GRAPH THEORY
Chapter 2 - Subgraphs

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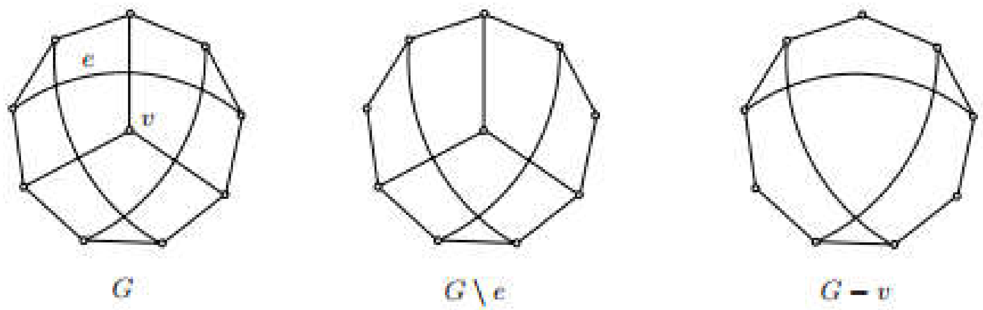
2.1. SUBGRAPHS AND SUPERGRAPHS

Edge and Vertex Deletion

- Given a graph G on n vertices and m edges:
  - The *edge deletion*, denoted by $G\backslash e$, is an operation on which a new graph is obtained by deleting the edge e from G but leaving the vertices and the remaining edges in tact. ($G\backslash e$ - an edge-deleted subgraph)
  - Similarly, the vertex deletion , denoted by $G - v$, is an operation on which a new graph is obtained by deleting from G the vertex v together with all the edges incident with v. ($G - v$ - a vertex deleted subgraph)
- Examples of subgraphs of G



Subgraph

- A *subgraph* of a graph G is a graph F such that $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$ and the assignment of endpoints to edges in F is the same as in G. We then write $F \subseteq G$ or $G \supseteq F$ and say that "G *contains* H" or "H is *contained in* G"
- A supergraph of a graph G is a graph H which contains G as a subgraph, that is, $H \supseteq G$.
- **Note that any graph is both a subgraph and a supergraph of itself.**
- All other subgraphs F and supergraphs H are referred to as proper; we then write $F \subset G$ or $H \supset G$, respectively.
- **The above definition apply also to digraphs, with the obvious modifications.**

A Graph Contains a Cycle

- **Theorem 2.1.1. Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.**

Maximality And Minimality

- Let $\mathcal{F}$ be a family of subgraphs of a graph G. A member F of $\mathcal{F}$ is *maximal* in $\mathcal{F}$ if no member of $\mathcal{F}$ properly contains F.
  - When $\mathcal{F}$ consists of the set of all paths of G, then a maximal member of $\mathcal{F}$ as a *maximal path* of G.
- F is *minimal* in $\mathcal{F}$ if no member of $\mathcal{F}$ is properly contained in F.
- In a graph G, which has at least one cycle,
  - the length of a longest cycle is called its *circumference*.
  - The length of a shortest cycle is called its *girth*.

Acyclic Graphs And Digraphs

- A graph is *acyclic* if does not contain a cycle.
  - It follows from Theorem 2.1 that an acyclic graph must have a vertex of degree less than two. In fact, every nontrivial acyclic graph has at least two vertices of degree less than two.
- A digraph is acyclic if it has no directed cycle.

Exercises

1) Show that every nontrivial acyclic graph has at least two vertices of degree less than two.
2) Deduce that every nontrivial connected acyclic graph has at least two vertices of degree one. When does equality hold?
3) Show that the maximal connected subgraphs of a graph are its components?
4) Given a graph G of order n and size m. Show that if $m \geq n$, then G contains a cycle.
5) For each positive integer n, find an acyclic graph with n vertices and $n-1$ edges?

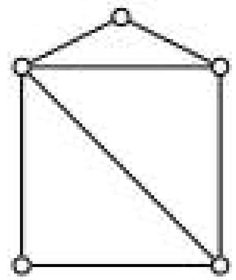
Exercises

6) Given a simple graph G of minimum degree δ
   a) Show that G contains a path of length δ
   b) For each $k \geq 0$, find a simple graph G with $\delta = k$ which contains no path of length greater than k
7) Given a simple graph of minimum degree $\delta \geq 2$
   a) Show that G contains a cycle of length at least $\delta + 1$
   b) For each $k \geq 2$, find a simple graph G with $\delta = k$ which contains no cycle of length greater than $k + 1$.
8) Show that a k-regular graph of girth four has at least $2k$ verties.

2.2.Spanning And Induced Subgraph

The Subgraphs

- A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$



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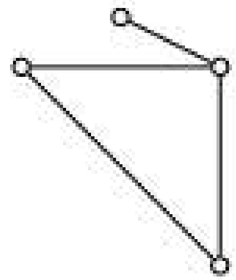
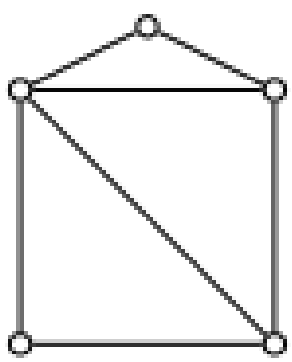


subgraph

Spanning Subgraphs and Spanning Supergraphs

- A *spanning subgraph* of a graph G is a subgraph obtained by edge deletion only (H is a spanning subgraph of G if every vertex of G is in H, i.e., $V(H) = V(G)$).
- Adding a set S of edges to a graph G yields a *spanning supergraph* of G, denoeted $G + S$.



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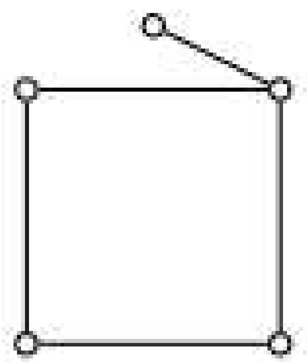
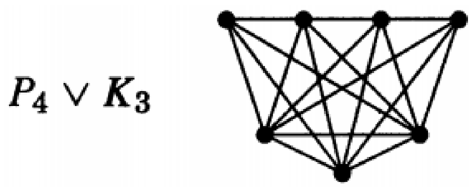


spanning

Join of Two Graphs

- The join of two graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy: x \in V(G), y \in V(H)\}$.



- The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel*, denoted W_n.
- Spanning paths and cycles are called *Hamilton paths* and *Hamilton cycles*, respectively.

Underlying Simple Graphs

- By deleting from a graph G all loops and, for every pair of adjacent vertices, all but one link joining them, we obtain a simple spanning subgraph called the *underlying simple graph* of G.
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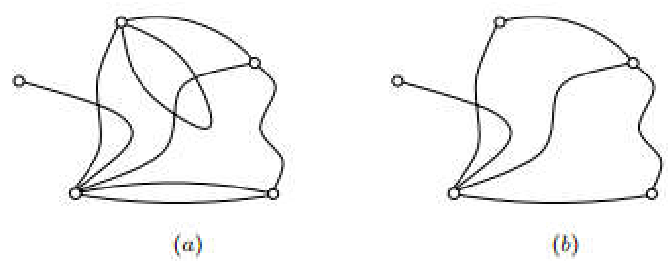
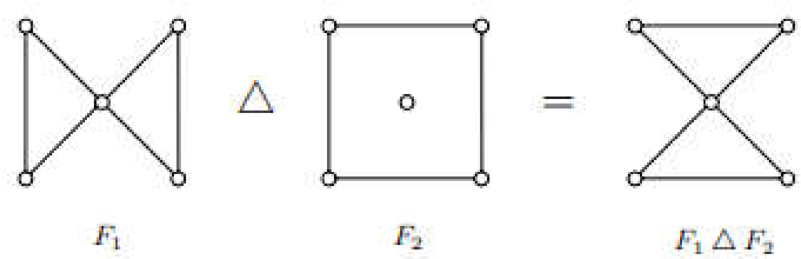


(a) A graph and (b) its underlying simple graph

The Symmetric Difference of Two Spanning Subgraphs

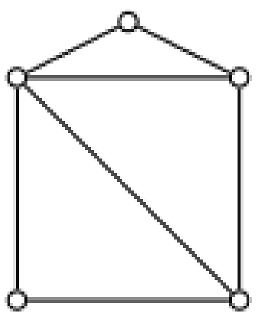
- Given spanning subgraphs $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ of a graph $G = (V, E)$. The *symmetric difference* of F_1 and F_2, denoted $F_1 \triangle F_2$, is the graph form the spanning subgraph of G whose edge set is the symmetric difference $E_1 \triangle E_2$



The symmetric difference of two graphs

Induced Subgraphs

- A subgraph obtained by vertex deletion only is called an *induced subgraph*. If X is the set of vertices deleted, the resulting subgraph is denoted by $G - X$.
- The subgraph is denoted by $G[Y]$ and referred to as the subgraph of G *induced by* Y ($G[Y]$ is the subgraph of G whose vertex set is Y and the edge set consists of all edges of G which have both endpoints in Y.



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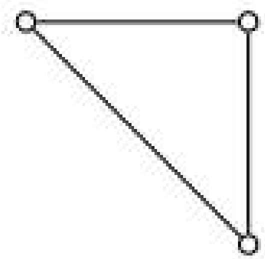


induced

Induced Subgraphs

- **Theorem 2.2.1 Every graph with average degree at least $2k$, where k is a positive integer, has an induced subgraph with minimum degree at least $k+1$**

The Edge-Induced Subgraph

- Given a graph G. If S is a set of edges, the *edge-induced subgraph* $G[S]$ is the subgraph of G whose edge is S and whose vertex set consists of all endpoints of edges of S.
- Any edge-induced subgraph $G[S]$ can be obtained by first deleting the edges in $E \backslash S$ and then deleting all resulting isolated vertices

Weighted Graphs And Subgraphs

- Given a graph G, with each edge e of G, let there be associated a real number $w(e)$, called its ***weight***. Then G, together with these weights on its edges, is called a, denoted by (G, w).
- If F is a subgraph of a weighted graph G, the weight $w(F)$ of F is the sum of the weights on its edges $\sum_{e \in E(F)} w(e)$.
- Example:
  - The travelling Salesman Problem TSP

Exercises

1) Let G be a simple graph on n vertices and m edges and c components
   a) How many spanning subgraphs has G?
   b) How many edges need to be added to G to obtain a connected spanning supergraph?
2) Let G be a graph on n vertices and m edges.
   a) How many induced subgraphs has G?
   b) How many edge-induced subgraphs has G?
3) Show that if G is simple and connected, but not complete, then G contains an induced path of length two?
4) Show that every shortest cycle in a simple graph is an induced subgraph?
5) Show that an induced subgraph of a line graph is itself a line graph?

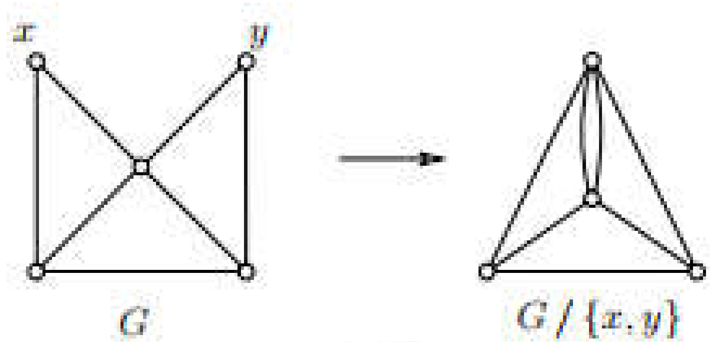
Exercises

6) Let G be a bipartite graph of maximum degree k
   a) Show that there is a k-regular bipartite graph H which contains G as an induced subgraph?
   b) Show, moreover, that G is simple, then there exists such a graph H which is simple?
7) Let G be a simple graph on n vertices, where $n \geq 4$, let k be an integer, $2 \leq k \leq n-2$. Suppose that all induced subgraphs of G on k vertices have the same number of edges. Show that G is either empty or complete.

2.3.Modifying Graphs

Vertex Identification

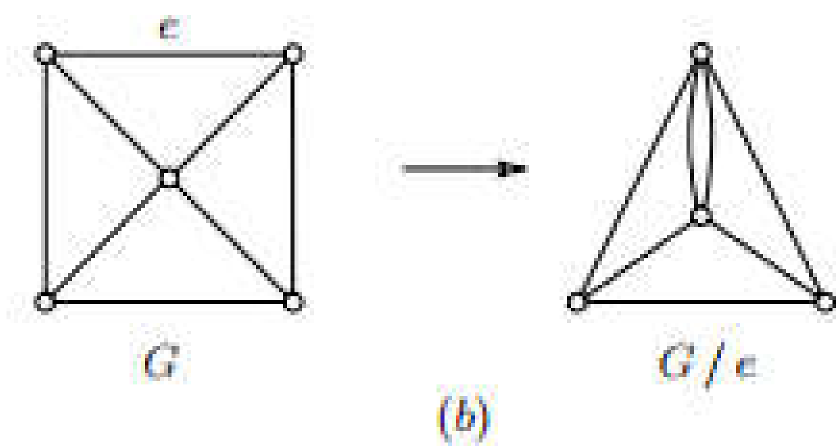
- To *identify* non-adjacent vertices x and of a graph G is to replace these vertices by a single vertex incident to all the edges which were incident in G to either x or y. We denote the resulting graph by $G/\{x,y\}$



(a)

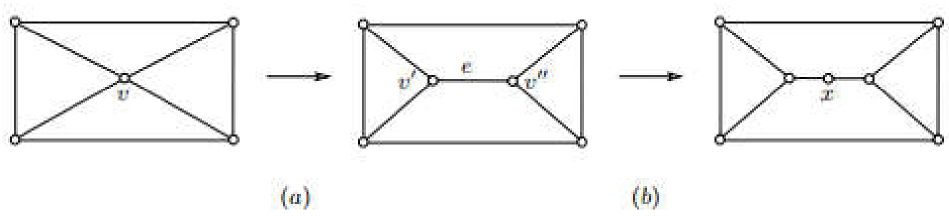
Edge Contraction

- To *contract* an edge e of a graph G is to delete the edge and then (if the edge is a link) identify its endpoints. The resulting graph is denoted by G/e.



Vertex Splitting And Edge Subdivision

- To *split* a vertex v is to replace v by two adjacent vertices, v' and v'', and to replace each edge incident to v by an edge incident to either v' or v'' (but not both, unless it is a loop at v), the other endpoints of the edge remaining unchanged.
- To subdivide an edge e is to delete e, add a new vertex x, and join x to the endpoints of e (when e is link, this amounts to replacing e by a path of length two)



(a) Splitting a vertex, and (b) subdividing an edge

Exercises

1) Show that $c(G/e) = c(G)$ for any edge e of a graph G.
2) Let G be an acyclic graph, and let $e \in E(G)$
   a) Show that $G\backslash\mathrm{e}$ is acylic
   b) Deduce that $m = n - c$, where n is the number of vertices, m is the number of edges and c is the number of components.

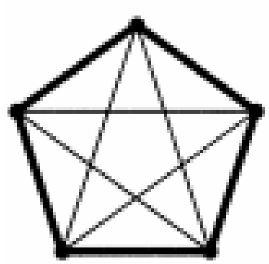
2.4.Decompositions and Coverings

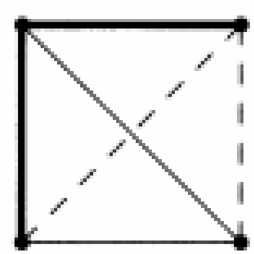
Decompositions

- A decomposition of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that:

$$U_{F\in\mathcal{F}}E(F) = E(G)$$

- If the family $\mathcal{F}$ consists entirely of paths or entirely of cycles, we call $\mathcal{F}$ a *path decomposition* or *cycle decomposition* of G.
- Observe that if a graph has a cycle decomposition $\mathcal{C}$, the degree of each vertex is twice the number of cycles of $\mathcal{C}$ to which it belongs, so is even. A graph in which each vertex has even degree is called an *even graph*.





Decompositions

- **Theorem 2.4.1 (Veblen's Theorem): A graph admits a cycle decomposition if and only if it is even.**
- **Theorem 2.4.2 Let $\mathcal{F} \coloneqq \{F_1, F_2, \dots, F_k\}$ be a decomposition of K_n into complete bipartite graphs. Then $k \geq n - 1$**

Coverings

- A *covering* or *cover* of a graph G is a family $\mathcal{F}$ of subgraphs of G, not necessarily edge-disjoint, satisfying
$$\bigcup_{F\in\mathcal{F}} E(F) = E(G)$$
- A covering is *uniform* if it covers each edge of G the same number of times; when this number is k, the covering is called a k-*cover*.
  - A 1-cover is thus simply a decomposition;
  - A 2-cover is usually called a *double cover*.
- If the family $\mathcal{F}$ consists entirely of paths or entirely of cycles, the covering is referred to as a *path covering* or *cycle covering*.

Exercises

1) Let e be an edge of an even graph G. Show that G/e is even.
2) Let n be a positive integer
   a) Describe a decomposition of K_{2n+1} into Hamilton cycles.
   b) Deduce that K_{2n} admits a decomposition into Hamilton paths
3) Show that K_n can be decomposed into copies of K_p only if $n-1$ is divisible by $p-1$ and $n(n-1)$ is divisible by $p(p-1)$.
4) Let G be a connected graph with an even number of edges. Deduce that G admits a decomposition into paths of length two.