Sparse Representations

Sargur N. Srihari
srihari@buffalo.edu

Regularization Strategies

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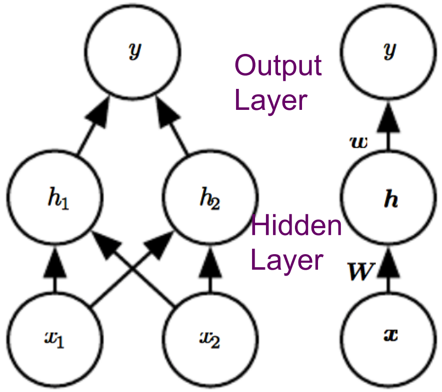
Direct and Indirect Penalties

- Direct Penalty
  - Weight decay penalizes parameters directly
  - L^1 penalization induces sparse parameterization
- Indirect Penalty
  - Another strategy is to place penalty on the activations of the units in the neural network
    - Encouraging their activations to be sparse
    - It imposes a complicated penalty on model parameters
  - Representational sparsity describes a representation where many of the elements of the representation are close to zero

Definition needs Matrix notation

- Given network drawn in two different styles
  - Matrix W describes mapping from $\boldsymbol{x}$ to $\boldsymbol{h}$
  - Vector $\boldsymbol{w}$ describes mapping from $\boldsymbol{h}$ to y
  - Intercept parameters b are omitted



- Layer 1 (hidden layer): $\boldsymbol{h}$ computed by function $f^{(1)}(\boldsymbol{x};\ W,\boldsymbol{c})=\boldsymbol{h}=g(W^T\boldsymbol{x}+\boldsymbol{c})$
  - $\boldsymbol{c}$ are bias variables
- Layer 2 (output layer) computes $\boxed{f^{(2)}(\boldsymbol{h};\boldsymbol{w},b)=\boldsymbol{h}^T\boldsymbol{w}+b}$
  - w are linear regression weights
  - Output is linear regression applied to $\boldsymbol{h}$ rather than to $\boldsymbol{x}$
- Complete model is

$$f(\boldsymbol{x};\ W,\boldsymbol{c},\boldsymbol{w},\boldsymbol{b})=f^{(2)}(f^{(1)}(\boldsymbol{x}))$$

Direct versus Representational Sparsity

– Parameter regularization

$$\underset{\boldsymbol{y} \in \mathbb{R}^m}{\begin{bmatrix} 18 \\ 5 \\ 15 \\ -9 \\ -3 \end{bmatrix}} = \underset{\boldsymbol{A} \in \mathbb{R}^{m \times n}}{\begin{bmatrix} 4 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 & -5 & 0 \end{bmatrix}} \underset{\boldsymbol{x} \in \mathbb{R}^n}{\begin{bmatrix} 2 \\ 3 \\ -2 \\ -5 \\ 1 \\ 4 \end{bmatrix}}$$

Weight matrix W is sparse

– Representational regularization

$$\underset{\boldsymbol{y} \in \mathbb{R}^m}{\begin{bmatrix} -14 \\ 1 \\ 19 \\ 2 \\ 23 \end{bmatrix}} = \underset{\boldsymbol{B} \in \mathbb{R}^{m \times n}}{\begin{bmatrix} 3 & -1 & 2 & -5 & 4 & 1 \\ 4 & 2 & -3 & -1 & 1 & 3 \\ -1 & 5 & 4 & 2 & -3 & -2 \\ 3 & 1 & 2 & -3 & 0 & -3 \\ -5 & 4 & -2 & 2 & -5 & -1 \end{bmatrix}} \underset{\boldsymbol{h} \in \mathbb{R}^n}{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -3 \\ 0 \end{bmatrix}}$$

$\boldsymbol{h}$ vector is sparse

Representational Regularization

- Accomplished using same sort of mechanisms used in parameter regularization
- Norm penalty regularization of representation
  - Performed by adding to the loss function J, a norm penalty on the representation.
    - The regularized loss function is

$$\tilde{J}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha\Omega(\boldsymbol{h})$$

where $\alpha \in [0,\infty)$

    - An L^1 penalty term induces sparsity: $\Omega(\boldsymbol{h}) = ||\boldsymbol{h}||_1 = \sum_i |h_i|$

Placing constraint on Activation Values

- Another approach to representational sparsity:
  - place a hard constraint on activation values
- Called Orthogonal matching pursuit (OMP)
  - Encode $\boldsymbol{x}$ with $\boldsymbol{h}$ that solves constrained optimization: $\boxed{\underset{\boldsymbol{h}, \|\boldsymbol{h}\|_0 < k}{\arg\min} \|\boldsymbol{x} - \boldsymbol{W}\boldsymbol{h}\|^2}$
    - where $\|\boldsymbol{h}\|_0$ is the number of zero entries of $\boldsymbol{h}$
    - Problem is solved efficiently when $\boldsymbol{W}$ is orthogonal
  - Often called OMP-k, where k is no. of zero entries
    - OMP-1 is very effective for deep architectures
- Essentially, any model with hidden units can be made sparse:
  - sparsity regularization is used in many contexts