rightarrow A_m = \frac{\lambda}{2}\cdot\frac{\lambda}{m2\pi}(0-0) = \lambda/2 \times \frac{\lambda}{m2\pi} \times 0$$

$$\therefore \boxed{A_m = 0}$$

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\, \sin mkx \, dx$$

$$= \frac{2}{\lambda}\int_0^{\lambda/2} f(x)\, \sin mkx\, dx + \frac{2}{\lambda}\int_{\lambda/2}^{\lambda} f(x)\, \sin mkx \, dx$$

$$= \frac{2}{\lambda}\int_0^{\lambda/2} (+1)\, \sin mkx\, dx + \frac{2}{\lambda}\int_{\lambda/2}^{\lambda} (-1)\, \sin mkx\, dx$$

$$= \frac{2}{\lambda}\left[-\frac{\cos mkx}{mk}\right]_0^{\lambda/2} - \frac{2}{\lambda}\left[-\frac{\cos mkx}{mk}\right]_{\lambda/2}^{\lambda}$$

$$= -\frac{2}{\lambda}\left[\frac{\cos mk\, \lambda/2}{mk} - \frac{\cos 0^\circ}{mk}\right] + \frac{2}{\lambda}\cdot\frac{1}{mk}\left[\cos mk\lambda - \cos mk\frac{\lambda}{2}\right]$$

$$= -\frac{2}{\lambda mk}\left[\cos m\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2} - 1\right] + \frac{2}{\lambda mk}\left[\cos m\frac{2\pi}{\lambda}\lambda - \cos m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right]$$

$$= -\frac{2}{\lambda mk}\left[\cos m\pi - 1\right] + \frac{2}{\lambda mk}\left[\cos m2\pi - \cos m\pi\right]$$

$$= + \frac{2}{\lambda mk}\left[\cancel{-\cos m\pi + 1 + \cos m2\pi - \cos m\pi}\right]$$

$$= \frac{2}{\lambda mk}\left[\cancel{1 + \cos m2\pi - 2\cos m\pi}\right]$$

$$= + \frac{2}{\lambda \cdot m\frac{2\pi}{\lambda}}\left[1 - \cos m\pi\right] + \frac{2}{\lambda m\frac{2\pi}{\lambda}}\left[\cos 2\pi m - \cos m\pi\right]$$

m এর সকল মানের জন্য +1 পাওয়া যায়

$$= \frac{1}{m\pi}\left[1 - \cos m\pi\right] + \frac{1}{m\pi}\left[1 - \cos m\pi\right]$$

$$\Rightarrow B_m = \frac{2}{m\pi}\left\{\cancel{\cos}\ 1 - \cos m\pi\right\}$$

$$\Rightarrow \boxed{B_m = \frac{2}{m\pi}\left\{1 - \cos m\pi\right\}}$$

$m=1$ হলে, $B_1 = \frac{4}{\pi}$ | $m=2$ হলে, $B_2 = 0$

$m=3$ হলে, $B_3 = \frac{4}{3\pi}$ | $m=4$ হলে, $B_4 = 0$

$m=5$ হলে, $B_5 = \frac{4}{5\pi}$ | $m=6$ হলে, $B_6 = 0$

∴ furier Siries এ A_0, A_m ও B_m এর মান বসিয়ে পাই,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

$$= 0 + 0 + B_1 \sin kx + B_2 \sin 2kx + B_3 \sin 3kx + B_4 \sin 4kx + \cdots$$

$$= \frac{4}{\pi}\sin kx + 0 + \frac{4}{3\pi}\sin 3kx + 0 + \frac{4}{5\pi}\sin 5kx + 0 + \cdots$$

$$= \frac{4}{\pi}\sin kx + \frac{4}{3\pi}\sin 3kx + \frac{4}{5\pi}\sin 5kx + \cdots$$

$$\therefore f(x) = \frac{4}{\pi}\left\{\sin kx + \frac{1}{3}\sin 3kx + \frac{1}{5}\sin 5kx + \cdots\right\}$$

ইহাই $(0 \to \lambda)$ ব্যবধিতে furier siries এর সমীকরণ

(৭)

যদি $f(x)=x$ হয়, তাহলে $0 \le x \le 2\pi$ ব্যবধিতে furier sries নির্ণয় কর।

আমরা জানি,

$$\lambda = b-a = 2\pi - 0 = 2\pi$$

furier sries,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

$$\therefore A_0 = \frac{2}{\lambda}\int_a^b f(x)\,dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}(4\pi^2 - 0)$$

$$= 2\pi$$

$$\therefore \boxed{A_0 = 2\pi}$$

$$A_m = \frac{2}{\lambda}\int_a^b f(x)\cos mkx\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x \cdot \cos mkx\,dx$$

$$\left[\begin{array}{l}\int u \cdot v\,dx \\ = u\int v\,dx - \int\left\{\frac{d}{dx}u \cdot \int v\,dx\right\}dx\end{array}\right]$$

LIAT

L

$$= \frac{1}{\pi}\int_0^{2\pi} \cos mkx \cdot x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \cos mkx\,dx$$

$$\Rightarrow A_m = \frac{1}{\pi}\left[x\cdot\int_0^{2\pi}\cos mkx\,dx - \int_0^{2\pi}\left\{\frac{d}{dx}(x)\cdot\int_0^{2\pi}\cos mkx\,dx\right\}dx\right]$$

$$= \frac{1}{\pi}\left[x\cdot\left[\frac{\sin mkx}{mk}\right]_0^{2\pi} - \int_0^{2\pi} 1\cdot\left[\frac{\sin mkx}{mk}\right]_0^{2\pi} dx\right]$$

$$= \frac{1}{\pi}\cdot x\cdot\left[\frac{\sin m\frac{2\pi}{\lambda}\cdot 2\pi}{mk}\right] - \int_0^{2\pi}\left[\frac{\sin m\frac{2\pi}{\lambda}\cdot 2\pi}{mk}\right]dx$$

$$= \frac{x}{\pi mk}\left[\sin\frac{m4\pi^2}{\lambda}\right] - \frac{1}{mk}\int_0^{2\pi}\left[\sin m\frac{4\pi^2}{\lambda}\right]dx$$

$$= \frac{x}{\pi mk}\left[\sin\frac{m4\pi^2}{2\pi}\right] - \frac{1}{mk}\int_0^{2\pi}\left[\sin\frac{m4\pi^2}{2\pi}\right]dx \qquad \boxed{\lambda = 2\pi}$$

$$= \frac{x}{\pi mk}\left[\sin m2\pi\right] - \frac{1}{mk}\int_0^{2\pi}\sin m2\pi\,dx$$

$$= \frac{x}{\pi mk}\times(\sin m2\pi) - \frac{1}{mk}\cdot\sin m2\pi\int_0^{2\pi}(1)\,dx$$

$$= \frac{x}{\pi mk}\sin m2\pi - \frac{1}{mk}\sin m2\pi\times[x]_0^{2\pi}$$

$$= \frac{x}{\pi mk}\sin m2\pi - \frac{2\pi}{mk}\sin m2\pi$$

$$= \left(\frac{x}{\pi mk} - \frac{2\pi}{mk}\right)\times\sin m2\pi$$

$$= \left(\frac{x}{\pi mk} - \frac{2\pi}{mk}\right)\times 0$$

$m = 1, 2, 3$ ~~যে~~ কোনো মানের জন্য
$\sin m2\pi = 0$ পাওয়া
যাবে।

$\therefore B_m = \cancel{\frac{2}{\lambda}\int_a^b f(x)\sin mkx\,dx}$

$\lambda = b-a = 2\pi - 0 = 2\pi$

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$

$= \cancel{\frac{2}{2\pi}\int_0^{2\pi} x\sin mx\,dx}$

$= \frac{1}{\pi}\left[x\int_0^{2\pi}\sin mx\,dx - \int_0^{2\pi}\left\{\frac{d}{dx}x\int_0^{2\pi}\sin mx\,dx\right\}dx\right]$

$= \frac{1}{\pi}\left[x\left[-\frac{\cos mx}{m}\right]_0^{2\pi} - \int_0^{2\pi}\left[-\frac{\cos mx}{m}\right]_0^{2\pi}dx\right]$

$= \frac{1}{\pi}\left[x\left[-\frac{\cos m2\pi}{m} + \frac{\cos 0^\circ}{m}\right] - \int_0^{2\pi}\left[-\frac{\cos m2\pi}{m} + \frac{\cos 0}{m}\right]dx\right]$

$= \frac{1}{\pi}\left[\frac{x}{m}(-1+1)\right] - \int_0^{2\pi} -\cos mx$

m গুঃ যে হওয়ানো
মানের জন্য / $m = 1,2,3$

$\lambda = b-a = 2\pi - 0 = 2\pi$, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$

$\therefore B_m = \frac{2}{\lambda}\int_a^b f(x)\sin mkx\,dx = \frac{1}{\pi}\int_0^{2\pi} x.\sin mx\,dx$

$= \frac{1}{\pi}\left\{-\frac{x\cos mx}{m} + \frac{d}{dx}x\int -\frac{\cos mx}{m}\,dx\right\}_0^{2\pi}$

$= \frac{1}{\pi}\left[-\frac{x\cos mx}{m} + \frac{1}{m}\frac{\sin mx}{m}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[\left[-\frac{x\cos mx}{m}\right]_0^{2\pi} + \left[\frac{\sin mx}{m^2}\right]_0^{2\pi}\right]$

$$= \frac{1}{\pi m}\left[\left[-x\cos mx\right]_0^{2\pi} + \left[\frac{\sin mx}{m}\right]_0^{2\pi}\right]$$

$$= \frac{1}{\pi m}\left[-2\pi\cos m2\pi + 0\times\cos 0^\circ + \frac{\sin m2\pi}{m} - \frac{\sin 0}{m}\right]$$

$$= \frac{1}{\pi m}\left[-2\pi\cos m2\pi + 0 + \frac{\sin m2\pi}{m} - 0\right]$$

উপরে m এর যেকোনো মানের জন্য $\sin m2\pi = 0$ এবং $\cos m2\pi = 1$ হয়।

$$= \frac{1}{\pi m}\left[-2\pi\times 1 + 0 + \frac{0}{m} - 0\right]$$

$$= \frac{-2\pi}{\pi m} = -2/m$$

$\therefore$ fourier Series হতে পাই, $\longrightarrow K=1$ হলে/

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m\cos mx + \sum_{m=1}^{\infty} B_m\sin mx$$

$$= \frac{2\pi}{2} + 0 + \sum_{m=1}^{\infty} -\frac{2}{m}\sin mx$$

$$\sum_{m=1}^{\infty} = \pi - 2\sin x - 1\sin 2x - \frac{2}{3}\sin 3x \cdots\cdots$$

$$= \pi - 2\left(\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x \cdots\cdots\right)$$

যদি $f(x) = x + x^2$ হয় তাহলে $(-\pi < x < \pi)$ ব্যবধিতে furier Siries নির্ণয় কর। $(0 < x < 2\pi)$

∴ আমরা জানি,

furier Siries।

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

$$\therefore \lambda = b - a = \pi - (-\pi) = 2\pi \quad \therefore k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$$

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx \quad \text{--- (i)}$$

$$\therefore A_0 = \frac{2}{\lambda} \int_a^b f(x)\,dx = \frac{2}{2\pi} \int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\left[\frac{x^2}{2} \right]_{-\pi}^{\pi} + \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2} + \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^3}{3} + \frac{\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^3 + \pi^3}{3} \right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\therefore \boxed{A_0 = \frac{2\pi^2}{3}}$$

$$\therefore A_m = \frac{2}{\lambda}\int_a^b f(x)\,\cancel{\sin mx}\ \cos mx\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi}(x+x^2)\cdot\cos mx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos mx\,dx + \int_{-\pi}^{\pi} x^2\cos mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[x\frac{\sin mx}{m} - \int \frac{d}{dx}x\int\frac{\sin mx}{m}\,dx\right]_{-\pi}^{\pi} + \left[x^2\frac{\sin mx}{m} - \int\frac{d}{dx}x^2\int\frac{\sin mx}{m}\,dx\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[x\frac{\sin mx}{m} - \frac{1}{m}\int\sin mx\,dx\right]_{-\pi}^{\pi} + \left[x^2\frac{\sin mx}{m} - \int 2x\cdot\left(-\frac{\cos mx}{m^2}\right)dx\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x}{m}\sin mx + \frac{1}{m^2}\cos mx\right]_{-\pi}^{\pi} + \left[\frac{x^2}{m}\sin mx + \frac{2}{m^2}\int x\cos mx\,dx\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x}{m}\sin mx + \frac{1}{m^2}\cos mx\right]_{-\pi}^{\pi} + \left[\frac{x^2}{m}\sin mx + \frac{2}{m^2}\left\{\frac{x}{m}\sin mx - \int\frac{\sin mx}{m}\,dx\right\}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x}{m}\sin mx + \frac{1}{m^2}\cos mx\right]_{-\pi}^{\pi} + \left[\frac{x^2}{m}\sin mx\right.\right.$$

$$\left.\left. + \frac{2}{m^2}\left\{\frac{x}{m}\sin mx + \frac{1}{m^2}\cos mx\right\}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x}{m}\sin mx\right]_{-\pi}^{\pi} + \frac{1}{m^2}\left[\cos mx\right]_{-\pi}^{\pi} + \frac{1}{m}\left[x^2\sin mx\right]_{-\pi}^{\pi}\right.$$

$$\left. + \frac{2}{m^3}\left[x\sin mx\right]_{-\pi}^{\pi} + \frac{2}{m^4}\left[\cos mx\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{m}\sin m\pi - \frac{-\pi}{m}\sin m(-\pi) + \frac{1}{m^2}\left\{\cos m\pi - \cos(m(-\pi))\right\}\right.$$

$$\left. + \frac{1}{m}\left[\pi^2\sin m\pi - (-\pi)^2\sin(-\pi m)\right]\right]$$

$$+ \frac{2}{m^3}\left[\pi\sin m\pi - (-\pi)\sin m(-\pi)\right] + \frac{2}{m^4}\left[\cos m\pi - \cos(-m\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{m}\sin m\pi - \frac{\pi}{m}\sin m\pi + \frac{1}{m^2}\left(\cos m\pi - \cos m\pi\right)\right.$$

$$+ \frac{1}{m}\left(\pi^2\sin m\pi + \pi^2\sin m\pi\right) + \frac{2}{m^3}\left(\pi\sin m\pi - \pi\sin m\pi\right)$$

$$\left. + \frac{2}{m^4}\left(\cos m\pi - \cos m\pi\right)\right]$$

$$= \frac{1}{\pi}\left[0 - 0 + \frac{1}{m^2}\times 0 + \frac{2\pi^2}{m}\sin m\pi + \frac{2}{m^3}\times 0 + \frac{2}{m^4}\times 0\right]$$

$$= \frac{2\pi^2\sin m\pi}{m\pi} = \frac{2\pi\sin m\pi}{m}$$

$\Rightarrow A_m = \frac{2\pi}{m} \sin m\pi$

$m = 1, 2 \ldots \infty$ পর্যন্ত যেকোন মানের জন্য

$\sin m\pi = 0$ হয়

$\therefore A_m = \frac{2\pi}{m} \times 0$

$\Rightarrow \boxed{A_m = 0}$

$\therefore B_m = \frac{2}{\lambda} \int_a^b f(x) \sin mkx \, dx$

$= \frac{2}{2\pi} \int_{-\pi}^{\pi} (x + x^2) \cdot \sin mx \, dx$

$= \frac{1}{\pi} \left[\left[(x+x^2)\left(-\frac{\cos mx}{m}\right) + \int \frac{d}{dx}(x+x^2) \frac{\cos mx}{m} dx \right] \right]_{-\pi}^{+\pi}$

$= \frac{1}{\pi} \left[(x+x^2)\left(-\frac{\cos mx}{m}\right) + \frac{(1+2x)}{m^2} \sin mx - \int \frac{d}{dx}(1+2x) \int \frac{\sin mx}{m^2} dx \, dx \right]_{-\pi}^{+\pi}$

$= \frac{1}{\pi} \left[\left[(x+x^2) \times \left(-\frac{\cos mx}{m}\right) + \left(\frac{1+2x}{m^2}\right) \sin mx \right]_{-\pi}^{\pi} - \int \frac{-2\cos mx}{m^3} dx \right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\left[(x+x^2)\left(\frac{-\cos mx}{m}\right) + \left(\frac{1+2x}{m^2}\sin mx\right) + \frac{2}{m^4}\sin mx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[-\frac{x\cos mx}{m}\right]_{-\pi}^{\pi} + \left[-\frac{x^2\cos mx}{m}\right]_{-\pi}^{\pi} + \frac{1}{m^2}\left[\sin mx\right]_{-\pi}^{\pi} + \frac{2x}{m^2}\left[x\sin mx\right]_{-\pi}^{\pi} + \frac{2}{m^4}\left[\sin mx\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{m}\cos m\pi - \frac{(-x-\pi)}{m}\cos(-\pi)m\right] - \left[\frac{\pi^2}{m}\cos m\pi - \frac{(-\pi)^2}{m}\cos(-\pi)m\right] + \frac{1}{m^2}\left[\sin m\pi - \sin(-\pi)m\right] + \frac{2}{m^2}\left[\pi\sin m\pi - (\pi)\sin m(-\pi)\right] + \frac{2}{m^4}\left[\sin m\pi - \sin m(-\pi)\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{\pi}{m}\cos m\pi - \frac{\pi}{m}\cos m\pi\right] - \left[\cancel{\frac{\pi^2}{m}\cos m\pi - \frac{\pi^2}{m}\cos m\pi}\right] + \frac{1}{m^2}\left[\sin m\pi + \sin m\pi\right] + \frac{2}{m^2}\left[\cancel{\pi\sin m\pi - \pi\sin m\pi}\right] + \frac{2}{m^4}\left[\sin m\pi + \sin m\pi\right]\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos m\pi}{m} + \frac{2\sin m\pi}{m^2} + \frac{4\sin m\pi}{m^4}\right]$$

∴ m এর যেই কোনো মানের জন্য $\sin m\pi = 0$ আসে

$$\therefore B_m = \frac{1}{\pi}\left[\frac{-2\pi \cos m\pi}{m}\right]$$

$$= \frac{-2\cos m\pi}{m}$$

∴ $m = 1$ হলে,

$$B_1 = \frac{-2 \times (-1)}{1} = 2$$

$m = 2$ হলে

$$B_2 = \frac{-2\times(1)}{2} = -1$$

$m = 3$ হলে,

$$B_3 = \frac{-2\times(-1)}{3} = \frac{2}{3}$$

$m = 4$ হলে,

$$B_4 = \frac{-2\times(1)}{4} = -1/2$$

∴ (i) নং সমীকরণে A_0, A_m, B_m এর মান বসিয়ে পাই

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$= \frac{2\pi^2}{2\times 3} + 0 + B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + B_4 \sin 4x + \cdots$$

$$= \frac{\pi^2}{3} + 2\sin x - 1\sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x$$

* obtain the fourier series of $F(x) = x - x^2$

From $\underset{(a,\ b)}{(-\pi,\ \pi)}$ $\therefore \lambda = b - a = \pi - (-\pi) = 2\pi$

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$

আমরা জানি,

fourier series,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx \quad \text{(1)}$$

$$\therefore \quad = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$A_0 = \frac{2}{\lambda}\int_a^b f(x)\,dx = \frac{2}{2\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} - \left[\frac{x^3}{3}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right] - \left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{2\pi^3}{3}\right] = -\frac{2\pi^2}{3}$$

$$\therefore A_m = \frac{2}{\lambda}\int_a^b f(x)\cos mx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cdot\cos mx\,dx$$

$$= \frac{1}{\pi}\left[\left[(x-x^2)\frac{\sin mx}{m} - \int(1-2x)\cdot\frac{\sin mx}{m}\,dx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[(x-x^2)\frac{\sin mx}{m} - \left[(1-2x)\left(-\frac{\cos mx}{m^2}\right) - \int(-2)\left(-\frac{\cos mx}{m^2}\right)dx\right]\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[(x-x^2)\frac{\sin mx}{m} + (1-2x)\left(\frac{\cos mx}{m^2}\right) - 2\frac{\sin mx}{m^3}\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[\frac{x\sin mx}{m}\right]_{-\pi}^{\pi} - \left[\frac{x^2\sin mx}{m}\right]_{-\pi}^{\pi} + \left[\frac{\cos mx}{m^2}\right]_{-\pi}^{\pi} - 2\left[\frac{x\cos mx}{m^2}\right]_{-\pi}^{\pi} - 2\left[\frac{\sin mx}{m^3}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin m\pi}{m} + \frac{\pi\sin m\pi}{m} - \cancel{\frac{\pi^2\sin m\pi}{m}} + \cancel{\frac{\pi^2\sin m\pi}{m}} + \cancel{\frac{\cos m\pi}{m^2}} - \cancel{\frac{\cos m\pi}{m^2}} - 2\left(\frac{\pi\cos m\pi}{m^2} + \frac{\pi\cos m\pi}{m^2}\right) - \frac{2\sin m\pi}{m^3} - \frac{2\sin m\pi}{m^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi\sin m\pi}{m} - \frac{4\pi\cos m\pi}{m^2} - \frac{4\sin m\pi}{m^3}\right]$$

m এর মান $1, 2, \ldots \infty$ পর্যন্ত যেই কোনো মানের জন্য $\sin m\pi = 0$ হয়,

$$= \frac{1}{\pi}\left[-\frac{4\pi\cos m\pi}{m^2}\right] = -\frac{4\cos m\pi}{m^2}$$

$$\Rightarrow A_m = -4\frac{\cos m\pi}{m^2} \qquad \left[\cos m\pi = (-1)^m \text{ পূর্ণসংখ্যা}\right]$$

$$= -\frac{4.(-1)^m}{m^2}$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2).\sin mx\, dx$$

$$= \frac{1}{\pi}\left[(x-x^2).\left(-\frac{\cos mx}{m}\right) - (1-2x)\left(-\frac{\sin mx}{m^2}\right) + (-2)\left(\frac{\cos mx}{m^3}\right)\right]_{-\pi}^{\pi}$$

$= \frac{1}{\pi}\Big[$ $\sin m\pi$ শূন্য হবে = [illegible] m পূর্ণ $m-1$ $\Big]$ $1^2, 2^3 \ldots \infty$ [illegible]

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos mx}{m}\right) - 2\left(\frac{\cos mx}{m^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[(\pi-\pi^2)\left(-\frac{\cos m\pi}{m}\right) - 2\frac{\cos m\pi}{m^3}\right] - \left[(-\pi-\pi^2)\left(-\frac{\cos m\pi}{m}\right) - 2\frac{\cos m\pi}{m^3}\right]\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos m\pi}{m} + \cancel{\frac{\pi^2\cos m\pi}{m}} - \cancel{\frac{2\cos m\pi}{m^3}} - \frac{\pi\cos m\pi}{m} - \cancel{\frac{\pi^2\cos m\pi}{m}} + \cancel{\frac{2\cos m\pi}{m^3}}\right]$$

$$= \frac{1}{\pi}\left(\frac{-2\pi\cos m\pi}{m}\right) = \frac{-2\cos m\pi}{m} = \frac{-2.(-1)^m}{m}$$

$\Rightarrow B_m = \frac{-2 \cdot (-1)^m}{m}$

$\therefore$ fourier series এ মান বসিয়ে পাই

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$= \frac{-2\pi^2}{2 \times 3} + \sum_{m=1}^{\infty} \frac{(-4) \cdot (-1)^m}{m^2} \cos mx + \sum_{m=1}^{\infty} \frac{(-2) \cdot (-1)^m}{m} \sin mx$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

$$+ 2\left[\frac{\sin x}{1^2} - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} - \cdots\right]$$

যদি $f(x) = x - x^2$ এ $x = 0$ বসিয়ে তাহলে
$f(x) = 0$

$$\Rightarrow -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right] + 2\left[\frac{0}{1^2} - \frac{0}{2^2} + \frac{0}{3^2} - \cdots\right]$$

$$\Rightarrow \frac{\pi^3}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right] = 0$$

$$\Rightarrow \frac{\pi^3}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

1

Chain Rule for Integration

in fourier Series

$$\int u \cdot v\, dx = \left\{ u \int v\, dx \right\} - \left\{ \frac{d}{dx} u \cdot \iint v\, dx\, dx \right\}$$
$$+ \left\{ \frac{d^2}{dx^2} u \cdot \iiint v\, dx\, dx\, dx \right\}$$
$$- \left\{ \frac{d^3}{dx^3} u \cdot \iiiint v\, dx\, dx\, dx\, dx \right\}$$
$$+ \left\{ \frac{d^4}{dx^4} u \cdot \int\!\!\int\!\!\int\!\!\int\!\!\int v\, dx\, dx\, dx\, dx\, dx \right\}$$
$$- - - - - -$$

এর চেয়ে খুব খুব সহজে আমরা Chain Rule ব্যবহার করতে পারবো.

Find the fourier series of $f(x) = x^2$

from $(-\pi, \pi)$.

$\therefore f(x)$ $\lambda = b - a = 2\pi$, $k = \frac{2\pi}{\lambda} = 1$

$\therefore \cancel{f(x) = \frac{A_0}{2} + \int_a^b A_m \cos mkx}$

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$\therefore A_0 = \frac{2}{\lambda} \int_a^b f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^3}{3\pi} + \frac{\pi^3}{3\pi}$$

$$= \frac{2\pi^2}{3}$$

$$A_m = \frac{2}{\lambda}\int_a^b f(x)\cos mx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \cos mx \, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin mx}{m} - (2x)\cdot\left(\frac{-\cos mx}{m^2}\right) + (2)\frac{-\sin mx}{m^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin mx}{m} + 2x\frac{\cos mx}{m^2} - 2\frac{\sin mx}{m^3}\right]_{-\pi}^{\pi}$$

m এর মান $1, 2, 3, \ldots \infty$ পূর্ণসংখ্যার জন্য $\sin m\pi = 0$ হয় এবং

$\cos m\pi = (-1)^m$

$$= \frac{1}{\pi}\left[2x\frac{\cos mx}{m^2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[2\pi\frac{\cos m\pi}{m^2} - \frac{2(-\pi)\cos m\pi}{m^2}\right]$$

$$= \frac{4\pi \cos m\pi}{\pi m^2} = \frac{4\cos m\pi}{m^2}$$

$$= \frac{4(-1)^m}{m^2} \qquad \left[\cos m\pi = (-1)^m\right]$$

$$\therefore A_m = \frac{4(-1)^m}{m^2}$$

$$B_m = \frac{2}{\pi}\int_a^b f(x) \sin mx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\downarrow}{x^2} \cdot \underset{\downarrow}{\sin mx} \, dx$$

even × odd = odd

∴ আমরা জানি $\int_{-a}^{a} \text{odd function} \, dx = 0$

~~(-a, a) ব্যবধিতে কোনো~~ odd function এর ~~অ~~

অভিকলন = 0 হয়,

$\therefore B_m = 0$

∴ fourier Series হবে,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$= \frac{2\pi^2}{2\times 3} + \sum_{m=1}^{\infty} \frac{4(-1)^m}{m^2} \cos mx + 0$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

* Find the fourier series of $f(x) = x + \frac{x^2}{4}$ from $(-\pi, \pi)$.

$\therefore \underset{(a,\,b)}{(-\pi, \pi)} \quad \therefore \lambda = b - a = 2\pi, \quad k = \frac{2\pi}{\lambda} = 1$

$$\therefore f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

$$\therefore A_0 = \frac{2}{\lambda} \int_a^b f(x)\, dx = \frac{2}{\pi} \int_{-\pi}^{\pi} \left(x + \frac{x^2}{4}\right) dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{12}\right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{12} - \frac{\pi^2}{2} + \frac{\pi^3}{12}\right]$$

$$= \frac{1}{\pi} \cdot \frac{2\pi^3}{12} = \frac{\pi^2}{6}$$

$\therefore \boxed{A_0 = \frac{\pi^2}{6}}$

$$\therefore A_m = \frac{2}{\lambda} \int_a^b f(x) \cos mx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + \frac{x^2}{4}\right) \cdot \cos mx\, dx$$

$$= \frac{1}{\pi} \left[\left(x + \frac{x^2}{4}\right) \cdot \frac{\sin mx}{m} - \left(1 + \frac{x}{2}\right)\left(-\frac{\cos mx}{m^2}\right) + \left(\frac{1}{2}\right) \cdot \left(-\frac{\sin mx}{m^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{x \sin mx}{m} + \frac{x^2 \sin mx}{4m} + \frac{\cos mx}{m^2} + \frac{x \cos mx}{2m^2} - \frac{\sin mx}{2m^3}\right]_{-\pi}^{\pi}$$

$\boxed{m \text{ পূর্ণ হলে কোনো মানের জন্য } \sin m\pi = 0}$

$$= \frac{1}{\pi} \left[\frac{\cos mx}{m^2} + \frac{x \cos mx}{2m^2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow A_m = \frac{1}{\pi}\left[\frac{2\cos m\pi + m\cos m\pi}{2m^2}\right]_{-\pi}^{\pi}$$

$$= \cancel{\frac{1}{\pi}\left[\frac{2\cos m\pi}{}\right.} = \frac{1}{\pi}\left[\frac{\cos m\pi\,(2+\pi)}{2m^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2+\pi)\cdot\cos m\pi}{2m^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2+\pi)\cos m\pi}{2m^2} - \frac{(2-\pi)\cdot\cos m\pi}{2m^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(2+\pi)\cos m\pi - (2-\pi)\cos m\pi}{2m^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{2\cos m\pi} + \pi\cos m\pi - \cancel{2\cos m\pi} + \pi\cos m\pi}{2m^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi\cos m\pi}{2m^2}\right] = \frac{1}{\pi}\cdot\frac{\pi\cos m\pi}{m^2}$$

$$= \frac{\cos m\pi}{m^2}$$

$$\Rightarrow A_m = \frac{\cos m\pi}{m^2} = \frac{(-1)^m}{m^2}$$

$$\left[\cos m\pi = (-1)^m\right]$$

$$\therefore \boxed{A_m = \frac{(-1)^m}{m^2}}$$

1

$$B_m = \frac{2}{\lambda}\int_a^b f(x)\sin mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+\frac{x^2}{4}\right)\cdot\sin mx\,dx$$

$$= \frac{1}{\pi}\left[\left(x+\frac{x^2}{4}\right)\left(-\frac{\cos mx}{m}\right)-\left(1+\frac{x}{2}\right)\left(-\frac{\sin mx}{m^2}\right)+\left(\frac{1}{2}\right)\left(\frac{\cos mx}{m^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(x+\frac{x^2}{4}\right)\left(-\frac{\cos mx}{m}\right)+\left(1+\frac{x}{2}\right)\left(\frac{\sin mx}{m^2}\right)+\frac{1}{2}\left(\frac{\cos mx}{m^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos mx}{m}-\frac{x^2\cos mx}{4m}+\frac{\sin mx}{m^2}+\frac{x\sin mx}{2m^2}+\frac{\cos mx}{2m^3}\right]_{-\pi}^{\pi}$$

যেহেতু [যেকোনো মানের জন্য $\sin m\pi = 0$]

$$= \frac{1}{\pi}\left[\frac{\cos mx}{2m^3}-\frac{x\cos mx}{m}-\frac{x^2\cos mx}{4m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos m\pi}{2m^3}-\frac{\pi\cos m\pi}{m}-\frac{\pi^2\cos m\pi}{4m}\right]-\left[\frac{\cos m\pi}{2m^3}-\frac{(-\pi)\cos m\pi}{m}-\frac{(-\pi)^2\cos m\pi}{4m}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\cos m\pi}{2m^3}}-\cancel{\frac{\cos m\pi}{2m^3}}-\frac{\pi\cos m\pi}{m}-\frac{\pi\cos m\pi}{m}-\cancel{\frac{\pi^2\cos m\pi}{4m}}+\cancel{\frac{\pi^2\cos m\pi}{4m}}\right]$$

$\Rightarrow B_m = \frac{1}{\pi}\left(-\frac{2\pi \cos m\pi}{m}\right) = -\frac{2\cos m\pi}{m}$

$\Rightarrow B_m = -\frac{2\cos m\pi}{m} = \frac{-2(-1)^m}{m}$ $\left[\cos m\pi = (-1)^m\right]$

$\therefore \boxed{B_m = \frac{-2(-1)^m}{m}}$

$\therefore A_0, A_m, B_m$ এর মান fourier Sities এ বসিয়ে পাই,

$\therefore f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$

$= \frac{A_0}{2}$

$= \frac{\pi^2}{12} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2}\cdot \cos mx + \sum_{m=1}^{\infty} \frac{-2(-1)^m}{m} \sin mx$

$= \cancel{\frac{\pi^2}{12} + \frac{(-1)^1}{1^2}\cos x + \frac{1}{2^2}\cos 2x} -$

$= \frac{\pi^2}{12}\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right]$

$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x \dots\right]$

$\therefore f(x) = \frac{\pi^2}{12}\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right]$

$+ 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right]$

এটাই নির্ণেয় fourier Sities

(1)

<u>Problem</u> দেখাও যে, জোড় ফাংশনের furier series এ কোনো sin পদ থাকে না।

<u>Solve</u> we know fourier cofficient,

$$A_0 = \frac{2}{\lambda}\int_a^b f(x)\,dx, \quad A_m = \frac{2}{\lambda}\int_a^b f(x)\cos mkx\,dx$$

& $$B_m = \frac{2}{\lambda}\int_a^b f(x)\sin mkx\,dx$$

<u>for $(-\pi, \pi)$ interval the cofficient will be,</u>

$$\boxed{\begin{aligned}\lambda &= b-a\\ &= 2\pi\\ k &= \frac{2\pi}{\lambda}\\ &= 1\end{aligned}}$$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

$$\therefore A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right] \quad \text{(i)}$$

Limit

	up	low			
x	0	$-\pi$	x	π	0
y	0	π	y	$-\pi$	0
				up	low

~~Let, $x = -y$, $dx = -dy$~~

$$= \frac{1}{\pi}\left[\cancel{\int_{+\pi}^{0} f(-y)(-dy) + \int_{0}^{-\pi} f(-y)(-dy)}\right]$$

limit

$$\therefore \int_{-\pi}^{0} f(x)\,dx \qquad x = -y, \quad dx = -dy$$

x	0	$-\pi$
y	0	π

বসিয়ে

$$\Rightarrow \int_{-\pi}^{0} f(x)\,dx = \int_{\pi}^{0} f(-y)\cdot(-dy)$$

$$= -\int_{\pi}^{0} f(-y)\,dy = \int_{0}^{\pi} f(-y)\,dy$$

$$= \int_{0}^{\pi} f(-x)\,dx \qquad \left[\int f(x)\,dx = \int f(y)\,dy\right]$$

$$= \int_{0}^{\pi} f(x)\,dx \qquad \left[\text{for even we can write, } f(-x) = f(x)\right]$$

$\therefore$ Put this value on (eq-1)

$$A_0 = \frac{1}{\pi}\left[\int_{0}^{\pi} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$\Rightarrow \boxed{A_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x)\,dx}$$

আবার,

$$\therefore A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos mx\,dx + \int_{0}^{\pi} f(x)\cos mx\,dx\right] \text{—(i)}$$

1

$\therefore \int_{-\pi}^{0} f(x) \cos mx \, dx$ এখানে $x = -y$; $dx = -dy$

বসিয়ে পাই & let limit (x) will be

x	0	$-\pi$
y	0	π
	Upper	Lower

$\therefore \int_{-\pi}^{0} f(x) \cos mx \, dx = \int_{+\pi}^{0} f(-y) \cdot (-dy) \cos my$ [মান & limit বসিয়ে]

$= -\int_{\pi}^{0} f(-y) \, dy = \int_{0}^{\pi} f(-y) \, dy \cos my$ [we can write $\int f(+x) dx = \int f(y) dy$]

$= \int_{0}^{\pi} f(-x) dx \cos(-mx)$ [$f(-x) = f(x)$ for even function]

$= \int_{0}^{\pi} f(x) dx \cos mx$ [$\cos(-mx) = \cos mx$]

$\therefore$ (iii) এতে ~~মান~~ বসিয়ে

$$A_m = \frac{1}{\pi} \left[\int_{0}^{\pi} f(x) dx + \int_{0}^{\pi} f(x) dx \right] \cos mx$$

~~$A_m = \frac{2}{\pi} \int_{0}^{\pi} f(x) dx$~~

$$A_m = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos mx \, dx$$

$$\therefore B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right] \quad \text{—(iv)}$$

$\therefore \int_{-\pi}^{0} f(x)\sin nx\,dx$; $x=-y$, $dx=-dy$ limit

	upper	lower
x	0	$-\pi$
y	0	π

হলে $\int_{-\pi}^{0} f(x)\sin nx\,dx$

$$= \int_{\pi}^{0} f(-y)\sin n(-y)\,(-dy) = -\int_{\pi}^{0} f(-y)\,(-\sin ny)\,dy$$

$$= -\int_{0}^{\pi} f(-y)\sin ny\,dy$$

$$= -\int_{0}^{\pi} f(-x)\sin nx\,dx$$

$$= -\int_{0}^{\pi} f(x)\sin nx\,dx$$

$\int f(x)\sin nx\,dx$
$= \int f(y)\sin ny\,dy$

$f(x) = f(-x)$ হলে যায়
for even function

$\therefore$ (iv) এতে বসালে পাই

$$B_n = \frac{1}{\pi}\left[-\int_{0}^{\pi} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$\therefore B_n = 0$, It proved, there have no term's of (sin) in fourier series for EVEN Function (Proved)

(1)

Problem // Proved that, for odd function, in fourier siries there have no terms of cosine.

save

we know // for $(-\pi,\pi)$ interval the cofficient of fourier siries are,

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$\boxed{B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx}$$

$$\therefore A_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right] \quad \text{--- (i)}$$

$= \frac{1}{\pi}\Big[$ Let, $x=-y$, $dx=-dy$

limit		
x	0	$-\pi$
y	0	π
	upper	lower

এখন $\int f(x)\,dx$ এ বসিয়ে

$\Rightarrow \cancel{A_0 = \frac{1}{\pi}}$

$$\Rightarrow \int_{\pi}^{0} f(-y)(-dy)$$

$$= \int_{-\pi}^{0} f(-y)\,dy = \int_{0}^{\pi} f(-y)\,dy = \int_{0}^{\pi} f(-x)\,dx$$

for odd function

$f(-x) = -f(x)$

$$= -\int_{0}^{\pi} f(x)\,dx$$

$\therefore$ = (i) নং এ বসিয়ে পাই,

$$\Rightarrow A_0 = \frac{1}{\pi}\left[-\int_0^{\pi} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$$

$$\Rightarrow \boxed{A_0 = 0}$$

আবার,

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos mx\,dx + \int_0^{\pi} f(x)\cos mx\,dx\right] \quad \text{(ii)}$$

$\therefore \int_{-\pi}^{0} f(x)\cos mx\,dx$ এখন $x = -y$, $dx = -dy$

x	0	$-\pi$
y	0	π

বসালে, $= \int_{+\pi}^{0} f(-y)\cos m(-y)(-dy)$

$$= -\int_{+\pi}^{0} f(-y)\cos my\,dy \quad [\cos(-mx) = \cos mx]$$

$$= \int_0^{\pi} f(-y)\cos my\,dy = \int_0^{\pi} f(-x)\cos mx\,dx$$

$$= -\int_0^{\pi} f(x)\cos mx\,dx \quad [f(-x) = -f(x) \text{ for odd function}]$$

এখন (ii) এ বসিয়ে পাই,

$$A_m = \frac{1}{\pi}\left[-\int_0^{\pi} f(x)\cos mx\,dx + \int_0^{\pi} f(x)\cos mx\,dx\right]$$

$$\Rightarrow \boxed{A_m = 0}$$

(1)

$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx \, f(x)\, dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin mx\, f(x)\, dx + \int_{0}^{\pi} \sin mx\, f(x)\, dx\right]$ — (iii)

$\int_{-\pi}^{0} \sin mx\, f(x)\, dx$ ধরি, $x = -y$; $dx = -dy$ limit

x	0	$-\pi$
y	0	π

— (ii)

$= \int_{\pi}^{0} \sin m(-y)\, f(-y)(-dy) = +\int_{\pi}^{0} \sin my\, f(-y)\, dy$

$= -\int_{0}^{\pi} \sin my\, f(-y)\, dy$

$= -\int_{0}^{\pi} \cancel{\sin my\, f(-y)}\ \sin mx\, f(-x)\, dx$

$= +\int_{0}^{\pi} f(x)\, \cancel{dx}\, \sin mx\, dx$ $\left[f(-x) = -f(x)\ \text{for odd function}\right]$

$= \int_{0}^{\pi} f(x) \sin mx\, dx$

$\therefore$ (iii) নং এ (ii) নং এর মান বসিয়ে পাই,

$B_m = \frac{1}{\pi}\left[\int_{0}^{\pi} f(x) \sin mx\, dx + \int_{0}^{\pi} f(x) \sin mx\, dx\right]$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin mx\, dx$

It's Proved that in odd function, there have not any cosine terms. (i)

Important // Expand the following function in fourier series, $f(x) = \begin{cases} -x & \text{when}, \ -\pi < x < 0 \\ x & \text{when}, \ 0 < x < \pi \end{cases}$

solve

আমরা জানি,

fourier series, formula

$$F(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m \cos mkx + B_m \sin mkx \right)$$ ①

$-\pi < x < 0$, $0 < x < \pi$

$\therefore \lambda = b - a$

$= \pi - (-\pi) = 2\pi$

$\therefore \lambda = 2\pi$ হলে $K = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi} = 1$

$$\therefore A_0 = \frac{2}{\lambda} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right]$$

$$\Rightarrow A_0 = \frac{1}{\pi} \left[\int_{-\pi}^{0} -x\, dx + \int_{0}^{\pi} x\, dx \right] = \frac{1}{\pi} \left[\int_{0}^{-\pi} x\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{x^2}{2} \right]_0^{-\pi} + \left[\frac{x^2}{2} \right]_0^{\pi} \right] = \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^2}{2} \right]$$

$$= \pi^2 / \pi$$

$$\therefore \boxed{A_0 = \pi} \text{ (i)}$$

(1)

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos mx + \int_{0}^{\pi} f(x)\cos mx\right]dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos mx\,dx + \int_{0}^{\pi} f(x)\cos mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{-\pi} x\cos mx\,dx + \int_{0}^{\pi} x\cos mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[(x)\left(\frac{\sin mx}{m}\right) - (1)\cdot\left(\frac{-\cos mx}{m^2}\right)\right]_0^{-\pi} + \left[(x)\cdot\left(\frac{\sin mx}{m}\right) - (1)\left(\frac{-\cos mx}{m^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\frac{\sin -m\pi}{m} + \frac{\cos m\pi}{m^2} - \frac{\cos 0}{m^2} + \pi\cdot\frac{\sin m\pi}{m} + \frac{\cos m\pi}{m^2} - \frac{\cos 0}{m^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi\sin m\pi}{m} + \frac{2\cos m\pi}{m^2} - \frac{2}{m^2}\right]$$

$\sin m\pi = 0$ for $m = 0, 1, 2, \ldots$

$\cos m\pi = (-1)^m$, $m = 0, 1, 2, \ldots$

$$= \frac{1}{\pi}\left[\frac{2\pi\times 0}{m} + \frac{2(-1)^m}{m^2} - 2/m^2\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^m - 2}{m^2}\right]$$

$$= \frac{2\{(-1)^m - 1\}}{\pi m^2} = \frac{2(-1)^m - 2}{\pi m^2}$$

$$\Rightarrow \boxed{A_m = \frac{2\left((-1)^m - 1\right)}{\pi m^2}}$$

(ii)

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin mx + \int_{0}^{\pi} f(x)\sin mx\right]dx$$

$$= \frac{1}{\pi}\left[\int_{0}^{-\pi} x\sin mx + \int_{0}^{\pi} x\sin mx\right]dx$$

$$= \frac{1}{\pi}\left[\left[(x)\left(-\frac{\cos mx}{m}\right)-(1)\left(-\frac{\sin mx}{m^2}\right)\right]_0^{-\pi} + \left[(x)\left(-\frac{\cos mx}{m}\right)-(1)\left(\frac{-\sin mx}{m^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[(-\pi)\left(-\frac{\cos -\pi m}{m}\right)+\frac{\sin -m\pi}{m^2}-0+0\right]+\left[\pi\left(-\frac{\cos m\pi}{m}\right)+\frac{\sin m\pi}{m^2}+0+0\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos m\pi}{m} - \frac{\sin m\pi}{m^2} - \frac{\pi\cos m\pi}{m} + \frac{\sin m\pi}{m^2}\right]$$

$$= \frac{1}{\pi}\times 0 = 0$$

$\therefore A_0, A_m, B_m$ এর মান (1) নং এ বসিয়ে পাই,

$$F(x) = \frac{\pi}{2} + \sum_{m=1}^{\infty}\left[\left(\frac{2(-1)^m - 2}{\pi m^2}\right).\cos mx + 0.\sin mx\right]$$

$$\Rightarrow F(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{(-1)^1 - 1}{1^2}.\cos 1x + 0 + \frac{(-1)^3 - 1}{\pi . 3^2}.\cos 3x \cdots\right]$$

$$\Rightarrow F(x) = \frac{\pi}{2} + \left(\frac{2\times(-2)}{\pi}\right)\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$\therefore F(x) = \pi/2 - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Problem || $F(x) = \begin{cases} 0 & \text{when } -\pi < x < 0 \\ \dfrac{\pi x}{4} & \text{when } 0 < x < \pi \end{cases}$ function of fourier..

$\therefore \lambda = b - a = \pi - (-\pi) = 2\pi$ এখন $\frac{2}{\lambda} = \frac{1}{\pi}$, $k = \frac{2\pi}{\lambda} = 1$

$$\therefore A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\,dx\right] = \frac{1}{\pi}\cdot\frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{4}\cdot\frac{\pi^2}{2} = \frac{\pi^2}{8}$$

$$\therefore \boxed{A_0 = \frac{\pi^2}{8}}$$

$$\therefore A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos mx\,dx + \int_{0}^{\pi} \frac{\pi x}{4}\cos mx\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{4}\int_{0}^{\pi} x\cos mx\,dx$$

$$= \frac{1}{4}\left[x\,\frac{\sin mx}{m} - (1)\,\frac{-\cos mx}{m^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\pi\,\overset{0 \text{ for } m=1,2,3\ldots}{\frac{\sin m\pi}{m}} + \frac{\cos m\pi}{m^2} - \frac{\cos 0}{m^2}\right]$$

$$= \frac{1}{4}\left[\frac{(-1)^m}{m^2} - \frac{1}{m^2}\right] = \frac{1}{4}\,\frac{(-1)^m - 1}{m^2}$$

$$\boxed{A_m = \frac{1}{4}\left\{\frac{(-1)^m - 1}{m^2}\right\}}$$

$$\therefore B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin mx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \sin mx \, dx + \int_{0}^{\pi} f(x) \sin mx \, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} \frac{\pi x}{4} \cdot \sin mx \, dx\right]$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{4} \int_{0}^{\pi} x \cdot \sin mx \, dx = \frac{1}{4}\int_{0}^{\pi} x \sin mx \, dx$$

$$= \frac{1}{4}\left[x\left(-\frac{\cos mx}{m}\right) - (1)\left(-\frac{\sin mx}{m^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{4}\left[-\pi\frac{\cos m\pi}{m} + \frac{\sin m\pi}{m^2} + 0 + \frac{\sin 0}{m^2}\right]$$

$$= \frac{1}{4}\left[-\pi\frac{(-1)^m}{m} + \frac{0}{m^2}\right] = \frac{1}{4}\left[-\frac{\pi(-1)^m}{m}\right]$$

$$\Rightarrow = \frac{1}{4}\left(\frac{1 - m\pi(-1)^m}{m^2}\right) = -\frac{\pi(-1)^m}{4m}$$

$$\Rightarrow \therefore \boxed{B_m = -\frac{\pi(-1)^m}{4m}}$$

∴ fourier series

$$F(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty}\left[A_m \cos mx + B_m \sin mx\right]$$

$$= \frac{\pi^2}{16} + \sum_{m=1}^{\infty}\left[\frac{(-1)^m - 1}{4m^2}\cos mx - \frac{\pi(-1)^m}{4m}\sin mx\right]$$

$$= \frac{\pi^2}{16} + \left(\frac{-2}{4(1)^2}\cos x + 0 \cdot \frac{-2}{4 \cdot 3^2}\cos 3x + \cdots\right)$$

$$\left(+\frac{\pi \sin x}{4(1)} + \frac{\pi}{4(2)}\sin 2x - \frac{\pi}{4(3)}\sin 3x + \frac{\pi}{4(4)}\sin 4x + \cdots\right)$$

$$\Rightarrow F(x) = \frac{\pi^2}{16} - \frac{2}{4(1)^2}\cos x - \frac{2}{4(3)^2}\cos 3x - \cdots$$

$$+ \frac{\pi \sin x}{4(1)} - \frac{\pi}{4(2)}\sin 2x + \frac{\pi}{4(3)}\sin 3x$$

Half Range furier series/ অর্ধবেৰুত্ত ফুরিয়ার ধারা

⇒ যদি ফুরিয়ার ধারায় কেবলমাত্র কোসাইন ধারা বা সাইন ধারা বিদ্যমান থাকে তবে ইহাকে অর্ধবেৰুত্ত ফুরিয়ার ধারা বলে।

যদি furier ধারায় কেবলমাত্র কোসাইন ধারা বিদ্যমান থাকে তবে ইহাকে অর্ধবেৰুত্ত ফুরিয়ার ধারা বলে।

যেমন, $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx$

for $(-\pi, \pi)$ interval

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad A_m = \frac{2}{\pi}\int_0^{\pi} f(x)\cos mx\,dx$$

আবার যদি furier series এ কেবলমাত্র সাইন ধারা বিদ্যমান থাকে তবে ইহাকে অর্ধবেৰুত্ত সাইন furier ধারা বলে।

যেখানে, $f(x) = \sum_{n=1}^{\infty} B_n \sin nx$

Here, $B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx$

$\therefore$ এখানে, interval $(0,\pi)$ থাকে যা fourier ব্যবধি $(-\pi,\pi)$ এর অর্ধেক, তাই একে Half Range fourier বলে।

1

fourier series formula for exponential function

1 $\cancel{\int e^{x} \cos\beta x\, dx = \frac{e^{x}}{\alpha^2+\beta^2}\left[\quad + \beta \sin\beta x\right]}$

1 $$\int e^{\alpha x} \cos\beta x \, dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha \cos\beta x + \beta \sin\beta x\right]$$

$\boxed{\text{for cosine}}$

2 $$\int e^{\alpha x} \sin\beta x \, dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha \sin\beta x - \beta \cos\beta x\right]$$

$\boxed{\text{for Sine}}$

important

Fourier transformation

(1) Integral transformation :-

⇒ যা একটি ফাংশনকে অন্যেকটি ফাংশনের মাধ্যমে প্রকাশ করা।

If we transfer a one dimentional continuous function of some space variable, then if $f(x)$ could be expressed as a linear combination of an infinite number of harmonic contributions,

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos kx\,dk + \int_0^\infty B(k)\sin kx\,dk\right] \quad \text{(i)}$$

If $A(k)$ & $B(k)$ are fourier cosine & sine transform of $f(x)$ given by,

$$\therefore A(k) = \int_{-\infty}^{\infty} f(x')\cos kx'\,dx'$$

$$B(k) = \int_{-\infty}^{\infty} f(x')\sin kx'\,dx' \quad \text{(ii)}$$

(1)

x' is a dummy variable over which the integration is carried out. (dummy variable | নির্দিষ্ট মান নেয়)
0 → male
1 → female

(ii) সংখ্যমান (i) নং এ বসিয়ে পাই,

$$f(x) = \frac{1}{\pi}\left[\left[\int_0^\infty \int_{-\infty}^{\infty} f(x')\cos kx' \cos kx \, dx' dk\right] + \left[\int_0^\infty \int_{-\infty}^{\infty} f(x')\sin kx' \sin kx \, dx' dk\right]\right]$$

$$= \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^{\infty} f(x')\cos k(x'-x) \, dx' dk\right]$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(x')\cos k(x'-x)\, dx'\right] dk \quad \text{—— (iii)}$$

if equation

The quantity in the square bracket is an even function of k, so we can write,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{\infty} f(x')\cos k(x'-x)\, dx'\right] dk \quad \text{—— (iv)}$$

→ যদি $\int_{-\infty}^{+\infty}$ কোনো ফাংশন even হলে, $= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\,dx$
(iv) থেকে (iii) হয় কেননা even হলে, $= 2\cdot\frac{1}{2\pi}\int_0^\infty f(x)\,dx$ লেখা যায়।

∴ Let us consider an odd function

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')\, \sin k(x'-x)\, dx'\right] dk = 0 \quad \text{—(v)}$$

(iv+v) করে পাই,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x') \left\{ i\sin k(x'-x) + \cos k(x'-x) \right\} dx'\right] dk$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')\, e^{ik(x'-x)}\, dx'\right] dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(x')\, e^{ikx'}\, dx'\right]}_{F(k)} e^{-ikx}\, dk \quad \text{—(vi)}$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{-ikx}\, dk} \quad \text{—(vii)}$$

where,

$$F(k) = \int_{-\infty}^{\infty} f(x')\, e^{ikx'}\, dx'$$

$$\therefore \boxed{F(k) = \int_{-\infty}^{\infty} f(x) \cdot e^{ikx} \cdot dx}$$ — (VIII)

Most important

equation

$f(x)$ এর fourier tranform হলো $F(k)$

$\therefore$ The transformation $F(k)$ is the fourier transformation of $f(x)$ which is symbolically denoted by, $\boxed{F(k) = \mathcal{F}\{f(x)\}}$

$f(x)$ it self is said to be the inverse fourier tramformation of $F(k)$

$$f(x) = \mathcal{F}^{-1}\{F(k) = \mathcal{F}^{-1}\mathcal{F}\{f(x)\}$$

for, spacial frequency, $x \to k = \frac{2\pi}{\lambda}$; Temporal frequency, $t \to \omega = \frac{2\pi}{T}$

$\therefore$ $x = t$, $k = \omega$ বসিয়ে (vii, viii) নং সমীকরণে বসিয়ে পাই

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} \cdot d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

Most Important

Fourier Transformation of the Gaussian Function

We know that,

The Gaussian probability function is, $f(x) = C e^{-ax^2}$

where, $C = \sqrt{\frac{a}{\pi}}$ ['a' is a constant]

$$\therefore f(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$$

We know that,

Fourier transformation of $f(x)$, $\boxed{F(k) = \int_{-\infty}^{\infty} f(x) \cdot e^{ikx}\, dx}$

$$F(k) = \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{a}{\pi}}\, e^{-ax^2}}_{f(x)} \cdot e^{ikx} \cdot dx$$

$$\Rightarrow F(k) = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2 + ikx} \cdot dx$$

$$\therefore F(k) = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2 + ikx} \cdot dx \text{ --- (i)}$$

এখানে, (e) এর Power কে ভাঙাতে হবে।

$$-ax^2 + ikx = -(\sqrt{a}x)^2 + 2 \cdot \sqrt{a}x \cdot \frac{ik}{2\sqrt{a}} + \left(\frac{ik}{2\sqrt{a}}\right)^2 - \left(\frac{ik}{2\sqrt{a}}\right)^2$$

$$= -1\left[(\sqrt{a}x)^2 - 2\sqrt{a}x\frac{ik}{2\sqrt{a}} + \left(\frac{ik}{2\sqrt{a}}\right)^2\right] + \left(\frac{ik}{2\sqrt{a}}\right)^2$$

$$= -\left(\sqrt{a} \cdot x - \frac{ik}{2\sqrt{a}}\right)^2 + \frac{i^2k^2}{4a}$$

$$\Rightarrow -ax^2+ikx = -\left(x\sqrt{a}-\frac{ik}{2\sqrt{a}}\right)^2-\frac{k^2}{4a} \quad \left[\begin{array}{l} i=\sqrt{-1} \\ i^2=-1 \end{array}\right]$$

এখানে, $x\sqrt{a}-\frac{ik}{2\sqrt{a}}$ কে β function বলে।

অর্থাৎ, $\beta = x\sqrt{a}-\frac{ik}{2\sqrt{a}}$ $\left[\frac{d}{d\beta}\right.$ দ্বারা গুণ করে$\left.\right]$

$$\Rightarrow \frac{d\beta}{d\beta} = \frac{d}{d\beta}\left(x\sqrt{a}-\frac{ik}{2\sqrt{a}}\right)$$

$$\Rightarrow 1 = \sqrt{a}\cdot\frac{dx}{d\beta} - 0$$

$$\Rightarrow \boxed{dx = \frac{1}{\sqrt{a}}\cdot d\beta \quad \beta = x\sqrt{a}-\frac{ik}{2\sqrt{a}}}$$

$$\therefore \boxed{-ax^2+ikx = -\beta^2-\frac{k^2}{4a}}$$

এই তিনটি মান (i) নং সমীকরণে বসিয়ে পাই,

$$F(k) = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} e^{-\beta^2-\frac{k^2}{4a}}\cdot\frac{1}{\sqrt{a}}\cdot d\beta$$

$$= \frac{\sqrt{a}}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{a}}\left[\int_{-\infty}^{\infty} e^{-\beta^2}\cdot d\beta\right]\cdot e^{-k^2/4a}$$

$$= \frac{1}{\sqrt{\pi}}\cdot e^{-k^2/4a}\cdot\left[\int_{-\infty}^{\infty} e^{-\beta^2}\cdot d\beta\right]$$

এটার value $\sqrt{\pi}$

$$= \frac{1}{\sqrt{\pi}}\cdot e^{-k^2/4a}\cdot\sqrt{\pi}$$

$$\therefore F(k) = e^{-k^2/4a}$$

$\therefore F(k) = e^{-k^2/4a}$

এটিও একটির Gaussian function কারণ, Gaussian probability function, $f(x) = c \cdot e^{-ax^2}$ ~~এমন~~ যা $F(k)$ ~~এর সাথে~~ সাথে মিল পাওয়া যায়।

সুতরাং,

~~এর~~ {একটি Gaussian Function এর fourier transform করলে আরেকটি Gaussian Function পাওয়া যায়।}

উপরে Gaussian probability function, $f(x) = c\,e^{-ax^2}$

$$= \sqrt{\frac{a}{\pi}} \cdot e^{-ax^2}$$

কে fourier transform করলে, $F(k) = e^{-k^2/4a}$ পাওয়া যায়.

যেটা কিনা নিজেও একটা Gaussian Function.

(1)

Find the fourier ~~function~~ Transfrom of, $f(t) = \begin{cases} \cos^2 \omega_p t & ; |t| \le T \\ 0 & ; \text{otherwise} \end{cases}$

এখানে জানি, $|t| \le T \Rightarrow -T \le t \le T \rightarrow$ Limit

∴ fourier tramsform হলো,

$$F(\omega) = \int_{-T}^{T} f(t) \cdot e^{i\omega t} \cdot dt$$

$$= \int_{-T}^{T} \cos^2 \omega_p t \cdot e^{i\omega t} \cdot dt + \int_{-T}^{T} 0 \cdot e^{i\omega t} \cdot dt$$

$$= \int_{-T}^{T} \underbrace{\cos^2 \omega_p t}_{\text{দুই-ঘাত থাকিলে}} \cdot e^{i\omega t} \cdot dt$$

$\therefore \cancel{\cos^2 \omega_p t = \frac{1}{2}(1+\cos 2\omega_p t)}$ 1 ঘাতে আনতে হবে

$= \cancel{\frac{1}{2}(1+\cos \omega_p t + \cos)}$

$= \frac{1}{2}\cancel{(1+\cos}$

We know,

$2\cos^2 A = 1+\cos 2A$

$\Rightarrow \cos^2 A = \frac{1+\cos 2A}{2}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\Rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\cos(2\omega_p t) = \frac{e^{i2\omega_p t} + e^{-i2\omega_p t}}{2}$

$\therefore \cos^2 \omega_p t = \frac{1}{2}(1+\cos 2\omega_p t)$

$\Rightarrow = \frac{1}{2}\left(1+\frac{e^{i2\omega_p t} + e^{-i2\omega_p t}}{2}\right)$

$\therefore \cos^2 \omega_p t = \frac{1}{2} + \frac{e^{i2\omega_p t}}{4} + \frac{e^{-i2\omega_p t}}{4}$

[exponential form এ নিয়ে solve করে আবার sin, cos এ আনতে হবে]

$$\therefore F(\omega) = \int_{-T}^{T}\left(\tfrac{1}{2} + \frac{e^{i2\omega_p t}}{4} + \frac{e^{-i2\omega_p t}}{4}\right) e^{i\omega t} \cdot dt$$

$$= \frac{1}{2}\int_{-T}^{T} e^{i\omega t} \cdot dt + \frac{1}{4}\int_{-T}^{T} e^{i(2\omega_p+\omega)t} \cdot dt + \frac{1}{4}\int_{-T}^{T} e^{i(\omega-2\omega_p)t} \cdot dt$$

$$= \frac{1}{2}\cdot\frac{1}{i\omega}\left[e^{i\omega t}\right]_{-T}^{T} + \frac{1}{4}\cdot\frac{1}{i(\omega+2\omega_p)}\left[e^{i(\omega+2\omega_p)t}\right]_{-T}^{T} + \frac{1}{4}\left[\frac{e^{i(\omega-2\omega_p)t}}{i(\omega-2\omega_p)}\right]_{-T}^{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2i\omega}\cdot\left(e^{i\omega T} - e^{-i\omega T}\right) + \frac{1}{4i(\omega+2\omega_p)}\left[e^{i(\omega+2\omega_p)T} - e^{-i(\omega+2\omega_p)T}\right] + \frac{1}{4i(\omega-2\omega_p)}\left[e^{i(\omega-2\omega_p)T} - e^{-i(\omega-2\omega_p)T}\right]$$

$$= \frac{1}{2i\omega}\left(\cancel{\cos\omega T} + i\sin\omega T - \cancel{\cos\omega T} + i\sin\omega T\right) + \frac{1}{4i(\omega+2\omega_p)}\left[\cancel{\cos(\omega+2\omega_p)T} + i\sin(\omega+2\omega_p)T - \cancel{\cos(\omega+2\omega_p)T} + i\sin(\omega+2\omega_p)T\right]$$

$$+ \frac{1}{4i(\omega-2\omega_p)}\left[\cancel{\cos(\omega-2\omega_p)T} + i\sin(\omega-2\omega_p)T - \cancel{\cos(\omega-2\omega_p)T} + i\sin(\omega-2\omega_p)T\right]$$

$$= \frac{2i\sin\omega T}{2i\omega} + \frac{2i\sin(\omega+2\omega_p)\cdot T}{4i(\omega+2\omega_p)} + \frac{2i\sin(\omega-2\omega_p)\cdot T}{4i(\omega-2\omega_p)}$$

$$= \frac{\sin\omega T}{\omega} + \frac{\sin(\omega+2\omega_p)\cdot T}{2(\omega+2\omega_p)} + \frac{\sin(\omega-2\omega_p)\cdot T}{2(\omega-2\omega_p)}$$

$$\therefore F(\omega) = \frac{\sin\omega T}{\omega} + \frac{\sin(\omega+2\omega p)T}{2(\omega+2\omega p)} + \frac{\sin(\omega-2\omega p)T}{2(\omega-2\omega p)}$$

this is fourier transformation of $f(t)$ function.

Physics Department

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(SI)

B

☆ Find the fourier transformation of, $f(t) = \begin{cases} \sin^2\omega_p t ; & |t| \le T \\ 0 & ; \text{otherwise} \end{cases}$

we know, $|t| \le T \Rightarrow -T \le t \le T$ → limit.

f(t) এর fourier transform.

$$F(\omega) = \int_{-T}^{T} f(t) \cdot e^{i\omega t} \cdot dt$$

$$= \int_{-T}^{T} \sin^2\omega_p t \cdot e^{i\omega t} \cdot dt + \int_{-T}^{T} 0 \cdot e^{i\omega t} \cdot dt$$

$$= \int_{-T}^{T} \sin^2\omega_p t \cdot e^{i\omega t} \cdot dt$$

we, eular equation

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore e^{i\theta} + e^{-i\omega} = 2\cos\theta$

$\Rightarrow \cos\theta = \left(\dfrac{e^{i\theta} + e^{-i\omega}}{2}\right)$

$\therefore \cos 2\omega_p t = \dfrac{e^{2i\omega pt} + e^{-2i\omega pt}}{2}$

we know, $2\sin^2\omega_p \cdot t = 1 - \cos 2\omega_p \cdot t$

$\Rightarrow \sin^2\omega_p \cdot t = \frac{1}{2}(1 - \cos 2\omega_p \cdot t)$

$\Rightarrow \sin^2\omega_p t = \frac{1}{2}\left(1 - \dfrac{e^{i2\omega pt} + e^{-i2\omega pt}}{2}\right)$

$\Rightarrow \sin^2\omega_p \cdot t = \frac{1}{2} - \dfrac{e^{i2\omega pt}}{4} - \dfrac{e^{-i2\omega p \cdot t}}{4}$

$\therefore \sin^2\omega_p t$ এর মান $F(\omega)$ তে বসিয়ে পাই,

$$F(\omega) = \int_{-T}^{T} \left(\frac{1}{2} - \frac{e^{i2\omega pt}}{4} - \frac{e^{-i2\omega_p \cdot t}}{4}\right) \cdot e^{i\omega t} \cdot dt$$

$$\Rightarrow F(\omega) = \frac{1}{2}\int_{-T}^{T} e^{i\omega t}\,dt - \frac{1}{4}\int_{-T}^{T} e^{i(2\omega_p+\omega)t}\,dt - \frac{1}{4}\int_{-T}^{T} e^{i(\omega-2\omega_p)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{i\omega t}}{i\omega}\right]_{-T}^{T} - \frac{1}{4}\left[\frac{e^{i(\omega+2\omega_p)t}}{i(\omega+2\omega_p)}\right]_{-T}^{T} - \frac{1}{4}\left[\frac{e^{i(\omega-2\omega_p)t}}{i(\omega-2\omega_p)}\right]_{-T}^{T}$$

$$= \frac{1}{2i\omega}\left(e^{i\omega T} - e^{-i\omega T}\right) - \frac{1}{4i(\omega+2\omega_p)}\left(e^{i(\omega+2\omega_p)T} - e^{-i(\omega+2\omega_p)T}\right)$$

$$- \frac{1}{4i(\omega-2\omega_p)}\left(e^{i(\omega-2\omega_p)T} - e^{-i(\omega-2\omega_p)T}\right)$$

$$= \frac{1}{2i\omega}\left[\cancel{\cos\omega T} + i\sin\omega T - \cancel{\cos\omega T} + i\sin\omega T\right] - \frac{1}{4i(\omega+2\omega_p)}$$

$$\left[\cancel{\cos(\omega+2\omega_p)T} + i\sin(\omega+2\omega_p).T - \cancel{\cos(\omega+2\omega_p).T} + i\sin(\omega+2\omega_p).T\right]$$

$$- \frac{1}{4i(\omega-2\omega_p)}\left[\cancel{\cos(\omega-2\omega_p).T} + i\sin(\omega-2\omega_p).T - \cancel{\cos(\omega-2\omega_p)} + i\sin(\omega-2\omega_p).T\right]$$

$$= \frac{\sin\omega T}{\omega} - \frac{\sin(\omega+2\omega_p).T}{2(\omega+2\omega_p)} - \frac{\sin(\omega-2\omega_p).T}{2(\omega-2\omega_p)}$$

$\therefore$ The fourier transform of $f(t)$ is

$$F(\omega) = \frac{\sin\omega T}{\omega} - \frac{\sin(\omega+2\omega_p).T}{2(\omega+2\omega_p)} - \frac{\sin(\omega-2\omega_p).T}{2(\omega-2\omega_p)}$$

Fourier transform of $f(x) = \begin{cases} E_0 \sin k_p x, & |x| < L \\ 0, & \text{otherwise} \end{cases}$

Limit, $|x| < L \Rightarrow -L < x < L$

$\therefore$ we know, fourier transform, $f(x)$ is

$$F(k) = \int_{-L}^{L} f(x)\, e^{ikx}\, dx$$

$$= \int_{-L}^{L} E_0 \sin k_p x \cdot e^{ikx}\, dx + \int_{-L}^{L} 0 \cdot e^{ikx}\, dx$$

$$= \int_{-L}^{L} E_0 \sin k_p x \cdot e^{ikx}\, dx$$

we know

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cancel{\Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta}}$

$\cancel{\therefore \cos\theta = \frac{1}{2}}(e^{i\theta} + e^{-i\theta})$

$\therefore e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$\therefore \sin k_p x = \frac{1}{2i}\left(e^{ik_p x} - e^{-ik_p x}\right)$

এখন $F(k)$ তে বসিয়ে পাই,

$$F(k) = \int_{-L}^{L} \frac{E_0}{2i}\left(e^{ik_p x} - e^{-ik_p x}\right) \cdot e^{ikx}\, dx$$

$$\Rightarrow F(k) = \frac{E_0}{2i}\left[\int_{-L}^{L} e^{i(k+k_p).x}\,.dx - \int_{-L}^{L} e^{i(k-k_p).x}\,dx\right]$$

$$= \frac{E_0}{2i}\left[\frac{1}{i(k+k_p)}\left[e^{i(k+k_p)x}\right]_{-L}^{L} - \frac{1}{i(k-k_p)}\left[e^{i(k-k_p).x}\right]_{-L}^{L}\right]$$

$$= \frac{E_0}{2i}\left[\frac{1}{i(k+k_p)}\left(e^{i(k+k_p)L} - e^{-i(k+k_p).L}\right) - \frac{1}{i(k-k_p)}\left(e^{i(k-k_p).L} - e^{-i(k-k_p).L}\right)\right]$$

$$= \frac{E_0}{2i^2(k+k_p)}\left(\cancel{\cos(k+k_p).L} + i\sin(k+k_p).L - \cancel{\cos(k+k_p).L} + i\sin(k+k_p).L\right) - \frac{E_0}{2i^2(k-k_p)}\left(\cancel{\cos(k-k_p).L} + i\sin(k-k_p).L - \cancel{\cos(k-k_p).L} + i\sin(k-k_p).L\right)$$

$$= \frac{E_0 \cdot 2i\sin(k+k_p).L}{2i^2.(k+k_p)} - \frac{E_0\, 2i\sin(k-k_p).L}{2i^2(k-k_p)}$$

$$= E_0\left(\frac{\sin(k+k_p).L}{i(k+k_p)} - \frac{\sin(k-k_p).L}{i(k-k_p)}\right)$$

$$\therefore F(k) = E_0\left[\frac{\sin(k+k_p).L}{i(k+k_p)} - \frac{\sin(k-k_p).L}{i(k-k_p)}\right]$$

This is Fourier transform of $f(x)$ function

☆ Fourier transform of $f(x) = \begin{cases} E_0 \cos k_p x \ ; & |x| < L \\ 0 & ; \text{ otherwise} \end{cases}$

$\therefore F(k) = \int_{-L}^{L} f(x).e^{ikx}.dx$

$= \int_{-L}^{L} E_0 \cos k_p.x .e^{ikx}.dx + \cancel{\int 0 . e^{ikx}.dx} \rightarrow 0$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta \ , \ e^{-i\theta} = \cos\theta - i\sin\theta$

$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\therefore \cos k_p.x = \frac{1}{2}(e^{ik_p x} + e^{-ik_p x})$

$\therefore F(k) = \int_{-L}^{L} \frac{E_0}{2}(e^{ik_p.x} + e^{-ik_p x}).e^{ikx}.dx$

$= \frac{E_0}{2}\left[\int_{-L}^{L} e^{i(k+k_p).x}.dx + \int_{-L}^{L} e^{i(k-k_p).x}.dx\right]$

$= \frac{E_0}{2}\left[\frac{1}{i(k+k_p)}\left[e^{i(k+k_p).x}\right]_{-L}^{L} + \frac{1}{i(k-k_p)}\left[e^{i(k-k_p)x}\right]_{-L}^{L}\right]$

$= \frac{E_0}{2i(k+k_p)}\left[e^{i(k+k_p)L} - e^{-i(k+k_p).L}\right] + \frac{E_0}{i(k-k_p)}$

$\left[e^{i(k-k_p)L} - e^{-i(k-k_p)L}\right]$

$$\Rightarrow F(k) = \frac{E_0}{2i(k+k_p)}\Big[\cancel{\cos(k+k_p)L} + i\sin(k+k_p).L - \cancel{\cos(k+k_p).L} + i\sin(k+k_p).L\Big] + \frac{E_0}{2i(k-k_p)}$$

$$\Big[\cancel{\cos(k-k_p).L} + i\sin(k-k_p).L - \cancel{\cos(k-k_p).L} + i\sin(k-k_p)L\Big]$$

$$= \frac{E_0\, 2i\sin(k+k_p).L}{2i(k+k_p)} + \frac{E_0\, 2i\sin(k-k_p).L}{2i(k-k_p)}$$

$$= E_0\left[\frac{\sin(k+k_p).L}{(k+k_p)} + \frac{\sin(k-k_p).L}{(k-k_p)}\right]$$

$$\therefore F(k) = E_0\left[\frac{\sin(k+k_p).L}{(k+k_p)} + \frac{\sin(k-k_p).L}{(k-k_p)}\right]$$

This is fourier transform of $f(x)$ Function.

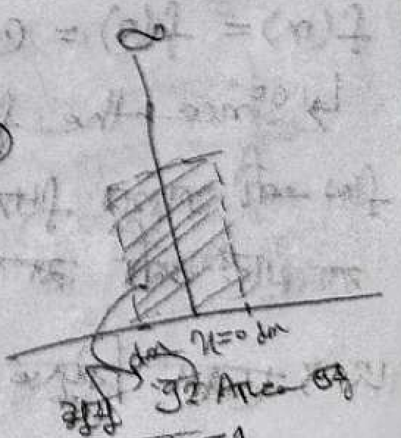
The direct Delta function

কোনো নির্দিষ্ট বিন্দু শুধু মান, অন্য স্থানের মান বিদ্যমান, এর আগে বা পরে তার মান শূন্য, সেই ফাংশনকেই ~~delta~~ ~~direct~~ Direc delta function বলে।

A ~~covenient~~ mathematicle expression representation of sharply peaked stimulus in the direct delta function $f(x)$. This is a quantity which is zero every where except at the region where it gose to infinity in a mannar so, as to encompas a unit area.

That is, $$\delta(x) = \begin{cases} 0 \; ; & x \neq 0 \\ \infty \; ; & x = 0 \end{cases} \qquad (1)$$

therefore $$\int \underbrace{\delta(x)}_{\text{height}} \, \underbrace{dx}_{\text{width}} = \underbrace{1}_{\text{Area}} \qquad (2)$$

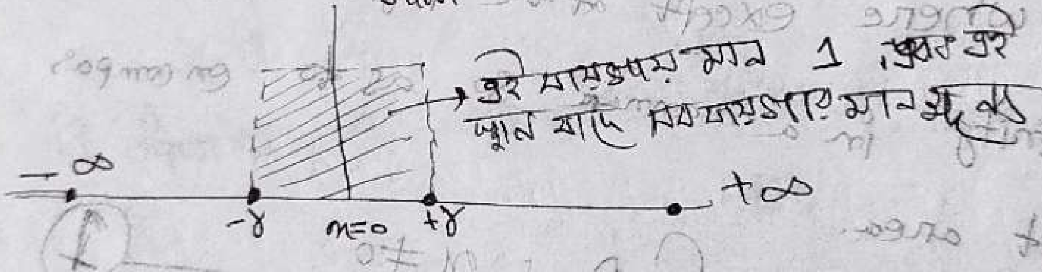


The most basic application of $\delta(x)$ is

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx .$$ where $f(x)$ crrosponds to any continous function.

Shifting property of delta function

Let us consider a tiny interval from $x = -\gamma$ to $+\gamma$ about the origin



$f(x) = f(0) = \text{constant}$

↳ Since the function is continous

$f(x)$ একটি যথেষ্ট ~~ফাংশন~~ যেখানে $x=0$ স্থানে continous মধ্যমানের একটি ~~মান~~ বিন্দু মান পাওয়া যাবে, তাছাড়া অন্য কোনো বিন্দুতে তার মান পাওয়া যাবে না।

Then, $x = -\infty$ to $-\gamma$ one from $+\gamma$ to $+\infty$

∴ The Integral,

$$\int_{-\infty}^{\infty} \delta(x) \cdot f(x)\, dx = 0,$$ because the δ-function is zero

→ এখানে Just $(-\gamma$ to $+\gamma)$ ব্যবধিতে $\delta(x)$ এর মান আছে। তাই যখন $(-\infty$ to $+\infty)$ নেয়া হয় তখন সেখানে অনেক জায়গা আছে যেখানে $\delta(x) = 0$ থাকে। যার জন্য সম্পূর্ণ Integral.

$$\therefore \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{+\infty} 0 \cdot f(x) \cdot dx = 0$$ আসে।

There for we can write, $$\boxed{\int_{-\gamma}^{+\gamma} \delta(x) \cdot dx = 1}$$

There fore, $$\int_{-\infty}^{\infty} \delta(x) \cdot f(x) \cdot dx \simeq f(0) = \text{Constant}$$

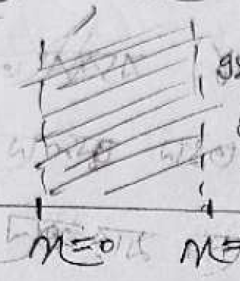
$f(0)$ মানে $x=0$ এর $x=0$ বিন্দুতে Integral করলে আমরা constant value পাই, তা ছাড়া অন্য সময়ে মান শূন্য

∴ This is called the Shifting property of delta (δ) function

Shifting origin property

Now shifting the origin at x_0 such that,

$$\delta(x-x_0) = \begin{cases} 0 & ; x \neq x_0 \\ 1 & ; x = x_0 \end{cases}$$



এখানে, $x=x_0$ ~~বিন্দুতে~~ function এর value = 1, তাছাড়া বাকি সব জায়গায় এর value শূন্য।

This is the spike reside rather ~~$x=0$~~ at $x=0$ at $x=x_0$ এখানে function এর অক্ষ $x=0$ হতে $x=x_0$ বিন্দুতে নেয়া হয়েছে। যেমনটা ডেল্টা ফাংশনের মান (1)। "এখানে ডেল্টা ও ফাংশনের অবস্থানের পরিবর্তনের মাধ্যমে যোজিতকরণ 1 হবে।"

$$\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)\, dx = 1}$$

Similarly এখানে ~~function~~ continuous function $f(x)$ গুণ করলে,

$$\boxed{\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)}$$

নতুন origin এর function এর মান পাওয়া যাবে তথা মূল $f(x_0)$ থেকে

"This is also a shifting property of δ function"

Propertise of fourier transform.

i) Linear propertise. If $F_1(k)$ & $F_2(k)$ are fourier transform of $f_1(x)$ & $f_2(x)$ respectively, then,

$$\mathcal{F}_c\{a f_1(x) + b f_2(x)\} = a F_1(k) + b F_2(k)$$ → where a & b are constant

Proof: $F_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} . f_1(x)\, dx$

$F_2(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} . f_2(x)\, dx$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} . dx$

L.H.S $= \mathcal{F}_c \{a f_1(x) + b f_2(x)\}$

$= \cancel{a F_1(k) + b F_2(k)}$

$= \frac{1}{\sqrt{2\pi}} \cancel{\int_{-\infty}^{\infty} e^{ikx} . a . f_1(x) . dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} . b . f_2(x) . dx}$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \{a f_1(x) + b f_2(x)\} . e^{ikx} . dx$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) \cdot e^{ikx} \cdot dx + b \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) \cdot e^{ikx} \cdot dx$$

$$= a\, F_1(k) + b\, F_2(k).$$

Gamma & Beta function

Mathematisian Ogler সর্বপ্রথম Gamma & beta function আবিষ্কার করেন।

1st Eulerian → beta function

2nd Eulerian → gamma function

গামা ফাংশন বা Second Eulerian: (এক চলক বিশিষ্ট)

একটি ইন্টিগ্রাল $\int_0^{\infty} e^{-x} x^{n-1} \cdot dx$ $(n>0)$ কে গামা ফাংশন বা second Eulerian বলে। ইহাকে $\sqrt{n}$ দ্বারা প্রকাশ করা হয়।

$$\therefore \boxed{\Gamma(n) = \sqrt{n} = \int_0^{\infty} e^{-x} x^{n-1} \cdot dx}$$

বিটা ফাংশনের first Eulerian : (২ টাকা মার্ক প্রশ্ন) (নির্দিষ্ট interval থাকে)

যদি থাকে $\int_0^1 x^{m-1}.(1-x)^{n-1}.dx$ কে বিটা ফাংশনের first eulerian হিসেবে $\beta(m,n)$ প্রকাশ করা হয়, অর্থাৎ:

$\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ যেখানে $m,n>0$

Star Gamma & beta function ~~question~~ formulas proved। / প্রমাণ করো ২টি

(formulas proved)

(i) $\Gamma 1 = 1$

(ii) $\Gamma(n+1) = n\Gamma n \; (=n!)$

(iii) $\beta(m,n) = \beta(n,m)$

(iv) $\beta(m,n) = \dfrac{\Gamma m . \Gamma n}{\Gamma m+n}$

formula প্রমাণ

$\beta(m,n) = \int_0^\infty \dfrac{y^{m-1}}{(1+y)^{m+n}}\,dy$; $\Gamma 1 = 1$

$\Gamma(n+m)+1 = (n+m)\Gamma(n+m) = (n+m)\Gamma(n+m)-1+1$

$= (n+m)(n+m-1)\Gamma n+m-1$

$= (n+m)(n+m-1)\Gamma(n+m-2)+1$

$= (n+m)(n+m-1)(n+m-2)\Gamma n+m-2$ লেখা যায়।

(v) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \dfrac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

(vi) $\Gamma \frac{1}{2} = \sqrt{\pi}$ ।

Question no.-1

$$\boxed{\int e^{mx}\,dx = \frac{e^{mx}}{m}}$$

$\Gamma 1 = 1$

আমরা জানি,

gamma function, $\Gamma n = \int_0^\infty e^{-x}\cdot x^{n-1}\cdot dx$

$$\Rightarrow \Gamma 1 = \int_0^\infty e^{-x}\cdot x^{1-1}\cdot dx$$

$$\Rightarrow \Gamma 1 = \int_0^\infty e^{-x}\cdot x^0\cdot dx = \int_0^\infty e^{-x}\cdot dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$\Rightarrow \Gamma 1 = \left(-e^{-\infty} + e^{-0}\right) = \left(\frac{1}{-e^{\infty}} + \frac{1}{e^0}\right) = \left(\frac{1}{-\infty} + \frac{1}{1}\right)$$

$$\Rightarrow \Gamma 1 = (0+1) \Rightarrow \boxed{\Gamma 1 = 1}$$ (Proved)

Question no. 2

$\Gamma(n+1) = n\Gamma n = n!$

আমরা জানি,

$$\Gamma n = \int_0^\infty e^{-x}\cdot x^{n-1}\cdot dx$$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x}\cdot x^{n+1-1}\cdot dx = \int_0^\infty e^{-x}\cdot x^n\cdot dx$$

$$\Rightarrow \cancel{\Gamma(n+1)}\ \Gamma(n+1) = \int_0^\infty e^{-x}\cdot x^n\cdot dx$$

আমরা জানি,

$$\int u.v.\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u.\int v.\,dx\right\}dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x}.x^n.\,dx = \int_0^\infty x^n.e^{-x}.\,dx$$

$$= x^n\int_0^\infty e^{-x}\,dx - \int_0^\infty\left\{\frac{d}{dx}x^n.\int_0^\infty e^{-x}\,dx\right\}dx$$

$$= \cancel{x^n.\left\{\left[\frac{e^{-x}}{-1}\right]_0^\infty\right\} - \int_0^\infty\left\{nx^{n-1}.\left[\frac{e^{-x}}{-1}\right]_0^\infty\right\}dx}$$

$$= x^n\left\{\cancel{\frac{-1}{e^\infty} + \frac{1}{e^0}}\right.$$

আমরা জানি,

$$\int u.v.\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u.\int v\,dx\right\}dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x}.x^n.\,dx = \int x^n.e^{-x}\,dx$$

$$= x^n\int_0^\infty e^{-x}.\,dx - \int_0^\infty\left\{\frac{d}{dx}x^n.\int e^{-x}\,dx\right\}dx$$

$$= \left[x^n e^{-x}\right]_0^\infty - \int_0^\infty n.x^{n-1}.(-e^{-x}).\,dx$$

$$=\left(-\infty^n \cdot e^{-\infty} + 0^n \cdot e^{-0}\right) + n\int_0^\infty e^{-x}\, x^{n-1} \cdot dx$$

$$= \left(\frac{-\infty^n}{e^{\infty}} + 0^n \cdot e^{0}\right) + n\,\Gamma(n)$$

$$= n\,\Gamma(n)$$

$$\Rightarrow \boxed{\Gamma(n+1) = n\,\Gamma(n)} \quad \text{(Proved)}$$

এটা প্রমান হলো যে, $\Gamma(n+1) = n\,\Gamma(n)$ অর্থাৎ,

$\Gamma(10) = \Gamma(9+1) = 9\,\Gamma(9)$ বা $\Gamma(7) = \Gamma(6+1) = 6\,\Gamma(6)$

$\left(\Gamma(10) = 9\,\Gamma(9)\right) \xleftrightarrow[\text{লেখা যায়,}]{} \left(\Gamma(7) = 6\,\Gamma(6)\right)$

$$\therefore \Gamma(n+1) = n\,\Gamma(n) = n\,\Gamma(n+1-1) = n\,\Gamma((n-1)+1)$$

$$= n(n-1)\,\Gamma(n-1) = n(n-1)\,\Gamma((n-1)-1+1)$$

$$= n(n-1)\,\Gamma((n-2)+1) = n(n-1)(n-2)\,\Gamma(n-2)$$

$$\Rightarrow \Gamma(n+1) = n(n-1)(n-2)(n-3)(n-4)\ldots 4 \cdot 3 \cdot 2 \cdot 1$$

এই ভাবে ধারাবাহিক চলতেই থাকবে.

$$\Rightarrow \boxed{\Gamma(n+1) = n!} \quad \text{(Proved)}$$

Question no. B

$\beta(m,n) = \beta(n,m)$

আমরা জানি,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx \quad \text{——— (1)}$$

Put $(1-x) = y \Rightarrow x = 1-y$ [অর্থাৎ অনুরীকরণ করেছি]

$\Rightarrow dx = 0 - dy$

$\Rightarrow dx = -dy$

$\therefore$ x এর upper limit = 1 & lower limit = 0, $\therefore$ x এর upper ও lower limit যদি আমরা $1-x=y$ সমীকরণে বসাই তাহলে y এর upper ও lower limit পাবো।

$\therefore$

eq- $(1-x)=y$		
	upper limit	lower limit
x	1 হলে,	0 হলে
y	0 হবে	1 হবে

$\therefore$ y এর upper limit zero & lower limit one হয়েছে।

(i) নং equation এর মধ্যে আমরা x এর জায়গায় 1 বসিয়ে পাই,

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} (y)^{n-1} (-dy)$$ এখানে limit ও y এর হবে,

$$= -\int_0^1 (1-y)^{m-1} y^{n-1} (-dy) \quad \left[\int_0^1 x \cdot dx = -\int_1^0 x \cdot dx\right.$$

$$\Rightarrow \beta(m,n) = \int_0^1 y^{n-1} \cdot (1-y)^{m-1} \cdot dy$$

$$\Rightarrow \boxed{\beta(m,n) = \beta(n,m)} \text{ (Proved)}$$

সুতরাং $\beta(m,n)$ এখানে m আগে n পরে তাই সূত্র

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ হবে।
আগে পরে

তাই যদি n আগে m পরে থাকে

$\int_0^1 x^{n-1} (1-x)^{m-1}$ এর ক্ষেত্রে তখন $\beta(n,m)$ হবে।

Question $\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$

$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \cdot dy$ প্রমাণ কর।

[where, $m,n > 0$]

Solve /

আমরা জানি,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \longrightarrow (1)$$

প্রমাণের জন্য,

এক্ষেত্রে যেহেতু $\left(\frac{1}{1+y}\right)$ একটি ধনাত্মক রাশি অবস্থায় আছে,

তাই ধরি,

$$x = \frac{1}{1+y} = (1+y)^{-1} \Rightarrow \frac{d}{dx} x = \frac{d}{dx}(1+y)^{-1}$$

$$\cancel{\Rightarrow dx = d(1+y \Rightarrow dx = -1}$$

$$\Rightarrow 1 = -1 \cdot (1+y)^{-1-1} \cdot \frac{d(1+y)}{dx}$$

$$\Rightarrow 1 = \frac{-1}{(1+y)^2} \cdot \frac{dy}{dx} \Rightarrow dx = -\frac{1}{(1+y)^2} dy$$

∴ x এর upper ও lower limit $(1, 0)$ এর জন্য y এর

$\cancel{y = \frac{1}{x} - 1}$ $x = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$ সমীকরণ হতে upper ও lower limit হবে $(0, \infty)$

∴ y এর মান ও limit (i) নং equation এ বসিয়ে পাই,

$$\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \cdot \left(1 - \frac{1}{1+y}\right)^{n-1} \cdot (-1)\frac{1}{(1+y)^2} \cdot dy$$

$$= (-) \int_{0}^{\infty} \frac{1}{(1+y)^{m-1}} \left(\frac{1+y-1}{1+y}\right)^{n-1} \cdot (-1) \cdot \frac{1}{(1+y)^2} dy$$

$$\Rightarrow \beta(m,n) = \int_0^\infty \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \frac{1}{(1+y)^2}\, dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}\, dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$\therefore \beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

আমরা জানি,

$$\beta(m,n) = \beta(n,m) \qquad \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

Question 4 (most most most important)

Relation between Gamma & beta function

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$$

আমরা জানি,

gamma function of $(n) = \Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx$ ——— (i)

gamma function দুটো (m,n) যুক্ত করার জন্য (i) নং সমীকরণের চলক (x) পরিবর্তন করে অন্য একটি ধ্রুবক এর সাথে গুণ করে রাখতে হবে।

ধরি, $x = \lambda y$ → নতুন চলক

x → পুরাতন চলক, λ → constant

$\Rightarrow dx = \lambda dy$ ——— (ii)

∴ নতুন (ii) নং সমীকরণের মান (i) নং সমীকরণে বসিয়ে পাই,

formula, $x = \lambda y$

	upper limit	lower limit
x	∞	0
y	∞	0

$$\Gamma n = \int_0^\infty e^{-\lambda y} \cdot (\lambda y)^{n-1} (\lambda dy)$$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} \cdot y^{n-1} \cdot \lambda^1 \cdot dy$$

$$\Rightarrow \overline{|n} = \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1+1} \cdot y^{n-1} \cdot dy$$

$$\Rightarrow \boxed{\overline{|n} = \int_0^\infty e^{-\lambda y} \cdot \lambda^n \cdot y^{n-1} \cdot dy} \rightarrow \text{(iii)}$$

→ একটি নতুন ~~function~~ সমীকরণ

λ^n → একটি constant

$$\Rightarrow \boxed{\frac{\overline{|n}}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot dy} \text{(iv)}$$

~~আমরা~~ আমাদের (iii) নং সমীকরণের মধ্যে λকে constant ধরিয়া ~~নিয়ে~~ $\overline{|m}$ এর জন্য সূত্র বের করব তবে λকেই ~~চলক~~ হিসাবে ~~ব্যবহার করব~~

$$\therefore ~~\overline{|m} = \int_0^\infty e^{-\lambda} \cdot \lambda^{m-1} \cdot d\lambda~~$$

$$\boxed{\overline{|m} = \int_0^\infty e^{-\lambda} \cdot \lambda^{m-1} \cdot d\lambda} \text{(v)}$$

→ λ এখানে চলক

(iii)×(v) করে পাই,

$$\overline{|m} \cdot \overline{|n} = \int_0^\infty \int_0^\infty e^{-\lambda y} \cdot e^{-\lambda} \cdot \lambda^n \cdot \lambda^{m-1} y^{n-1} \, dy \, d\lambda$$

$$\Rightarrow \overline{|m}\,\overline{|n} = \int_0^\infty \left[\int_0^\infty e^{-\lambda(y+1)} \cdot \lambda^{(m+n)-1} \cdot d\lambda\right] y^{n-1} \cdot dy \quad \text{(vi)}$$

(iv ও vi নং সমীকরণ তুলনা করে পাই)

$$\frac{\Gamma m}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot dy \text{ — (iv)}$$

$$\frac{\Gamma m \, \Gamma n}{1} = \int_0^\infty \left[\int_0^\infty e^{-\lambda(y+1)} \lambda^{(m+n)-1} \, d\lambda \right] y^{n-1} \cdot dy \text{ — (vi)}$$

(iv) নং সমীকরণে y ~~হলো চলক তাই~~ y এর power $(n-1)$ তাই L.H.S এ (Γm) রয়েছে। আবার $e^{-\lambda y}$ এখানে ~~constant~~ λ তাই L.H.S এ λ^n ভাগ আকারে আছে ঠিক একই ভাবে।

$\left[\int_0^\infty e^{-\lambda(y+1)} \cdot \lambda^{(m+n)-1} \cdot d\lambda \right.$ এখানে λ হলো চলক তাই $\lambda^{(m+n)-1}$ এর $(m+n)$ কি L.H.S এ $\Gamma m+n$ আকারে থাকবে এবং $e^{-\lambda(y+1)}$ এখানে $(y+1)$ constant তাই L.H.S এ $(y+1)^{m+n}$ ভাগ আকারে থাকবে কারণ ~~(iv)~~

(iv) নং সমীকরণের Γm n ও λ যেই কাজ করেছে ঠিক তেমনি (vi) নং সমীকরণে $(m+n)$ ও $(y+1)$ একই কাজ করেছে।

(bravo?)

$$\therefore \int_0^\infty e^{-x(y+1)}\, x^{(m+n)-1}\, dx = \frac{\Gamma(m+n)}{(y+1)^{m+n}}$$ লেখা যাবে।

∴ এই মান (vi) নং সমীকরণে বসিয়ে পাই,

$$\Gamma m\,\Gamma n = \int_0^\infty \left[\int_0^\infty e^{-x(y+1)}\, x^{(m+n)-1}\, dx\right] y^{n-1}\, dy$$

$$\Rightarrow \Gamma m\,\Gamma n = \int_0^\infty \left[\frac{\Gamma(m+n)}{(y+1)^{m+n}}\right] y^{n-1}\, dy$$

$$\Rightarrow \Gamma m\,\Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(y+1)^{m+n}}\, y^{n-1}\, dy$$

$$\Rightarrow \Gamma m\,\Gamma n = \Gamma(m+n) \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}}\, dy$$

$$\Rightarrow \Gamma m\,\Gamma n = \Gamma(m+n) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

→ কে $\beta(m,n)$ লেখা যায়

$$\Rightarrow \Gamma m\,\Gamma n = \Gamma(m+n)\ \beta(m,n)$$

যা আমাদের math [illegible]

কিবা হতো প্রমান করা আছে

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}}$$

(Proved)

Question no. 5

(i) $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} \text{ --- (i)}$$

আমরা জানি,

$$B(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\, dx \text{ --- (ii)}$$

আমরা জানি, Sin, cos এর Relation,
$\sin^2\theta + \cos^2\theta = 1$

Let $x = \sin^2\theta \Rightarrow dx = 2\sin^{2-1}\theta \cdot \cos\theta \cdot d\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$\Rightarrow \cancel{d\theta = \frac{dx}{2\sin\theta\cos\theta}}$

For limit,

	upper limit	lower limit
x	0	1
θ	0	$\pi/2$

$x = \sin^2\theta$

∴ (ii) নং সমীকরণের Interval ও dx এর মানগুলো বসিয়ে পাই,

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \left(1-\sin^2\theta\right)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta \cdot \cos\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \cdot d\theta \text{ --- (iii)}$$

(i ও iii) নং সমীকরণের মধ্যে p, q ~~এর মানের~~ তুলনা করলে, লেখা যেতে পারে যে,

$$2m-1=p \qquad \Big| \qquad 2n-1=q$$

$$\Rightarrow m=\frac{p+1}{2} \qquad \Big| \qquad \Rightarrow n=\frac{q+1}{2}$$

$$\therefore \beta(m,n)=2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$ এ m, n

এর মান বসালে,

$$\Rightarrow \beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta=\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$\Rightarrow 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} \qquad \left[\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$$

$$\boxed{\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}}$$

(Proved)

Question no. 6) most most most Important.

(6) $\Gamma\frac{1}{2} = \sqrt{\pi}$

আমরা জানি,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ এখানে π বসিয়ে

$m,n = \frac{1}{2}, \frac{1}{2}$ হলে

$$\Rightarrow \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}}\frac{1}{\sqrt{1-x}}\,dx = \int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = \int_0^1 \frac{1}{\sqrt{x-x^2}}\,dx$$

$$= \int_0^1 \frac{dx}{\sqrt{-(x^2-x)}} = \int_0^1 \frac{dx}{\sqrt{-\left(x^2-2.x\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)+\frac{1}{4}}}$$

$$= \int_0^1 \frac{dx}{\sqrt{-\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

formula,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^1 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{1}{2}}\right]_0^1$$

$$= \left[\sin^{-1} \frac{\frac{2x-1}{2}}{1/2}\right]_0^1 = \left[\sin^{-1}(2x-1)\right]_0^1$$

$$= \sin^{-1}(2-1) - \sin^{-1}(0-1) = \sin^{-1}1 - \sin^{-1}(-1)$$

$$= \sin^{-1}1 + \sin^{-1}1 = \pi/2 + \pi/2 = \pi$$

$$\Rightarrow \beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \pi$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \pi$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} = \pi$$

$$\Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\Rightarrow \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

(Proved)

formula

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\Gamma\tfrac{1}{2}\cdot\Gamma\tfrac{1}{2} = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\right) = \Gamma 1$$

$$\Gamma 1 = \int_0^\infty e^{-x}\cdot x^{1-1}\,dx$$

$= 1$ আসে

যা 1নং Questionএ প্রমান করা আছে

Some math Problem | Proved, $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Some imp math

Solve|| $\int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}} d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta \, d\theta$

$= \frac{\Gamma\frac{\frac{1}{2}+1}{2}\, \Gamma\frac{-\frac{1}{2}+1}{2}}{2\Gamma\frac{\frac{1}{2}-\frac{1}{2}+2}{2}} = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\Gamma\frac{2}{2}} \quad 2\Gamma 1$

$= \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2} = \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{2}$

$= \frac{\pi}{2\sin n\pi} \quad \frac{\pi}{2\sin\frac{1}{4}\cdot\pi} = \frac{\pi}{2\sin\frac{\pi}{4}}$

$= \frac{\pi}{\sqrt{2}}$ (2)

formula,

$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$= \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$

$\Gamma 1 = 1$

শর্ত প্রমাণ /

$\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Problem (2)

$I = \int_0^2 \sqrt{x(2-x)}\, dx$ এর মান নির্ণয় কর.

	upper	lower
x	2	0
z	1	0

$=$ Let, $x = 2z \Rightarrow dx = 2dz$

$I = \int_0^1 \sqrt{2z(2-2z)}\; 2dz$

$= 4\int_0^1 \sqrt{z(1-z)}\; dz$

$= 4\int_0^1 z^{1/2}(1-z)^{1/2}\, dz$

formula,

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$$= 4\int_0^1 z^{3/2-1/2}(1-z)^{3/2-1/2}\,dz = 4\,\beta(3/2, 3/2)$$

$$= 4\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}} = 4\cdot\frac{(\Gamma\frac{1}{2})^2}{\Gamma 1}$$

$$= 4\,\frac{(\sqrt{\pi})^2}{1} = 4\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\frac{3}{2}+\frac{3}{2}}$$

$$= 4\,\frac{(\Gamma\frac{3}{2})^2}{\Gamma 3}$$

$$= 4\,\frac{\left[\frac{3}{2}\,\Gamma\frac{1}{2}\right]^2}{\cancel{3}\cdot 2\cdot\Gamma 1} = \frac{4\cdot\left(\frac{3}{2}\right)^2(\sqrt{\pi})^2}{\cancel{6}\,2}$$

$$= \frac{\cancel{8}}{2}\cdot\pi \quad 2$$

formula

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Gamma 1 = 1$$

$$\Gamma n = n\,\Gamma n = n\,(n-1)\,\Gamma n-1$$

$$\cancel{= n!}$$

$$\Gamma n = \Gamma(n-1)+1$$

$$= (n-1)\,\Gamma n-1$$

অথবা

$$\Gamma n = (n-1)!$$

Proved if $\int_0^{\pi/2} \sin^5\theta\cos^4\theta\,d\theta$

$$= \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{4+1}{2}}{2\,\Gamma\frac{5+4+2}{2}} = \frac{\Gamma 3\,\Gamma\frac{5}{2}}{2\,\Gamma\frac{11}{2}}$$

$$= \frac{2\cdot\Gamma 1\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}}{2\cdot\left(\frac{11}{2}-1\right)\left(\frac{11}{2}-2\right)\left(\frac{11}{2}-3\right)\Gamma\frac{11}{2}-3}$$

formula

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta$$

$$= \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\Gamma n = (n-1)!$$

$$\Gamma n = (n-1)(n-2)(n-3)\,\Gamma n-3$$

$$\Gamma\tfrac{5}{2} = \left(\tfrac{5}{2}-1\right)\left(\tfrac{5}{2}-2\right)\Gamma\tfrac{5}{2}-2$$

$$= \frac{2 \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\tfrac{1}{2}}}{2\left(\frac{11}{2}-1\right)\left(\frac{11}{2}-2\right)\left(\frac{11}{2}-3\right)\left(\frac{11}{2}-4\right)\left(\frac{11}{2}-5\right)\sqrt{\tfrac{11}{2}-5}}$$

$$= \frac{3 \times \frac{1}{2} \times \sqrt{\pi}\ \cancel{\sqrt{\tfrac{1}{2}}}}{2 \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\ \cancel{\sqrt{\tfrac{1}{2}}}}$$

$$\boxed{\begin{aligned} \sqrt{\tfrac{1}{2}} &= \sqrt{\pi} \\ \Gamma\tfrac{1}{2} &= \sqrt{\pi} \end{aligned}}$$

$$= \frac{8}{315}$$

math $\int_0^{\pi/2} \sin^6 x \, dx = \cancel{\int_0^{\pi} \sin^4 x \sin^2 x \, dx}$

$$= \int_0^{\pi} \sin^6 x \cos^0 x \, dx$$

$$\boxed{\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\sqrt{\frac{p+1}{2}}\ \sqrt{\frac{q+1}{2}}}{2\sqrt{\frac{p+q+2}{2}}}}$$

$$= \frac{\sqrt{\frac{6+1}{2}}\ \sqrt{\frac{0+1}{2}}}{2\sqrt{\frac{6+0+2}{2}}} = \frac{\sqrt{\tfrac{7}{2}}\ \sqrt{\tfrac{1}{2}}}{2\sqrt{4}}$$

$$= \frac{\left(\tfrac{7}{2}-1\right)\left(\tfrac{7}{2}-2\right)\left(\tfrac{7}{2}-3\right)\sqrt{\tfrac{7}{2}-3}\ \sqrt{\tfrac{1}{2}}}{2\,(4-1)(4-2)(4-3)\sqrt{4-3}}$$

$$\boxed{\begin{aligned} \sqrt{\tfrac{1}{2}} &= \sqrt{\pi} \\ \sqrt{1} &= 1 \end{aligned}}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}}}{2 \times 3 \times 2 \times 1 \times \sqrt{1}}$$

$$= \frac{5}{32}\pi = \frac{32}{5}\pi$$

$$= \frac{5\pi}{32}$$

Problem $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$ Proved that

Solve

$$\Rightarrow \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta\, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{-1/2+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{-1/2+0+2}{2}} \times \frac{\Gamma\frac{1/2+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{1/2+0+2}{2}}$$

$$= \frac{\Gamma 1/4\, \Gamma 1/2}{2\,\Gamma 3/4} \cdot \frac{\Gamma 3/4\, \Gamma 1/2}{2\,\Gamma 5/4} = \frac{\Gamma 1/4 \left(\Gamma 1/2\right)^2}{4\,\Gamma 5/4} = \frac{\Gamma 1/4 \left(\Gamma 1/2\right)^2}{4\,\Gamma 1/4 + 1}$$

$$= \frac{\Gamma 1/4 \left(\Gamma 1/2\right)^2}{4 \cdot \frac{1}{4} \cdot \Gamma 1/4} = \frac{\left(\Gamma 1/2\right)^2}{4/4} = \frac{\left(\sqrt{\pi}\right)^2}{1}$$

$$= \pi$$

formula

$B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$, $\Gamma n = (n-1)!$
$= (n-1)\,\Gamma n-1$

$\Gamma 1 = 1$; $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Gamma n \cdot \Gamma 1-n = \frac{\pi}{\sin n\pi}$

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

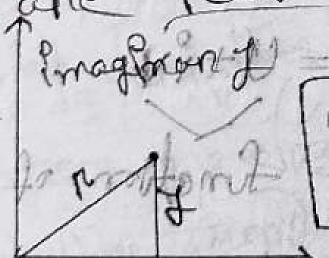
Chapter 2

Tecniques of complex Variable

Complex variable : Z কে আমরা একটি complex variable বলবো যদি তার মধ্যে বাস্তব ও অবাস্তব দুটি অংশ থাকে। $\therefore Z = \underbrace{x}_{\text{Real variable}} + i\underbrace{y}_{\text{imaginary part}}$

complex number : A number of the form $(a+ib)$ is called complex number where a, b are real number and $i = \sqrt{-1}$;

কাতেসিয়ান formula



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \quad \therefore \boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \rightarrow \text{modulas} \\ \theta &= \tan^{-1}\left|\frac{y}{x}\right| \rightarrow \text{Argument} \end{aligned}}$$

(i) Rectengular form : $Z = x + iy$

(ii) Polar form : $Z = re^{i\theta}$

Complex function / $z = x + iy = f(x,y)$

আবার $f(u,v) = u + iv = f(z)$

$= u(x,y) + i\,v(x,y)$

এই ভাবে complex function লেখা যেতে পারে।

Analytic function / A function $f(z)$ is said to be analytic at a point (z_0) if it is diffrentiable not only at (z_0) but also every point of some ~~neibret~~ neighbourhood values. (z_0)

$\therefore$ a complex function $f(z) = u + iv$

function of (x,y)

$u(x,y) + i\,V(x,y)$

Sufficient condition for f(z) to be analytic

Cauchy – Riemann relation যদি কোনো complex function মানে, তাহলে তাকে Analytic function বলা হবে।

$\therefore$ (i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

if it satisfies then the function is analytic.

Cauchy-Riemann Relation proof

$f = f(z)$, where $z = x+iy$ using the rules of partial derivatives.

$\therefore$ একবার x ও আরেকবার y এর সাপেক্ষে $\frac{\partial}{\partial}$ Partial derivatives নিয়ে পাই,

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial}{\partial x}(x+iy) = \frac{\partial f}{\partial z} \cdot (1)$$

$$\therefore \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial}{\partial y}(x+iy) = \frac{\partial f}{\partial z} \cdot (i)$$

(1)

যেখানে, $f(z) = u(x,y) + i\,v(x,y)$, Partial derivative এর

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \end{aligned}\right\} \text{--- (2)}$$

A complex function has a derivative with respect to real variable if real and imaginary part do. Thus by (2) (u) and (v) also have partial derivative with respect to (x and y).

অর্থাৎ complex function এর ও অন্তরীকরণ আছে ~~যদি~~ বাস্তব চলকের সাপেক্ষে যদি function এর বাস্তব ও অবাস্তব অন্তরীকরণ করে। একই ভাবে (U, V) ও (x, y) এর দুটি function যার কারণে তাদের ও পার্শিয়াল অন্তরীকরণ করা যাবে (x ও y) এর সাপেক্ষে।

(i) নং (1 ও 2) নং সমীকরণ combined করে পাই,

$$\therefore \frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--- (3)}$$

$$\therefore i \cdot \frac{\partial f}{\partial z} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{i^4}{i}\cdot\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y}$$

$$= i^3 \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial z} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \quad \text{--- (4)}$$

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$\therefore$ (3 ও 4) নং equation এর বাস্তব ও অবাস্তব অংশগুলিকে তুলনা করে পাই

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

ইহাই প্রমাণিত যা কচি-রাম্যান এর Relation নামে পরিচিত।

Cauchy-Riemann। Relation বলে।

Example 1

$f(z) = 2xy + i(x^2 - y^2)$ is it a analytic function?

Ans $f(z) = u + iv$

$\therefore u = 2xy$, $v = x^2 - y^2$

$\therefore$ ~~কচি - রামনের মত সমীকরণ~~ Relation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — (i) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ — (ii)

$\frac{\partial}{\partial x} u = \frac{\partial}{\partial x} 2xy = 2y \therefore \frac{\partial}{\partial y} v = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$

$\therefore$ (i) নং Relation $\boxed{\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}}$

আবার, $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} 2xy = 2x \therefore \frac{\partial}{\partial x} v = \frac{\partial}{\partial x}(x^2 - y^2) = 2x$

$\therefore \boxed{\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial y}}$ (ii) নং Relation

$\therefore$ This function is not a analytic function

Cause it's doesn't obey cauchy-Riemann Relation

Ex 2 $f(z) = \dfrac{x - iy}{x^2+y^2}$ হল analytic কিনা

Ans $\therefore f(z) = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$; $f(z) = u + iv$

$\therefore u = \dfrac{x}{x^2+y^2}$, $v = -\dfrac{y}{x^2+y^2}$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x} \cdot \dfrac{x}{x^2+y^2} = \dfrac{(x^2+y^2)\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$

$= \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ — (1)

আবার $\dfrac{\partial v}{\partial y} = \dfrac{\partial}{\partial y}\dfrac{-y}{x^2+y^2} = -\left[\dfrac{(x^2+y^2)\frac{\partial}{\partial y}y - y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}\right]$

$= \dfrac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ — (2)

$\therefore \boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ প্রমাণ

আবার,

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).0 - 2xy}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \quad \text{--- (3)}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2).0 - 2xy}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \quad \text{--- (4)}$$

∴ (3, 4) হতে পাই,

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \text{ হয়}$$

Cauchy-Riemann Relation মানে ~~তাই~~

ইহা একটি Analytic function.

Ex 3 $f(z) = z^3$ Analytic function কিনা?

আমরা জানি, $z = x + iy$ $\therefore z^3 = (x+iy)^3$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$z^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2y\,i - 3xy^2 - iy^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$\therefore f(z) = u + iv$

$\therefore u = x^3 - 3xy^2$; $v = 3x^2y - y^3$

$\therefore \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$ —— (1)

$\dfrac{\partial u}{\partial y} = 0 - 6xy$ —— (2)

$\dfrac{\partial v}{\partial x} = 6xy$ —— (3)

$\dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$ —— (4)

$(1=4)\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ → Cauchy-Riemann Relation মানে তাই

$(2=3)\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ $f(Z) = Z^3$ function টি Analytic function.

Harmonic function

ল্যাপলাস সমীকরণ যদি মেটে হয় তখন এই function কে Harmonic function বলা হবে।

$\therefore$ ল্যাপলাসিয়ান সমীকরণ, $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$

$\therefore f(z) = u + iv$ = complex function

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \end{aligned}\right\}$$ if it satisfy, then it is harmonic function.

Ex// $u = x^2 - y^2$, $v = \frac{y}{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} = 2x$ | $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial^2 u}{\partial x^2} = 2$ | $\frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

∴ এখন Imaginary part হতে পাই

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \left[\frac{(x^2+y^2)\cdot 0 - y\,2x}{(x^2+y^2)^2}\right]$$

$$= \frac{+2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2\cdot 2y - 2xy\left[2\cdot(x^2+y^2)\cdot 2x\right]}{(x^2+y^2)^4}$$

$$= \frac{2y\left[(x^2+y^2)^2 - 4x^2(x^2+y^2)\right]}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\cdot 2y\left[x^2+y^2-4x^2\right]}{(x^2+y^2)^4}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

Now, $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = -\left[\frac{(x^2+y^2) - y.2y}{(x^2+y^2)^2}\right]$

$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 . 2y - (y^2-x^2) . 2 . (x^2+y^2) . 2y}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)^2 . 2y - 4y(x^2+y^2)(y^2-x^2)}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2) . 2y\left[(x^2+y^2) - 2(y^2-x^2)\right]}{(x^2+y^2)^4}$

$= \frac{2y\left[x^2+y^2-2y^2+2x^2\right]}{(x^2+y^2)^3}$

$= \frac{2y\left[3x^2-y^2\right]}{(x^2+y^2)^3}$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

So this is a Harmonic funtion (Proved)

Find imaginary part

Complex variable, $f(z) = u(x,y) + iv(x,y)$

যদি $u(x,y)$ দেওয়া থাকে তবে $v(x,y) = ?$

$\therefore v = v(x,y)$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Cauchy Rimann Relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

v এর মান নিয়ে u বসাতে হবে

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\Rightarrow \boxed{v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy}$$

→ for finding imaginary part

Should not contain ~~x~~ with respect to ~~y~~

Find Real Part

যদি $v(x,y)$ দেওয়া থাকে তবে $u(x,y) = ?$

$u = u(x,y)$

$$\Rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

Relation

$$\Rightarrow \boxed{u = \int \frac{\partial v}{\partial y}\,dx - \int \underbrace{\frac{\partial v}{\partial x}\,dy}}$$ for finding Real part

y যুক্ত মানের Integrand, তাহা x যুক্ত পদ থাকলে = 0 হবে

Ex 1 // The real part of analytic function is $e^{-y}\cos x$. Find imaginary part.

∴ দেয়া আছে, $u = e^{-y}\cos x$ $\quad v(x,y) = ?$

∴ $v = v(x,y)$

$$\Rightarrow dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

C-R-Relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\Rightarrow \int dv = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

$$\Rightarrow v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy \quad \text{——— (i)}$$

$$\therefore \frac{\partial}{\partial y}u = \frac{\partial}{\partial y}e^{-y}\cos x = e^{-y}\frac{\partial}{\partial y}\cos x + \cos x\frac{\partial}{\partial y}e^{-y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\cos x\, e^{-y} \quad \text{——— (ii)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^{-y}\cos x\right) = e^{-y}\frac{\partial}{\partial x}\cos x + 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = -e^{-y}\sin x \quad \text{—(iii)}$$

(i) নং সমীকরণে (ii, iii) এর মান বসিয়ে পাই,

$$V = \int e^{-y}\cos x \, dx - \int e^{-y}\sin x \, dy$$

$$\Rightarrow V = e^{-y}\sin x + c$$ x যুক্ত পদ থাকলেই Integral ZERO হবে

বিদ্র:

$$\boxed{\text{আমরা জানি, } V = -\int \frac{\partial u}{\partial y}\,dx + \int \frac{\partial u}{\partial x}\,dy}$$

∴ এখানে সূত্রের দ্বিতীয় পদে V, u তে থাকা সূত্রের মধ্যে যদি y বা x এর ~~সাপেক্ষে~~ অন্তরীকরণ করার সময় x বা y যুক্ত পদ থাকে তাহলে Integral মান ০ হবে

Construction of analytic function using milne Thomson method

Prob-1

$f(z)$ function এর যদি Real part দেওয়া থাকে.

$\therefore$ $u(x,y)$ = given

$v(x,y) = ?$

$$\left[\begin{array}{l} f(z) = u + iv \\ f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \end{array}\right.$$

Step 1 | $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ বের করতে হবে।

Step 2 | C-R Relation $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Step 3 | $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ শুধু (u) এর মান দেওয়া আছে

$= \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ তাই convert করা হয়েছে

Step 4 | milne thomson: $x = z, y = 0 \rightarrow$ Put in $f'(z)$

Step 5 | "$\int f'(z)$" $= f(z) = ?$

$f'(z)$ এর Integral $f(z)$ হবে হলো Answer.

Prob – 2

Imaginary part of $f(z)$ is given

$v(x,y)$ – given. $u(x,y) = ?$

Step 1 // $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

Step 2 // C-R Relation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Step 3 // $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ → শুধু যার যার দেয়া থাকবে তাকে বসাতে হবে

Step 4 //

mile thomson: $x = z$ & $y = 0$ → put $f'(z)$

and the f' answer

Step 5 // integral $f'(z)$

is $f(z)$

Ex// Find analytic function $f(z)$ whose imaginary part $v = e^x(x\sin y + y\cos y)$ is given.

∴ Step 1// $\frac{\partial v}{\partial x} = \left\{e^x(x*0+\sin y)+0\right\} + (x\sin y + y\cos y)e^x$

$\frac{\partial v}{\partial y} = e^x(\cos y\, x + y(-\sin y) + \cos y.1) + 0$

Step 2// C-R Relation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Step 3// $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ | $f(z) = u + iv$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

Step 4// $f'(z) = e^x(\cos y . x - y\sin y + \cos y) + i\left\{e^x(\sin y) + e^x(x\sin y + y\cos y)\right\}$

Let

$x = z$, $y = 0$ by milne thomson

$f'(z) = e^z(\cos 0 . (z) - 0\sin 0 + \cos 0) + i\{e^z(\sin 0) + e^z(z\sin 0 + 0\cos 0)\}$

$\Rightarrow f'(z) = e^z(z+1) + i \times 0$

$\Rightarrow f'(z) = e^z(z+1)$

Step 5// $\int f'(z) = \int e^z (z+1)\, dz$

$\Rightarrow f(z) = (z+1)\int e^z - \frac{d}{dz}(z+1) \int\int e^z + c$

$= (z+1)\, e^z - e^z + c$

$\Rightarrow f(z) = z e^z + c$

Ex 2// find the Analytic function whose Real number is, $u = e^x(x.\cos 2y - y \sin 2y)$ given

solve

Step 1// $\frac{\partial u}{\partial x} = e^x \left[\cos 2y - 0\right] + (x.\cos 2y - y \sin 2y).e^x$

$\Rightarrow \frac{\partial u}{\partial x} = e^x (\cos 2y) + e^x (x.\cos 2y - y \sin 2y)$

Similarly, $\frac{\partial u}{\partial y} = e^x \left[-2x \sin 2y - 2y \cos 2y - \sin 2y\right] + 0$

Step 2 // C-R Relation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(z) = u + iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= e^x(\cos 2y) + e^x(x.\cos 2y - y\sin 2y) - i\left[e^x(-2x\sin 2y - 2y\cos 2y - \sin 2y\right]$$

Step 3 // Let $x = z, y = 0$ by mile thomson

$$\therefore f'(z) = e^z(\cos 0) + e^z(z.\cos 0 - 0\sin 0) - i\left[e^z(-2z\sin 0 + 2.0.\cos 0 - \sin 0)\right]$$

$$= e^z + ze^z - (i \times 0) = e^z + ze^z$$

$$\Rightarrow f'(z) = e^z + ze^z \quad \text{--- (1)}$$

Now integral eq (1)

$$\Rightarrow \int f'(z) = \int e^z.dz + \int ze^z\, dz$$

$$\Rightarrow f(z) = \cancel{e^z + e^z \int z\, dz +}$$

$$e^z + z\int e^z.dz - \frac{d}{dz}z\iint e^z\, dz\, dz$$

$$= e^z + ze^z - e^z + c = ze^z + c$$

∴ The Analytic function by milne thomson method is,

$$f(z) = a\, z e^{z} + c$$

uv formula

$$\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$$

$$\int u.v = u\int v - \frac{d}{dx}u\iint v + \frac{d^2}{dx^2}u\iiint v - \frac{d^3}{dx^3}u\iiiint v + \cdots$$

Question॥ কসি রিমেন সমীকরণ প্রতিপাদন কর এবং দেখাও যে সমীকরণ দুটো ল্যাপলাসের সমীকরণ মেনে চলে।

Solve॥ We know complex function. $f(z) = u + iv$

where, u & v are the function of x, y, here,

$z = x + iy$.

$\therefore f = f(z) = u(x, y) + i\,v(x, y)$

Let partial differentiation with respect to $(x \,\&\, y)$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \end{aligned}\right\} \text{---} (1)$$

আবার,

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}\cdot\frac{\partial}{\partial x}(x+iy) = \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}\cdot\frac{\partial}{\partial y}(x+iy) = i\frac{\partial f}{\partial z} \end{aligned}\right\} \text{---} (2)$$

$\therefore$ (1 ও 2) হতে পাই,

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{---} (3), \quad \cancel{i\frac{\partial f}{\partial z} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}}$$

$i\frac{\partial f}{\partial z} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial f}{\partial z} = \frac{i^4}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ $\quad [i^3 = -i$

$\Rightarrow \frac{\partial f}{\partial z} = i^3\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial f}{\partial z} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ — (4)

$\therefore$ (3 & 4) ~~के~~ distributed the Real & Imaginary part

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

here is the equation & Relation of of cauchy - Relation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (5) , $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (6)

$\therefore$ Again let's partial diffrentiation with respect to x (5) & y (6)

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ — (7) $\quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x . \partial y}$ — (8)

(7+8)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) . u = 0 \Rightarrow \nabla^2 . u = 0$ — (9)

Similarly, $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ —— (10)

Here, (9, 10) equation are laplasian equation.

If Laplasian eq is zero then the function will be called Harmonic function.

Proved: $f(z) = z^2 + 5iz + 3 - i$ function is obey cauchy rieman equation

$\therefore z = (x+iy)$ then $f(z) = (x+iy)^2 + 5i(x+iy) + 3 - i$

$= x^2 + 2ixy + (iy)^2 + 5xi + 5yi^2 + 3 - i$

$\Rightarrow f(z) = x^2 + 2ixy - 1y^2 + 5xi - 5y + 3 - i$

$\Rightarrow f(z) = x^2 + 2xyi - 1y^2 + 5xi - 5y + 3 - i$

$\Rightarrow f(z) = (x^2 - y^2 - 5y + 3) + i(2xy + 5x - 1)$

$\Rightarrow f(z) = u + iv$

So, the real part, $u = x^2 - y^2 - 5y + 3$

imaginary part, $v = 2xy + 5x - 1$

$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y - 5$

$\frac{\partial v}{\partial x} = 2y + 5, \quad \frac{\partial v}{\partial y} = 2x$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ obey the C-R Relation

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

C-R-Relation.

Proved: $f(z) = e^{z^2}$ obey

$\boxed{z = x+iy}$

$\therefore f(z) = e^{(x+iy)^2}$

$\Rightarrow f(z) = e^{x^2+2xyi+(iy)^2} = e^{x^2-y^2+2xyi}$

$\Rightarrow f(z) = e^{x^2-y^2} \cdot e^{i2xy}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow f(z) = e^{x^2-y^2}\cdot(\cos 2xy + i\sin 2xy)$

$\Rightarrow f(z) = (e^{x^2-y^2}\cos 2xy + i\, e^{x^2-y^2}\cdot\sin 2xy)$

$\Rightarrow f(z) = u + iv$

$\therefore$ Real part, $u = e^{x^2-y^2}\cdot\cos 2xy$

Imaginary part, $v = e^{x^2-y^2}\cdot\sin 2xy$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{x^2-y^2}\cdot\cos 2xy)$

$= e^{x^2-y^2}(-\sin 2xy)\cdot 2y + \cos 2xy\cdot e^{x^2-y^2}\cdot(2x)$

$\frac{\partial u}{\partial x} = 2x\cdot\cos 2xy\cdot e^{x^2-y^2} - 2y\cdot\sin 2xy\cdot e^{x^2-y^2}$

$\therefore \frac{\partial u}{\partial y} = -2y\cdot\cos 2xy\, e^{x^2-y^2} - 2x\sin 2xy\, e^{x^2-y^2}$

$\therefore \frac{\partial v}{\partial x} = 2y\cos 2xy\, e^{x^2-y^2} + 2x\cdot\sin 2xy\, e^{x^2-y^2}$

$\therefore \frac{\partial v}{\partial y} = 2x\cos 2xy\, e^{x^2-y^2} - 2y\sin 2xy\, e^{x^2-y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ follows C-R Equation. (Proved)

It's mean $f(z) = e^{z^2}$ (অপেক্ষককে বিশ্লেষক Analytic function)

Problem// ডি-ময়েভার ও আয়লারের সূত্র বর্ণনা কর।

ডি-ময়েভার// If (n) is a negative or positive integer, then $(\cos\theta + i\sin\theta)^n$ will be $(\cos n\theta + i\sin n\theta)$, And if (n) is a fraction number then $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.

$\therefore n = m$ হলে, $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

$n = -m$ হলে, $(\cos\theta + i\sin\theta)^{-m} = \cos(-m\theta) + i\sin(-m\theta)$

$n = p/q$ হলে, $(\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$

অয়লারের সূত্র ঃ from Trigonometric expansion,

we know $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ — (i)

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots$ — (ii)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

in (eq-2) let, $x = i\theta$

$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} \cdots$

$\Rightarrow e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \cdots + \cdots$

$\Rightarrow e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$

$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

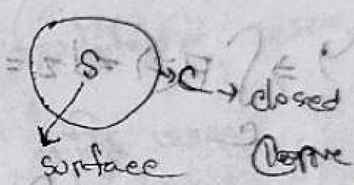
$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ This is the formula of ~~Eul~~ Eular.

~~Cauchy's integral formula proof~~

Cauchy's theorem proof

If $f(z)$ is an analytic function and $f'(z)$ is continuous within and on the closed curve (C) then

$$\int_C f(z)\, dz = 0$$



Solve//

∴ $f(z) = u + iv$

$z = x + iy \Rightarrow dz = dx + i\,dy$

$$\therefore \int_C f(z)\, dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C v\,dx + u\,dy \quad \text{——} (1)$$

∴ Green theorem অনুযায়ী formula

$$\int_C M\,dx + N\,dy = \int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

C → closed loop S → surface

Green theorem এর formula অনুযায়ী বের করা যায়,

$$\Rightarrow \int_C F(z)\,dz = \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\therefore$ Cauchy Riman Relation অনুযায়ী

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ এখানে}$$

$$\Rightarrow \int_C F(z)\,dz = \int_S \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\int_S \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$\Rightarrow \int_C F(z)\,dz = \int_S 0\,dx\,dy + i\int_S 0\,dx\,dy$$

$$\therefore \boxed{\int_C F(z)\,dz = 0}$$ Proved

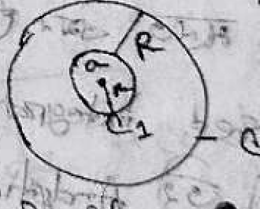
Cauchy's integral formula proof

If $f(z)$ is analytic function within and on the closed curve (c). If (a) be any point inside (c),

Then, $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a)$

Solve// ~~আমরা~~ $f(z)$ হলো Analytic function যার Integral zero.

But closed curve এর মধ্যে একটি বিন্দু আছে (a), যার কারণে $\frac{f(z)}{z-a}$ Analytic নয় এবং এর Integral ~~এটা~~ অসম্ভব।



C_1 = Non Analytic Region
R = Analytic Region

~~টি curve~~

Plot (a) এর small circle C_1, inside C, with radius r and center "a",

let, $(z-a) = re^{i\theta}$ → polar form of circle (r)

$\Rightarrow dz = re^{i\theta} \cdot i \cdot d\theta$

f(z) যেহেতু Analytic তাই এর closed curve integral

$$\int_C \frac{f(z)}{z-a}\,dz = \int_C \frac{f(a+re^{i\theta})}{re^{i\theta}}\, re^{i\theta}.\,i\,d\theta$$

$$= \int_C f(a+re^{i\theta}).\,i\,d\theta$$

$$= i\int_C f(a+re^{i\theta}).\,d\theta$$

$$= i\left[\int_{C_1} f(a+re^{i\theta})\,d\theta + \int_R f(a+re^{i\theta})\,d\theta\right]$$

$z-a = re^{i\theta}$

$z = (a+re^{i\theta})$

$dz = re^{i\theta}.\,i\,d\theta$

(C) curve কে দুই ভাগে ভাগ করা হয়েছে (C_1, R) Region এ এই অংশকে আলাদা ভাবে লেখা যায়।

∴ R-Region এ মধ্যে এমন কোনো point নেই যেখানে ফাংশনের closed integral zero হবে না, তাই (R) Region এর সকল point এই Analytic function থাকায়,

Cauchy's theorem অনুসারে closed integral zero হয়।

অর্থাৎ $\int_R f(a+re^{i\theta})\,d\theta = 0$

$$= i\oint_{C_1} f(a + re^{i\theta})\, d\theta$$

$$= i\oint_{C_1} f(a)\, d\theta$$

$$= i \cdot f(a) \oint_{C_1} d\theta$$

$$= i \cdot f(a) \int_0^{2\pi} d\theta$$

$$= i \cdot f(a) \left[\theta\right]_0^{2\pi}$$

$$\Rightarrow 2\pi i \cdot f(a)$$

For small circle with radius "r", $r \approx 0$

for that $re^{i\theta}$ term will be also zero

closed integral (C_1) এর ~~radius~~ interval $0 \to 2\pi$

$f(a)$ = constant.

(a) point. যেখানে function

f Analytic নয়।

$$\Rightarrow \oint_{C} \frac{f(z)}{z-a}\, dz = 2\pi i \cdot f(a)$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz}$$

Cauchy's Integral formula proof.

(a) হলো একটি closed curve এর point যেখানে f(a) function Not analytic, তাহলে এর closed integral শূন্য নয়।

Problem// If, $f(z) = u + iv$ function is an Analytic function then find the imaginary part, when real part is, $u =$

$u = e^{-x}(x \sin y - y \cos y)$.

Solve// we know,

v is function of (x, y)

$\therefore v = v(x, y)$

$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ —— (i)

Cauchy's Relation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

~~According mille thomsom theory, Integral eq (i), and let $x = z, y = 0$~~

$\Rightarrow \int dv = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$ —— (ii)

এই part এ যদি x যুক্ত না থাকে তাহলে তার Integral zero হবে।

$\Rightarrow V = -$

$\therefore \dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y} e^{-x}(x\sin y - y\cos y) = e^{-x}\left[\dfrac{\partial}{\partial y}\{x\sin y - y\sin y\}\right]$
$+ (x\sin y - y\sin y)\dfrac{\partial}{\partial y} e^{-x}$

$\Rightarrow \dfrac{\partial u}{\partial y} = e^{-x}\left[x\cos y - y\cos y - \sin y\right]$ —— (iii)

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}\left[e^{-x}(x\sin y - y\cos y)\right] =$

$= e^{-x}\dfrac{\partial}{\partial x}(x\sin y - y\cos y) + e^{-x}(x\sin y - y\cos y)\dfrac{\partial}{\partial x} e^{-x}$

$\Rightarrow \dfrac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$
$= (e^{-x}\sin y - e^{-x}.x\sin y + e^{-x}y\cos y)$ —— (iv)

(iii, iv) কে মান (ii) এ বসিয়ে

$V = -\int (e^{-x}.x\cos y - e^{-x}y\cos y - e^{-x}\sin y)\,dx$
$+ \int (e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y)\,dy$

(কোনো চলক নাই তাই integral zero হবে।)

$\Rightarrow V = \cancel{-x\cos y}$

$\Rightarrow v = -\cos y \int x e^{-x} dx + y \cos y \int e^{-x} dx + \sin y \int e^{-x} dx$

$\Rightarrow v = -\cos y \left[x(-e^{-x}) - (1)(e^{-x})\right] + y \cos y (-e^{-x}) + \sin y (-e^{-x})$

$\Rightarrow v = + x e^{-x} \cos y + e^{-x} \cos y - e^{-x} y \cos y - e^{-x} \sin y$

$\therefore v = e^{-x} \cos y (x - y + 1) - e^{-x} \sin y$

Problem // Proved, $u = e^{x}(x\cos y - y \sin y)$ function is a Harmonic function.

Solve

If, Laplasian equation become zero,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then function (u) should be called a Harmonic function.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} e^{x}(x\cos y - y \sin y)$

$= e^{x} \frac{\partial}{\partial x}(x\cos y - y\sin y) + (x\cos y - y \sin y) \frac{\partial}{\partial x} e^{x}$

$= e^{x} \cos y + e^{x} . x\cos y - e^{x} y \sin y$

$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y + e^x x \cos y - e^x y \sin y$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^x \cos y + e^x x \cos y - e^x y \sin y)$

$= e^x \cos y + \cos y[x e^x + e^x] - e^x y \sin y$

$= e^x \cos y + e^x \cos y + x e^x \cos y - y e^x \sin y$

$\therefore \frac{\partial^2 u}{\partial x^2} = 2e^x \cos y + x e^x \cos y - y e^x \sin y$ —— (ii)

আবার

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} e^x (x \cos y - y \sin y)$

$= e^x [-x \sin y - y \cos y - \sin y]$

$= -e^x . x \sin y - y e^x \cos y - e^x \sin y$

আবার

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-e^x . x \sin y - y e^x \cos y - e^x \sin y)$

$= -e^x . x \cos y - e^x (y(-\sin y) + \cos y) - e^x \cos y$

$= -x e^x \cos y + y e^x \sin y - e^x \cos y - e^x \cos y$

$\therefore \frac{\partial^2 u}{\partial y^2} = -2e^x \cos y + y e^x \sin y - x e^x \cos y$ —— (iii) (ii+iii) করলে

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (2e^x \cos y) - (2e^x \cos y) + (x e^x \cos y) - (x e^x \cos y)$$
$$+ (y e^x \sin y) - (y e^x \sin y)$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Proved) that, function (u) is a harmonic function.

~~Problem~~ Proved, $u = 3x^2 y + 2x^2 - 2y^2 - y^3$ function is a harmonic function.

~~Solve~~:

$\therefore \frac{\partial u}{\partial x} = 6xy + 4x$ | $\frac{\partial u}{\partial y} = 3x^2 - 4y - 3y^2$

$\therefore \frac{\partial^2 u}{\partial x^2} = 6y + 4$ (i) | $\frac{\partial^2 u}{\partial y^2} = -4 - 6y$ (ii)

$(i+ii)$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 4 - 6y = 0$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, the laplasian equation zero means that the function (u) is Harmonic function.

Ex// Evaluate $\int_C \frac{z^2}{z-1}\,dz$ where, (C) is the circle

$|z| = 1/2$ । By Cauchy's Theorem .

আমরা জানি // $|z| = 1/2 \Rightarrow |x+iy| = 1/2 \Rightarrow \sqrt{x^2+y^2} = 1/2$

$\Rightarrow x^2+y^2 = (1/2)^2$ → This is a circle equation where

center point is $(0, 0)$ & radius is $(r = 1/2)$

আবার main equation এর function হলো $\frac{z^2}{z-1}$ অতএব,

$z-1 \neq 0 \Rightarrow (z \neq 1) \Rightarrow z \neq a$ তাহলে $a = 1$ point

টি হলো singular point বা ঐ point এ function টি

Analytic নয়, যেহেতু উপরের circle টির Radius $(1/2)$

এবং $a = 1/2$ বিন্দুটি $(r = 1/2)$ এর বাহিরে যার জন্য

closed circle এ সম্পূর্ণ সমতলটি Analytic । তবে

close circle এ Analytic function এর integral zero.

$$\therefore \int_C \frac{z^2}{z-1}\,dz = 0$$

Problem) $\int_C \frac{e^z}{(z+2i)(z-2)} dz$ where C is the circle

$|z| = 1$.

$\therefore f(z) = \frac{e^z}{(z+2i)(z-2)}$ } where, $z = -2i, 2$ singular point of $f(z)$.

Singular point of $f(z)$ mean that, on that's point the function is not Analytic.

$\therefore |z| = 1 \Rightarrow |x+iy| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$

$\Rightarrow x^2+y^2 = 1^2$ } here the closed circle,
center $= (0,0)$ (radius $= 1$)

Since both singular point of $f(z)$ are outside the circle (C). By Cauchy's theorem

$$\int_C \frac{e^z}{(z+2i)(z-2)} dz = 0$$

Q. Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along (i) the line $(y = x/2)$

(ii) the real axis to (2) & then vertically to $(2+i)$

$$\boxed{\begin{matrix} z = x+iy \\ dz = dx + i\,dy \end{matrix}} \quad \boxed{\bar{z} = x - iy}$$

$$\therefore \int_0^{2+i} (\bar{z})^2 \, dz = \int_0^{2+i} (x - iy)^2 \, dz = \int_{0+0}^{2+i} (x^2 - y^2 - 2xyi)(dx + i\,dy)$$

$\boxed{i}$

পুরো Integrate x বা y তে convert করতে হবে।

Since, $y = x/2 \Rightarrow x = 2y \Rightarrow dx = 2dy$

$\therefore \int_{0+0}^{2+i}$ → x এর limit, y এর limit; y এর মান বসালে limit $\int_0^1$ হবে $x = 2y$, $dx = 2dy$ হবে

$$\therefore \int_0^1 (4y^2 - y^2 - 4y^2 i)(2dy + i\,dy)$$

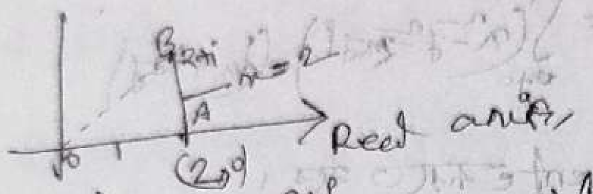
$$\Rightarrow \int_0^1 (3-4i)\,y^2 \cdot (2+i)\,dy = (3-4i)(2+i) \int_0^1 y^2 \, dy$$

$$= (3-4i)(2+i) \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{6 + 3i - 8i + 4}{3} = \frac{10 - 5i}{3} = \boxed{\frac{5}{3}(2-i)}$$

(iii)

The real axis, $x=2$, & vertically $(2+i)$ ~~২০নং~~



$$\therefore \int_0^{2+i} (\bar{z})^2 dz = \int_{0+0}^{2+i} (x^2-y^2-2xyi)(dx+idy)$$

$$= \int_{OA} (x^2-y^2-2xyi)(dx+idy) + \int_{AB} (x^2-y^2-2xyi)(dx+idy) \quad \text{---(i)}$$

আবার দিকে, OA line এ $y=0$, $dy=0$

x এর limit $\boxed{x = 0 \to 2}$

আবার, AB line এ $x=2$, $dx=0$

y এর limit $\boxed{y = 0 \to 1}$

২. (এ) এখন মূল্য বসিয়ে,

$$= \int_{OA} (x^2 - y^2 - 2xyi)(dx + idy) + \int_{AB} (x^2 - y^2 - 2xyi)(dx + idy)$$

$$= \int_0^2 (x^2 - 0^2 - 0)(dx + 0) + \int_0^1 (4 - y^2 - 4yi)(dx + idy)$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\left[4y - \frac{y^3}{3} - \frac{4y^2 i}{2}\right]_0^1$$

$$= \left[\frac{2^3}{3}\right]_0^2 + i\left(4 - \frac{1}{3} - \frac{4}{2}i\right) = \frac{8}{3} + \left(\frac{11}{3}\right)i + 2$$

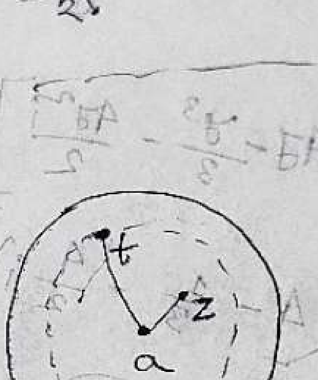
$$= \frac{14}{3} + i\,\frac{11}{3}$$

Taylor's series theorem for complex numbers

Let $f(z)$ be a analytic function inside the circle (C) with a center (a) and (z) is any point inside (C) then

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

Proof// given z is any point inside the circle 'c' and $f(z)$ is analytic in 'c'.



Draw a circle "c_1" with center 'a' such that it enclose the point "z" consider "a" point "t" on circle "c_1".

Now// $$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \cancel{\frac{1}{(t-a)*(z-a}}$$

$$= \frac{1}{(t-a)\left[1-\frac{(z-a)}{(t-a)}\right]} = \frac{1}{(t-a)} \cdot \left(1 - \frac{z-a}{t-a}\right)^{-1}$$

আমরা জানি, binomial expansion formula,

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$\therefore \frac{1}{t-z} = \frac{1}{t-a}\left(1-\frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\left(\frac{z-a}{t-a}\right)^3+\cdots\right]$$

$$\Rightarrow \frac{1}{t-z} = \frac{1}{t-a}+\frac{z-a}{(t-a)^2}+\frac{(z-a)^2}{(t-a)^3}+\cdots$$

Series এর value যদি decrease হয় তাহলে এটা convergence হবে

এবং এর Integration হবে।

$\therefore |t-a| > |z-a| \Rightarrow \frac{z-a}{t-a} < 1$, Series is convergent

So multiplying both side by $f(t)$ and integrating along the circle "C_1"

$$\oint_C \frac{f(t)}{t-z}\,dt = \oint_{C_1} \frac{f(t)}{t-a}\,dt + (z-a)\oint_{C_1}\frac{f(t)}{(t-a)^2}\,dt$$

$$+ (z-a)^2\oint_{C_1}\frac{f(t)}{(t-a)^3}\,dt + \cdots$$

Cauchy's Integral formula;

$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a)$ | $\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{1}{n!} f^n(a)$

$f^n = n$ time differentiation

∴ formula gulan मानवरिए

$$\Rightarrow \oint_{C} \frac{f(t)}{t-z}\,dt = \oint_{C_1} \frac{f(t)\cdot dt}{t-a} + (z-a)\oint_{C_2} \frac{f(t)\cdot dt}{(t-a)^2}$$

$$+ (z-a)^2 \oint_{C_3} \frac{f(t)\cdot dt}{(t-a)^3} + \cdots$$

$$\Rightarrow \oint f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2 \frac{f''(a)}{2!}$$

$$+ (z-a)^3 \frac{f'''(a)}{3!} + \cdots$$

$$\therefore \boxed{f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f'''(a)}{3!} + \cdots}$$

Tylor Series proved
for complex variable function.

Residue theorm & formula

Residue at infinity for f(z) function.

$$Res(\infty) = \lim_{z \to \infty} [-z f(z)]$$

Residue at a simple pole (a) for f(z) function

$$\cancel{Res(a)}\ \underset{z=a}{Res}[f(z)] = \lim_{z \to a} (z-a) f(z)$$

Residue theorm | If f(z) is an analytic function on a closed curve 'c', except at a finite number of singular points with 'C' then,

$$\int_C f(z)\,dz = 2\pi i \cdot \left\{ \begin{array}{c} \text{sum of Residues at all singular} \\ \text{points with in C} \end{array} \right\}$$

Q// Evaluate : $\oint_C \frac{z^2+1}{(z-2)(z+3)}\,dz$, where (c) is $|z+1|=2$

$f(z) = \frac{z^2+1}{(z-2)(z+3)}$

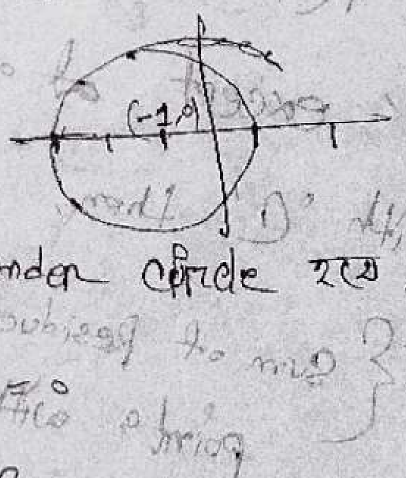
∴ Solution// দেয়া আছে,

$|z+1|=2 \Rightarrow |x+iy+1| = 2 \Rightarrow |(x+1)+iy| = 2$

$\Rightarrow \sqrt{(x+1)^2+y^2} = 2 \Rightarrow (x+1)^2+(y-0)^2 = 2^2$

$\Rightarrow \{x-(-1)\}^2+(y-0)^2 = 2^2$

∴ বৃত্তাকারটির কেন্দ্র $(-1,0)$ এবং ব্যাসার্ধ $= 2$



অতএব,
Singular point under circle হবে,

$z-2=0$, $z+3=0$

$\Rightarrow z=2$, $z=-3$

∴ $z=-3$ ~~বিন্দুটি বৃত্ত~~ circle এর মধ্যে অবস্থিত

এর জন্য $z=-3$ হবে singular point, ওই point এ functionটি Analytic হবে না।

$$\therefore \int_C \frac{z^2+4}{(z-2)(z+3)} dz = 2\pi i \times Res(-3)$$

$$= \oint_C = 2\pi i \cdot \lim_{z\to -3} (z-(-3)) \frac{z^2+4}{(z-2)(z+3)}$$

$$= 2\pi i \cdot \lim_{z\to -3} \frac{z^2+4}{(z-2)} = 2\pi i \frac{(-3)^2+4}{(-3-2)}$$

$$= \frac{26\pi i}{-5}$$

$$\therefore \int_C \frac{z^2+4}{(z-2)(z+3)} = \frac{26\pi i}{-5}$$

Important Show that $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \pi/2$ (By Residue theorem)

Let $\boxed{z = e^{i\theta}} \Rightarrow dz = i \cdot e^{i\theta} \cdot d\theta$

$\Rightarrow dz = iz \cdot d\theta$

$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

আবার, $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$ $\} \therefore e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\Rightarrow \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$

$\therefore \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{e^{i\theta}-\frac{1}{e^{i\theta}}}{2i} = \dfrac{z-\frac{1}{z}}{2i}$

$\Rightarrow \boxed{\sin\theta = \dfrac{z^2-1}{2iz}}$

∴ মান গুলো উপরে বসিয়ে বসিয়ে /

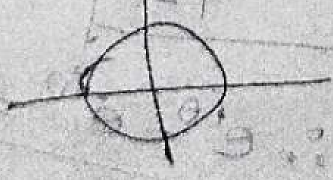
$$= \int_C \frac{\frac{dz}{iz}}{5+3\cdot\frac{z^2-1}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{10iz+3z^2-3}{2iz}} = \int_C \frac{2dz}{3z^2+10iz-3}$$

$$= 2\int_C \frac{dz}{3z^2+10iz+3i^2} = 2\int_C \frac{dz}{3z^2+9iz+iz+3i^2}$$

$$= 2\int_C \frac{dz}{3z(z+3i)+i(z+3i)} = 2\int_C \frac{dz}{(z+3i)(3z+i)}$$

∴ ছিল আবার, $e^{i\theta} = z = |x+iy| = 1$, হবে cause limit $0 \to 2\pi$ ছিল।

$\therefore x^2+y^2=1^2$



অর্থাৎ কেন্দ্র $(0,0)$, radius $= 1$ ছিলো।

$\therefore 2\oint_C \frac{dz}{(z+3i)(3z+i)}$ এর singular point Here, $f(z) = \frac{1}{(z+3i)(3z+i)}$

$z = -3i$, $z = -\frac{i}{3}$

এখানে, $z = -3i$ হলো out of the circle এবং $z = -\frac{i}{3}$ হলো

inside circle cause Radius হলো 1

$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = 2\oint_C \frac{dz}{(z+3i)(3z+i)} = 2\left[2\pi i \cdot (\text{Residue of singular point under the circle})\right]$

$= 2\left[2\pi i \cdot \lim_{z \to -i/3} (z+i/3) \cdot \frac{1}{(z+3i)(3z+i)}\right]$

$= 2\left[2\pi i \cdot \lim_{z \to -i/3} \frac{(3z+i)}{3(z+3i)(3z+i)}\right]$

$= 4\pi i \cdot \lim_{z \to -i/3} \left(\frac{1}{3z+9i}\right) = 4\pi i \cdot \frac{1}{\left(-3 \cdot \frac{i}{3} + 9i\right)}$

$= 4\pi i \frac{1}{8i} = \frac{\pi}{2}$

$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = 2\oint_C \frac{dz}{(z+3i)(3z+i)} = \pi/2$

$= 2\left[2\pi i \cdot \left(\text{Residue}\left(-\frac{i}{3}\right)\right)\right]$

Problem: $\int_0^{2\pi} \frac{d\theta}{2+3\cos\theta}$ solve it by Residue theorem.

Solve: let, $z = e^{i\theta}$

$\Rightarrow dz = i \cdot e^{i\theta} \cdot d\theta$

$\Rightarrow dz = i \cdot z \cdot d\theta$

$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\Rightarrow \cos\theta = \frac{z + \frac{1}{z}}{2}$

$\Rightarrow \boxed{\cos\theta = \frac{z^2+1}{2z}}$

$\therefore$ new integral is

$$\int_0^{2\pi} \frac{d\theta}{2+3\cos\theta} = \int_C \frac{\frac{dz}{iz}}{2+3\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_C \frac{\frac{dz}{iz}}{\frac{4z+3z^2+3}{2z}} = \int_C \frac{2dz/i}{3z^2+4z+3}$$

$$= \frac{2}{i} \int_C \frac{dz}{3z^2+4z+3}$$

$\therefore 3z^2+4z+3$ is singular points

$\therefore z = -\frac{2}{3} + \frac{\sqrt{5}}{3} i$, $z = -\frac{2}{3} - \frac{\sqrt{5}}{3} i$

$\therefore e^{i\theta} = z = |x+iy| = 1$

$\Rightarrow x^2+y^2 = 1$

এখানে কেন্দ্র (0,0) Radius 1

$\therefore z = -\frac{2}{3} + \frac{\sqrt{5}}{3}i$ হলো singular point, cause এর point

যা বৃত্তের Radius এর মধ্যে।

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+3\cos\theta} = \frac{2}{i}\int_C \frac{dz}{3z^2+4z+3} = \frac{2}{i}\left[2\pi i \cdot \text{Resedu}\left(-\tfrac{2}{3}+\tfrac{\sqrt{5}}{3}i\right)\right]$$

$$= 4\pi \cdot \lim_{z\to\left(-\frac{2}{3}+\frac{\sqrt{5}}{3}i\right)} \frac{\left(z+\frac{2}{3}-\frac{\sqrt{5}}{3}i\right)}{3z^2+4z+3}$$

$$= 4\pi \lim_{z\to\left(-\frac{2}{3}+\frac{\sqrt{5}}{3}i\right)} \frac{\left(z+\frac{2}{3}-\frac{\sqrt{5}}{3}i\right)}{\left(z+\frac{2}{3}+\frac{\sqrt{5}}{3}i\right)\left(z+\frac{2}{3}-\frac{\sqrt{5}}{3}i\right)}$$

$$= 4\pi \lim_{z\to\left(-\frac{2}{3}+\frac{\sqrt{5}}{3}i\right)} \frac{1}{\left(z+\frac{2}{3}+\frac{\sqrt{5}}{3}i\right)}$$

$$= 4\pi \frac{1}{-\frac{2}{3}+\frac{\sqrt{5}}{3}i+\frac{2}{3}+\frac{\sqrt{5}}{3}i} = 4\pi \cdot \frac{1}{2\frac{\sqrt{5}}{3}i}$$

$$= \frac{6}{\sqrt{5}\,i}$$

Important

Proved, $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \pi/12$ by Residue method.

$\therefore$ Let, ~~cosθ~~

$z = e^{i\theta}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore 2\cos\theta = e^{i\theta} + \frac{1}{e^{i\theta}} = z + \frac{1}{z} = \frac{z^2+1}{z}$

$\Rightarrow \cos\theta = \boxed{\frac{z^2+1}{2z}}$ — ②

So, $z = e^{i\theta}$.

$\Rightarrow dz = i e^{i\theta} . d\theta$

$\Rightarrow dz = i . z . d\theta$

$\therefore \boxed{d\theta = \frac{dz}{iz}}$ — ①

আবার, $e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$e^{-i3\theta} = \cos 3\theta - i\sin 3\theta$

$\therefore 2\cos 3\theta = e^{i3\theta} + \frac{1}{e^{i3\theta}} = (e^{i\theta})^3 + \left(\frac{1}{e^{i\theta}}\right)^3$

$\Rightarrow 2\cos 3\theta = z^3 + \frac{1}{z^3} = \frac{z^6+1}{z^3}$

$\Rightarrow$ ~~$\cos 3\theta = \frac{z^6+1}{2z^3}$~~ — ③

$\Rightarrow \cos 3\theta = \frac{z^6+1}{2z^3}$

(1,2,3) এর মান প্রদত্ত সমাকল Integral এ বসিয়ে পাই

$$\int_C \frac{\frac{z^6+1}{2z^3}}{5-4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} = \int_C \frac{\frac{z^6+1}{2z^3}}{\frac{5z-2z^2-2}{z}} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{(z^6+1) \cdot dz \cdot z}{2 \cdot i \cdot z \cdot z^3 \cdot (5z-2z^2-2)} = \cancel{\int_C \frac{z^3(z^3+1) \cdot z \, dz}{2i \cdot z^3 \cdot z \, (5z-2z^2-2)}}$$

$$\cancel{\frac{1}{2i}\int_C \frac{z^3+1}{5z-2z^2-2}\,dz} = \frac{1}{2i}\int_C \frac{(z^6+1)\,dz}{z^3(5z-2z^2-2)}$$

$$= -\frac{1}{2i}\int_C \frac{(z^6+1)\,dz}{z^3(2z^2-5z+2)} = -\frac{1}{2i}\int_C \frac{(z^6+1)\,dz}{z^3(2z-1)(z-2)}$$

$$= -\frac{1}{2i}\int_C \frac{(z^6+1)}{(z^3)(2z-1)(z-2)}\,dz$$

∴ আমরা জানি, $|z| = 1 \Rightarrow |x+iy| = 1 \Rightarrow x^2+y^2 = 1^2$

ও কেন্দ্র $(0,0)$ হলে,

∴ Circle এর Radius = 1

∴ singular point গুলো হলো,

$z^3 = 0 \Rightarrow \boxed{z=0}$ | $2z-1=0 \Rightarrow \boxed{z=1/2}$ | $z-2=0 \Rightarrow \boxed{z=2}$

এটি singular ~~হবে~~

$$\Rightarrow \pi \left\{ \lim_{z\to 0} \frac{z^6+1}{(2z-1)(z-2)} + \lim_{z\to 1/2} \frac{z^6+1}{2 \cdot z^3 \cdot (z-2)} \right\}$$

$$= \pi \left\{ \frac{0+1}{(0-1)(0-2)} + \frac{(1/2)^6+1}{2\cdot\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}-2\right)} \right\}$$

$$= \pi \left\{ \frac{1}{2} - \frac{65}{24} \right\} = \pi. \qquad f(z) = \frac{z^6+1}{z^3(2z-1)(z-2)}$$

If $z = z_0$ is simple pole,

$$\text{Res } f(z_0) = \lim_{z\to z_0} (z-z_0) f(z)$$

Formula

If $z = z_0$ is pole of order m

$$\text{Res } f(z_0) = \frac{1}{\lfloor m-1} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]$$

$\therefore$ zero $z = 0$ is pole of order (3)

$\therefore$ Residue$(0) = \frac{1}{\lfloor 3-1} \lim_{z\to 0} \frac{d^{3-1}}{dz^{3-1}} \left[(z-0)^3 f(z)\right]$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 \cdot \frac{(z^6+1)}{z^3(2z-1)(z-2)}$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\frac{z^6+1}{2z^2-4z-z+2} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-6z+2}\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\frac{d}{dz}\left[\frac{(2z^2-6z+2)(6z^5)-(z^6+1)(4z-6)}{(2z^2-6z+2)^2}\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\frac{12z^7-36z^6+12z^5-4z^7-4z+6z^6+6}{(2z^2-6z+2)^2}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\frac{8z^7-30z^6+12z^5-4z+6}{(2z^2-6z+2)^2}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{(2z^2-6z+2)^2\cdot(56z^6-180z^5+60z^4-4)-}{(2z^2-6z+2)^4}$$

$$(8z^7-30z^6+12z^5-4z+6)\cdot 2\cdot(2z^2-6z+2)(4z-6)$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{(2z^2-6z+2)^2\cdot(56z^6-180z^5+60z^4-4)-(8z^7-30z^6+12z^5-4z+6)\cdot 2\cdot(2z^2-6z+2)\cdot(4z-6)}{(2z^2-6z+2)^4}$$

$$= \frac{1}{2}\cdot\frac{2^2\cdot(-4)-(6)\cdot 2\,(2)(-6)}{(2)^4}$$

$\therefore$ Residue $= \frac{21}{8}$ উহা মান সম্পন্ন করাতে

আবার $\text{Residue}\left(\frac{1}{2}\right) = \lim_{z\to 1/2} \left(z-\frac{1}{2}\right) \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}$

$$= -\frac{65}{24}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{-1}{2i}\left[\oint \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz\right]$$

$$= \frac{-1}{2i}\left[2\pi i \left\{\text{Sum of singular points Residues}\right\}\right]$$

$$= -\pi \cdot \left\{\text{Residue}(0) + \text{Residue}(1/2)\right\}$$

$$= -\pi \left\{\cancel{\frac{67}{24}}\ \frac{21}{8} - \frac{65}{24}\right\}$$

$$= \frac{\pi}{12}$$

বিঃদ্রঃ

Residue (0) এর মান = বের করতে হবে কিন্ত this math. Cause calculation করে সঠিক মান পাওয়া যায় না।

Problem/1

(1) Find solution by Residue method

(i) $\oint \frac{e^{z}}{(z^2+a^2)} dz$ when C is circle, $|z|=4$

(ii) $\oint \frac{dx}{x^2+1}$ $\oint \frac{e^{imz}}{z^2+1}$

∴ (i) $|z|=4 \Rightarrow x^2+y^2=2^2$
Radius (2)

∴ $\oint \frac{e^{z}}{z^2+a^2} dz$

∴ নিকালন হবে $z^2+a^2=0$ হলে

$z^2=-a^2$ ∴ ~~$z=+ia, -ia$~~
$\Rightarrow z=\pm ia$

∴ $z=ia$ তে Residue হবে

$\lim_{z\to ia} (z-ia) \frac{e^{z}}{(z+ia)(z-ia)} = \lim_{z\to ia} \frac{e^{z}}{z+ia}$

$\Rightarrow \frac{e^{ia}}{2ia} = Res(z=ia)$

$\therefore \oint \frac{e^z}{(z^2+a^2)} dz = \oint \frac{e^z}{(z+ia)(z-ia)} dz$

$= 2\pi i \cdot \{\text{Res}(z = ia)\}$

$= 2\pi i \cdot \frac{e^{ia}}{2ia}$

$= \frac{\pi}{a} \cdot e^{ia}$

$= \frac{\pi}{a} \left[1 + ia + \frac{(ia)^2}{2!} + \frac{(ia)^3}{3!} - - - \right]$

$\therefore$ (vi) $\oint_C \frac{e^{imz}}{z^2+1} dz$

$\therefore z^2+1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$

$\therefore z = +i$ এর জন্য Residue $= \lim_{z \to i} (z-i) \frac{e^{imz}}{(z-i)(z+i)}$

$= \frac{e^{i^2 m}}{(i+i)} = \frac{e^{-m}}{2i}$

$\therefore \oint \frac{e^{imz}}{z^2+1} dz$

$= \oint \frac{e^{imz}}{(z+i)(z-i)} dz = 2\pi i \{\text{Res}(z=i)\}$

$= 2\pi i \cdot \frac{e^{-m}}{2i} = \pi \cdot e^{-m}$

(11) $\int_0^{\pi} \frac{1}{x^2+1}\,dx$

So $x^2+1=0$
$\Rightarrow x = \pm i$

$\int_0^{\pi} \frac{1}{(x+i)(x-i)}\,dx$

$2\pi i \cdot \{\text{Res}(x=i)\}$

$\therefore \int_0^{\pi} \frac{1}{(x+i)(x-i)}\,dx =$

$= 2\pi i \cdot \left\{ \lim_{x \to i} (x-i) \frac{1}{(x+i)(x-i)} \right\}$

$= 2\pi i \lim_{x \to i} \frac{1}{(x+i)}$

$= 2\pi i \cdot \frac{1}{2i}$

$= \pi$

$\therefore \int_0^{\pi} \frac{1}{x^2+1} = \pi$

shotcut method — most important

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$

$$= Re\int_0^{2\pi} \frac{e^{i3\theta}}{5-4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)} d\theta$$

$$= Re \oint_C \frac{z^3}{5-2\left(z-\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= Re\left[\frac{1}{-i}\oint \frac{z^3}{(2z-1)(z-2)}dz\right]$$

$$= Re\left[\frac{1}{-i}\oint \frac{\left(z^3/z-2\right)}{2\left(z-1/2\right)}dz\right]$$

$$= Re\left[-\frac{1}{2i}\oint \frac{\left(z^3/z-2\right)}{\left(z-1/2\right)}dz\right]$$

(1) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

(2) $\cos 3\theta$ = Real part of $(e^{i3\theta})$
$= Re(e^{i3\theta})$

$z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$|z| = 1$

২এর

Circle of Radius

(1) ২এর

Cauchy Riemann Integral formula

$$\oint \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a)$$

এখানে $a = 1/2$ ও $f(z) = \frac{z^3}{z-2}$

$\therefore f(a) = \frac{a^3}{a-2}$

$$\therefore \Rightarrow Re\left[-\frac{1}{2i}\cdot 2\pi i \cdot \frac{(1/2)^3}{(1/2)-2}\right] = Re\left[\frac{\pi}{12}\right] = \frac{\pi}{12}$$

Bessel's function

Bessel's function is

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{(r+1)}\,\sqrt{n+(r+1)}}$$

$\sqrt{r+1} = r! = \lfloor r$

$\sqrt{n+r+1} = (n+r)! = \lfloor n+r$

যদি $k = -n$ হয়,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{(r+1)}\,\sqrt{-n+(r+1)}}$$

Math Problem

(1) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$\therefore$ L.H.S $= \frac{d}{dx}\left[x^n J_n(x)\right]$

$$= \frac{d}{dx}\left[x^n \cdot \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{1}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}} \left\{\frac{d}{dx} x^{2n+2r}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}} (2n+2r)\, x^{2n+2r-1}$$

$$= x^{n} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2(n+r) \cdot x^{n+2r-1}}{\lceil r+1 \lceil n+r+1 \; 2^{n+2r}}$$

$$= x^{n} \sum_{r=0}^{\infty} \frac{(-1)^r (n+r) \cdot x^{n+2r-1}}{\lceil r+1 \, (n+r) \lceil n+r \;\; 2^{-1} \cdot 2^{n+2r}}$$

$$= x^{n} \sum_{r=0}^{\infty} \frac{(-1)^r \; x^{n+2r-1}}{\lceil r+1 \lceil n+r \;\; 2^{n+2r-1}}$$

$$= x^{n} \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{(n-1)+2r}}{\lceil r+1 \lceil (n-1)+r+1}$$

$$= x^{n} \; J_{n-1}(x) = R.H.S \quad (\text{Proved})$$

Problem-2

$$\frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = -x^{-n} \; J_{n+1}(x) \quad (\text{Proved})$$

$$\therefore L.H.S = \frac{d}{dx}\left[x^{-n} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\lceil r+1 \lceil n+r+1}\right]$$

$$= \frac{d}{dx}\left[\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r} \cdot x^{2r}}{\lceil r+1 \lceil n+r+1}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r}}{\lceil r+1 \lceil n+r+1} \left(\frac{d}{dx} x^{2r}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2 \cdot r \cdot x^{2r-1}}{\Gamma(r+1)\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot r \cdot x^{2r-1}}{r!\,\Gamma(n+r+1)\, 2^{n-1+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, r \cdot x^{2r-1}}{r\cdot(r-1)!\,\Gamma(n+r+1)\, 2^{n-1+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)\, 2^{n-1+2r}} = \sum_{r=1}^{\infty} \frac{(-1)^r \cdot x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)\, 2^{n-1+2r}}$$

এখানে, $r=0$ এর জন্য $(r-1)! = (-1)!$ আসে যেটা ∞ হয়.

তাই $r-1 = m$ ধরি, $r = m+1$ $\quad [m = 0, 1, 2, \ldots$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2(m+1)-1}}{(m+1-1)!\,\Gamma(n+m+1+1)\, 2^{n-1+2m+2}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)(-1)^m \cdot x^{2m+1}\,\{x^{-n}\cdot x^{n}\}}{m!\,\Gamma((n+1)+(m+1))\, 2^{(n+1)+2m}}$$

$\{x^{-n}\cdot x^{n}\} \rightarrow$ math মিলানোর জন্য করতে হয়

$$= (-1)\cdot x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{(n+1)+2m}}{\Gamma(m+1)\,\Gamma((n+1)+(m+1))\, 2^{(n+1)+2m}}$$

এখন, $m = r$ ধরি, ~~math মিলানোর জন্য~~

$$= (-1)\, x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{x^{(n+1)+2r}} \left(\frac{x}{2}\right)^{(n+1)+2r}}{\Gamma(r+1)\,\Gamma((n+1)+(r+1))}$$

$$= (-1)\, x^{-n}\, J_{n+1}(x)$$

Problem || Proved that $J_{-n}(x) = (-1)^n J_n(x)$

we know, from bessel's function's that,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lceil r+1 \lceil n+r+1} \quad (1)$$

let putting $n=-n$ in equation (1)

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-n+2r}}{\lceil r+1 \lceil -n+r+1}$$

here, n is integer number and must be need ~~greater~~ $r>n$ otherwise $\lceil -n+r+1 = \infty$, then ~~zero~~ r. will be zero

And the function limit $\left[\begin{matrix} r=n, & k=0 \\ r=\infty, & k=\infty \end{matrix}\right]$

So, let, $r = n+k$

$$\therefore J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} (x/2)^{-n+2n+2k}}{\lceil n+k+1 \lceil -n+n+k+1}$$

$$\Rightarrow J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \left(\frac{x}{2}\right)^{n+2k}}{\lceil k+1 \lceil n+k+1}$$

$$\Rightarrow J_{-n}(x) = (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\lceil r+1 \lceil n+r+1} \quad \boxed{\text{let } k=r}$$

$\therefore$ ~~$J_{-n}(x) = (-1)^n \sum_{r=0}^{\infty} (-1)^r.$~~ $\Rightarrow J_{-n}(x) = (-1)^n \cdot J_n(x)$

Proved // $x \cdot J_n'(x) = n J_n(x) \pm x J_{n+1}(x)$

আমরা জানি,

We know

bassel's function,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}}$$

$$\therefore \frac{d}{dx} J_n(x) = J_n'(x)$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{1}{2}\right)^{n+2r} x^{n+2r-1} \cdot (n+2r)}{\sqrt{r+1}\,\sqrt{n+r+1}}$$

$$\Rightarrow x \cdot J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}} (n+2r)$$

$$\Rightarrow x \cdot J_n'(x) = n \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}} + 2r \sum_{r=0}$$

$$+ 2r \cdot \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\sqrt{r+1}\,\sqrt{n+r+1}}$$

$$\Rightarrow x \cdot J_n'(x) = n J_n(x) + 2 \sum_{r=0}^{\infty} \frac{(-1)^r \cdot r \left(\frac{x}{2}\right)^{n+2r}}{r \sqrt{r}\,\sqrt{n+r+1}}$$

$$\Rightarrow x \cdot J_n'(x) = n J_n(x) + 2 \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{(r-1)!\,\sqrt{n+r+1}}$$

$\therefore$ $r=0$ হলে $(r-1)!$ অসংজ্ঞায়িত হয়।

তাই $r=1$ হতে ∞ এবং $r-1=0$, $\Rightarrow m=0$

$r-1=m$, $r=m+1$ ~~ধরে~~ হলে

$$\Rightarrow x \cdot J_n'(x) = n J_n(x) + 2\sum_{m=0}^{\infty} \frac{(-1)^{m+1} \left(\frac{x}{2}\right)^{n+2m+2}}{(m+1-1)!\,\Gamma(n+m+1+1)}$$

$$\Rightarrow x \cdot J_n'(x) = n J_n(x) - 2\sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{x}{2}\right)^{n+1+2m} \cdot \left(\frac{x}{2}\right)^{1}}{m!\,\Gamma((n+1)+m+1)}$$

$$\Rightarrow x \cdot J_n'(x) = n J_n(x) - x\sum_{m=0}^{\infty} \frac{(-1)^{r} \left(\frac{x}{2}\right)^{(n+1)+2r}}{\Gamma(r+1)\,\Gamma((n+1)+r+1)} \quad \boxed{m=r}$$

$$\Rightarrow \boxed{x \cdot J_n'(x) = n J_n(x) - x\, J_{n+1}(x)}$$

(Proved) $x\, J_n'(x) = -n J_n(x) + x\, J_{n-1}(x)$

we know

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r} \cdot (n+2r)\, x^{n+2r-1}}{\Gamma(r+1)\,\Gamma(n+r+1)}$$

$$\Rightarrow x \cdot J_n'(x) = x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right) \left(\frac{x}{2}\right)^{n+2r}}{\Gamma r+1 \, \Gamma n+r+1} (2n+2r-n)$$

$$\Rightarrow x J_n'(x) = -n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\Gamma r+1 \, \Gamma n+r+1} + 2 \sum_{r=0}^{\infty} \frac{(-1)^r (n+r) \left(\frac{x}{2}\right)^{n+2r}}{\Gamma r+1 \, \Gamma (n+r)+1}$$

$$\Rightarrow x J_n'(x) = -n J_n(x) + 2 \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma r+1 \, \Gamma n+r}$$

$$\Rightarrow x J_n'(x) = -n J_n(x) + 2 \cdot \frac{x}{2} \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r-1}}{\Gamma r+1 \, \Gamma (n-1)+r+1}$$

$$\Rightarrow x J_n'(x) = -n J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{(n-1)+2r}}{\Gamma r+1 \, \Gamma (n-1)+r+1}$$

$$\Rightarrow x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

Problem

Proved $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma r+1 \, \Gamma n+r+1}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{\Gamma r+1 \, \Gamma 1/2+r+1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{1/2} \cdot (x/2)^{2r}}{\Gamma r+1 \, \Gamma 3/2+r}$$

$$\Gamma n = (n-1)(n-2)\Gamma n-2$$

$$= \sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{\Gamma r+1 \, \Gamma \frac{3}{2}+r}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma 3/2} - \frac{\left(\frac{x}{2}\right)^2}{\Gamma 2 \, \Gamma 5/2} + \frac{\left(\frac{x}{2}\right)^4}{\Gamma 3 \, \Gamma 7/2} \right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma 3/2} - \frac{x^2}{\Gamma 1 . 2^2 . \left(\frac{3}{2}\right) \Gamma 3/2} + \frac{x^4}{2 \, \Gamma 1 . \left(\frac{5}{2}\right) . \left(\frac{3}{2}\right) \Gamma 3/2 \; 2^4} - \right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\Gamma 3/2} \left[1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{1/2 . \Gamma 1/2} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - - - \right]$$

$$= \sqrt{\frac{x}{2}} \frac{1 . 2}{x . \sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - \right]$$

~~expansion~~

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \right]$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{x}} . \frac{1}{\sqrt{\pi}} \sin x$$

$$\left[\cancel{\sin x = 0} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \right]$$

$$= \sqrt{\frac{2}{x}} (\sqrt{\pi})^{-1} \sin x$$

→ ~~expansion~~ expansion

$$\Rightarrow \sqrt{\frac{2}{\pi x}} \sin x$$

Prove $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

We know the Bessel's functions

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{1/2+2r}}{\Gamma(r+1)\,\Gamma(\frac{1}{2}+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{1/2} \left(\frac{x}{2}\right)^{2r}}{\Gamma(r+1)\,\Gamma(\frac{3}{2}+r)} = \sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{\Gamma(r+1)\,\Gamma(\frac{3}{2}+r)}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(3/2)} - \frac{(x/2)^2}{\Gamma 2\,\Gamma(5/2)} + \frac{(x/2)^4}{\Gamma 3\,\Gamma(7/2)} - \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\Gamma(3/2)} \left[1 - \frac{x^2}{2^2 (1\cdot\Gamma 1)(\frac{3}{2}\Gamma\frac{3}{2})} + \frac{x^4}{2^4 (2\cdot 1\cdot \Gamma 1)(\frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2})} \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{1}{\Gamma(3/2)} \left[1 - \frac{x^2}{1.2.3} + \right.$$

$$\Rightarrow J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-1/2+2r}}{\Gamma(r+1)\,\Gamma(r-\frac{1}{2}+1)}$$

$$\Rightarrow J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{-1/2} (-1)^r \left(\frac{x}{2}\right)^{2r}}{\Gamma(r+1)\,\Gamma(r+\frac{1}{2})}$$

$$= \left(\frac{x}{2}\right)^{-1/2} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{\Gamma(r+1)\,\Gamma(r+\frac{1}{2})}$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma 1/2} - \frac{(x/2)^2}{\Gamma 2\,\Gamma 3/2} + \frac{(x/2)^4}{\Gamma 3\,\Gamma 5/2} - \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma 1/2} - \frac{x^2}{2^2(1.\Gamma 1).(1/2.\Gamma 1/2)} + \frac{x^4}{2^4(2.1.\Gamma 1).(\frac{3}{2}.\frac{1}{2}.\Gamma 1/2)} - \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{-1/2}\frac{1}{\Gamma 1/2}\left[1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \cdots\right]$$

$$= \left(\frac{2}{x}\right)^{1/2}\frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\,.\cos x$$

Expansion of $\cos x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

Expansion of $\sin x$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\therefore \left(J_{1/2}(x)\right)^2 + \left(J_{-1/2}(x)\right)^2$$

$$= \left(\sqrt{\frac{2}{\pi x}}\sin x\right)^2 + \left(\sqrt{\frac{2}{\pi x}}\cos x\right)^2$$

$$\Rightarrow \frac{2}{\pi x}\left(\sin^2 x + \cos^2 x\right)$$

$$\Rightarrow \frac{2}{\pi x}$$

Proved || $\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$

$J_{n-1}(x)$ এর ~~...~~ আনতে হলে $(2n+2r-n)$ নিয়ে করতে হবে

$J_{n+1}(x)$ " " $(+1)$ " " $(n+2r)$ নিয়ে করতে হবে।

We know |

Bessel's function, $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)}$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (1/2)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)} \cdot x^{n+2r-1}(n+2r)$$

$$\Rightarrow x\cdot J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)}\cdot(n+2r)$$

$$\Rightarrow x\cdot J_n'(x) = n\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)} + 2r\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(r+1)\,\Gamma(n+r+1)}$$

$$\Rightarrow x\cdot J_n'(x) = n\,J_n(x) + 2\sum_{r=0}^{\infty} \frac{(-1)^r \cdot r \cdot (x/2)^{n+2r}}{r\,\Gamma(r)\,\Gamma(n+r+1)}$$

$$\Rightarrow x\cdot J_n'(x) = n\,J_n(x) + 2\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(r)\,\Gamma(n+r+1)}$$

ধরি, $\Gamma(r) = (r-1)!$ $\therefore r=0$ এর জন্য, $(-1)!$ হয় যা সম্ভব নয়,

$r-1 = m = 0$ এবং $r = m+1$

$$\Rightarrow x J_n'(x) = n J_n(x) + 2\sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(\frac{x}{2}\right)^{n+2m+2}}{\Gamma(m+1)\,\Gamma(m+2)+(n+1)}$$

$$\Rightarrow x J_n'(x) = n J_n(x) + 2\left(\frac{x}{2}\right)^2 \cdot \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(\frac{x}{2}\right)^{n+2m+1}}{\Gamma(m+1)\,\Gamma(n+1)+(m+1)}$$

$$\Rightarrow x J_n'(x) = n J_n(x) + (-1)^1 x \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{(n+1)+2m}}{\Gamma(m+1)\,\Gamma(n+1)+m+1}$$

$$\Rightarrow x J_n'(x) = n J_n(x) - x \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+1+2r}}{\Gamma(r+1)\,\Gamma(n+1)+r+1}$$

$$\Rightarrow x J_n'(x) = n J_n(x) - x \cdot J_{n+1}(x)$$

$$\Rightarrow \cancel{x^{n-1} J_n'(x) = x^{n-1} n J_n(x) - x^{n-1} \cdot x J_{n+1}(x)}$$

$$\Rightarrow \cancel{x^{n} J_n'(x) = n \cdot x^{n-1} J_n(x) - x^{n} J_{n+1}(x)}$$

~~এর~~ x^{-n} এবং তার x^{-n-1} গুণ করলে

$$\Rightarrow x^{-n-1} \cdot x J_n'(x) = x^{-n-1} n J_n(x) - x^{-n-1} \cdot x J_{n+1}(x)$$

$$\Rightarrow x^{-n} J_n'(x) = x^{-n-1} \cdot n J_n(x) - x^{-n} J_{n+1}(x)$$

$$x^{-n-1} \cdot J_n(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow x^{-n} J_n'(x) + (-n) \cdot x^{-n-1} \cdot J_n(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow -\int_0^x x^{-n} J_{n+1}(x)\,dx = x^{-n} J_n(x)$$

$$\Rightarrow \int_0^x x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$$ (Proved)

Prove that // $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$

we know bessel's function is,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot (x/2)^{n+2r}}{\Gamma r+1\, \Gamma n+r+1}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{1}{2}\right)^{n+2r} \cdot (n+2r)\, x^{n+2r-1}}{\Gamma r+1\, \Gamma n+r+1}$$

$$\Rightarrow x \cdot J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\Gamma r+1\, \Gamma n+r+1} \left\{(2n+2r)-n\right\}$$

$$\Rightarrow x \cdot J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2(n+r) \left(\frac{x}{2}\right)^{n+2r}}{\Gamma(r+1)\, (n+r)\, \Gamma n+r} - n\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\Gamma r+1\, \Gamma n+r+1}$$

$$\Rightarrow x \cdot J_n'(x) = 2\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma r+1\, \Gamma n+r} - n J_n(x)$$

$$\Rightarrow x \cdot J_n'(x) = 2\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r} \left(\frac{x}{2}\right)^1}{\Gamma r+1\, \Gamma (n-1)+r+1} - n J_n(x)$$

$$\Rightarrow x J_n'(x) = x \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r}}{\Gamma r+1\, \Gamma (n-1)+r+1} - n J_n(x)$$

$$\Rightarrow x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

$$\Rightarrow x J_n'(x) + n J_n(x) = x J_{n-1}(x)$$

[উভয় পক্ষকে এখন, x^n আসতে হবে তাই x^{n-1} দ্বারা গুণ করে]

$$\Rightarrow x^{n-1} x J_n'(x) + n x^{n-1} J_n(x) = x^{n-1} x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n'(x) + (n x^{n-1}) J_n(x) = x^n J_{n-1}(x)$$

$$\Rightarrow \frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n(x) = \int_0^x x^n J_{n-1}(x)\, dx$$

$$\Rightarrow \int_0^x x^n J_{n-1}(x) = x^n J_n(x)$$

গোলকীয় স্থানাঙ্ক পদ্ধতিতে Matrix আকারে Conjugate matrix হিসেবে বা 2nd Rank matrix নির্ণয় কর।

The ~~spherical~~ Matrix tensor in form of spherical co-ordinates is

$$[g^{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$$\therefore |g^{ij}| = g = r^4 \sin^2\theta$$

From the definition of conjugate matrix of tensors, is,

$$g_{ij} = \frac{\text{Cofactor of } g^{ij}}{g} = \frac{G_{ij}}{g}$$

$$\therefore g_{11} = \frac{G_{11}}{g} = \frac{1}{r^4 \sin^2\theta} \begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} = 1$$

$$\therefore g_{22} = \frac{G_{22}}{g} = \frac{1}{r^4 \sin^2\theta} \begin{vmatrix} 1 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} = (r^2)^{-1}$$

$$\therefore g_{33} = \frac{G_{33}}{g} = \frac{1}{r^4 \sin^2\theta} \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = \left(\frac{1}{r^2 \sin^2\theta}\right)$$

Cofactor ~~Conjugate~~ of,

$$\therefore g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = g_{32} = 0$$

$\therefore$ For the spherical co-ordinates matrix conjugate will be

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$