KEY TERMS & MAIN RESULTS – DISCRETE MATHEMATICS

<table>
<tr><th>Key terms</th><th>Examples</th><th>Exercises –
Do yourself</th></tr>
<tr><td colspan="3">Chapter 1 – Logic & Proofs</td></tr>
<tr><td>Propositions</td><td>Ex. Determine whether the proposition is TRUE or FALSE.
a/ 1 + 1 = 2 and 2 + 2 = 1.
b/ 1 + 1 = 2 or 2 + 2 = 1
c / 1 + 1 = 2 if and only if 2 + 2 = 1.
d/ 1 + 1 = 2 if 2 + 2 = 1.
e/ If it is snowing, then it is snowing.
Solution.
a/ FALSE ($T \wedge F$)
b/ TRUE ($T \vee F$)
c/ FALSE ($T \leftrightarrow F$)
d/ TRUE ($F \rightarrow T$)
e/ TRUE ($p \rightarrow p$)</td><td>1/ Determine whether the proposition is TRUE or FALSE.
a/ 1 + 1 = 2 if and only if pigs can fly.
b/ I am a superman if 1 + 1 = 2.
c/ If 1 + 1 = 2 or 1 + 1 = 3 but not both, then I can fly.
d/ For every nonnegative integer, n, the value of $n^2 + n + 41$ is prime.</td></tr>
<tr><td>Truth tables</td><td>Ex. Write the truth table for the proposition $\neg(r \rightarrow \neg q) \vee (p \wedge \neg r)$.
Solution.
<table>
<tr><th>p</th><th>q</th><th>r</th><th>$\neg(r \rightarrow \neg q) \vee (p \wedge \neg r)$</th></tr>
<tr><td>T</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>T</td><td>F</td></tr>
<tr><td>T</td><td>F</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>F</td><td>F</td></tr>
<tr><td>F</td><td>F</td><td>T</td><td>F</td></tr>
<tr><td>F</td><td>F</td><td>F</td><td>F</td></tr>
</table></td><td>2/ Construct the truth tables for the propositions:
a/ $(p \wedge \neg q) \vee (\neg p \wedge q)$
b/ $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$
c/ $p \wedge r \rightarrow \neg q \vee p$
d/ $p \rightarrow (q \oplus p)$</td></tr>
<tr><td>Connectives / Operations</td><td>Ex. Let p and q be the propositions
p : It is below freezing.
q : It is snowing.</td><td>3/ Let p, q, and r be the propositions:
p :You get an A on the</td></tr>
</table>

<table>
<tr>
<td></td>
<td>Write these propositions using p and q and logical connectives (including negations).
a/ It is below freezing but not snowing.
b/ It is either snowing or below freezing (or both).
c/ That it is below freezing is necessary and sufficient for it to be snowing.
Solution.
a/ $p \wedge \neg q$
b/ $p \vee q$
c/ $p \leftrightarrow q$</td>
<td>final exam.
q :You do every exercise in this book.
r :You get an A in this class.
Write these propositions using p, q, and r and logical connectives (including negations).

a/ You get an A in this class, but you do not do every exercise in this book.
b/ You get an A on the final, you do every exercise in this book, and you get an A in this class.
c/ To get an A in this class, it is necessary for you to get an A on the final.
d/ Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.</td>
</tr>
<tr>
<td>Tautology</td>
<td>Ex. Determine whether this proposition is a tautology: $\left[(p \to q) \wedge \neg q\right] \to \neg p$
Solution.
Truth table of $\left[(p \to q) \wedge \neg q\right] \to \neg p$:
<table>
<tr><td>p</td><td>q</td><td>p→q</td><td>(p→q)∧¬q</td><td>(p→q)∧¬q → ¬p</td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>F</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>T</td><td>T</td><td>T</td></tr>
</table>
⇨ $\left[(p \to q) \wedge \neg q\right] \to \neg p$ is a tautology.</td>
<td>4/ Determine whether each of these propositions is a tautology:
a/ $p \wedge q \to p$
b/ $(p \to q) \vee (q \to p)$
c/ $\left[(p \to q) \wedge \neg p\right] \to \neg q$
d/ $\neg(p \to \neg p) \to q$.</td>
</tr>
<tr>
<td>If-then
Necessary
Sufficient</td>
<td>Ex. Write each of these statements in the form “if p, then q” in English.
a/ To get a good grade it is necessary that</td>
<td>5/ Write each of these statements in the form “if p, then q” in English.</td>
</tr>
</table>

	you study. b/ Studying is **sufficient** for passing. ***Solution.*** a/ If you get a good grade, then you study. (Equivalently, if you don't study, then you don't get a good grade.) b/ If you study, then you pass.	a/ It is **necessary** to walk 8 miles to get to the top of Long's Peak. b/ A **sufficient** condition for the warranty to be good is that you bought the computer less than a year ago. c/ I will remember to send you the address **only if** you send me an e-mail message. (Hint: "**if** p, then q" can be written as "p **only if** q").
If and only if	**Ex.** Write each of these propositions in the form "p if and only if q" in English. a/ If it is hot outside, you buy an ice cream cone, and if you buy an ice cream cone, it is hot outside. b/ For you to win the contest it is **necessary and sufficient** that you have the only winning ticket. c/ If you watch television, your mind will decay, and **conversely**. ***Solution.*** a/ It is hot outside **if and only if** you buy an ice cream cone. b/ You win the contest if and only if you have the only winning ticket. c/ Your mind will decay **if and only if** you watch television.	**6/** Write each of these propositions in the form "p if and only if q" in English. a/ If you read the newspaper every day, you will be in formed, and **conversely**. b/ For you to get an A in this course, it is **necessary and sufficient** that you learn how to solve discrete mathematics problems. c/ It rains if it is a weekend day, and it is a weekend day if it rains.
Negation	**Ex.** Find the **negation** of the propositions a/ It is Thursday **and** it is cold. b/ I will go to the play **or** read a book. c/ If it is rainy, then we go to the movies. ***Solution.*** a/ It is not Thursday **or** it is not cold. (Keep in mind, $\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$) b/ I won't go to the play **and** I won't read a book.	**7/** Find the **negation** of the propositions. a/ If you study, then you pass. b/ Alex and Bob are absent. c/ He is young or strong.

(Keep in mind, $\overline{p\vee q}\equiv\overline{p}\wedge\overline{q}$)

c/ It is not rainy but we don’t go to the movies.

(Keep in mind, $\overline{p\to q}\equiv p\wedge\overline{q}$)

Equivalence

Ex1. a/ Write a proposition **equivalent** to $p\vee\overline{q}$ that uses only *p*, *q*, $\neg$ and the connective $\wedge$.

b/ Write a proposition **equivalent** to $(p\to q)\wedge(p\to\overline{q})$.

Solution.

a/ $p\vee\overline{q}\equiv\overline{\overline{p\vee\overline{q}}}$ (double negation)

$\equiv\overline{\overline{p}\wedge\overline{\overline{q}}}\equiv\overline{\overline{p}\wedge q}$ (see De Morgan's laws)

So, $p\vee\overline{q}\equiv\neg(\neg p\wedge q)$.

b/ $(p\to q)\wedge(p\to\overline{q})\equiv(\overline{p}\vee q)\wedge(\overline{p}\vee\overline{q})$

$\equiv\overline{p}\wedge(q\vee\overline{q})$ (distributive law)

$\equiv\overline{p}\wedge(T)$

$\equiv\overline{p}$.

(Keep in mind, $p\to q\equiv\overline{p}\vee q$)

Ex2. Determine whether two propositions are **equivalent**.

a/ $p\to q$ and $\overline{p}\to\overline{q}$

b/ $(p\to q\vee r)$ and $(p\to q)\vee(p\to r)$

Solution.

a/ Use a truth table

p	q	p→q	¬p → ¬q
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

⇨ NOT EQUIVALENT.

b/ Starting from the right-hand side,

8/ a/ Write a proposition **equivalent** to $p\to\overline{q}$ that uses only *p*, *q*, $\neg$ and the connective $\wedge$.

b/ Write a proposition **equivalent** to $(p\to q)\wedge(\overline{p}\to q)$.

c/ Write a proposition **equivalent** to (¬p ∨ ¬q) → (p ∧ ¬q).

9/ Determine whether two propositions are **equivalent**.

a/ $p\to q$ and $\overline{q}\to\overline{p}$

b/ $(p\to q\wedge r)$ and $(p\to q)\wedge(p\to r)$

c/ $\overline{p\oplus q}$ and q ↔ q

<table>
<tr><td></td><td>$(p \to q) \vee (p \to r) \equiv (\overline{p} \vee q) \vee (\overline{p} \vee r)$
$\equiv \overline{p} \vee q \vee \overline{p} \vee r$
$\equiv (\overline{p} \vee \overline{p}) \vee q \vee r$ (commutative and associative laws)
$\equiv \overline{p} \vee q \vee r$ (idempotent law)
$\equiv \overline{p} \vee (q \vee r)$ (associative law)
$\equiv p \to (q \vee r)$
➔ EQUIVALENT.</td><td></td></tr>
<tr><td>Predicates Quantifiers</td><td>Ex1. What is the truth values of these propositions? (the domain for variable x is {-3, -2, -1, 0, 1, 2})
a/ $\forall x(x > 1 \wedge x^2 > 1)$
b/ $\forall x(x > 1 \vee x^2 > 1)$
c/ $\forall x(x > 1 \to x^2 > 1)$
Solution.
a/ FALSE (counter example: x = -2)
b/ FALSE (counter example: x = 0)
c/ TRUE (no counter example)

Ex2. Suppose P(x, y) is a predicate and the universe for the variables x and y is {1, 2, 3}.
Suppose P(1, 3), P(2, 1), P(2, 2), P(2, 3), P(2, 3), P(3, 1), P(3, 2) are true, and P(x, y) is false otherwise.
Determine whether the following statements are true.
a/ $\forall x \exists y P(x, y)$
b/ $\exists y \forall x P(x, y)$
c/ $\forall x \exists y (P(x, y) \to P(y, x))$
Solution.
a/ TRUE (we can see P(1, 3), P(2, 2), P(3, 2) are true ➔ for each x in {1, 2, 3}, there is at least one y in {1, 2, 3}.)
b/ FALSE (we can see that no y in {1, 2, 3} for all x in {1, 2, 3}, details are in below:
y = 1: P(2, 1), P(3, 1) are true only (true with x = 2, 3, all x in {1, 2, 3}).
y = 2: P(2, 2), P(3, 2) are true only.</td><td>10/ What is the truth values of these propositions? (the domain for variable x is the set of all real numbers.)
a/ $\forall x(x > 1 \wedge x^2 > 1)$
b/ $\forall x(x > 1 \vee x^2 > 1)$
c/ $\forall x(x > 1 \to x^2 > 1)$

11/ Suppose P(x, y) is a predicate and the universe for the variables x and y is {1, 2, 3}.
Suppose P(1,3), P(2,1), P(2,2), P(2, 3), P(2, 3), P(3, 1), P(3, 2) are true, and P(x, y) is false otherwise.
Determine whether the following statements are true.
a/ $\forall y \exists x P(x, y)$
b/ $\forall y \exists x (P(x, y) \to P(y, x))$

12/ Find a negation of each of these statements:
a/ $\forall x(P(x) \to Q(x))$
b/ $\exists x(P(x) \wedge \neg Q(x))$
c/ $\forall x \exists y(\neg P(x, y) \vee \neg Q(x, y))$
d/ $\forall x \in R(x < 2 \to x^2 < 4)$</td></tr>
</table>

	• y = 3: P(1, 3), P(2, 3) are true only. c/ TRUE • x = 1: P(1, 3) → P(3, 1) • x = 2: P(2, 2) → P(2, 2) • x = 3: P(3, 1) → P(1, 3) **Ex3.** Find the negation of each of these statements. a/ b/ c/	
Translation	**Ex.** Suppose the variable *x* represents students and *y* represents courses, and: • $A(y)$: *y* is an advanced course • $M(y)$: *y* is a math course • $F(x)$: *x* is a freshman • $B(x)$: *x* is a full-time student • $T(x, y)$: student *x* is taking course *y*. Write these statements using these predicates and any needed quantifiers. a/ Linh is taking MAD101. b/ No math course is an advanced course. c/ Every freshman is a full-time student. d/ There is at least one course that every full-time student is taking. ***Solution.*** a/ T(Linh, MAD101) b/ $\forall y\left(M(y) \to \overline{A(y)}\right)$ or equivalently, $\neg\exists y\left(M(y) \wedge A(y)\right)$ c/ $\forall x\left(F(x) \to B(x)\right)$ d/ $\exists y \forall x\left(B(x) \to T(x, y)\right)$.	**13/** Suppose the variable *x* represents students and *y* represents courses, and: • $A(y)$: *y* is an advanced course • $M(y)$: *y* is a math course • $F(x)$: *x* is a freshman • $B(x)$: *x* is a full-time student • $T(x, y)$: student *x* is taking course *y*. Write these statements using these predicates and any needed quantifiers. a/ Nam is taking a math course. b/ There are some freshmen who are not taking any course. c/ There are some full-time students who are not taking any advanced course.
Arguments Valid/invalid Rules of inference	**Ex.** Determine whether the following argument is valid. "Rainy days make gardens grow. Gardens don't grow if it is not hot. It always rains on a day that is not hot. Therefore, if it is not hot, then it is hot."	**14/** Determine whether the following argument is valid. Dong is an AI Major or a CS Major but not both. If he does not know discrete

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Solution.
Consider the statements:
r : it a rainy day
g: gardens grow
h: it is hot
Then,

Rainy days make gardens grow can be written as “r → g” (1)
“Gardens don't grow if it is not hot” is denoted by “h → ¬g” (2)
“It always rains on a day that is not hot” becomes “¬h → r” (3)

From (3), ¬h → r and from (1), r → g. So, ¬h → g (4) can be drawn.
From (2), h → ¬g, this is equivalent to g → ¬h (5).
From (4) and (5), ¬h → g and g → ¬h, we can conclude that ¬h → h, or in words “if it is not hot, then it is hot”.
⇨ VALID ARGUMENT.
</td>
<td>math, he is not an AI Major. If he knows discrete math, he is smart. He is not a CS Major. Therefore, he is smart.</td>
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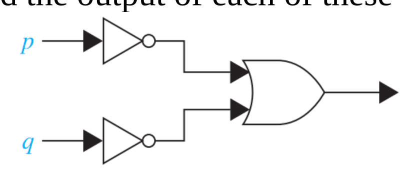
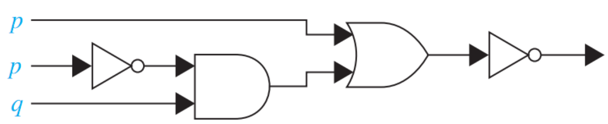
Applications.

Logic Circuits. (readings – pages ____)
Find the output of each of these combinatorial circuits.
a)


b)


The goal of this exercise is to translate some assertions about binary strings into logic notation.

The domain of discourse is the set of all finite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, ... (Here λ denotes the empty string.)
Consider a string like 10x1y, if the value of x is 110 and the value of y is 11, then the value of 10x1y is the binary string 10110111.
Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as I do in the definition of the predicate NO-1S below).

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Meaning	Formula	Name
x is a prefix of y	$\exists z\ (xz = y)$	PREFIX(x, y)
x is a substring of y	$\exists u \exists v\ (uxv = y)$	SUBSTRING(x, y)
x is empty or a string of 0's	NOT(SUBSTRING$(1, x)$)	NO-1S(x)

a) x consists of three copies of some string.
b) x is an even-length string of 0's.
x does not contain both a 0 and a 1.

Chapter 2 – sets, sequences, sums

Sets Elements Empty set Subsets	**Ex1.** Determine whether each of these statements is true or false. a/ $2 \in \{2,\{2\}\}$. b/ $2 \in \{\{2\},\{\{2\}\}\}$. c/ $\varnothing \in \{0\}$. d/ $\varnothing \in \{\varnothing, \{\varnothing\}\}$. e/ $\varnothing \subseteq \{0\}$. ***Solution.*** a/ True b/ False c/ False d/ True e/ True	**15/** Determine whether each of these statements is true or false. a/ $2 \in \{\{\{2\}\}\}$. b/ $2 \in \{\{2\},\{2,\{2\}\}\}$. c/ $\varnothing \in \{x\}$. d/ $\varnothing \subseteq \{x\}$.
Cardinality of a set	**Ex.** What is the **cardinality** of each of these sets? a/ $\{a, \{a\}\}$. b/ $\{\varnothing, a, \{a, \{a\}\}\}$. ***Solution.*** a/ $\lvert\{a, \{a\}\}\rvert = 2$. b/ $\lvert\{\varnothing, a, \{a, \{a\}\}\}\rvert = 3$.	**16/** What is the **cardinality** of each of these sets? a/ $\{\varnothing, \{\varnothing\}\}$. b/ $\{ \{a, \{a\}, b\} \}$.
Power set	*The **power set** of a set A, denoted by P(A), is the set of all subsets of A.* For example, if A = {1, 2}, then the power set of A is the set $P(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. If A contains n elements, P(A) contains 2^n elements. **Ex1.** Determine whether each of these sets is the **power set** of a set, where a and b are distinct elements. a/ $\{\varnothing, \{a\}\}$	**17/** Determine whether each of these sets is the **power set** of a set? a/ $\varnothing$. b/ $\{\varnothing\}$. c/ $\{\varnothing, \{a\}, \{\varnothing\}\}$. d/ $\{\varnothing, \{\{1\}\}, \{2\}, \{\{1\}, 2\}\}$. **18/** How many elements does each of these sets have?

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b/ {∅, {a}, {∅,a}}
c/ {∅, {a}, {b}, {a, b}}
Solution.
a/ {∅, {a}} is the power set of the set {a}.
b/ {∅, {a}, {∅,a}} cannot be a power set of any set.
c/ {∅, {a}, {b}, {a, b}} is the power set of the set {a, b}.

Ex2. How many elements does each of these sets have?
a/ P({a, {a}})
b/ P({∅, a, {a}, {{a}}})
c/ P(P(∅))
Solution.
a/ |{a, {a}}| = 2 ➔ | P({a, {a}})| = 2^2 = 4.
b/ |{∅, a, {a}, {{a}}}| = 4
➔ |P({∅, a, {a}, {{a}}})| = 2^4 = 16.
c/ |∅| = 0 ➔ |P(∅)| = 2^0 = 1 ➔ | P(P(∅))| = 2^1 = 2
</td>
<td>
a/ P({∅, {a}}) .
b/ P({a, {a}, {a, {a}}}).
c/ P(P({∅})).
</td>
</tr>
<tr>
<td>
Union ∪

Intersection ∩

Difference –

Symmetric difference ⊕

Complement
</td>
<td>
Ex1. Prove that, for all sets A, B:
a/ $A - B = A \cap \overline{B}$
b/ A – B ⊆ A.
c/ A = (A – B) ∪ (A ∩ B)
Solution.
a/ We use a membership table:
<table>
<tr><td>A</td><td>B</td><td>$\overline{B}$</td><td>A – B</td><td>$A \cap \overline{B}$</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
</table>
Based on the agreement of two latest columns, an element belongs to A – B if and only if it belongs to $A \cap \overline{B}$.
So, $A - B = A \cap \overline{B}$.
b/ Membership table:
<table>
<tr><td>A</td><td>B</td><td>A – B</td><td>A</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>
</td>
<td>
19/ Show that if A and B are sets with A ⊆ B, then
a/ A ∪ B = B.
b/ A ∩ B = A.
c/ A ∩ B ⊆ A.
d/ A ⊕ B = B – A.
e/ $\overline{B} \subseteq \overline{A}$.

20/ Find the sets A and B if A ⊆ B and A ∪ B = {1, 3, 4, 5, 7, 9}, and A ∩ B = {3, 4, 7}.

21/ Find the sets A and B if A − B ={2, 3, 5, 7}, B − A = {1, 4}, and A ∩ B ={8, 6}.
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<td>From the table, if an element belongs to A – B (the corresponding number is 1), then it also belongs to A (the corresponding number is also 1).

Ex2. Find the sets A and B if A − B ={1, 5, 7, 8}, B − A = {2, 10}, and A ∩ B ={3, 6, 9}.
Solution.
From Ex1 (c):
• A = (A – B) ∪ (A∩B) = {1, 5, 7, 8, 3, 6, 9}
• B = (B – A) ∪ (A∩B) = {2, 10, 3, 6, 9}.</td>
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<tr>
<td>A×B</td>
<td>Ex. Given the sets C = {red; blue; yellow} and S = {small, medium, large}.
a/ Construct Cartesian product C×S.
b/ What is the cardinality of the set C×S ? How many subsets does C×S have?
Solution.
a/ C×S = {(red, small), (red, medium), (red, large), (blue, small), (blue, medium), (blue, large), (yellow, small), (yellow, medium), (yellow, large)}.
b/ |C×S | = 3·3 = 9 ➔ A×B has 2^9 subsets.</td>
<td>22/ Given the sets A = {0, 1}.
a/ Construct the set A×A.
b/ Find the complement of the set {(0, 1)} in A×A.
c/ What is the cardinality of the set A×A? List all subsets of A×A.</td>
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<tr>
<td>Set representation</td>
<td>Ex1. Let U = {a, b, c, d, e, f, g} be the universal set. Find the bit string representing the subset A = {a, c, d, g}.
Solution.
<table>
<tr><td>U</td><td>a</td><td>b</td><td>c</td><td>d</td><td>e</td><td>f</td><td>g</td></tr>
<tr><td>U</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>A</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
</table>

Ex2. Let U = {1, 2, 3, 4, 5, 6, 7, 8}. Given the subsets A = {1, 2, 3, 5, 7}, B = {2, 4, 5}. Find the bit string representing the subset A – B.
Solution.
<table>
<tr><td>U</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td>A</td><td>1</td><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>B</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>A-</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td></tr>
</table></td>
<td>23/ Suppose that the universal set is U ={1, 2, 3, 4, 5, 6, 7, 8}. Express each of these sets with bit strings.
a/ {3, 4, 5}
b/ {1, 3, 6, 8}
c/ {1, 2, 3, 5} ⊕ {2, 3, 4, 6, 7}

24/ Let U = {a, b, c, d, e, f, g} be the universal set. Suppose A and B are sets given by bit strings 1010101 and 1100111. List all elements in the set $\overline{A \cap B}$.</td>
</tr>
</table>

	B	
functions	**Ex1.** Determine which rules are functions. a/ f: Z $\rightarrow$ Z; f(x) = 1/(2x-1) b/ f: Z $\rightarrow$ R; f(x) = 1/(2x-1) c/ f: R $\rightarrow$ R; f(x) = 1/(2x-1) ***Solution.*** a/ f: Z $\rightarrow$ Z; f(x) = 1/(2x-1) This rule is not a function, because f(2) = 1/3 does not belong to the set Z (the set of integers). b/ f: Z $\rightarrow$ R; f(x) = 1/(2x-1) This rule is a function, we determine exactly one output value for every input value. c/ f: R $\rightarrow$ R; f(x) = 1/(2x-1) This rule is not a function because f(1/2) is not defined. **Ex2.** Determine whether f is a function from the set of all bit strings to the set of integers if **f(S) is the position of a 0 bit** in S. ***Solution.*** Consider the string S = "10011" as an input, f(S) = the position of a 0 bit in S ➔ f(10011) can be 2 or 3 ➔ f is NOT a function.	**25/** Determine which rules are functions. a/ f: Z $\rightarrow$ Z; f(x) = 1/(x^2 - 2) b/ f: Z $\rightarrow$ R; f(x) = 1/(x^2- 2) c/ f: R $\rightarrow$ R; f(x) = 1/(x^2- 2) **26/** Determine whether f is a function from the set of all bit strings to the set of integers if f(S) is the number of 0 bits in S. **27/** Let R be the set {(a, b) | a - 1 = b or b - 1 = a}, where a and b are in {-2, -1, 0, 1, 2}. a/ List all ordered pairs of R. b/ Is R a function? Explain your answer.
One-to-one Onto Bijection Inverse functions (f^{-1}) Invertible	**Ex1.** a/ Determine whether the function from N = {0, 1, 2, …} to N is **one-to-one.** a/ f(n) = $(n – 1)^2$ b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **one-to-one.** $f(n)=\begin{cases}-2n & \text{if } n<0\\ 2n+1 & \text{if } n\geq 0\end{cases}$ ***Solution.*** a/ f(2) = f(0) = 1 ➔ f is not one-to-one. b/ • If n, m are different negative integers ➔ f(n) $\neq$ f(m) because f(n) = -2n $\neq$ -2m = f(m). • If n, m are different non-negative	**28**a/ Determine whether the function from f(n) = $(n + 1)^2$ N = {0, 1, 2, …} to N is **one-to-one.** b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **one-to-one.** $f(n)=\begin{cases}n^2 & \text{if } n<0\\ 2n & \text{if } n\geq 0\end{cases}$ **29/** Determine whether the function f from the set of all bit strings to the set of integers is one-to-one

integers ➔ $f(n) \neq f(m)$ because $2n+1 \neq 2m + 1$.

- If n is negative and m is non-negative ➔ $f(n) = -2n$ (even) and $f(m) = 2m + 1$ (odd) ➔ $f(n) \neq f(m)$

➔ $\forall n \forall m \left(n \neq m \rightarrow f(n) \neq f(m)\right)$

➔ f is one-to-one.

Ex2. Determine whether the function f from the set of all bit strings to the set of integers is one-to-one if f(S) is the number of 1 bits in S.

Solution.

f(01011) = f(1110) = 3 ➔ f is not one-to-one.

Ex3. a/ Determine whether the function $f(n) = (n – 1)^2$ from N = {0, 1, 2, …} to N is **onto.**

b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **onto.**

$$f(n) = \begin{cases} -2n & \text{if } n < 0 \\ 2n+1 & \text{if } n \geq 0 \end{cases}$$

Solution.

a/ Because $f(n) = (n-1)^2 \neq 2$ for all values of n ➔ f is not onto.

b/ Because $f(n) \neq 0$ for all n ➔ f is not onto.

Ex4. Determine whether each of these functions is a **bijection** from R to R. In case f is a bijection, find the inverse function f^{-1}.

a/ $f(x) = −3x + 4$

b/ $f(x) = −3x^2 + 7$

Solution.

a/ For every y in R, we can find **exactly one** x in R such that $y = −3x + 4$. In this case, $x = (y – 4)/(-3)$.

And the **inverse function** is $f^{-1}(y) = (y – 4)/(-3)$.

b/ For some y in R, we cannot find x (or can

if f(S) is the number of 0 bits in S.

30a/ Determine whether the function from $f(n) = (n + 1)^2$ N = {0, 1, 2, …} to N is **onto.**

b/ Determine whether the function from N = {0, 1, 2, …} to Z = {…, -2, -1, 0, 1, 2, …} is **onto**

$$f(n) = \begin{cases} n/2 & \text{if n is even} \\ -(n+1)/2 & \text{if n is odd} \end{cases}$$

31a/ List all ***functions*** from {□, ς} to {SHOOT, PASS, SPRINT}.

b/ List all ***one-to-one*** functions from {□, ς} to {SHOOT, PASS, SPRINT}.

c/ List all ***onto*** functions from {□, ς} to {SHOOT, PASS, SPRINT}.

32/ Determine whether each of these functions is a **bijection** from R to R. In case f is a **bijection**, find the **inverse function** f^{-1}.

a/ $f(x) = 2x − 5$

b/ $f(x) = (x – 3)(x + 1)$

	find more than one values of x) in R such that y = -3x + 4. For example, no value of x in R such that $10 = -3x^2 + 7$ or $1 = -x^2$. ⇨ f is not a **bijection**.	
Composite function	**Ex1.** Find fog and gof, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from R to R. ***Solution.*** • $(fog)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1$ • $(gof)(x) = g(f(x)) = g(x^2 + 1) = (x^2 + 1) + 2 = x^2 + 3$. **Ex2.** Let f = {(a, 1); (b, 3); (c, 2)} be a function from {a, b, c} to {1, 2, 3}. a/ Find f^{-1}. b/ Find fof^{-1} and $f^{-1}of$. ***Solution.*** a/ f^{-1} = {(1, a); (3, b); (2, c)} b/ fof^{-1} = {(1, 1); (2, 2); (3, 3)} and $f^{-1}of$ = {(a, a); (b, b); (c, c)}.	**33/** Find fog and gof, where $f(x) = 2x + 1$ and $g(x) = 1 – x^3$, are functions from R to R. **34/** Let g = {(1, c); (2, b); (3, a)} be a function from {1, 2, 3} to {a, b, c}. a/ Find g^{-1}. b/ Find gog^{-1} and $g^{-1}og$.
Sequences	**Ex1.** List the first 6 terms of each of these sequences. a/ the sequence that lists each positive integer three times, in increasing order b/ the sequence whose n^{th} term is $2^n – n^2$ c/ the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two previous terms. ***Solution.*** a/ 111, 222, 333, 444, 555, 666 b/ 1, 0, -1, 0, 7, 28 c/ 2, 4, 6, 10, 16, 26 **Ex2.** Find the first four terms of the sequence defined by each of these recurrence relations and initial conditions. a/ $a_n = −2a_{n−1}$, $a_0 = −1$ b/ $a_n = a_{n−1} − a_{n−2}$, $a_0 = 2$, $a_1 = −1$ c/ $a_n = a_{n−1}$, $a_0 = 5$. ***Solution.*** a/ $a_0 = -1$ $a_1 = -2a_0 = -2.(-1) = 2$	**35/** List the first 6 terms of each of these sequences. a/ the sequence whose n^{th} term is the sum of the first n odd positive integers b/ the sequence whose n^{th} term is $n! – 2^n$ c/ the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms. **36/** Find the first four terms of the sequence defined by each of these recurrence relations and initial conditions. a/ $a_n = −a_{n−1}$, $a_0 = 5$ b/ $a_n = a_{n−1} − n$, $a_0 = 4$ c/ $a_n = a_{n-2}$, $a_0 = 3$, $a_1 = 5$

	$a_2 = -2a_1 = -2(2) = -4$ $a_3 = -2a_2 = -2(-4) = 8$ b/ $a_0 = 2$, $a_1 = -1$ $a_2 = a_1 - a_0 = -1 - 2 = -3$ $a_3 = a_2 - a_1 = -3 - (-1) = -2$ c/ $a_0 = 5$ $a_1 = a_0 = 5$ $a_2 = a_1 = 5$ $a_3 = a_2 = 5$	
Special sums	Special sum: $$\sum_{i=1}^{n} i = 1+2+3+...+n = \frac{n(n+1)}{2}$$ **Ex1.** Find the value of each of these sums. a/ $\sum_{i=1}^{10} i$ b/ $\sum_{i=1}^{10} 3$ c/ $\sum_{i=1}^{10} (i+3)$ d/ $\sum_{i=1}^{10} (3i+1)$ ***Solution.*** a/ $\sum_{i=1}^{10} i = \frac{10(10+1)}{2} = 55$ b/ $\sum_{i=1}^{10} 3 = 3+3+...+3 = 3 \cdot 10 = 30$ c/ $\sum_{i=1}^{10} (i+3) = \sum_{i=1}^{10} i + \sum_{i=1}^{10} 3 = 55 + 30 = 85$ d/ $\sum_{i=1}^{10} (3i+1) = \sum_{i=1}^{10} (3i) + \sum_{i=1}^{10} 1$ $= 3\sum_{i=1}^{10} i + 10$ $= 3 \cdot 55 + 10$ $= 175$ **Ex2.** Compute each of these double sums. a/ $\sum_{i=0}^{2} \sum_{j=1}^{3} (i+j)$	**37/** Find the value of each of these (double) sums. a/ $\sum_{k=10}^{20} k$ b/ $\sum_{k=1}^{7} (2k-1)$ c/ $\sum_{i=1}^{3} \sum_{j=0}^{2} (2i-j)$ d/ $\sum_{i=1}^{3} \sum_{j=0}^{2} j$ e/ $\sum_{i=1}^{10} \sum_{j=1}^{20} (i \cdot j)$

b/ $\sum_{i=1}^{3}\sum_{j=1}^{4} i$

c/ $\sum_{i=1}^{20}\sum_{j=1}^{30} (i \cdot j)$

Solution.

a/

$$\sum_{i=0}^{2}\sum_{j=1}^{3} (i+j) =$$

$(0+1)+(0+2)+(0+3)$ // i = 0

$+(1+1)+(1+2)+(1+3)$ // i = 1

$+(2+1)+(2+2)+(2+3)$ //i = 3

= 27

b/

$$\sum_{i=1}^{3}\sum_{j=1}^{4} i =$$

1+1+1 + 1 // i = 1

+ 2 + 2 + 2 + 2 // i = 2

+ 3 + 3 + 3 + 3 // i = 3

= 24

c/ $\sum_{i=1}^{20}\sum_{j=1}^{30} (i \cdot j) =$

$\sum_{j=1}^{30} j$ // i = 1

$+\sum_{j=1}^{30} (2j)$ // i = 2

$+\sum_{j=1}^{30} (3j)$ // i = 3

...

$+\sum_{j=1}^{30} (20j)$ // i = 20

$$= (1+2+3+...+20)\sum_{j=1}^{30} j$$

$$= \frac{20(20+1)}{2} \cdot \frac{30(30+1)}{2}$$

= 97650

Chapter 3 – Algorithms & Integers

<table>
<tr>
<td>Algorithms</td>
<td>Ex1. List all the steps used to search for 9 in the sequence 2, 3, 4, 5, 6, 8, 9, 11 using a linear search. How many comparisons required to search for 9 in the sequence?
Solution.
Below is the linear search algorithm in pseudocode
procedure linear search(x: integer, $a_1, a_2, \ldots, a_n$: distinct integers)
i := 1
while ($i \le n$ and $x = a_i$)
    i := i + 1
if $i \le n$ then location := i
else location := 0
return location{location is the subscript of the term that equals x, or is 0 if x is not found}
All the steps used to search for 9 using a linear search:
i = 1
$(1 \le 8$ and $9 \ne 2)$ ➔ i:=i+1 = 2
i = 2
$(2 \le 8$ and $9 \ne 3)$ ➔ i:= i+1 = 3
i = 3
$(3 \le 8$ and $9 \ne 4)$ ➔ i:= i+1 = 4
i = 4
$(4 \le 8$ and $9 \ne 5)$ ➔ i:= i+1 = 5
i = 5
$(5 \le 8$ and $9 \ne 6)$ ➔ i:= i+1 = 6
i = 6
$(6 \le 8$ and $9 \ne 8)$ ➔ i:= i+1 = 7
i = 7
$(7 \le 8$ and $9 \ne 9)$ // the condition is false
$7 \le 9$ ➔ location = 7.

Based on the steps above, there are 15 comparisons ($\le$, $\ne$) required.</td>
<td>38/ List all the steps used to search for 8 in the sequence 3, 5, 6, 8, 9, 11, 13, 14 using a binary search. How many comparisons required to search for 8 in the sequence?

39/ Josephus problem.
This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive.
However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive.
Devise an algorithm to determine the alive positions if the number of rebels is n and an alive rebel will be killed after counting to k ($k < n$).</td>
</tr>
<tr>
<td>Big-O
Big-Omega
Big-theta</td>
<td>Ex1. In the table below, check ✓if the fact is true and check ✗ otherwise.
<table><tr><td>function</td><td>$= O(x^2)$</td><td>$= \Omega(x^2)$</td><td>$= \Theta(x^2)$</td></tr></table></td>
<td>40/ Determine whether each of these functions is $O(x^2)$.</td>
</tr>
</table>

2x + 11			
$x^2 + 3x + 1$			
$x^2\log x + 2018$			
$x^3 - 5x^2 + 3$			

Solution.

function	$= O(x^2)$	$= \Omega(x^2)$	$= \Theta(x^2)$
2x + 11	✓	✗	✗
$x^2 + 3x + 1$	✓	✓	✓
$x^2\log x + 2018$	✗	✓	✗
$x^3 - 5x^2 + 3$	✗	✓	✗

Ex2. Find the least integer k such that $\frac{(\sqrt{x^8+x^4+1}+1)(\log x+3)}{x^2+1}$ is $O(x^k)$.

Solution.

$\sqrt{x^8+x^4+1} \sim \sqrt{x^8} = x^4$

$\log x + 3 \sim \log x$

$and\ x^2+1 \sim x^2$

So, $\frac{(\sqrt{x^8+x^4+1}+1)(\log x+3)}{x^2+1} \sim \frac{x^4\log x}{x^2} = x^2\log x$

In other hand, $x^2\log x$ is $O(x^3)$ ➔ the least integer k is 3.

Ex3. a/ Show that $\log_{10}n$ is O(logn)

b/ Show that log(n!) is O(nlogn).

Solution.

a/ $\log_{10}n = \log_{10}2.\log n$ ➔ $\log_{10}n$ is O(logn)

b/ $\log(n!) = \log(1\cdot2\cdot3\cdots n) \le \log(n\cdot n\cdot n\cdots n) = \log(n^n) = n\log n$

⇨ log(n!) is O(nlogn).

a/ f(x) = 3x + 7.

b/ $f(x) = \log(x^3) + 2x$.

c/ $f(x) = (2x^3 + x^2\log x)/(x+2)$.

d/ $f(x) = 2^x + 1$.

41/ Find the least integer k such that f(x) is $O(x^k)$ for each of these functions.

a/ $f(x) = 2x^2 + x^2\log x$.

b/ $f(x) = x^3 + (\log x)^4$.

c/ $f(x) = (x\log x + 3x)(x^2 + 100x + 1)$.

42/ Show that $1 + 2 + 3 + \dots + n$ is $O(n^2)$.

Complexity of an algorithm

Ex1. Consider the **algorithm:**

procedure giaithuat($a_1, a_2, \dots, a_n$: integers)
count:= 0
for i:= 1 to n do
    if $a_i > 0$ then count: = count + 1
print(count)

43/ Consider the **algorithm:** procedure thuattoan($a_1, a_2, \dots, a_n$: positive real numbers).
m := 0
for i := 1 to n-1

	Give the **best big-O complexity** for the algorithm above. ***Solution.*** With one "for loop" in the algorithm, the complexity of the algorithm is O(n). **Ex2.** How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, each requiring 10^{-9} seconds, with these values of n? a/ 10 b/ 50 ***Solution.*** a/ n = 10 ➔ the algorithm uses $2.10^2 + 2^{10}$ operations, each requiring 10^{-9} seconds ➔ need $(2.10^2 + 2^{10}).10^{-9} = 0.000001224$ seconds. b/ n = 50 ➔ the algorithm uses $2.50^2 + 2^{50}$ operations, each requiring 10^{-9} seconds ➔ need $(2.50^2 + 2^{50}).\ 10^{-9} = 1125900$ seconds.	for j := i + 1 to n m := max($a_i.a_j$, m) Give the **best big-O complexity** for the algorithm above. **44/** How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, each requiring 10^{-9} seconds, with these values of n? a/ 30. b/ 100.
Divide Divisor Division Quotient Remainder mod and div	**Ex1.** Show that if a | b and b | c, then a | c, where a, b, c are integers. ***Solution.*** a | b ➔ $\exists k \in Z$ (b = ka) b | c ➔ $\exists m \in Z$ (c = mb) ⇨ c = m(ka) = (mk)a, where mk is an integer ⇨ a | c. **Ex2.** Prove or disprove that if ab | c, where a, b, and c are positive integers, then a | c and b | c. ***Solution.*** ab | c ➔ $\exists k \in Z$ (c = kab) ⇨ c = (kb)a and c = (ka)b, where ka, kb are integers ⇨ a | c and b | c. **Ex3.** What are the quotient and remainder when a/ 1001 is divided by 13? b/ −111 is divided by 11?	**45/** Show that if a | b and b | a, then a = b or a = −b, where a, b are integers. **46/** Prove or disprove that if a | bc, then a | b or a | c, where a, b, and c are positive integers and a ≠ 0. **47/** What are the **quotient** and **remainder** when a/ −1 is divided by 3? b/ 3 is divided by 13? c/ −123 is divided by 19? **48/** Evaluate these quantities. a/ −17 **mod** 2. b/ 144 **mod** 7. c/ −101 **div** 13. d/ 199 **div** 19.

	Solution. a/ 1001 = 13.77 + 0 ⇨ quotient = 77 and remainder = 0. b/ -111 = 11.(-11) + 10 ⇨ quotient = -11 and remainder = 10. **Ex4.** Suppose a ***mod*** 4 = 3 and b ***mod*** 8 = 7, find ab ***mod*** 4. ***Solution.*** • We have, b mod 8 = 7 ➔ b = 8k + 7, where k is an integer ⇨ b = 4(2k + 1) + 3 ⇨ b mod 4 = 3 • So, ab mod 4 = ((a mod 4).(b mod 4)) mod 4 = (3.3) mod 4 = 1.	 **49/** Suppose a ***mod*** 3 = 2 and b ***mod*** 6 = 4, find ab ***mod*** 3.
Congruence	**Ex.** Decide whether each of these integers is **congruent to 5 modulo 17**. a/ 80 b/ 103 c/ -29 d/ -122 ***Solution.*** Recall that *a is congruent to b modulo m* if and only if m **divides** a – b. Or equivalently, $a \equiv b \pmod{m} \Leftrightarrow m \mid (a-b)$ a/ 17 □ (80 – 5) ➔ 80 is not congruent to 5 modulo 17. b/ 17 □ (103 – 5) ➔ 103 is not congruent to 5 modulo 17. c/ 17 | (-29 – 5) ➔ -29 is congruent to 5 modulo 17. d/ 17 □ (-122 – 5) ➔ -122 is not congruent to 5 modulo 17.	**50/** Decide whether each of these integers is **congruent to 3 modulo 7**. a/ 37. b/ 66. c/ −17. d/ −67. **51/** Find an integer x in {0, 1, 2, …, 6} such that: a/ $5.x \equiv 1 \pmod 7$. b/ $x.x^2 \equiv 1 \pmod 7$.
Encryption Decryption Hashing functions Pseudo random	**Ex1.** Suppose ***pseudo-random numbers*** are produced by using: $x_{n+1} = (3x_n + 11) \bmod 13$. If $x_3 = 5$, find x_2 and x_4. ***Solution.*** • $x_4 = (3x_3 + 11) \bmod 13$ $= (3.5 + 11) \bmod 13 = 0$ • $x_3 = (3x_2 + 11) \bmod 13$	**52/** Suppose ***pseudo-random numbers*** are produced by using: $x_{n+1} = (2x_n + 7) \bmod 9$. a/ If $x_0 = 1$, find x_2 and x_3. b/ If $x_3 = 3$, find x_2 and

numbers

So, $5 = (3x_2 + 11) \bmod 13$

$\Leftrightarrow 13 \mid (3x_2 + 11 - 5)$

$\Leftrightarrow 13 \mid (3x_2 + 6)$ (*)

Note that x_2 is in 0..12 ➔ $x_2 = 11$ is the solution of (*).

Ex2. Using the function

$f(x) = (x + 10) \bmod 26$

to **encrypt** messages. Answer each of these questions.

a/ ***Encrypt*** the message STOP

b/ ***Decrypt*** the message LEI.

Solution.

A	B	C	…	Z
0	1	2		25

S	T	O	P
18	19	14	15

x	18	19	14	15
$f(x) = (x+10) \bmod 26$	2	3	24	25

2	3	24	25
C	D	Y	Z

⇨ STOP has been encrypted to CDYZ.

b/ We will **decrypt** the message LEI using the inverse function $f^{-1}(x) = (x - 10) \bmod 26$.

Encrypted form	L	E	I
x	11	4	8
$f^{-1}(x) = (x - 10) \bmod 26$	1	20	24
Original message	B	U	Y

Ex3. Which memory locations are assigned by the **hashing function** $h(k) = k \bmod 101$ to the records of insurance company customers with these Social Security

x_4.

53 a/ **Encrypt** the message SELL using the function $f(x) = (x + 21) \bmod 26$.

b/ **Decrypt** these messages "CFMV L" that were encrypted using the $f(x) = (x + 17) \bmod 26$.

54/ A parking lot has 31 visitor spaces, numbered from 0 to 30. Visitors are assigned parking spaces using the **hashing function** $h(k) = k \bmod 31$, where k is the number formed from the first three digits on a visitor's license plate. Which spaces are assigned by the **hashing function** to cars that have these first three digits on their license plates: 317, 918, 007, 111?

	Numbers? a/ 104578690 b/ 432222187 ***Solution.*** a/ h(104578690) = 104578690 mod 101 = 58. ⇨ The memory location 58 is assigned to the customer with the Social Security number 104578690. b/ h(501338753) = 501338753 mod 101 = 3. So, the memory location 3 is assigned to the customer with the Social Security number 501338753.	
Prime, relatively prime Gcd, lcm	**Ex1.** Which positive integers less than 30 are **relatively prime** to 30? ***Solution.*** Recall that two positive integers a and b are called **relatively prime** if and only if their greatest common divisor is 1. So, positive integers less than 30 are **relatively prime** to 30 are: 1, 7, 11, 13, 17, 19, 23, 29. **Ex2.** The value of the **Euler φ-function** at the positive integer n, φ(n), is defined to be the number of positive integers less than or equal to n that are **relatively prime** to n. Find these values of the Euler φ-function. a/ φ(6) b/ φ(7) ***Solution.*** a/ n = 6: positive integers less than or equal to 6 that are relatively prime to 6 are: 1, 5 ⇨ φ(6) = 2 b/ n = 7: positive integers less than or equal to 6 that are relatively prime to 6 are: 1, 2, 3, 4, 5, 6 ⇨ φ(7) = 6 **Ex3.** If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their **greatest common divisor** is $2^3 3^4 5$, what is their **least**	**55/** Which positive integers less than 18 are **relatively prime** to 18? **56/** Find these values of the **Euler φ-function**. a/ φ(4). b/ φ(5). c/ φ(11). **57/** If the product of two integers is 3072 and their **least common multiple** is 384, what is their **greatest common divisor**?

	common multiple? ***Solution.*** If a and b are positive integers, then **ab = gcd(a, b).lcm(a, b)**. So, $2^7 3^8 5^2 7^{11}$ = gcd(a, b).lcm(a, b) = $2^3 3^4 5$.lcm(a, b) ➔ lcm(a, b) = $2^7 3^8 5^2 7^{11}/2^3 3^4 5 = 2^4 3^4 5(7^{11})$	
Euclidean algorithm	**Ex.** Use the **Euclidean algorithm** to find a/ gcd(8, 28) b/ gcd(100, 101). ***Solution.*** a/ 28 mod 8 = 4 ➔ gcd(8, 28) = gcd(4, 8) 8 mod 4 = 0 ➔ gcd(4, 8) = gcd(0, 4) = 4. b/ 101 mod 100 = 1 ➔ gcd(100, 101) = gcd(1, 100) 100 mod 1 = 0 ➔ gcd(1, 100) = gcd(0, 1) = 1.	**58/** Use the **Euclidean algorithm** to find a/ gcd(12, 18). b/ gcd(111, 201).
Integer representation Decimal Binary Octal Hexadecimal Expansions Base b expansions	**Ex1.** Convert 96 to a/ a binary expansion. b/ a base 5 expansion. c/ a base 13 expansion. ***Solution.*** a/ 96 = $(1100000)_2$ b/ • 96 = 19.5 + 1 • 19 = 3.5 + 4 • 3 = 0.5 + 3 ⇨ 96 = 19.5 + 1 = (3.5 + 4).5 + 1 ⇨ 96 = $3.5^2 + 4.5^1 + 1.5^0$ ⇨ 96 = $(341)_5$ c/ • 96 = 7.13 + 5 • 7 = 0.13 + 7 ⇨ 96 = $7.13^1 + 5.13^0$ ⇨ 96 = $(75)_{13}$ **Ex2.** Convert each of the following expansions to **decimal expansion**. a/ $(102)_3$ b/ $(325)_7$ c/ $(A3)_{12}$ ***Solution.***	**59/** Convert 69 to a/ a binary expansion. b/ a base 6 expansion. c/ a base 9 expansion. **60/** Convert each of the following expansions to **decimal expansion**. a/ $(401)_5$ b/ $(12B7)_{13}$

	a/ $(1021)_3 = 1.3^3 + 0.3^2 + 2.3^1 + 1.3^0 = 34$ b/ $(325)_7 = 3.7^2 + 2.7^1 + 5.7^0 = 166$ c/ $(A3)_{12} = A.12^1 + 3.12^0 = 10.12 + 3 = 123$.	

Applications: Check digits.

1/ **UPCs.** Retail products are identified by their Universal Product Codes (UPCs). The most common form of a UPC has 12 decimal digits: the first digit identifies the product category, the next five digits identify the manufacturer, the following five identify the particular product, and the last digit is a **check digit**. The check digit is determined by the congruence

$3x_1 + x_2 + 3x_3 + x_4 + 3x_5 + x_6 + 3x_7 + x_8 + 3x_9 + x_{10} + 3x_{11} + x_{12} \equiv 0 \pmod{10}$.

For example, if the first 11 digits of a UPC are 79357343104, then the check digit is $x_{12} = 2$.

In fact, let x_{12} be check digit, we have

$3 \cdot 7 + 9 + 3 \cdot 3 + 5 + 3 \cdot 7 + 3 + 3 \cdot 4 + 3 + 3 \cdot 1 + 0 + 3 \cdot 4 + x_{12} \equiv 0 \pmod{10}$

Simplifying, we have $98 + x_{12} \equiv 0 \pmod{10} \Rightarrow x_{12} = 2$.

a/ Find the check digit for the **USPS** money orders that have identification number that start with these ten digits 7555618873 and 6966133421.

b/ Determine whether 74051489623 and 88382013445 are valid **USPS** money order identification number.

2/ **Parity Check Bits**. Digital information is represented by bit string, split into blocks of a specified size. Before each block is stored or transmitted, an extra bit, called a **parity check** bit, can be appended to each block. The parity check bit x_{n+1} for the bit string $x_1x_2...x_n$ is defined by $x_{n+1} = x_1 + x_2 + \cdots + x_n \bmod 2$.

(It follows that xn+1 is 0 if there are an even number of 1 bits in the block of n bits and it is 1 if there are an odd number of 1 bits in the block of n bits). When we examine a string that includes a parity check bit, we know that there is an error in it if the parity check bit is wrong. However, when the parity check bit is correct, there still may be an error. For example, if we receive in a transmission the bit string 11010110, we find that $1 + 1 + 0 + 1 + 0 + 1 + 1 \equiv 1 \pmod 2$, so the **parity check** is incorrect. So, we reject the string.

Suppose you received these bit strings over a communications link, where the last bit is a **parity check** bit. In which string are you sure there is an error?

a/ 00100111111

b/ 10101010101

Chapter 4 – Induction & Recursion

Mathematical induction Strong induction	**Ex1.** Prove the statement "6 divides $n^3 - n$ for all integers $n \geq 0$", using ***mathematical induction*** method. ***Solution.*** *Basis step*. The statement is true for $n = 0$, since 6 divides 0.	**61/** Prove that 2 divides $n^2 + n$ whenever n is a positive integer. **62/** Prove that $2^n < n!$ if n is an integer greater than

	Inductive step. • Suppose for every integer $k \geq 0$, the statement is true, that is, "6 divides $k^3 - k$" • We have, $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k + 1) = k^3 - k + 3(k^2 + k)$. As 6 divides $k^3 - k$ and $3(k^2 + k)$ is a multiple of 6, we conclude that $(k+1)^3 - (k+1)$ is also a multiple of 6. By induction, 6 divides $n^3 - n$ for all integers $n \geq 0$. **Ex2.** Suppose you wish to prove that the following is true for all positive integers *n* by using the Principle of Mathematical Induction: $P(n)$ = "$1 + 3 + 5 + \ldots + (2n - 1) = n^2$" (a) Write *P*(1) (b) Write *P*(12) (c) Write *P*(13) (d) Use the fact "*P*(12) is true" to prove "*P*(13) is true" (e) Write *P*(*k*) (f) Write *P*(*k* + 1) (g) Use the Principle of Mathematical Induction to prove that *P*(*n*) is true for all positive integers *n*. ***Solution.*** a/ "$1 = 1^2$" b/ "$1 + 3 + 5 + \ldots + (2 \cdot 12 - 1) = 12^2$" c/ "$1 + 3 + 5 + \ldots + (2 \cdot 13 - 1) = 13^2$" d/ We have P(12) is true, or "$1 + 3 + 5 + \ldots + (2 \cdot 12 - 1) = 12^2$" is true. So, $1 + 3 + 5 + \ldots + (2 \cdot 13 - 1)$ $= 1 + 3 + 5 + \ldots + (2 \cdot 12 - 1) + (2 \cdot 13 - 1)$ $= 12^2 + (2.13 - 1)$ (due to the truth of P(12)) $= 12^2 + (2.12 + 1)$ $= (12 + 1)^2$ $= 13^2$. Hence, $1 + 3 + 5 + \ldots + (2 \cdot 13 - 1) = 13^2$ and	3. **63/** Suppose you wish to use the Principle of Mathematical Induction to prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$, for all $n \geq 1$. (a) Write *P*(1). (b) Write *P*(5). (c) Use *P*(5) to prove P(6). (d) Write *P*(*k*). (e) Write *P*(*k* + 1). (f) Use the Principle of Mathematical Induction to prove that *P*(*n*) is true for all $n \geq 1$. **64/** Suppose that the only currency were 2-VND bills and 5-VND bills. Use **strong induction** to show that any amount greater than 3 VND could be made from a combination of these bills.

P(13) is true.
e/ "$1 + 3 + \ldots + (2k – 1) = k^2$"
f/ "$1 + 3 + \ldots + [2(k + 1) – 1] = (k + 1)^2$"
g/

- BASIC STEP.

"$1 = 1^2$" ➔ P(1) is true.

- INDUCTIVE STEP.

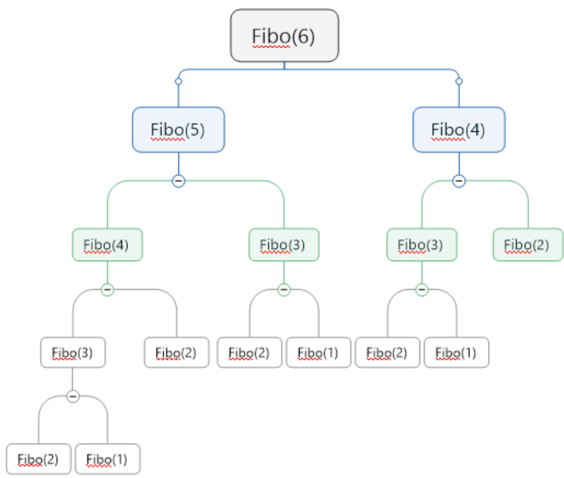
Suppose for each positive integer k, P(k) is true, that is,
"$1 + 3 + \ldots + (2k – 1) = k2$" is true.
Then, $1 + 3 + \ldots + (2k – 1) + [2(k + 1) – 1]$
$= k^2 + [(2(k + 1) – 1)]$ (due to the truth of P(k))
$= k^2 + 2k + 1$
$= (k + 1)^2$
Hence, $1 + 3 + 5 + \ldots + [2\cdot(k+1) – 1] = (k + 1)^2$ and P(k + 1) is true.
By induction, P(n) is true for all positive integers n.

Ex3. Use **strong induction** to prove that every amount of postage of six cents or more can be formed using 3-cent and 4-cent stamps.
Solution.

- BASIS STEP.
  - 6 cents: two 3-cent stamps
  - 7 cents: one 3-cent stamp and one 4-cent stamp.
  - 8 cents: two 4-cent stamps.
- INDUCTIVE STEP.

Assume every amount of postage of j cents ($6 \leq j \leq k$, $k \geq 8$) can be formed using 3-cent and 4-cent stamps.
We need to show that an amount of postage of (k + 1) cents can be formed using 3-cent and 4-cent stamps.
In fact, $k + 1 = (k – 2) + 3$, and since $6 \leq (k – 2) \leq k$, it follows that (k – 2) cents can be formed using 3-cent and 4-cent stamps (by the assumption above).
So, (k – 2) + 3 cents can be formed using 3-

	cent and 4-cent stamps.	
Recursive definitions	**Ex1.** Give a **recursive definition** of each of these functions. a/ $f(n) = n$, $n = 1, 2, 3, \ldots$ b/ $f(n) = 3n + 5$, $n = 0, 1, 2, \ldots$ ***Solution.*** a/ $f(n) = n$, $n = 1, 2, 3, \ldots$ BASIS STEP. $f(1) = 1$ RECURSIVE STEP. For $n > 1$, $f(n) = n$ $\Rightarrow$ $f(n-1) = n-1$ $\Rightarrow$ $f(n) = f(n-1) + 1$ b/ $f(n) = 3n + 5$, $n = 0, 1, 2, \ldots$ BASIS STEP. $f(0) = 5$ RECURSIVE STEP. For $n > 0$, $f(n) = 3n + 5$ $\Rightarrow$ $f(n-1) = 3(n-1) + 5 = 3n + 2$ $f(n) = f(n-1) + 3$ **Ex2.** Give a **recursive definition** of each of these sets. a/ $A = \{2, 5, 8, 11, 14, \ldots\}$. b/ $B = \{\ldots, -5, -1, 3, 7, 10, \ldots\}$. c/ $C = \{3, 12, 48, 192, 768, \ldots\}$. ***Solution.*** a/ $A = \{2, 5, 8, 11, 14, \ldots\}$ BASIS STEP. $2 \in A$ RECUSIVE STEP. $x \in A \rightarrow x + 3 \in A$. b/ $B = \{\ldots, -5, -1, 3, 7, 10, \ldots\}$ BASIS STEP. $3 \in B$ RECUSIVE STEP. $x \in B \rightarrow (x + 4 \in B$ and $x - 4 \in B)$. c/ $C = \{3, 12, 48, 192, 768, \ldots\}$ BASIS STEP. $3 \in C$ RECUSIVE STEP. $x \in C \rightarrow 4x \in C$.	**65/** Give a **recursive definition** of each of these functions. a/ $f(n) = (-1)^n$, $n = 0, 1, 2, 3, \ldots$ b/ $f(n) = 7$, for all $n = 1, 2, 3, \ldots$ c/ $f(n) = 1 + 2 + 3 + \ldots + n$, $n = 1, 2, 3, \ldots$ **66/** Find $f(3)$, $f(4)$ if: a/ $f(1) = 3$ and $f(n) = 2f(n-1) + 5$. b/ $f(n) = f(n-1).f(n-2)$ and $f(0) = 1$, $f(1) = 2$. c/ $f(n) = (f(n-1))^2 - 1$ and $f(1) = 2$. **67/** Give a **recursive definition** of each of these sets. a/ $A = \{0, 3, 6, 9, 12, \ldots\}$. b/ $B = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$. c/ $C = \{0.9, 0.09, 0.009, 0.0009, \ldots\}$.

Recursive algorithms	**Ex1.** Consider an **recursive algorithm** to compute the n^{th} Fibonacci number: procedure Fibo(n : positive integer) if n = 1 return 1 else if n = 2 return 1 else return Fibo(n – 1) + Fibo(n – 2) How many additions (+) are used to find Fibo(6) by the algorithm above? ***Solution.*** From the tree below, there are 7 additions.  **Ex2.** a/Give a **recursive algorithm** to find $S_m(n) = m + n$, where n is a non-negative integer and m is an integer. b/ Use mathematical induction to show that the algorithm is correct. ***Solution.*** a/ • Recursive definition of $S_m(n)$: BASIS STEP. $S_m(0) = m + 0 = m$ RECURSIVE STEP. For n > 0, $S_m(n) = m + n$ ⇨ $S_m(n - 1) = m + (n – 1)$ ⇨ $S_m(n) = S_m(n - 1) + 1$ • Recursive algorithm to find $S_m(n)$: procedure sum(m: integer; n: non-negative integer) if n = 0 then sum(m, n) := m else then sum(m, n): =sum(m, n – 1) + 1	**68/** Consider an algorithm: procedure Fibo(n : positive integer) if n = 1 return 1 else if n = 2 return 1 else if n = 3 return 2 else return Fibo(n – 1) + Fibo(n – 2) + Fibo(n – 3) How many additions (+) are used to find Fibo(6) by the algorithm above? **69**a/ Write a **recursive algorithm** to find the sum of first n positive integers. b/ Use **mathematical induction** to prove that the algorithm in (a) is correct. c/ Write a **recursive algorithm** to find the value of the function f(n) = 7, for n = 1, 2, 3, … **70/** Consider the following algorithm: procedure tinh(a: real number; n: positive integer) if n = 1 return a else return a·tinh(a, n-1). a/ What is the output if inputs are: n = 4, a = 2.5? Explain your answer. b/ Show that the algorithm computes n·a using Mathematical Induction.

	b/ Prove the correctness of the algorithm: BASIS STEP. If n = 0: sum(m, n) := m = m + 0 = m + n = $S_m(n)$. INDUCTIVE STEP. Suppose for every integer k ≥ 0, sum(m, k) returns m + k. We need to show that sum(m, k + 1) returns m + k + 1. In fact, from the algorithm, k + 1 > 0 and sum(m, k + 1): = sum(m, k) + 1 and then returns m + k + 1. (by the assumption, sum(m, k) returns m + k).	

Applications.

1. Determine whether each of the following bit strings belongs to the set S **recursively defined** by:

- BASIS STEP: 0 ∈ S
- RECURSIVE STEP: 1w ∈ S or 0w ∈ S if w ∈ S

a/ λ (the empty string)
b/ 0
c/ 110
d/ 10110

2. Let S be set of all bit strings of any length. Define the number $\#_0(s)$ **recursively** by:

- *Basis step:* $\#_0(s) = 0$, where λ is the empty string.
- *Recursive step:*

$$\#_0(xs) = \begin{cases} \#_0(s)\ if\ x \neq 0 \\ 1 + \#_0(s)\ if\ x = 0 \end{cases}.$$

a) Find $\#_0(111)$
b) Find $\#_0(010)$
c) What can we say about s if $\#_0(s) = 0$?
d) If s and w are two bit strings, show that $\#_0(sw) = \#_0(s) + \#_0(w)$.

Chapter 5 – Counting

Product rule & sum rule Counting functions Counting one-to-one	**Ex1.** Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with a vowel. ***Solution.*** • Keep in mind a row of *seven blanks*: - - - - - - -. • There are *five ways* in which the first letter in the string can be a vowel.	**71/** There are three available flights from Hanoi to Bangkok and, regardless of which of these flights is taken, there are five available flights from Bangkok to Manila. In how many

functions	• Once the vowel is placed in the first blank, there are 25 ways in which to fill in the second blank, 24 ways to fill in the third blank, etc. • Using the product rule, we obtain $\underset{\text{place vowel}}{5} \cdot \underbrace{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20}_{\text{place other letters}}$ **Ex2.** Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with C or V and end with C or V. ***Solution.*** Using a row of 7 blanks, we first count the number of strings belonging one of two cases: • Case 1: Strings begin with C and end with V: C - - - - - V. ⇨ By the product rule, the number of ways to fill in the five interior letters is 24 · 23 · 22 · 21 · 20. • Case 2: Strings begin with V and end with C: V - - - - - C. ⇨ By the product rule, the number of ways to fill in the five interior letters is 24 · 23 · 22 · 21 · 20. Therefore, by the **sum rule,** the answer is (24 · 23 · 22 · 21 · 20) + (24 · 23 · 22 · 21 · 20) = 2(24 · 23 · 22 · 21 · 20). **Ex3.** How many subsets of the set {1, 2, 3, 4, 5} a/ contain 2 and 3? b/ do not contain 3? c/ have more than one element? ***Solution.*** a/ Suppose A is a subset of {1, 2, 3, 4, 5}, then A contains members chosen from {1, 2, 3, 4, 5}. We can see: • 1 may belong to A or not. • 2 may belong to A or not.	ways can a person fly from Hanoi to Manila via Bangkok? **72/** Find the number of strings of length 7 of letters of the alphabet, with repeated letters allowed, that have vowels in the first two positions. **73/** Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with E and end with V or a vowel. **74/** A final test of the course MAD101 contains 50 multiple choice questions. There are four possible answers for each question. a/ In how many ways can a student answer the questions if the student answers every question? b/ In how many ways can a student answer the questions on the test if the student can leave answers blank? **75/** How many **subsets** of the set {(0, 0), (0, 1), (1, 0), (1, 1)} a/ are there in total? b/ contain (0, 0) and (1, 1)? **76/** **a/** How many functions are there from

	3 may belong to A or not. 4 may belong to A or not. 5 may belong to A or not. Therefore, there 2·2·2·2·2 ways to construct A. ⇨ There are $2^5 = 32$ subsets of {1, 2, 3, 4, 5}. b/ Similarly to the part a/, there are $2^4 = 16$ subsets of {1, 2, 3, 4, 5} do not contain 3. c/ number of subsets having more than one element = number of all subsets – number of subsets having no element = $2^5 – 1 = 31$. **Ex4.** **a/** How many functions are there from the set {a, b, c, d} to the set {1, 2, 3, 4}? **a/** How many one-to-one functions are there from the set {a, b, c, d} to the set {1, 2, 3, 4}? ***Solution.*** a/ A function corresponds to a choice of one of the 4 elements in the codomain{1, 2, 3, 4} for each of elements {a, b, c, d}in the domain. Therefore, we have: 4 ways to choose the value of the function at a. 4 ways to choose the value of the function at b. 4 ways to choose the value of the function at c. 4 ways to choose the value of the function at d. ⇨ By the product rule, there are 4^4 functions. b/ An one-to-one function corresponds to a choice of one of the 4 elements in the codomain{1, 2, 3, 4} for each of elements {a, b, c, d}in the domain so that no value of the codomain can be used again. Therefore, we have: 4 ways to choose the value of the function at a. 3 ways to choose the value of the	the set {a, b, c, d} to the set {1, 2, 3}? **b/** How many **one-to-one** functions are there from the set {a, b, c, d} to the set {1, 2, 3}?

	function at b (because the value used for a cannot be used again for b). • 2 ways to choose the value of the function at c. • 1 ways to choose the value of the function at d. ⇨ Therefore, there are 4·3·2·1 one-to-one functions.	
$\|A \cup B\| =$ $\|A\| + \|B\| - \|A\cap B\|$ (in A or in B)	**Ex1.** Find the number of integers from 100 to 1000 inclusive that are a/ divisible by 7. b/ divisible by 7 or 11. ***Solution.*** a/ When we divide 1000 by 7, we obtain 142 + 6/7. Then, the largest integer in our range that is divisible by 7 is 142·7, or 994. And if we divide 100 by 7, the result is about 14 + 2/7. So, the smallest integer in 100..1000 that is divisible by 7 is 21, not 14. Therefore, the number of integers between 100 and 1000 inclusive that are divisible by 7 is (994 – 21)/7 + 1, or 139. b/ • From the part a/, there are 139 integers that are divisible by 7. • Similarly, there are (990 – 110)/11 + 1, or 81 integers between 100 and 1000 inclusive that are divisible by 11. • And again, there are (924 – 154)/77 + 1, or 11 integers between 100 and 1000 inclusive that are divisible by 77. By Inclusion – Exclusion principle, the answer is 139 + 81 – 11 = 209. **Ex2.** Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with E or end with a vowel. ***Solution.***	**77/** Find the number of integers from 999 to 9999 inclusive that are: a/ divisible by 13 or 17. b/ divisible by 13 but not by 17. **78/** Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that a/ begin with V or end with a vowel. b/ begin or end with a vowel. c/ begin or end with a vowel (but not both).

	Using a row of seven blanks: - - - - - - - • There are 25·24·23·22·21·20 strings of the form E- - - - - -. • There are 25·24·23·22·21·20·5 strings of the form - - - - - - (a vowel) • There are 24·23·22·21·20·4 strings of the form E- - - - - (a vowel, not E) By Inclusion – Exclusion principle, the answer is 25·24·23·22·21·20 + 25·24·23·22·21·20·5 – 24·23·22·21·20·4.	
Counting problems and Recurrence relations	**Ex1.** A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills. a/ Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters. b/ Find a_0, a_1, a_2, a_3, a_4. c/ How many ways are there to deposit \$7 for a book of stamps? ***Solution.*** a/ Let a_n be the number of ways to deposit n dollars in the vending machine. Some ways to deposit n dollars: • One \$1 coin first, then (n – 1) dollars. In this case, there are a_{n-1} ways corresponding to (n – 1) remaining dollars. • One \$1 bill first, then (n – 1) dollars. In this case, there are also a_{n-1} ways corresponding to (n – 1) remaining dollars. • One \$5 bill first, then (n – 5) dollars (if n > 5). In this case, there are a_{n-5} ways corresponding to (n – 5) remaining dollars. So, we have the recurrence relation $a_n = 2a_{n-1}$, if $5 > n \geq 1$ $a_n = 2a_{n-1} + a_{n-5}$, if $n \geq 5$ b/ • $a_0 = 1$ // the only way to deposit zero dollar is depositing nothing.	**79/** How many bit strings of length eight do not contain three consecutive 0s? **80/** Verify that $a_n = 3^n$ and $a_n = 3^n + 1$ are solutions to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$. **81/** You take a job that pays \$10,000 annually. a/ How much do you earn *20* years from now if you receive a ten percent raise each year? b/ How much do you earn *20* years from now if each year you receive a raise of \$1000 plus four percent of your previous year's salary?

- $a_1 = 2a_0 = 2$.
- $a_2 = 2a_1 = 4$

$1-coin, $1-bill
$1-bill, $1-coin
$1-coin, $1-coin
$1-bill, $1-bill

- $a_3 = 2a_2 = 2.4 = 8$
- $a_4 = 2a_3 = 16$.

c/ $a_5 = 2a_4 + a_0 = 32 + 1 = 33$
$a_6 = 2a_5 + a_1 = 66 + 2 = 68$
$a_7 = 2a_6 + a_2 = 136 + 4 = 140$.

Ex2. Find a **recurrence relation** for the number of bit strings of length n that do not contain three consecutive 0s.

Solution.

Let a_n be the number of bit strings of length n that do not contain three consecutive 0s.
For example, $a_1 = 2$ (two bit strings "0" and "1" of length 1), $a_2 = 4$ and $a_3 = 7$ (except for the string "000").
Strings of length n we want to count are of exactly one of three cases:

- 1 (n – 1 remaining bits satisfying the condition). For example, with n = length = 4, 1001 and 1100 are strings of this type, but 1~~000~~ or ~~0~~110 are not. ➔ there are a_{n-1} such strings.
- 01 (n – 2 remaining bits satisfying the condition) ➔ there are a_{n-2} such strings.
- 001(n – 3 remaining bits satisfying the condition) ➔ there are a_{n-3} such strings.

Therefore, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

Ex3. Verify that $a_n = 3^{n+2}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.

Solution.

$a_n = 3^{n+2}$

⇨ $a_{n-1} = 3^{(n-1)+2} = 3^{n+1}$

⇨ $a_{n-2} = 3^{(n-2)+2} = 3^n$

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⇨ $4a_{n-1} - 3a_{n-2} = 4.(3^{n+1}) – 3(3^{n})$
⇨ $4a_{n-1} - 3a_{n-2} = 3.(3^{n+1}) + (3^{n+1}) – 3(3^{n}) = 3.(3^{n+1}) = 3^{n+2}$
⇨ $4a_{n-1} - 3a_{n-2} = a_n$.
Therefore, $a_n = 3^{n+2}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.

Ex4. You take a job that pays $25,000 annually. How much do you earn *n* years from now if you receive a four percent raise each year?
Solution.
a/ Let S_n be the salary after n years.
Then, $S_n = (1 + 0.04)S_{n-1}$
⇨ $S_n = (1 + 0.04)^n S_0 = 25{,}000\cdot(1.04)$.

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Applications.

1/ Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.
a/ Find a **recurrence relation** for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.
b/ What are the initial conditions?
c/ How many different messages can be sent in 10 microseconds using these two signals?

2/ Suppose inflation continues at five percent annually. (That is, an item that costs $1.00 now will cost $1.05 next year). Let a_n = the value (that is, the purchasing power) of one dollar after *n* years.
a/ Find a **recurrence relation** for a_n.
b/ What is the value of $1000 after 10 years?
c/ What is the value of $1000 after 50 years?
d/ If inflation were to continue at ten percent annually, find the value of $1000 after 50 years.

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Chapter 8 - Relations

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Binary relation

Properties of relations

Combination of relations

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Ex1. List the ordered pairs in the **relation** R from A ={0, 1, 2, 3, 4} to B ={0, 1, 2, 3}, where (a, b) ∈ R if and only if
a/ a + b = 4.
b/ a | b.
Solution.
a/ R = {(1, 3); (2, 2); (3, 1); (4, 0)}

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82/ List the ordered pairs in the **relation** R from A ={0, 1, 2, 3, 4} to B ={0, 1, 2, 3}, where (a, b) ∈ R if and only if
a/ a > b.
b/ a – b = 1.

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Composite relation	b/ R = {(1, 1); (1, 2); (1, 3); (1, 0); (2, 0); (2, 2); (3, 0); (3, 3)}. **Ex2.** Determine whether the relation R on the set of all real numbers is **reflexive, symmetric, antisymmetric, and/or transitive**, where $(x, y) \in R$ if and only if a/ $(x, y) \in R \Leftrightarrow x = 2y$. b/ $x = 1$. ***Solution.*** a/ $x = 2y$. • $(1, 1) \notin R$ (because $1 \neq 2.1$) ➔ R is not reflexive. • $2 = 2.1$ ➔ $(2, 1) \in R$ but $(1, 2) \notin R$ (because $1 \neq 2.2$) ➔ R is not symmetric. • If xRy and yRx ➔ $x = 2y$ and $y = 2x$ ➔ $x = y (= 0)$ ➔ R is antisymmetric. • $(4, 2) \in R$ and $(2, 1) \in R$ but $(4, 1) \notin R$ ➔ R is not transitive. b/ $(x, y) \in R \Leftrightarrow x = 1$. • $(2, 2) \notin R$ ➔ R is not reflexive. • $(1, 2) \in R$ but $(2, 1) \notin R$ ➔ R is not symmetric. • If $(x, y) \in R$ and $(y, x) \in R$, then $x = 1$ and $y = 1$ ➔ $x = y$. Hence, R is antisymmetric. • If $(x, y) \in R$ and $(y, z) \in R$, then $x = 1$ and $y = 1$ ➔ $(x, z) \in R$. Hence, R is transitive. **Ex3.** Let R be the relation on the set of ordered pairs of positive integers such that $(a, b)R(c, d) \Leftrightarrow a + d = b + c$. Show that a/ R is reflexive. b/ R is symmetric. c/ R is transitive. ***Solution.*** a/ For every positive integer a, (a, a) R (a, a) because $a + a = a + a$. b/ $(a, b)R(c, d) \Leftrightarrow a + d = b + c$	c/ $a = 2b$. **83/** Determine whether the relation R on the set of all real numbers is **reflexive, symmetric, antisymmetric, and/or transitive**, where $(x, y) \in R$ if and only if a/ $xy = 0$. b/ $x = y + 1$ or $x = y – 1$. c/ $x \equiv y \pmod 5$. **84/** Let R be the relation on the set of ordered pairs of positive integers such that $(a, b)R(c, d) \Leftrightarrow ad = bc$. Show that a/ R is reflexive. b/ R is symmetric. c/ R is transitive. **85/** Let R = {(1, 2), (1, 3), (2, 3), (3, 1)}, and S = {(2, 1), (3, 1), (3, 2)} be relations on the set {1, 2, 3}. Find a/ $R – S$. b/ $R \cap S$. c/ $R \cup S$. d/ $R \oplus S$. e/ $\bar{R}$ f/ S^{-1}. g/ SoR. **86/** List the 16 different relations on the set {0, 1}. **87/** Which of the 16 relations on {0, 1}, are

	$\Leftrightarrow$ c + b = d + a $\Leftrightarrow$ (c, d)R(a, b). Hence, R is symmetric. c/ For all positive integers a, b, c, d, m and n, if (a, b)R(c, d) and (c, d)R(m, n), then a + d = b + c and c + n = d + m $\Rightarrow$ a + d + c + n = b + c + d + m $\Rightarrow$ a + n = b + m $\Rightarrow$ (a, b)R(m, n). Therefore, R is transitive. **Ex4.** Let R = {(1, 1), (3, 3), (2, 3)}, and S = {(1, 2), (3, 1), (2, 2)} be relations on the set {1, 2, 3}. Find a/ R – S. b/ R $\cap$ S. c/ R $\cup$ S. d/ R $\oplus$ S. e/ $\bar{R}$ f/ S^{-1}. g/ SoR. ***Solution.*** V	a/ reflexive? b/ ir-reflexive? c/ symmetric? d/ anti-symmetric? e/ asymmetric? f/ transitive?
Counting relations	**Ex1.** How many different relations on {a, b} contain the pair (a, b)? ***Solution.*** Every relation on the set {a, b} is a subset of the Cartesian product {a, b}$\times$ {a, b}. On other hand, {a, b}$\times$ {a, b} = {(a, a); (a, b); (b, a); (b, b)}, which has 2^4 subsets. $\Rightarrow$ There are 2^4 = 16 relations. **Ex2.** How many different **reflexive relations** are there on the set {a, b}? ***Solution.*** Every relation on the set {a, b} is a subset of the Cartesian product {a, b}$\times$ {a, b}. And a **reflexive relations** on the set {a, b} is a set containing both (a, a); and (b, b). By the product rule, there are 1.1.2.2 such subsets. Therefore, there are 4 **reflexive** relations on	**88/** a/ How many different relations are there on the set {a, b, c}? b/ How many different relations on the set {a, b, c} do not contain (a, a)? c/ How many different **ir-reflexive relations** are there on the set {a, b, c}?

	the set {a, b}. **Ex3.** How many different relations are there from {a, b, c, d} to {1, 2, 3}? ***Solution.*** There are $2^{4 \cdot 3} = 2^{12}$ relations from {a, b, c, d} to {1, 2, 3}.	
Representations of relations	**Ex1.** Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). a/ {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}. b/ {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}. ***Solution.*** a/ $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$ b/ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$ **Ex2.** How many 1-entries does the matrix representing the relation R on A ={1, 2, 3,..., 100} consisting of the first 100 positive integers have if R is a/ $\{(a, b) \mid a \le b\}$? b/ $\{(a, b) \mid a + b = 100\}$? ***Solution.*** a/ $aRb \Leftrightarrow a \le b$. $\begin{array}{cccccc} R & 1 & 2 & \ldots & 99 & 100 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ 2 & 0 & 1 & \ldots & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 99 & 0 & 0 & 0 & 1 & 1 \\ 100 & 0 & 0 & 0 & 0 & 1 \end{array}$ ⇨ The number of 1-entries is (1 + 2 + 3 + … + 100) = 5050. b/ $aRb \Leftrightarrow a + b = 100$. The matrix has the size of 100x100.	**89/** Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). a/ {(1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 1), (4, 1), (4, 2)} b/ {(1, 4), (3, 1), (3, 2), (3, 4)}. **90/** How many 1- entries does the matrix representing the relation R on A ={1, 2, 3,..., 100} consisting of the first 100 positive integers have if R is a/ $\{(a, b) \mid a = b \pm 1\}$? b/ $\{(a, b) \mid a + b < 101\}$? **91/** Let R and S be relations on a set represented by the matrices $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$ Find the matrices that represent

Since the (row i, column 100 – i)-position in the matrix is the only 1-entry of row i, it follows that the matrix has n 1-entries.

Ex3. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find the matrix representing

a/ R^{-1}.

b/ $\overline{R}$.

c/ R^2.

d/ $R - R^2$.

e/ $R \oplus R^2$.

Solution.

a/ Let M_R and $M_{R^{-1}}$ be the matrices representing relations R and R^{-1}.

Recall that $(i, j) \in R^{-1} \Leftrightarrow (j, i) \in R$, or equivalently,

(i, j)-entry = 1 in $M_{R^{-1}} \Leftrightarrow$ (j, i)-entry = 1 in M_R.

⇨ $M_{R^{-1}}$ is the transpose of M_R.

$$⇨ \quad M_{R^{-1}} = (M_R)^T = M_R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

b/ Let M_R and $M_{\overline{R}}$ be the matrices representing relations R and $\overline{R}$.

Recall that $(i, j) \in \overline{R} \Leftrightarrow (i, j) \notin R$, or equivalently,

(i, j)-entry = 1 in $M_{\overline{R}} \Leftrightarrow$ (i, j)-entry = 0 in M_R.

$$⇨ \quad M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

c/ Let M_{R^2} be the matrix representing the relation R^2.

The matrix of R^2 (= RoR) can be computed

a/ $R \cup S$.

b/ $R \cap S^{-1}$.

c/ $R - \overline{S}$.

d/ $R \oplus S$.

e/ RoS.

92/ Suppose that the relation R on a set is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

a/ Is R **reflexive**?

b/ Is R **symmetric**?

c/ Is R **antisymmetric**?

by Boolean product of $M_R ε M_R$

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

d/ Let M_{R-R^2} be the matrix representing relation $R - R^2$.

Recall that A – B is the set of elements that belong to A but not belong to B.

So, the relation $R - R^2$ contains only ordered pairs (a, b) where $(a, b) \in R$ but $(a, b) \notin R^2$.

So, the (i, j)-entry of M_{R-R^2} is 1 $\Leftrightarrow$ the (i, j)-entry of M_R is 1 and the (i, j)-entry of M_{R^2} is 0.

Therefore, $M_{R-R^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$

e/ Let $M_{R \oplus R^2}$ be the matrix representing the relation $R \oplus R^2$.

Recall that $R \oplus R^2$ contains only ordered pairs (a, b) that belong to exactly one of $(R - R^2)$ and $(R^2 - R)$.

So, (i, j)-entry of $M_{R \oplus R^2}$ is 1 $\Leftrightarrow$ (i, j)-entry of M_R is 1 OR (i, j)-entry of M_{R^2} is 1 (BUT NOT BOTH).

Therefore, $M_{R \oplus R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

Ex4. Suppose that the relation R on a set is represented by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

a/ Is R reflexive?

	b/ Is R symmetric? c/ Is R antisymmetric? ***Solution.*** a/ Recall that a relation R on a set A is reflexive if and only if $\forall a \in A, (a, a) \in R.$ Or equivalently, in the matrix M_R, the (row i, column i)-entry is 1 for every value of i. We can see (3, 3)-entry of M_R is 0 ➔ (3, 3) $\notin R$ ➔ R is not reflexive. b/ Recall that a relation R on a set A is symmetric if and only if $\forall a \forall b, (a, b) \in R \to (b, a) \in R.$ Based on this definition, R is symmetric if and only if the matrix M_R is symmetric, that is, the (i, j)-entry of M_R equals to the (j, i)-entry of M_R. Since M_R is not symmetric ((1, 2)-entry of M_R is 1 and (2, 1)-entry of M_R is 0), we can conclude that R is not symmetric. c/ Recall that the relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that a = b. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. So, it is easy to see that R is not antisymmetric ($m_{23} = m_{32} = 1$).	
Equivalence relations Partitions & equivalence classes	**Ex1.** Let R be the relation on the set of real numbers such that aRb *if and only if a − b is an integer*. Show that R is an **equivalence relation**. ***Solution.*** • Because a − a = 0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. • Now suppose that aRb. Then a −b is an integer, so b − a is also an integer. Hence, bRa. It follows that R is symmetric.	**93/** Which of these relations on {0, 1, 2, 3} are **equivalence relations**? What are the **equivalence classes** of that equivalence relation? a/ R = {(0, 0), (1, 1), (2, 2), (3, 3)}. b/ S = {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}. c/ T = {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3,

	- If aRb and bRc, then a − b and b − c are integers. Therefore, a −c = (a − b) + (b − c) is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an **equivalence relation**. **Ex2.** **(Congruence Modulo m).** Let m be an integer with m > 1. Show that the relation R ={(a, b) | a ≡ b(mod m)} is an **equivalence relation** on the set of integers. ***Solution.*** Recall that a ≡ b (mod m) if and only if m divides a − b. - Note that a − a = 0 is divisible by m, because 0 = 0 · m. Hence, a ≡ a (mod m), so congruence modulo m is reflexive. - Now suppose that a ≡ b (mod m). Then a − b is divisible by m, so a − b = km, where k is an integer. It follows that b − a = (−k)m, so b ≡ a (mod m). Hence, congruence modulo m is symmetric. - Next, suppose that a ≡ b (mod m) and b ≡ c (mod m). Then m divides both a − b and b − c. Therefore, there are integers k and l with a − b = km and b − c = lm. Adding these two equations shows that a − c = (a − b) + (b − c) = km + lm = (k + l)m. Thus, a ≡ c (mod m). Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an **equivalence relation**. **Ex3.** List the ordered pairs in the **equivalence relation** R produced by the **partition** A_1 ={1, 2}, A_2 ={3}, and A_3 ={4} of S ={1, 2, 3, 4}. ***Solution.*** R = {(1, 1); (1, 2); (2, 1); (2, 2); (3, 3); (4,	2), (3, 3)}. **94/** Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that f(x) = f(y). Show that R is an **equivalence relation** on A. **95/** Which of these relations on the set of all people are **equivalence relations**? Determine the properties of an equivalence relation that the others lack. a/ {(a, b) | a and b are the same age}. b/ {(a, b) | a and b share a common parent}. c/ {(a, b) | a and b speak a common language}. **96/** What is the **congruence class** $[3]_m$ (that is, the **equivalence class** of 4 with respect to congruence modulo m) when m is a/ 2 b/ 3 c/ 4 d/ 5 **97/** List the ordered pairs in the **equivalence relations** produced by the **partition** {a, b}, {c, d},

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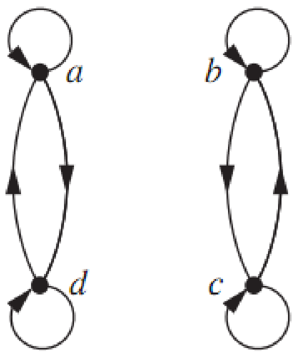
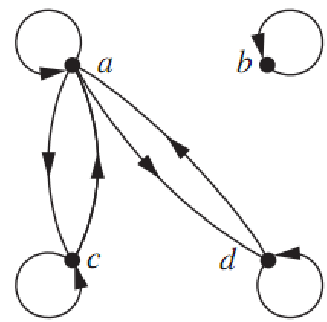
Ex4. Determine whether the relation with the directed graph shown is an **equivalence relation** on the set {a, b, c, d}.


Solution.
- R is reflexive (there are loops at every vertex).
- R is symmetric (there is an edge from v1 to v2 whenever there is an edge from v2 to v1).
- R is transitive (if there is an edge from v1 to v2 and an edge from v2 to v3, then there is an edge from v1 to v3).
So, R is an equivalence relation.

Ex5. Which of these relations on {0, 1, 2, 3} are **equivalence relations**? What are the **equivalence classes** of that equivalence relation?
a/ R = {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}.
b/ S = {(0, 0), (1, 2), (2, 1), (2, 2), (2, 3), (3, 3)}.
Solution.
a/ R is reflexive, symmetric and transitive. So, R is an equivalence relation.
Equivalence classes are: {0}; {1, 2}; {3}.
b/ S is not an equivalence relation because S is not symmetric ((2, 3)$\in$S but (3, 2)$\notin$S).
Therefore, S is not an equivalence relation.</td>
<td>{e} of {a, b, c, d, e}.

98/ Determine whether the relation with the directed graph shown is an **equivalence relation**.

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<td>n-ary relations</td>
<td>**Ex1.** The 3-tuples in a 3-ary relation</td>
<td>**99/** The 4-tuples in a 4-</td>
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and application to database	represent the following attributes of a student database: student ID number, name, phone number. What is a likely **primary key** for this relation? ***Solution.*** • Two students may have the same name ➔ is not a likely **primary key** for this relation. • Some students may not have phone numbers ➔ phone number is also not a likely **primary key** for this relation. • Students have different ID numbers ➔ student ID number is a likely **primary key** for this relation. **Ex2.** What do you obtain when you apply the **projection** $P_{2,3,5}$ to the 5-tuple (a, b, c, d, e)? ***Solution.*** $P_{2,3,5}$(a, b, c, d, e) = (a, d).	ary relation represent these attributes of published books: title, ISBN, publication date, number of pages. What is a likely **primary key** for this relation? **100/** Which **projection** mapping is used to delete the first, second, and fourth components of a 6-tuple?
	Applications.	

Chapter 9 – Graph Theory

Simple graphs Edge Vertex/vertice s Special simple graphs: K_n, C_n, W_n, Q_n Handshaking theorem. Degree of a vertex Adjacent Incident	**Ex1.** The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in non-increasing order. How many **edges** does a graph have if its **degree** sequence is 4, 3, 3, 2, 2? ***Solution.*** Based on the **handshaking theorem**, number of edges = (½)(the sum of degrees of vertices) = (½)(4 + 3 + 3 + 2 + 2) = 7. **Ex2.** A sequence $d_1, d_2, ..., d_n$ is called **graphic** if it is the **degree sequence** of a **simple graph**. Determine whether each of these sequences is **graphic**. For those that are, draw a graph having the given degree sequence. a/ 3, 3, 3, 3, 2, 0. b/ 5, 4, 3, 2, 1. c/ 7, 6, 5, 4, 4, 2, 1, 1.	**101/** Draw these **simple graphs**. a/ K_7. b/ C_7. c/ W_7. d/ $K_{3,4}$. **102/** Find the **degree sequence** of each of the graphs in the exercise 101. **103/** An undirected graph has five **vertices** of **degree** three and three **vertices** of **degree** five. How many **edges** does the graph have?

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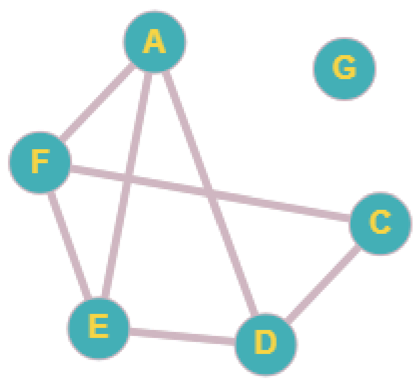
Solution.
Recall that a simple graph has no any a multiple edge or a loop.
a/ Below is a simple graph having the degree sequence 3, 3, 3, 3, 2, 0.



b/ Recall that a graph cannot have an odd number of vertices that have odd degrees. So, no graph having the degree sequence **5**, 4, **3**, 2, **1** (3 vertices that have odd degrees).
c/ Suppose there is such a **simple graph** with vertices a, b, c, d, e, f, g, h where deg(a) = 7, deg(b) = 6, deg(c) = 5, deg(d) = 4, deg(e) = 4, deg(f) = 3, deg(g) = 1 and deg(h) = 1.

- First, vertex a must be adjacent to 7 other vertices. Hence, vertex a is adjacent to both g and h.
- Next, there are 6 vertices that are adjacent to b. From 7 remaining vertices beside b, at least one of g and h is adjacent to b. In this situation, at least one of g and h must have degree 2 or larger. It is a contradiction with the fact deg(g) = deg(h) = 1.

So, there is no such a simple graph.

Ex4. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in G if and only if they are not adjacent in G.

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104/ Determine whether each of these sequences is **graphic**. For those that are, draw a graph having the given degree sequence.
a/ 5, 4, 3, 2, 1, 0.
b/ 1, 1, 1, 1.
c/ 4, 4, 3, 2, 1.
b/ 8, 8, 4, 4, 2, 2, 0, 0.

105/ The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in G if and only if they are not adjacent in G.
a/ If G is a **simple graph** with 9 **vertices** and $\overline{G}$ has 11 **edges**, how many **edges** does G have?
b/ If the **degree sequence** of the **simple graph** G is 4, 2, 2, 1, 1, what is the **degree sequence** of G? Draw the graphs G and $\overline{G}$.

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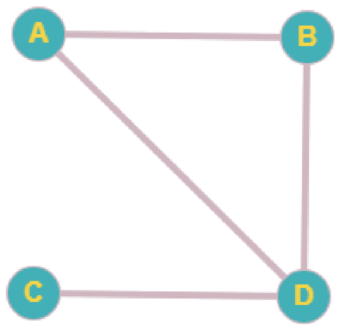
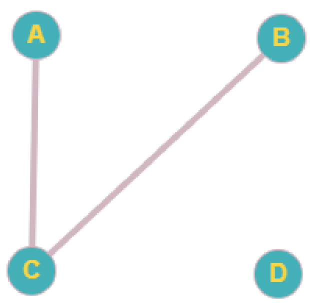
	Draw the **complementary graph** of the graph below.  ***Solution.*** By the definition, the **complementary graph** $\overline{G}$ is given below:  Note that the union of G and $\overline{G}$ is the complete graph K_n, where n is the number of vertices of G. So, if G has m edges, then $\overline{G}$ has n(n-1)/2 – m edges.	
Bipartite graphs	**Ex1.** For which values of n is the graph C_n **bipartite**? a/ C_n. b/ W_n. ***Solution.*** a/ C_n. • If n is even, “odd-position” vertices and “even-position” vertices must be colored by different color (e.g., red for “odd-position” and black for “even-position” vertices). Since no edge connects an “odd-position” vertex to an “even-position” vertex, the graph C_n is bipartite if n is even. • If n is odd, the first and the final vertices in the cycle are both in “odd-	**106/** For which values of n are these graphs **bipartite**? a/ K_n. b/ Q_n. **107/** Determine whether the graph represented by the adjacency matrix is bipartite. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

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positions” and they connect each other. This means, they are colored by the same color while they are adjacent. Therefore, C_n is not bipartite if n is odd.

b/ W_n.

Since the vertex at the center connecting to all other vertices around the cycle, the graph W_n is a non-bipartite graph.

Ex2. Determine whether the graph represented by the adjacency matrix is bipartite.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution.

From the matrix, the corresponding graph is bipartite.

In fact, the set of vertices can be divided into two parts $\{v_1, v_4\}$ and $\{v_2, v_3\}$, where v_1 (row 1) and v_4 (row 4) are not adjacent; and v_2 and v_3 are not adjacent.

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Connected graphs

Cut vertex

Cut edge

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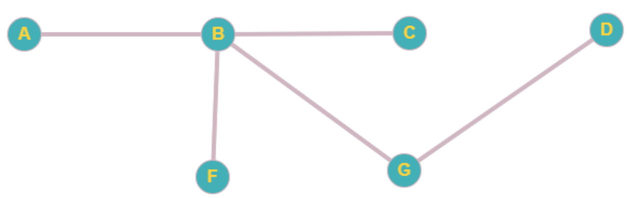
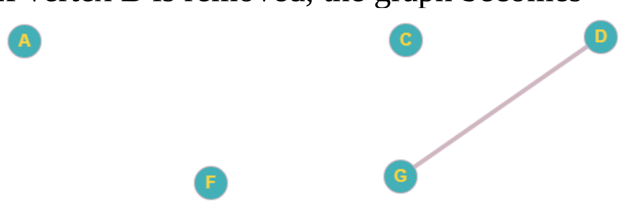
Ex1. Find all **cut vertices** of the given graph.



Solution.

If vertex B is removed, the graph becomes



and **disconnected**. So, B is a cut vertex.

Similarly, if we remove vertex G, the graph becomes a **disconnected** graph. Hence, G is also a **cut vertex**.

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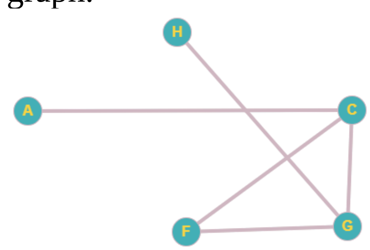
108/ Find all **cut vertices** and all **cut edges** (or **bridges**) of the given graph.



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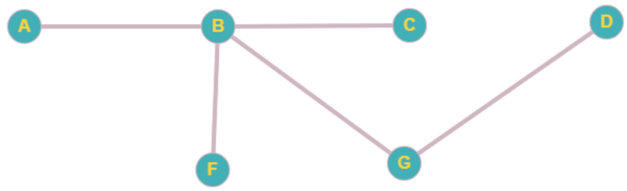
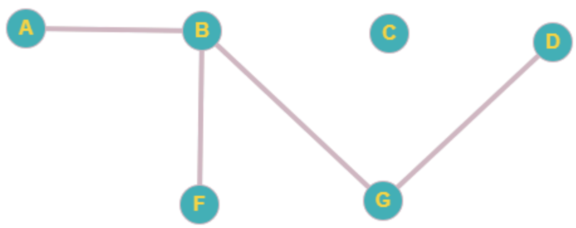
	Ex2. Find all **cut edges** (or **bridges**) of the given graph.  ***Solution.*** The given graph is connected and a removing a cut edge (or bridge) makes a disconnected graph. For example, if BC is removed, the graph becomes a disconnected graph as below  So, BC is a bridge of the given graph. Similarly, other bridges are BF, BG, GD, and AB.	
Representing graphs Adjacency matrix Incidence matrix	**Ex1.** Find an **adjacency matrix** for each of these graphs. a/ K_5. b/ W_5. c/ $K_{2,3}$. ***Solution.*** a/ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ b/ $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$	**109/** Find an **adjacency matrix** for each of these graphs. a/ K_6. b/ C_6. c/ W_6. d/ $K_{2,4}$. e/ Q_3. **110/** Find the number of 1-entries in the **incidence matrix** of each of these graphs. a/ K_n. b/ W_n. c/ $K_{m,n}$. **111/** Draw a graph that has

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c/ $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

Ex2. Find the number of 1-entries in the incidence matrix of each of these graphs.
a/ K_7.
b/ $K_{2,5}$.
Solution.
a/ Recall that the incidence matrix has the number of columns equaling to the number of edges of the graph.
So, the incidence matrix of the graph K_7 has 7(6)/2 = 21 columns.
By the definition of an incidence matrix and because K_7 is a simple graph, each column of this incidence matrix has exactly two 1-entries. Hence, there are $21 \cdot 2 = 42$ 1-entries in this matrix.
b/ Similarly to part a/, since the graph $K_{2,5}$ has $2 \cdot 5 = 10$ edges, the incidence matrix of this graph has 20 1-entries.
</td>
<td>
$\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}$
as its adjacency matrix.
Is this graph bipartite?
</td>
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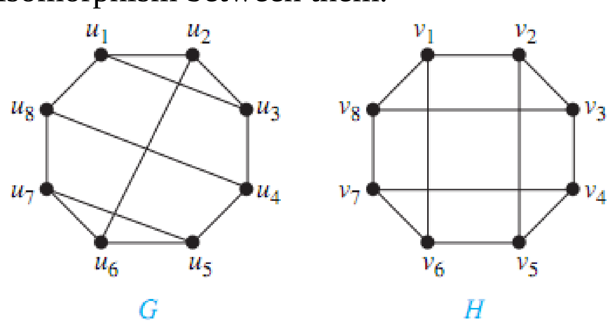
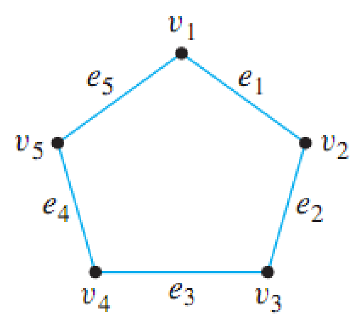
<td>
Isomorphism

Path of length n

Counting paths
</td>
<td>
Ex1. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.


Solution.
We will show that two graphs are not isomorphic by using a special path the one graph has but another graph has not.
In fact, the left-hand side graph (G) has one path making a “triangle” (e.g. u_1-u_2-u_3) while the graph H has no the same property.
</td>
<td>
112/ Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.


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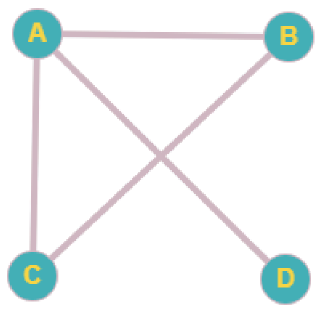
So, two graphs are not isomorphic.

Ex2. How many paths of length 3 between A and B does the graph have?


Solution.
The **adjacency matrix** of the graph is
$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
To find the number of paths of length 3 between A and B, we can multiply the (1, 2)-entry of the matrix M^3.
First, we will compute M^2.
$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
Next, to find the (1, 2)-entry of M^3, we multiply the first row of M^2 by the second column of M. The result is 4.
</td>
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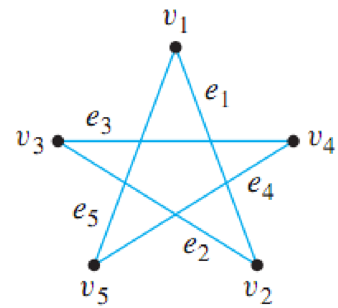
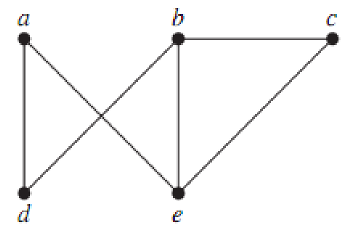
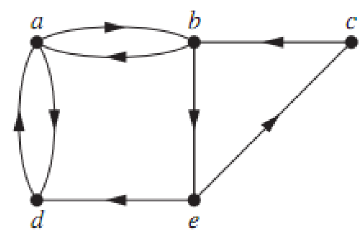


113/ How many **paths of length 2** between a and b does each of these graph have?
a/


b/


</td>
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<tr>
<td>Euler paths/circuits

Hamilton paths/circuits</td>
<td>
Ex1. For which values of n do these graphs have an **Euler circuit**?
a/ K_n.
b/ C_n.
Solution.
Recall that a connected graph has an **Euler circuit** if and only if every vertex of this
</td>
<td>
114/ For which values of n do these graphs have an **Euler circuit**?
a/ W_n.
b/ Q_n.

115/ For which values of
</td>
</tr>
</table>

graph has even degree.
a/ Every vertex of K_n has degree n – 1. So, K_n has an **Euler circuit** if and only if n is an odd integer and n > 1.
b/ Every vertex of C_n has degree 2. So, C_n has an **Euler circuit** for every integer n > 2.

Ex2. Does the undirected graph represented by the adjacency matrix

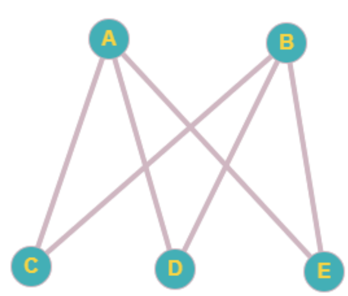
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 0 & 4 \\ 1 & 0 & 2 & 3 \\ 1 & 4 & 3 & 0 \end{bmatrix}$$

have an **Euler circuit**? And what is the length of an **Euler circuit** in this graph?

Solution.
The graph is connected and its degree sequence is 8, 8, 6, 6. So, it has an Euler circuit. The length of an **Euler circuit** equals to the number of edges the graph has, which is (8 + 8 + 6 + 6)/2 = 14 by the handshaking theorem.

Ex3. a/ Determine whether $K_{2,3}$ has a **Hamilton circuit** or a **Hamilton path**.
b/ Determine whether $K_{3,3}$ has a **Hamilton circuit** or a **Hamilton path**.
c/ Determine whether $K_{2,4}$ has a **Hamilton circuit** or a **Hamilton path**.

Solution.
a/ The graph $K_{2,3}$ has a **Hamilton path**, has no **Hamilton circuit**.



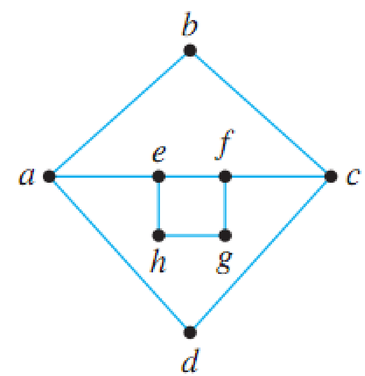
(**Hamilton path**, not a **Hamilton circuit** C-A-D-B-E)

m and n does the complete bipartite graph $K_{m,n}$ have an
a/ Euler circuit?
b/ Euler path?

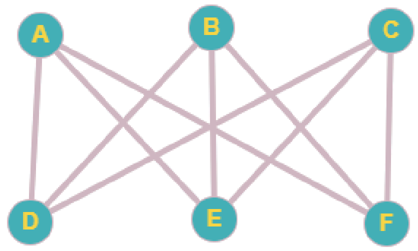
116/ For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

117/ What is the length of the longest simple **circuit** in the graph W_7?

118/ Find a **Hamilton circuit** in the graph or explain that it does not have.



b/ $K_{3,3}$ has a **Hamilton circuit** and a **Hamilton path**.



(**Hamilton circuit** and **Hamilton path**: A-D-B-E-C-F-A)

c/ $K_{2,4}$ has no a **Hamilton circuit** or a **Hamilton path**.

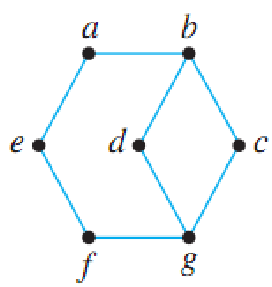
Ex4. What is the length of the longest simple **circuit** in the graph K_{11}?

Solution.

K_{11} has an **Euler circuit** (because every vertex in this graph has degree 10).

Recall that an **Euler circuit** is a simple circuit containing every edge of the graph K_{11}. So, an **Euler circuit** in K_{11} is also the longest simple circuit. Therefore, the length of the longest simple circuit equals to the number of edges of the graph K_{11}, which is 55.

Ex5. Find a **Hamilton circuit** in the graph or explain that it does not have.



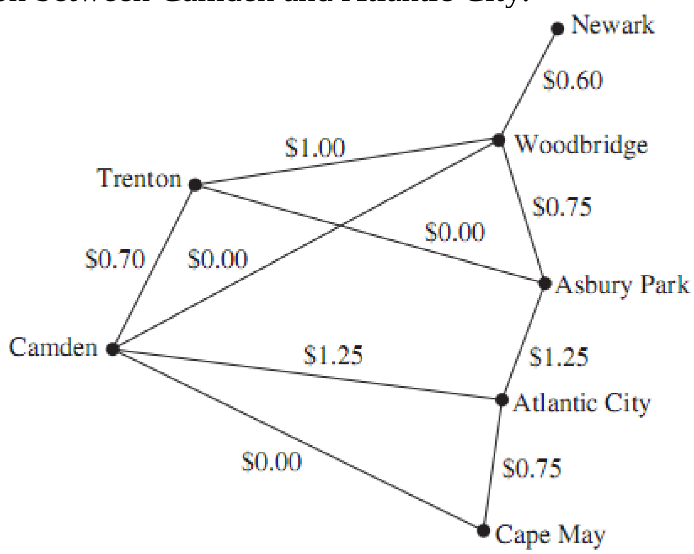
Solution.

Suppose the graph has a **Hamilton circuit,** call it (H). If a vertex has degree 2, then (H) must pass through both two edges incident with this vertex. So, (H) passes through the edges a-b, a-e, f-e, f-g, c-b and c-g exactly

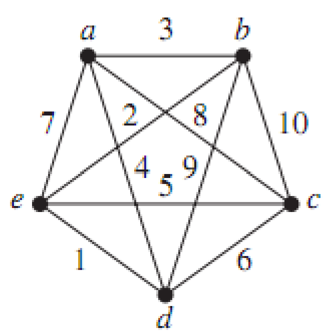
	once. Because b (g) has degree 3, (H) cannot pass through all edges incident with b (g). Hence, (H) cannot pass through d-b two edges and d-g. It follows that (H) cannot pass through d. It is a contradiction.	

Applications.

1/ Use **Dijkstra's algorithm** to find a least expensive route in terms of total dollars using the roads in the graph between Camden and Atlantic City.



2/ Solve the **traveling salesman problem** for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



3/ The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E? If so, how can this be done?

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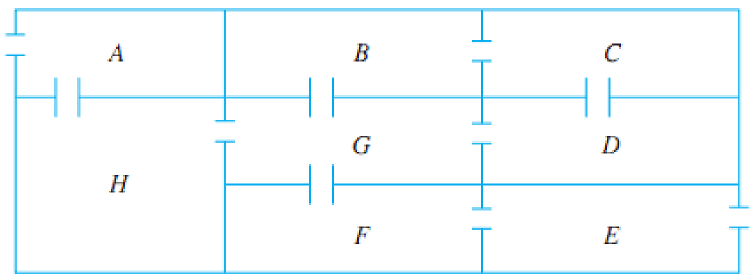
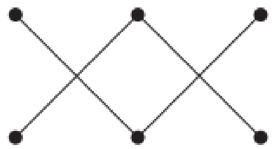
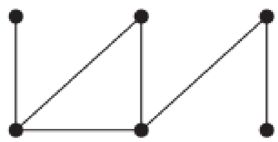
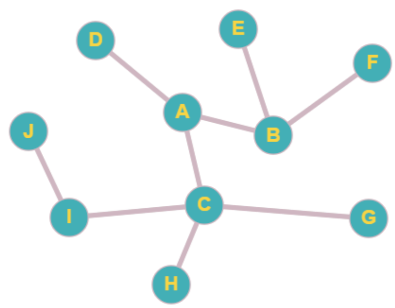
<tr><td colspan="3">
</td></tr>
<tr><th colspan="3">Chapter 10 – Trees</th></tr>
<tr>
<td>Tree - Definition

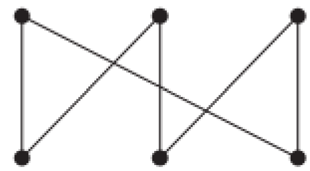
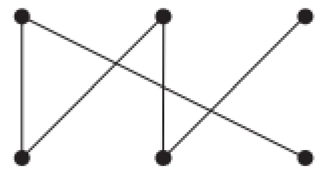
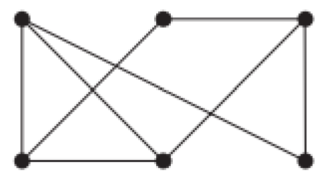
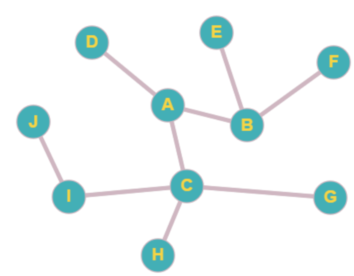
Leaf

Internal nodes

Child
Parent

Height</td>
<td>Ex1. Which of these graphs are trees?
a/

b/

c/


Solution.
a/ The graph is disconnected. So, it is not a tree.
b/ The graph is connected, but it has a simple circuit (see a triangle). So, it is not a tree.
c/ This graph is a tree because it is a connected graph with no simple circuit.

Ex2. Given the tree</td>
<td>119/ Which of these graphs are trees?
a/

b/

c/

d/


120/ Given the tree</td>
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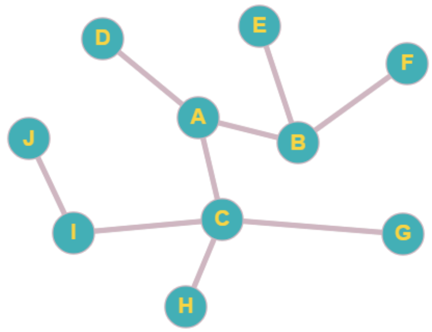
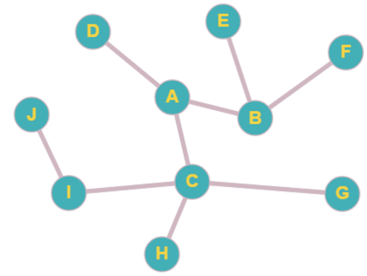
<td>

If B is chosen as the root of the tree, what is the height of the tree?
Solution.
If B is the root of the tree, the levels of nodes in the tree are shown as below:
Level 0: B
Level 1: E, F, A
Level 2: C, D
Level 3: I, G, H
Level 4: J
Recall that the height of the tree is the maximum level of the tree. Hence, the height of the tree is 4.</td>
<td>

If J is chosen as the root of the tree, what is the height of the tree?</td>
</tr>
<tr>
<td>m-ary trees full m-ary trees and properties</td>
<td>Ex1. a/ How many edges does a tree with 10,000 vertices have?
b/ How many edges does a full binary tree with 1000 internal vertices have?
Solution.
a/ number of edges = number of vertices – 1 = 10000 – 1 = 9999.
b/ Keep in mind, in a full m-ary tree we have n = mi + 1, where n is number of nodes and i is the number of internal nodes.
In this case, n = 2·1000 + 1 = 2001.
⇨ Number of edges = 2001 – 1 = 2000.

Ex2. Suppose a full binary tree has 35 nodes.
a/ How many leaves does the tree have?
b/ How large can the height of the tree possibly be?
c/ How small can the height of the tree possibly be?</td>
<td>121a/ How many vertices does a full 5-ary tree with 100 internal vertices have?
b/ How many leaves does a full 3-ary tree with 100 vertices have?

122/ Suppose a full ternary tree has 16 leaves.
a/ How large can the height of the tree possibly be?
b/ How small can the height of the tree possibly be?</td>
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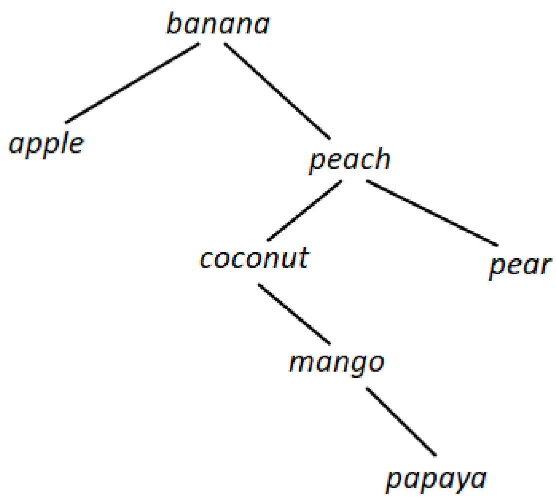
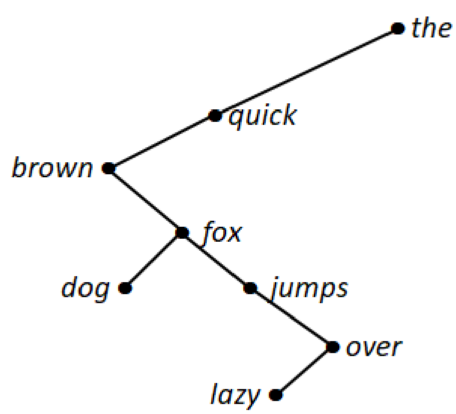
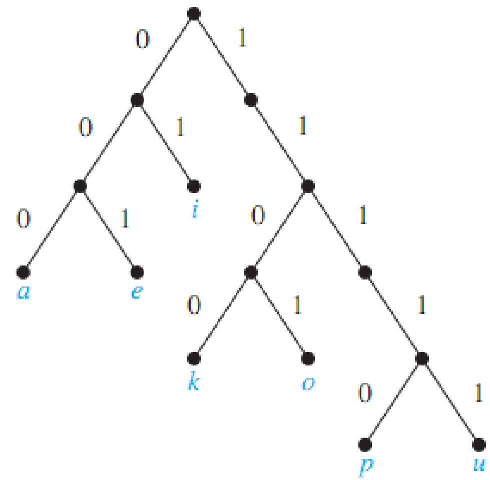
	Solution. a/ n = mi + 1 and n = i + *l* ⇨ 35 = 2·i + 1 and 35 = i + *l* ⇨ i = 17 and *l* = 18. b/ maximum height of the tree = number of internal nodes of the tree = 17. c/ minimum height of the tree = $\lceil \log_m(l) \rceil$ = $\lceil \log_2(18) \rceil$ = 5.	
Binary search trees	**Ex1.** Build a **binary search tree** for the words *banana*, *peach*, *apple*, *pear*, *coconut*, *mango*, and *papaya* using alphabetical order. ***Solution.***  **Ex2.** How many comparisons are needed to locate or to add each of these words in the **binary search tree** for Ex1, starting fresh each time? a/ pear b/ banana c/ orange. ***Solution.*** a/ Staring from the root of the tree to locate/insert the word pear: banana < pear ➔ go to the right peach < pear ➔ go to the right pear = pear ➔ locating/inserting successfully. ⇨ 3 comparisons. b/ banana = banana ➔ strop locating after	**123/** Build a **binary search tree** for the words EAGLE, ANT, BAT, DUCK, BEAR, PIG, CAT and DOG using alphabetical order. **124/** How many comparisons are needed to locate or to add each of these words in the **binary search tree** for exercise **123**, starting fresh each time? a/ BEAR b/ DOG c/ MONKEY. **125/** Using alphabetical order, construct a **binary search tree** for the words in the sentence **"**Let the cat out of the bag.**"**

	one comparison. c/ banana < orange ➔ go to the right peach > orange ➔ go to the left coconut < orange ➔ go to the right mango < orange ➔ go to the right papaya > orange ➔ go to the left left child of papaya is null ➔ fail to locate the word orange. ⇨ We fail to locate orange by comparing it successively to banana, peach, coconut, mango, and papaya. Once we determine that orange should be in the left subtree of papaya, and find no vertices there, we know that orange is not in the tree. Thus 5 comparisons were used. **Ex3.** Using alphabetical order, construct a **binary search tree** for the words in the sentence "The quick brown fox jumps over the lazy dog." ***Solution.***  	
Prefix codes Huffman code	**Ex1.** Which of these codes are **prefix codes**? a/ a: 11, e: 00, t : 10, s: 010. b/ a: 01, e: 101, t : 110, s: 1101. ***Solution.*** a/ This code scheme is a prefix code. b/ From the code scheme, we can see that t is encoded by **110** which is also the first part of the string **110**1 used for s. So, this code scheme is not a prefix code.	**126/** Which of these codes are **prefix codes**? a/ a: 101, e: 11, t : 001, s: 011, n: 010 b/ a: 010, e: 11, t : 011, s: 1011, n: 1001, i: 10101 **127/** Construct the binary tree with **prefix codes** representing these coding

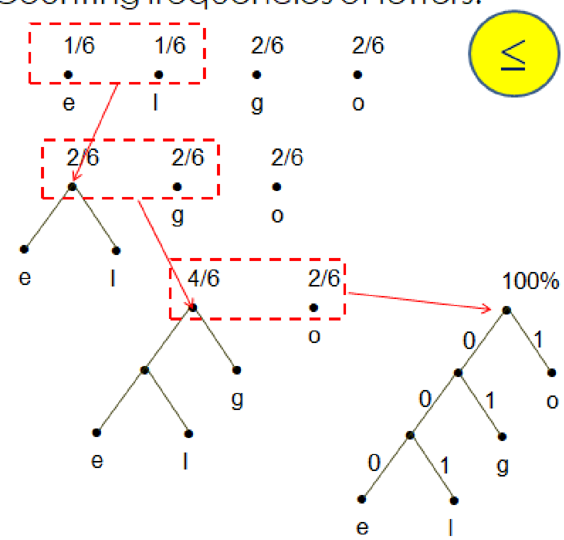
Ex2. What are the codes for a, e, i, k, o, p, and u if the coding scheme is represented by this tree?



Solution.
Moving from the root of the tree to each leaf and writing the bits labeled on edges, we obtain the codes:
a: 000, e: 001, i: 01, k: 1100, o: 1101, p: 11110, u: 11111.

Ex3. Use Huffman coding to encode the word "google". What is the average number of bits required to encode a symbol?
Solution.

Counting frequencies of letters:

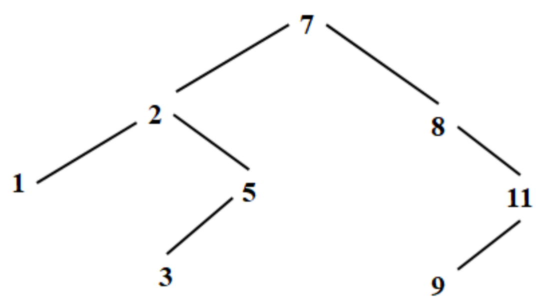


From the last tree, we obtain the codes for

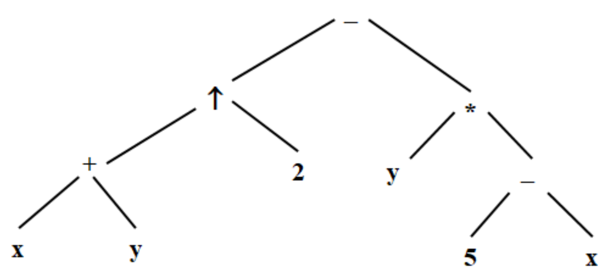
schemes.
a/ a: 11, e:0, t : 101, s: 100.
b/ a:1, e: 01, t : 001, s: 0001, n: 00001.

128/ Use **Huffman coding** to encode the word "success". What is the average number of bits required to encode a symbol?

129/ Use **Huffman coding** to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol?

	"google": e: 000, l: 001, g: 01, o: 1 ⇨ The word "google" can be coded as string "011101001000" ⇨ The average number of bits required to encode a symbol is 12/5 = 2.4.	
Tree traversal Pre-order In-order Post-order	**Ex1.** a/ Construct the **binary search tree** for the sequence: 7, 8, 2, 9, 5, 1, 3, 11, 9. Given the order of numbers after applying b/ pre-order traversal. c/ in-order traversal. d/ post-order traversal. ***Solution.*** a/ The binary search tree for the given sequence of numbers:  b/ A preorder traversal produces the list: 7, 2, 1, 5, 3, 8, 11, 9. c/ A preorder traversal produces the list: 1, 2, 3, 5, 7, 8, 9, 11. d/ A preorder traversal produces the list: 1, 3, 5, 2, 9, 11, 8, 7.	**130**a/ Construct the binary search tree for the sequence: 11, 13, 5, 4, 7, 15, 10, 6, 9. Given the order of numbers after applying a/ pre-order traversal. b/ in-order traversal. c/ post-order traversal.
Expression tree Prefix Infix Postfix notations	**Ex1.** What is the value of each of these **expressions**? a/ $+ - \uparrow 3\ 2 \uparrow 2\ 3\ /\ 6 - 4\ 2$ b/ $6\ 2\ /\ 5 + 5\ 2 - *$ ***Solution.*** a/ This is a **prefix notation** $+ - \uparrow 3\ 2 \uparrow 2\ 3\ /\ 6\ \underbrace{-\ 4\ 2}_{2}$ $= + - \uparrow 3\ 2 \uparrow 2\ 3\ \underbrace{/\ 6\ 2}_{3}$ $= + - \uparrow 3\ 2\ \underbrace{\uparrow 2\ 3}_{8}\ 3$ $= + -\ \underbrace{\uparrow 3\ 2}_{9}\ 8\ 3$	**131/** What is the value of each of these **expressions**? a/ $7\ 4\ 2 - - 1\ 3\ 4 + + *$ b/ $* + 3 + 3 \uparrow 3 + 3\ 3\ 3$ c/ $2\ 1 * 2 \uparrow 7\ 7 - 9\ 3\ / * -$ **132**a/ Represent the expressions (2*x + y)*((y - x))↑2 using **binary trees**. Write these expressions in b/ **prefix notation**.

<table>
<tr>
<td></td>
<td>= + − 9 8 3
1
= + 1 3
4
= 4
b/ This is a postfix notation
6 2 / 5 + 5 2 − ∗
= 6 2 / 5 + 5 2 − ∗
3
= 3 5 + 5 2 - *
8
= 8 5 2 - *
3
= 8 3 *
= 24.

Ex2. a/ Represent the expressions (x + y)↑2 − y*(5 − x) using binary trees.
Write these expressions in
b/ prefix notation.
c/ postfix notation.
Solution.
a/ Expression tree for the given expression:


b/ prefix notation:
 – ↑ + x y 2 * y – 5 x
c/ postfix notation:
x y + 2 ↑ y 5 x - * -</td>
<td>c/ postfix notation.</td>
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<tr>
<td colspan="3">Applications.

1/ a/ Use Huffman coding to encode these symbols with frequencies a: 0.4, b: 0.2, c: 0.2, d: 0.1, e: 0.1 in two different ways by breaking ties in the algorithm differently. First, among the trees of minimum weight select two trees with the largest number of vertices to combine at each stage of the algorithm. Second, among the trees of minimum weight select two trees with the smallest number of vertices at each stage.
b/ Compute the average number of bits required to encode a symbol with each code and compute the variances of this number of bits for each code. Which tie-breaking procedure</td>
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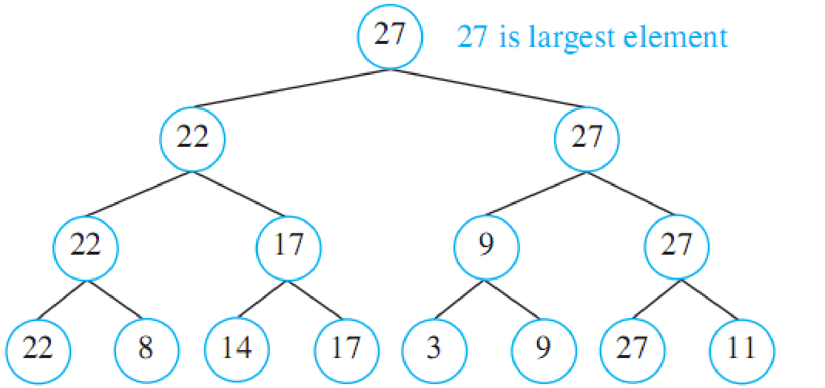
produced the smaller variance in the number of bits required to encode a symbol?

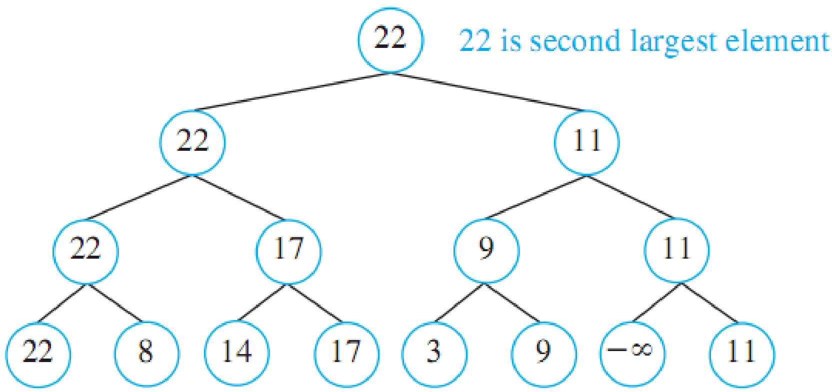
2/ Suppose that m is a positive integer with $m \geq 2$. An **m-ary Huffman code** for a set of N symbols can be constructed analogously to the construction of a **binary Huffman code**. At the initial step, $((N - 1) \bmod (m - 1)) + 1$ trees consisting of a single vertex with least weights are combined into a rooted tree with these vertices as leaves. At each subsequent step, the m trees of least weight are combined into an m-ary tree.
Using the symbols 0, 1, and 2 use ternary ($m = 3$) Huffman coding to encode these letters with the given frequencies: A: 0.25, E: 0.30, N: 0.10, R: 0.05, T: 0.12, Z: 0.18.

3/ Use a **decision tree** to give the best way to find the lighter counterfeit coin among 24 coins.

4/ The **tournament sort** is a sorting algorithm that works by building an ordered binary tree. We represent the elements to be sorted by vertices that will become the leaves. We build up the tree one level at a time as we would construct the tree representing the winners of matches in a tournament. Working left to right, we compare pairs of consecutive elements, adding a parent vertex labeled with the larger of the two elements under comparison. We make similar comparisons between labels of vertices at each level until we reach the root of the tree that is labeled with the largest element. The tree constructed by the tournament sort of 22, 8, 14, 17, 3, 9, 27, 11 is illustrated in part (a) of the figure. Once the largest element has been determined, the leaf with this label is relabeled by $-\infty$, which is defined to be less than every element. The labels of all vertices on the path from this vertex up to the root of the tree are recalculated, as shown in part (b) of the figure. This produces the second largest element. This process continues until the entire list has been sorted.



(a)



(b)

Use the **tournament sort** to sort the list 17, 4, 1, 5, 13, 10, 14, 6.