BUILDING A YIELD CURVE GENERATOR

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1. Libor and Swap Rates. Libor rates are quoted every day for standard maturities 1 month, 3 months, ... They are quoted in the form of an *annualized rate* L, and an *accrual basis* which is "actual/360" except in the case of GBP where it is "actual/365". The interest paid in cash terms is $N \times L \times \theta$ where N is the principal amount and θ is the day-count fraction:

$$\theta = \frac{\text{Number of days of deposit}}{360}.$$

Since everything is linear in N, we always normalize to $N = 1$, so the Libor payment is θL.

If the Libor rate is L over an interval $[T_1, T_2]$ then the corresponding zero-coupon bond (ZCB) value is

$$p(T_1, T_2) = \frac{1}{1 + \theta L}, \tag{1.1}$$

i.e. if we deposit $p(T_1, T_2)$ at T_1 we will receive \$1 at T_2. Inverting this we see that

$$L = \frac{1}{\theta}\left(\frac{1}{p(T_1, T_2)} - 1\right).$$

A *floating rate note (FRN)* is a coupon bond, entered at time T_0, that pays interest $\theta_k L_k$ at each of a sequence of coupon dates $\{T_k, k = 1, \ldots, n\}$ and repays the principal 1 at T_n. Here θ_k is the accrual fraction for $[T_{k-1}, T_k]$ and L_k is the Libor rate set at T_{k-1} and paid at T_k.

PROPOSITION 1.1. *A floating rate note has value 1 at every coupon date.*

Proof. Using (1.1) we see that a Libor payment $\theta_k L_k$ at T_k is equivalent to a payment at T_{k-1} of

$$\frac{\theta_k L_k}{1 + \theta_k L_k} = \left(\frac{1}{p(T_{k-1}, T_k)} - 1\right) p(T_{k-1}, T_k) = 1 - p(T_{k-1}, T_k). \tag{1.2}$$

If, at time $T_j \le T_{k-1}$ we buy a T_{k-1} ZCB and sell a T_k ZCB then the value of our position is $p(T_j, T_{k-1}) - p(T_j, T_k)$, and its value at T_{k-1} will be $p(T_{k-1}, T_{k-1}) - p(T_{k-1}, T_k) = 1 - p(T_{k-1}, T_k)$, exactly the amount at (1.2). If we do this for each $k = j+1, \ldots, n$ then the value of the position at T_j is

$$(p(T_j, T_j) - p(T_j, T_{j+1})) + (p(T_j, T_{j+1}) - p(T_j, T_{j+2})) + \cdots + (p(T_j, T_{n-1}) - p(T_j, T_n)) = 1 - p(T_j, T_n).$$

We see that a payment of $1 - p(T_j, T_n)$ perfectly hedges all the Libor payments of the FRN, and a further payment of $p(T_j, T_n)$ hedges the principal payment 1 at T_n, so the total value of the FRN at T_j is 1. $\square$

A fixed-rate note pays a coupon $\theta_k K$ at T_k where K is the fixed annualized rate (the same for every k). The value of this note at T_j is

$$K \sum_{k=j+1}^{n} \theta_k p(T_j, T_k) + p(T_j, T_n). \tag{1.3}$$

A (plain vanilla) *interest rate swap* is a contract whereby one party (the payer) pays the other party (the receiver) the net payment $\theta_k(K - L_k)$ at each coupon date. If we artificially add a payment of 1 from each side to the other at T_n (these payments net to zero and are not actually made) then we see that the swap amounts to exchanging a FRN for a fixed-rate note. Hence the value is the difference between the value of the FRN (= 1) and the fixed-rate value given by (1.3). The contract is initiated at T_0 and the fixed-side rate K is chosen so that the contract has value 0 at that time. From (1.3), this means that K is equal to the *swap rate*

$$S_n = \frac{1 - p(T_0, T_n)}{\sum_{k=1}^{n} \theta_k p(T_0, T_k)}. \tag{1.4}$$

2. Backing out ZCB values from market data. Zero coupon bonds are not directly traded assets. Their value must be inferred from the prices of contracts actually traded in the markets. These are

- Libor rates
- Interest rate futures
- Swap rates

As mentioned above, Libor rates are directly quoted for various maturities up to 1 year and maybe more. There is also a huge market in swaps of maturities up to 10 years, and significant trading in longer-dated swaps out to 30 years or more. In this note we will not deal with interest rate futures: unlike the other two classes, their pricing is not model-independent and presents a more delicate problem.

2.1. Use of swap data to infer ZCB values. Let us first consider the simple, but unrealistic, case where swap rates $S_1, \ldots, S_n$ are quoted in the market for all maturity times $T_1, \ldots, T_n$ where we assume that the coupon dates are the same for all swaps. From (1.4) with $n = 1$

$$S_1 = \frac{1 - p(T_0, T_1)}{\theta_1 p(T_0, T_1)} = \frac{1}{\theta_1}\left(\frac{1}{p(T_0, T_1)} - 1\right) = L_1.$$

That is, the 1-period swap rate is equal to the Libor rate, and hence from (1.1)

$$p(T_0, T_1) = \frac{1}{1 + \theta_1 S_1}.$$

Now suppose $p(T_0, T_i)$ have been determined for $i = 1, \ldots, k$, and let $A_k = \sum_{i=1}^{k} \theta_1 p(T_0, T_i)$. Then from (1.4)

$$S_{k+1} = \frac{1 - p(T_0, T_{k+1})}{A_k + \theta_{k+1} p(T_0, T_{k+1})},$$

so that

$$p(T_0, T_{k+1}) = \frac{1 - S_{k+1} A_k}{1 + \theta_{k+1} S_{k+1}}.$$

We can thus recursively determine all the $p(T_0, T_k)$ values.

2.1.1. Interpolation. To determine values $p(T_0, t)$ for times t that are not coupon dates, the normal procedure is to use *linear interpolation of rates*. Specifically, suppose $t = (1 - \lambda)T_k + \lambda T_{k+1}$ for some $\lambda \in (0, 1)$. We write $p(T_0, T_k) = e^{-r_k T_k}$ and $p(T_0, T_{k+1}) = e^{-r_{k+1} T_{k+1}}$ (defining r_k, r_{k+1}), set $r = (1 - \lambda) r_k + \lambda r_{k+1}$ and define

$$p(T_0, t) = e^{-rt}.$$

2.1.2. Missing data. Invariably, some of the S_k values will not be available. Suppose we have determined $p(T_0, T_k)$ and the next swap rate available is S_{k+j}. For $i = 1, \ldots, j - 1$ let $\hat{P}_i(p)$ be the interpolation of $P(T_0, T_k)$ and the as yet unknown $p = P(T_0, T_{k+j})$ as given above. Then using interpolated values where necessary we have

$$S_{k+j} = \frac{1 - p}{A_k + \theta_{k+1} \hat{P}_{k+1}(p) + \cdots + \theta_{k+j-1} \hat{P}_{k+j-1}(p) + \theta_{k+j} p} \tag{2.1}$$

We now have to solve equation (2.1) for p (given S_{k+j} and A_k). The solution is $p = P(T_0, T_{k+j})$ and then all intermediate discount factors are defined by interpolation as in (2.1). Note that the right-hand side of (2.1) (call it $f(p)$) is monotone decreasing in p and $f(1) = 0$. Assuming $f(0) > S_{k+j}$ (this is highly likely—why?) we can solve the equation $f(p) = S_{k+j}$ by binary search:

1. Set $l = 0, r = 1$
2. Set $m = (l + r)/2$
3. If $f(m) < S_{k+1}$ then set $r = m$, else $l = m$
4. Repeat 2 and 3 until $|f(m) - S_{k+1}| < \epsilon$
5. Return $P(T_0, T_{k+j})) = m$.

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3. Yield curve construction. The data is typically similar to that shown in Table 3.1. (Interest rate futures are not included.) Libor rates are available up to 1 year maturity and these directly define the corresponding discount factors via (1.1). Values $P(T_0, t)$ for other times t less than 1 year are determined by interpolation. For $t > 1$ year, apply the procedure described in section 2.1. Note that we only have swap quotes for integer numbers of years, but there is a coupon every 6 months, so in the notation of Section 2.1.2 we will have $j = 2$ at each stage.

	Maturity	Rate (%, annualized)
Libor rates	Overnight	0.29
	1 month	0.38
	3 months	0.60
	6 months	0.91
	1 year	1.21
Swap rates	2	1.51
Maturity in years	3	1.90
	4	2.23
	5	2.51
	6	2.76
	7	2.96
	8	3.13
	9	3.27
	10	3.39

Table 3.1

EUR yield curve, 17.02.10. Rates up to and including 1 year are Libor rates, rates beyond 1 year are swap rates, semi-annual coupon, 30/360 accrual basis.