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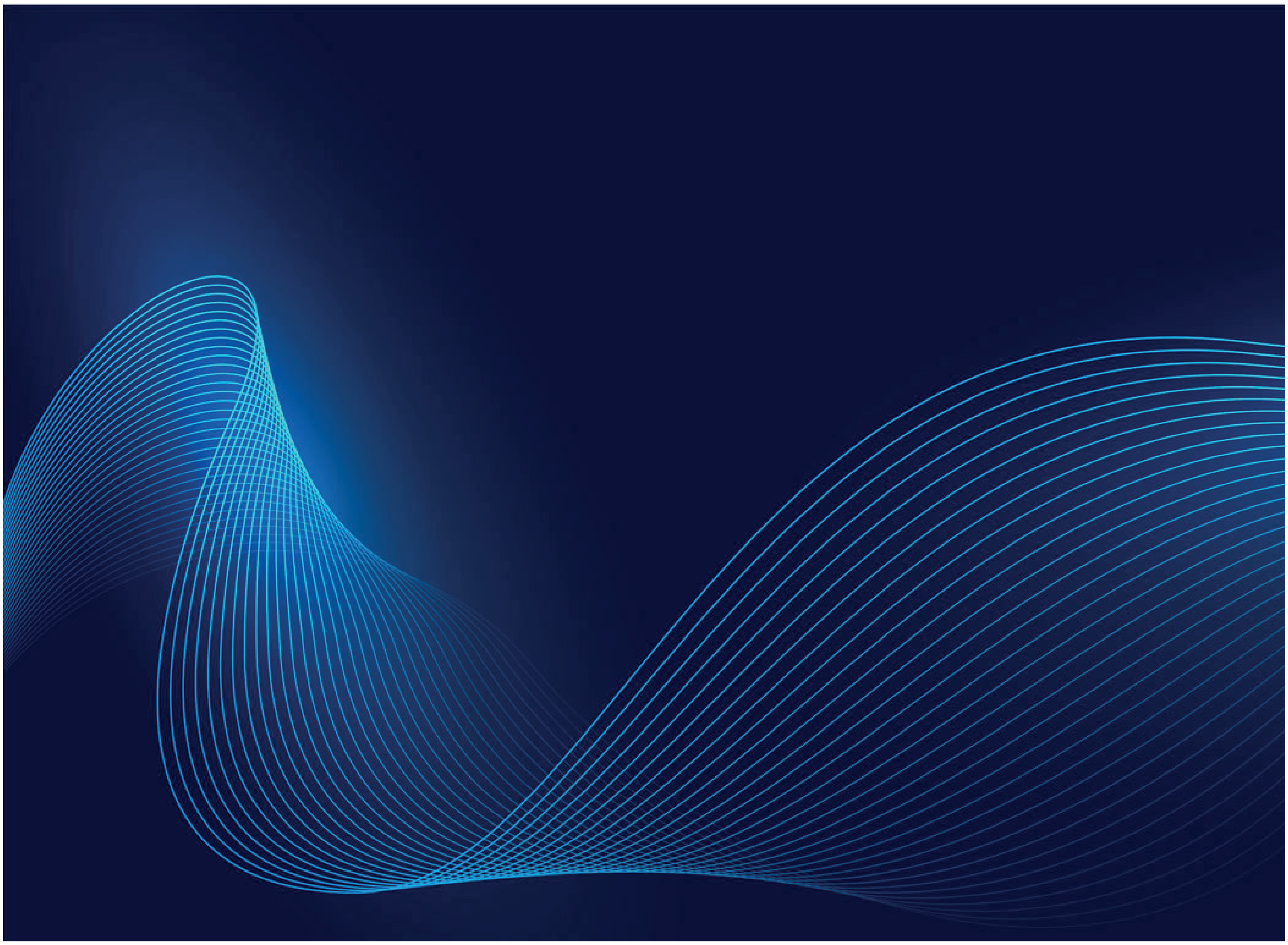

Multi-curve Cheyette-style models with lower bounds on tenor basis spreads

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This article presents a general multi-curve Cheyette-style model that allows precise control over tenor basis spreads. The specification was proposed by Grbac and Runggaldier, but a solution for the no-arbitrage drift function has remained elusive. Michael Konikov and Andrew McClelland recover the drift function via an ansatz and proceed to fully develop the model, providing an example with a level-dependent volatility function to secure lower bounds on spreads

Tenor basis swap spreads reflect the differences between forward reference rates of different tenors. For example, the 3M Ibor-versus-6M Ibor spread reflects the difference between forward 3M and 6M Ibors. We use Ibor here to refer to an inter-bank offered rate, of which Libor (London's Ibor) is a well-known example. Since the financial crisis of 2007–8, there has been significant volatility in observed spreads; as discussed momentarily, how these spreads are modelled is of paramount importance to the credit valuation adjustment (CVA) for a tenor basis swap, among other things.

Libor itself is being phased out in favour of risk-free reference rates, such as the Secured Overnight Financing Rate (SOFR) in USD markets. The modelling framework developed here is of course applicable to similar references rates that are not being phased out.[1] Examples include Euribor or Tibor, which are being reformed to make heavier use of transaction data, and BBSW or Cora, which are already transaction based.[2] Similarly, the framework could be used for the Bank Bill Index (BBI) that is being developed by the Intercontinental Exchange (Ice) as an alternative to USD Libor.[3] In short, the framework is applicable to any reference rate that resembles a money-market term rate, and which is thus sensitive to credit and liquidity concerns.

Denote the forward 3M and 6M Ibors by $R_3(t, T)$ and $R_6(t, T)$, where t is the calendar date and T is the fixing date. A 'textbook' 3M Ibor-versus-6M Ibor tenor basis swap exchanges 3M Ibor-plus-spread at a 3M frequency for 6M Ibor-flat at a 6M frequency, and we will denote the spread by $\phi_{3,6}$. Letting $P(t, T)$ denote discount factors computed off the overnight (ON) curve, its value is:

$$\begin{aligned} V_{3,6}(t) = {} & \sum_j P(t, T_{6,j+1}) R_6(t, T_{6,j}) \tau_{6,j} \\ & - \sum_i P(t, T_{3,i+1})(R_3(t, T_{3,i}) + \phi_{3,6}) \tau_{3,i} \end{aligned} \tag{1}$$

assuming that the trade pays the 3M leg, and where we ignore convention- and calendar-related complications for notational convenience. In the above, $T_{3,i}$ and $T_{6,j}$ refer to 3M and 6M fixing dates, respectively, and $\tau_{3,i}$ and $\tau_{6,j}$ are the relevant day-count fractions, ie, $\tau_{3,i} \approx \frac{3}{12}$ and $\tau_6 \approx \frac{6}{12}$. The dominant risk factor for such a trade is the spread between the forward 6M Ibor curve and the forward 3M Ibor curve. Indeed, the value of the trade roughly satisfies:

$$V_{3,6}(t) \propto R_6^{\text{av}}(t) - R_3^{\text{av}}(t) - \phi_{3,6} \tag{2}$$

where $R_3^{\text{av}}(t)$ and $R_6^{\text{av}}(t)$ are average levels of the respective forward curves. The model selected for the 3M Ibor-versus-6M Ibor tenor basis spread will thus have a greater bearing than any other factor on the CVA for such a basis swap. Indeed, computing the CVA for a tenor basis swap would resemble valuing a tenor basis swaption.

There is an existing literature on tenor basis modelling, and managing CVA for tenor basis swaps has indeed been an impetus for its growth. However, some important issues remain open. In particular, a well-documented feature of tenor basis swap spreads is that they rarely take negative values. Thus, a desirable property of tenor basis models is their ability to control the lower bounds of spreads, but there is an absence of specifications with this property within the popular 'multiplicative' framework developed by Henrard (2013) and others.

To develop such a model, we build upon the existing work of Martinez (2009), Grbac & Runggaldier (2015) and Miglietta (2015), who recast multiplicative models in terms of instantaneous forward spreads. These authors used level-independent volatility functions for (Gaussian) spread processes, while we will use level-dependent volatility functions as a means of imposing lower bounds. As we shall see, the no-arbitrage drift restriction for the spread curve process is more complicated than it is for the discounting curve process. A solution for the drift function was not derived in the earlier works, and, in turn, a Markov representation for the spread process was unavailable. Despite this, the earlier authors were able to make progress in characterising Ibor dynamics, but, as we will discuss, this is *only* possible in the level-independent case. In this article, we both solve the drift restriction and derive a Markov representation under the general case of level-dependent volatility. With this, we arrive at an intuitive and tractable multi-curve specification in the spirit of Cheyette (1992) that supports lower bounds and is easy to implement.

We also discuss the calibration of multi-curve models, which is another issue underdeveloped in the literature. We argue for using a mix of historical and implied data, as opposed to relying on swaption volatilities alone. Specifically, we argue that loading the observed swaption skew onto the spread process can lead to spread dynamics that poorly match empirical behaviour. A calibration strategy informed by this view is formalised and the results of a calibration exercise are presented.

[1] *A catalogue of various reform efforts or terminations of references rates is provided in FSB (2018).*

[2] *Euribor and Tibor are reference rates in Europe and Japan that use a Libor-like methodology. The Bank Bill Swap Rate (BBSW) of Australia and the Canadian Dollar Offered Rate (CDOR) rely on liquid bank-bill markets.*

[3] *See Ice (2019) for motivations for the BBI and a review of the proposed methodology.*

The model

Ibor specification. We focus here on a two-curve setup consisting of the base curve, taken to be the ON curve, along with the 3M Ibor curve. We model the ON curve and the ON-versus-3M Ibor spread curve, which links the ON curve and the 3M Ibor curve and governs the behaviour of ON-versus-3M Ibor tenor basis swap spreads. The analysis is readily extended to a larger set of curves including, for example, the 1M and 6M Ibor curves, as is done in Konikov & McClelland (2019). We begin with the ON curve, modelled via its instantaneous forward curve, $f_0(t,T)$. Discount factors are $P(t,T) = \exp(-\int_t^T f(t,u)\,\mathrm{d}u)$, and in a 'single-curve world' the 3M Ibor off the ON curve would be:

$$R_{0,3}(t,T) = \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)\,\mathrm{d}u} - 1) \tag{3}$$

where we employ the subscript $0,3$ to emphasise that this is not the true 3M Ibor.

To compute the true 3M Ibor off the 3M Ibor curve, we follow Grbac & Runggaldier (2015), Miglietta (2015) and Martinez (2009), who employ an intuitive extension of (3):

$$R_3(t,T) = \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)+s_3(t,u)\,\mathrm{d}u} - 1) \tag{4}$$

Here, $s_3(t,T)$ is a fictitious process, playing the role of the instantaneous forward 3M Ibor spread curve over the ON curve. The specification can be motivated and justified theoretically by a simple credit-based argument. Let $P^*(t,T)$ denote the value of a hypothetical bond for a representative Ibor panel bank. For simplicity, we assume zero recovery and a deterministic hazard process $\lambda(t)$. It is straightforward to demonstrate under these conditions that the 3M Ibor would be equal to:

$$\frac{1}{\tau_3}\mathbb{E}_t^{T+\tau_3}\bigg[\frac{1}{P^*(T,T+\tau_3)}-1\bigg] = \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)+\lambda(u)\,\mathrm{d}u}-1) \tag{5}$$

which resembles (4) with $s_3(t,u)$ replaced by $\lambda(u)$. We note also that liquidity factors could easily be worked into $P^*(t,T)$ with similar effect.

The specification in (4) may be viewed as a special case of the 'multiplicative' specification:

$$R_3(t,T) = \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)\,\mathrm{d}u} S_3^{\times}(t,T) - 1) \tag{6}$$

as has been popularised by Henrard (2013) and others; here, $S_3^{\times}(t,T)$ is a fictitious multiplicative discrete-tenor spread factor. The specification in (6) can be recast to that in (4) by setting:

$$S_3^{\times}(t,T) = \mathrm{e}^{\int_T^{T+\tau_3} s_3(t,u)\,\mathrm{d}u} \tag{7}$$

The advantage of working with models of $s_3(t,T)$ is that the behaviour of the observed tenor basis swap spread curve will roughly mimic that of $s_3(t,T)$. Indeed, a crude approximation of $\mathrm{d}R_3(t,T)$ is simply $\mathrm{d}f_0(t,T)+\mathrm{d}s_3(t,T)$, which follows from a simple Taylor expansion. Behaviours inherited from $s_3(t,T)$ include level dependence in volatility and lower bounds, which are of central interest to our objectives. We note that it is possible to control these aspects of spread behaviour by manipulating the diffusion function of $S_3^{\times}(t,T)$, but with far less transparency than in the case of $s_3(T,t)$.[4]

[4] *Eberlein et al (2019) use a lognormal model of $S_3^{\times}(t,T)$ with a shift of one to enforce lower bounds at zero. Shifts allow for control over the lower bound, but controlling level dependence requires more, eg, a constant elasticity of variance (CEV) specification.*

Moreover, extending the setup in (4) to a general multi-spread setting (ie, for 6M Ibors) is much simpler than doing so in (6), as discussed momentarily.

Another intuitive setup is the 'additive' specification of Mercurio (2010), namely:

$$R_3(t,T) = R_{0,3}(t,T) + S_3^{+}(t,T) \tag{8}$$

where $R_{0,3}(t,T)$ is the single-curve 3M Ibor off the ON curve in (3), and where $S_3^{+}(t,T)$ is a fictitious additive discrete-tenor spread. We feel that modelling $s_3(t,T)$ retains the best of both worlds. It is as intuitive as the additive approach in that we simply add $s_3(t,T)$ to $f_0(t,T)$ and evaluate a standard Ibor formula as per (4), and we have the strong credit-based justification for the specification in (4) presented earlier, which is not available for the additive specification in (8).

Curve dynamics and drift restrictions. We model $f_0(t,T)$ and $s_3(t,T)$ as each having separable Heath-Jarrow-Morton (HJM) (Heath *et al* 1992) dynamics. This allows us to adapt the insights of Cheyette (1992), who explored such dynamics in the single-curve context. To ease exposition, we use a simple specification wherein each curve has a loading onto a dedicated factor such that the curves are uncorrelated, and where each factor loading involves a single exponential. These assumptions are without any loss of generality. General factor structures allowing for non-zero correlations and composite factor loadings are treated fully in Konikov & McClelland (2019).

Our system evolves under the spot measure as:

$$\left.\begin{aligned} \mathrm{d}f_0(t,T) &= \mu_0(t,T)\,\mathrm{d}t + \psi_0(t)\mathrm{e}^{-\kappa_0(T-t)}\,\mathrm{d}W_0(t) \\ \mathrm{d}s_3(t,T) &= \mu_3(t,T)\,\mathrm{d}t + \psi_3(t)\mathrm{e}^{-\kappa_3(T-t)}\,\mathrm{d}W_3(t) \end{aligned}\right\} \tag{9}$$

with $\mathrm{d}W_0(t)\,\mathrm{d}W_3(t) = 0\,\mathrm{d}t$. In order to enforce lower bounds on $s_3(t,T)$, we will adopt a diffusion function of the CEV type:

$$\psi_3(t) = \nu_3(t)(s_3(t,t) - l_3(t))^{\beta} \tag{10}$$

where $s_3(t,t)$ is the 'short spread', and where $l_3(t)$ governs lower bounds. The use of the CEV specification is well understood in the single-curve context, so we do not elaborate upon it here. We note only that when $s_3(t,t) = l_3(t)$, ie, when the short spread attains its lower bound, the curve $s_3(t,T)$ is unable to diffuse to lower values. For the forthcoming analysis, we do not impose (10), leaving $\psi_0(t)$ and $\psi_3(t)$ as general state-dependent quantities so as to obtain general results. We will also use the notation $\sigma_0(t,T) = \psi_0(t)\mathrm{e}^{-\kappa_0(T-t)}$ and $\sigma_3(t,T) = \psi_3(t)\mathrm{e}^{-\kappa_3(T-t)}$ where convenient.

Models such as (9) were proposed in Grbac & Runggaldier (2015), Miglietta (2015) and Martinez (2009). See also Geelhar *et al* (2017), who extend such models for stochastic volatility. The typical no-arbitrage restriction applies for $\mu_0(t,T)$:

$$\int_t^T \mu_0(t,u)\,\mathrm{d}u = \frac{1}{2}\bigg(\int_t^T \sigma_0(t,u)\,\mathrm{d}u\bigg)^2 \tag{11}$$

which upon differentiation against T yields the no-arbitrage drift function, as below.

Base curve drift restriction: conclusive.

$$\mu_0(t,T) = \bigg(\int_t^T \sigma_0(t,u)\,\mathrm{d}u\bigg)\sigma_0(t,T) \tag{12}$$

An analogous no-arbitrage restriction can be derived for $\mu_3(t,T)$. To this end, consider a contract paying $R_3(T,T)\tau_3$ at $T+\tau_3$, ie, a standard 3M Ibor payment. Denoting its value by $V_3(t,T)$, we clearly have:

$$V_3(t,T) = P(t,T+\tau_3)R_3(t,T)\tau_3 \\ = e^{-\int_t^T f_0(t,u)\,du + \int_T^{T+\tau_3} s_3(t,u)\,du} - e^{-\int_t^{T+\tau_3} f_0(t,u)\,du} \tag{13}$$

which must grow on average at the risk-free rate under the spot measure. Taking the differential of (13) and enforcing the restriction that $\mathbb{E}_t[dV_3(t,T)] = r(t)V_3(t,T)\,dt$ gives rise to the drift restriction:

$$\int_T^{T+\tau_3} \mu_3(t,u)\,du = -\frac{1}{2}\bigg(\int_T^{T+\tau_3} \sigma_3(t,u)\,du\bigg)^2 \tag{14}$$

We note that under general conditions where $s_3(t,T)$ and $f_0(t,T)$ are correlated, covariance terms will appear in this restriction. Differentiating (14) against T yields the following necessary condition.

■ **Spread curve drift restriction: inconclusive.**

$$\mu_3(t,T+\tau_3) - \mu_3(t,T) \\ = -(\sigma_3(t,T+\tau_3) - \sigma_3(t,T))\bigg(\int_T^{T+\tau_3} \sigma_3(t,u)\,du\bigg) \tag{15}$$

This is a difference equation in $\mu_3(t,T)$. It stands in contrast to the restriction on $\mu_0(t,T)$ in (11), which resolves the nature of $\mu_0(t,T)$ conclusively. Grbac & Runggaldier (2015), Miglietta (2015) and Martinez (2009) encountered the restriction in (15), but they did not obtain a solution to it for $\mu_3(t,T)$. In spite of this, Grbac & Runggaldier (2015) were able to make progress in characterising Ibor dynamics. This was possible as they adopted a Gaussian specification, $\psi_3(t) = \nu_3(t)$, for a deterministic parameter curve $\nu_3(t)$. To appreciate the importance of this, note that all we require to describe 3M Ibors as per (4) is the quantity $\int_T^{T+\tau_3} s_3(t,u)\,du$, which evaluates to:

$$\int_T^{T+\tau_3} s_3(t,u)\,du \\ = \int_T^{T+\tau_3} s_3(0,u)\,du + \int_0^t \int_T^{T+\tau_3} \mu_3(v,u)\,du\,dv \\ + \int_0^t \int_T^{T+\tau_3} \sigma_3(v,u)\,du\,dW_3(v) \\ = \int_T^{T+\tau_3} s_3(0,u)\,du - \frac{1}{2}\int_0^t \bigg(\int_T^{T+\tau_3} \sigma_3(v,u)\,du\bigg)^2 dv \\ + \int_0^t \int_T^{T+\tau_3} \sigma_3(v,u)\,du\,dW_3(v) \tag{16}$$

with the use of (14). It is thus clear that the integrated spread has no direct dependence upon $\mu_3(t,T)$. However, for state-dependent $\sigma_3(t,T)$ such as our CEV specification, $\nu_3(t)(s_3(t,t) - l_3(t))^\beta$, the quantity $s_3(t,t)$ is present, and this does depend on $\mu_3(t,T)$. It is only in the special case of a state-independent $\sigma_3(t,T)$ that $s_3(t,t)$ is not present and there is no need for $\mu_3(t,T)$.

To obtain a solution for $\mu_3(t,T)$, we return to (14). As was also noted by the earlier authors, closer inspection reveals some flexibility in choosing a solution. Indeed, we observe that $\mu_3(t,T)$ enters under an integral, and the region of integration is the Ibor coverage period, which slides with the fixing date T. Thus, we have a restriction on the behaviour of $\mu_3(t,T)$ *over* an Ibor coverage period, but not *within* such a period. To understand this, note that constructing an Ibor could have been achieved with the discrete-tenor multiplier $S_3^\times(t,T)$. Introducing the infinitesimal tenor $s_3(t,T)$ affords modelling advantages, but introducing an integral was ultimately artificial; we discuss alternative specifications momentarily. An immediate consequence of this is flexibility in specifying $s_3(t,T)$, eg, we could initialise it with any $s_3(0,T)$ agreeing with $S_3^\times(0,T)$, though in practice $s_3(0,T)$ is of course determined by a financially meaningful interpolation scheme. Flexibility in the drift $\mu_3(t,T)$ derives from the same source as flexibility in $s_3(t,T)$.

For the separable volatility structure $\sigma_3(t,T) = \psi_3(t)e^{-\kappa_3(T-t)}$, there is in fact a solution for $\mu_3(t,T)$ that is 'natural' in the sense that it satisfies (14) and requires no arbitrary parameterisation. To obtain this solution, substitute the volatility structure into (15) for:

$$\mu_3(t,T+\tau_3) - \mu_3(t,T) = \psi_3(t)^2 \frac{(e^{-\kappa_3\tau_3} - 1)^2}{\kappa_3} e^{-2\kappa_3(T-t)} \tag{17}$$

The form of (17) motivates an ansatz of the type:

$$\mu_3(t,T) = \psi_3(t)^2 A_3 e^{-2\kappa_3(T-t)}$$

Substituting this into (17) and solving for A_3 yields the solution for $\mu_3(t,T)$:

$$\mu_3(t,T) = \psi_3(t)^2 \frac{(e^{-\kappa_3\tau_3} - 1)^2}{\kappa_3(e^{-2\kappa_3\tau_3} - 1)} e^{-2\kappa_3(T-t)} \tag{18}$$

which can be confirmed to satisfy the sufficient condition in (14). Importantly, this solution is separable in t and T, and thus a representation of $s_3(t,T)$ in terms of Markov states is available.

We can readily extend (4) to produce a 6M Ibor via the introduction of $s_6(t,T)$, which captures the credit and liquidity factors driving the basis between the 3M Ibor and 6M Ibor curves:

$$R_6(t,T) = \frac{1}{\tau_6}(e^{\int_T^{T+\tau_6} f_0(t,u)+s_3(t,u)+s_6(t,u)\,du} - 1) \tag{19}$$

Note that we integrate $s_6(t,T)$ over an interval of length τ_6 along with both $f_0(t,T)$ and $s_3(t,T)$. This enforces that $(1+\tau_6 R_6(t,T)) \geqslant (1+\tau_3 R_3(t,T))(1+\tau_3 R_3(t,T+\tau_3))$, or rather that a forward 6M Ibor is no less than the 'average' of the two adjacent forward 3M Ibors spanning its coverage period. Further discussion of multiple spread curves and the derivation of drift functions analogous to (18), ie, for $\mu_6(t,T)$, are provided in Konikov & McClelland (2019).

■ **An alternative specification and a related specification.** We consider here an alternative to the formulation in (4). We have adopted this formulation, wherein the instantaneous spread curve $s_3(t,T)$ is modelled, as it has already been developed within the literature and adopted by others for practical applications. However, with minimal effort we could move to a formulation wherein one models the mean spread curve:

$$\bar{s}_3(t,T) = \frac{1}{\tau_3}\int_T^{T+\tau_3} s_3(t,u)\,du \tag{20}$$

which is akin to a discrete-tenor spread. Modelling $\bar{s}_3(t,T)$ with the same dynamics as those prescribed for $s_3(t,T)$, ie, with $\bar{s}_3(t,t)$ appearing in the diffusion function, is an equivalent way of obtaining control over level dependence in spreads and enforcing lower bounds.

In most practical settings, the two approaches will lead to near-identical implementations in terms of Markov representations, approximate swaption prices, etc. As such, one may be indifferent to which is adopted. Models of $s_3(t,T)$ are slightly more intuitive and accessible, as we are simply analogising instantaneous forward rates $f_0(t,T)$. Moreover, extensions to the many-curve case as per (19) are straightforward, as we are simply integrating instantaneous spreads over common intervals. Conversely, the drift function of $\bar{s}_3(t,T)$ is entirely straightforward, and while we are able to solve for the drift function of $s_3(t,T)$, some may prefer to avoid the issue entirely.

In more advanced settings involving, eg, jump discontinuities or time-varying reversion coefficients, recasting from $s_3(t,T)$ to $\bar{s}_3(t,T)$ offers more tangible advantages, as complications arising from the restriction on the drift of $s_3(t,T)$ become more pronounced. These issues are discussed in greater detail in Konikov & McClelland (2019), where the formulation in (20) is developed further.

A related specification to that considered here appears in Cuchiero *et al* (2016) and Fontana *et al* (2019). These authors also work within the multiplicative framework and re-cast in terms of an instantaneous spread:

$$S_3^{\times}(t,T) = S_3^{\times}(t,t)\mathrm{e}^{\int_t^T s_3'(t,u)\,\mathrm{d}u} \tag{21}$$

where we have taken some liberties with notation. One may easily verify that there is an equivalence between our $s_3(t,T)$ and their $s_3'(t,T)$:

$$s_3'(t,T) = s_3(t,T+\tau_3) - s_3(t,T) \tag{22}$$

or rather that $s_3'(t,T)$ is a difference in the curve $s_3(t,T)$ at a distance of τ_3. This mirrors the drift restriction in (15), and thus the drift of $s_3'(t,T)$ may be recovered straightforwardly. However, we would argue that $s_3(t,T)$ is a much more intuitive quantity than $s_3'(t,T)$, eg, it is directly analogous to $f_0(t,T)$. Moreover, the resulting form of Ibors is standard for $s_3(t,T)$, as in (4), while for $s_3'(t,T)$ it is somewhat exotic:

$$\begin{aligned}\frac{1}{\tau_3}&(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)+s_3(t,u)\,\mathrm{d}u} - 1)\\ &= \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_3(t,u)\,\mathrm{d}u} - 1) \text{ versus}\\ &\qquad \frac{1}{\tau_3}(\mathrm{e}^{\int_T^{T+\tau_3} f_0(t,u)\,\mathrm{d}u + \int_t^T s_3'(t,u)\,\mathrm{d}u} - 1)\end{aligned} \tag{23}$$

We provide a more complete literature review in Konikov & McClelland (2019).

Markov state processes, swap rates and swaption pricing. We present the Markov representation of $s_3(t,T)$ here for reference:[5]

$$\begin{aligned}s_3(t,T) &= s_3(0,T) + \mathrm{e}^{-\kappa_3(T-t)}X_3(t)\\ &\quad + A_3\mathrm{e}^{-\kappa_3(T-t)}(\mathrm{e}^{-\kappa_3(T-t)}-1)Y_3(t)\end{aligned} \tag{24}$$

$$\left.\begin{aligned}\mathrm{d}X_3(t) &= \kappa_3\left(\frac{A_3}{\kappa_3}\psi_3(t)^2 - A_3Y_3(t) - X_3(t)\right)\mathrm{d}t + \psi_3(t)\,\mathrm{d}W_3(t)\\ \mathrm{d}Y_3(t) &= 2\kappa_3\left(\frac{1}{2\kappa_3}\psi_3(t)^2 - Y_3(t)\right)\mathrm{d}t\end{aligned}\right\}$$

$$A_3 = \frac{(\mathrm{e}^{-\kappa_3\tau_3}-1)^2}{\kappa_3(\mathrm{e}^{-2\kappa_3\tau_3}-1)} \tag{25}$$

In Konikov & McClelland (2019) there is a full derivation of this result, and there is a full derivation for the general framework with multiple spread curves, multiple factors and composite factor loadings. One interesting feature of this representation is that there is a convexity term in the drift of $X_3(t)$, namely $A_3\psi_3(t)^2$. For models of $f_0(t,T)$, such a term does not feature. Its presence here owes to the fact that $\mu_3(t,t) \neq 0$, which contrasts with $\mu_0(t,t) = 0$.

We briefly present approximate swap rate dynamics, and we discuss how approximate basis swap spreads and swaption prices can be obtained by extension; derivations and results for the general case are provided in Konikov & McClelland (2019). Consider a forward-starting 3M Ibor-versus-fixed swap, with fixing/payment dates of $T_i \in \{T_0, T_1, \ldots\}$. The 3M Ibor fixes at T_i and exchanges against a fixed rate at T_{i+1}, with $T_{i+1} - T_i = \tau_{3,i}$. The at-the-money (ATM) swap rate is:

$$\omega_3(t;T_0) = \sum_i \frac{P(t,T_{i+1})R_3(t,T_i)\tau_{3,i}}{L(t;T_0)}$$

$$\text{where } L(t;T_0) = \sum_j P(t,T_{j+1})\tau_{3,i} \tag{26}$$

The well-known 'frozen coefficients' technique for deriving approximate (annuity-measure) swap rate dynamics can be easily adapted for our modelling framework. For the model in (9), one obtains:

$$\begin{aligned}\mathrm{d}\omega_3(t;T_0) &\approx \gamma_0(T_0)\psi_0(t)\mathrm{e}^{-\kappa_0(T_0-t)}\,\mathrm{d}\bar{W}_0^{T_0}(t)\\ &\quad + \gamma_3(T_0)\psi_3(t)\mathrm{e}^{-\kappa_3(T_0-t)}\,\mathrm{d}\bar{W}_3^{T_0}(t)\end{aligned} \tag{27}$$

where the loadings $\gamma_0(T_0)$ and $\gamma_3(T_0)$ are functions of $P(0,T_i)$, $R_3(0,T_i)$ and $L(0;T_0)$. One can also derive approximate dynamics for tenor basis swap spreads, which have the same structure as those in (27), but the loading onto $\mathrm{d}\bar{W}_0^{T_0}(t)$ is of course far smaller for spreads.

In general, pricing swaptions under (27) requires numerical techniques, complicating calibration. For affine specifications, however, eg, $\psi_0(t) = \nu_0(t)$ and $\psi_3(t) = \nu_3(t)(s_3(0,t) + X_3(t) - l_3(t))^{1/2}$, the vector process $[\omega_3(t;T_0), X_3(t), Y_3(t)]$ is affine and amenable to standard Fourier techniques.[6] Note that this is a general result and holds for models of multiple spread curves and multiple factors, provided that the joint diffusion function is affine in the relevant short-rate and short-spread factors. For such models, it is possible to approximate the volatilities of swap rates and basis swap spreads in closed form; this is useful to us, as these quantities are used towards calibration.

Calibration

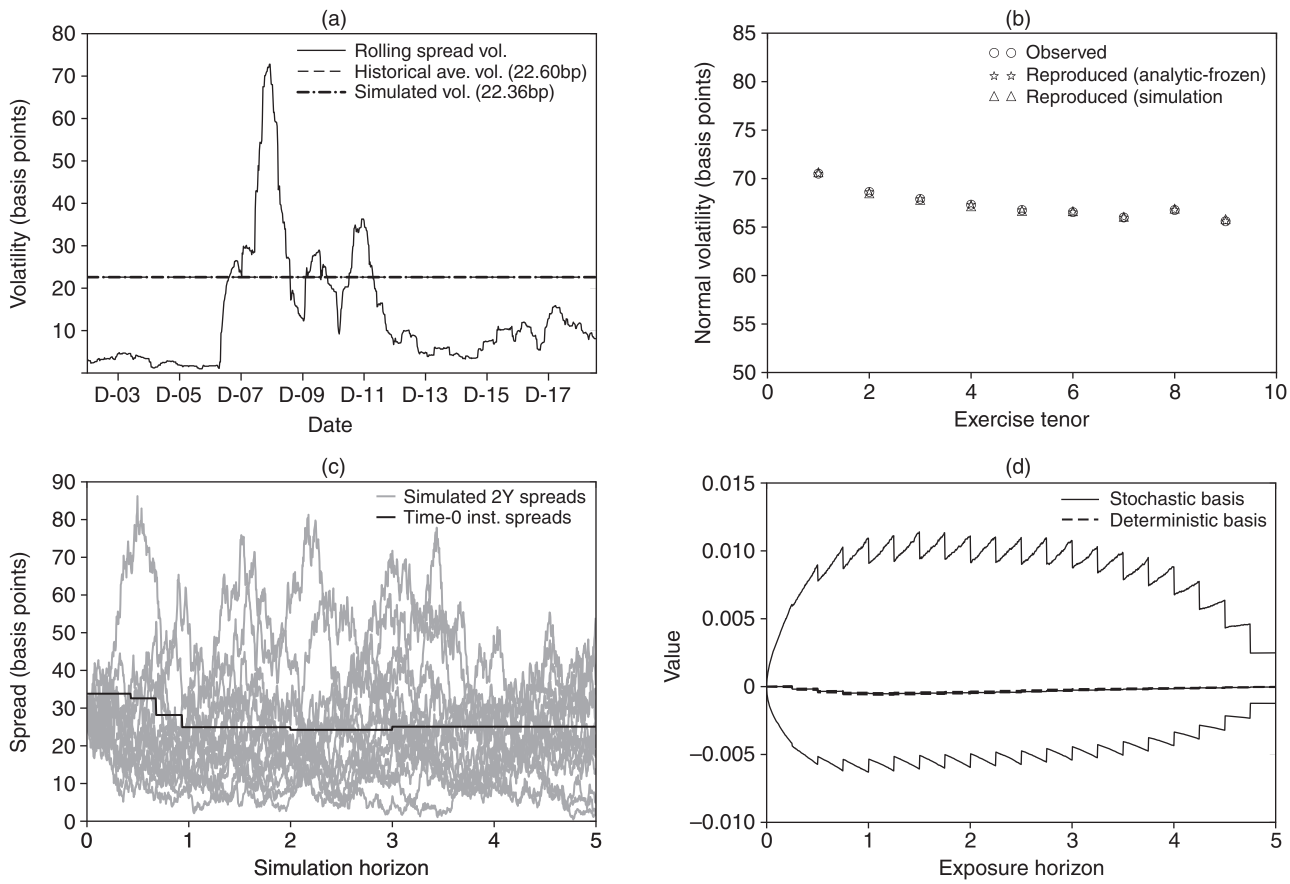
Historical versus implied data. The use of historical data has already been considered by Henrard (2018), but this was not the central subject of his article, and there remains much room for development. Of particular benefit would be further motivation and discussion of the benefits of such an approach vis-á-vis relying solely on swaption volatilities, and also a formalisation of calibration strategies.

When discussing swaption (or capfloor) volatilities, we assume for simplicity that swaptions reference 3M Ibor-paying underlyings. In our setup, model-based swaption volatilities will be sensitive to both the ON volatility, $\nu_0(t)$, and the ON-versus-3M Ibor spread volatility, $\nu_3(t)$. This is clear from (27), wherein swap rate innovations have a $\mathrm{d}\bar{W}_3^{T_0}(t)$ contribution. As such, it is formally impossible to calibrate $\nu_0(t)$ from swaption volatilities

[5] *Setting $Y_3'(t) = -A_3Y_3(t)$ would be more in line with the classical Cheyette (1992) approach. We separate $-A_3(t)$ so that $Y_3(t)$ can be reused in the representation for $s_6(t,T)$, etc, in the general case of multiple spread curves.*

[6] *A Feller condition can be derived for the case where $\beta = \frac{1}{2}$, as is presented in Konikov & McClelland (2019).*

1 Numerical tests visualisation



(a) Historical rolling ON-versus-3M Ibor tenor basis spread volatilities (2Y tenor) with the average volatility and the fitted volatility. (b) Calibrated model's fit to observed swaption volatilities, for both the analytic frozen-coefficient approximation and a full simulation under the true dynamics. (c) Simulation of ON-versus-3M Ibor basis swap spreads (2Y tenor) with the initial bootstrapped instantaneous spread curve ($s_3(0,T)$) for reference. (d) Exposure percentiles for a unit-notional ON-versus-3M Ibor tenor basis swap (5Y tenor) with zero basis volatility ($\nu_3 = 0$) profiles for reference

without reference to $\nu_3(t)$. The latter must be accounted for when calibrating the former. If $\nu_3(t)$ is ignored (set to zero) during calibration of $\nu_0(t)$ and reintroduced afterwards, the fit of the model to swaption volatilities will clearly be affected. More broadly, this would overstate the ON curve volatility, affecting things such as the calculation of ON futures convexity, CVA for overnight index swaps, long-dated equity trades where discount-curve convexity is material, and so on.

Many authors calibrate multi-curve models to swaption volatilities in isolation, but we feel that for CVA purposes we are far better served by incorporating historical data. Indeed, under ideal settings the calibrated model will reproduce historically observed spread behaviour. To understand our position, consider a simple set of ATM co-terminal implied volatilities. It is known that with a deterministic $s_3(t,T)$, or rather just $s_3(T)$, we have sufficient information in this set to calibrate $\nu_0(t)$ at a given set of interpolation knots. Now consider volatilities for swaptions with the same tenor structure but struck at ATM $\pm$ 0.5%. Technically, these additional implied volatilities will render the volatility of a stochastic $s_3(t,T)$, namely $\nu_3(t)$, identifiable because $\nu_3(t)$ will affect the skew under CEV dynamics.[7]

However, as $f_0(t,T)$ is Gaussian and produces almost no skew, this would be akin to assuming the factors driving observed skew in swaption volatilities, ie, level dependence and stochastic volatility in the 3M Ibor curve, are owed entirely to the behaviour of the ON-versus-3M Ibor spread curve. This is refuted empirically, and thus such a calibration is unlikely to be consistent with historically observed ON-versus-3M Ibor spreads. Of course, if we had supreme confidence in our model specification, we could possibly calibrate to swaption volatilities alone, but in practical CVA settings this is rarely true. Note also that the use of historical data is essentially unavoidable for the volatilities of other tenor basis spreads. Indeed, the 3M Ibor-versus-6M Ibor spread curve has no bearing upon swaptions over 3M Ibor-versus-fixed swaps.

Criterion. A method to incorporate historical data into our calibration criteria is now presented. A minimal setup is employed in order to convey the general concept. We assume the quantities to be calibrated are $\{\nu_0(t), \nu_3\}$. The rationale for a constant ν_3 is that it will be inferred from the historical basis swap spread volatility, ie, a single number as opposed to a curve. This could be relaxed if we had a forecast of basis spread volatility,

[7] *Although in Konikov & McClelland (2019) we demonstrate that for values of $\nu_3(t)$ (or ν_3) that reproduce historical spread volatility, the skew produced is minimal relative to that typically observed in markets.*

eg, via exponentially weighted moving average (EWMA). We only assume that $\{\kappa_0, \kappa_3\}$ are known in order to simplify the presentation; in practice, these could be calibrated to (discrete-horizon) correlations between swap rates across tenors and between basis swap spreads across tenors, respectively, requiring only a mild extension of the calibration criteria. Similarly, we set the CEV coefficient to $\beta = \frac{1}{2}$, but it can of course be calibrated by incorporating other (higher-order) historical moments of basis swap spreads into the calibration criteria.

We require sufficient information to identify $\{\nu_0(t), \nu_3\}$. To this end, we select an ON-versus-3M basis swap spread of a given tenor τ^{bs}, eg, 2Y, and compute its historical volatility $\hat{\varphi}_{0,3}(\tau^{\mathrm{bs}})$. In general, one could choose a set of such basis swaps with varying tenors and attempt a bestfit, but we keep matters simple here. We also select a set of observed swaption volatilities, $\hat{\upsilon}_3(\tau_i^{\mathrm{ex}}, \tau_i^{\mathrm{sw}}, k_i)$, where τ_i^{ex} is the exercise tenor, τ_i^{sw} is the underlying tenor and k_i is the moneyness. Let $\varphi_{0,3}(\tau^{\mathrm{bs}})$ and $\upsilon_3(\tau_i^{\mathrm{ex}}, \tau_i^{\mathrm{sw}}, k_i)$ denote the model-based analogues of our calibration targets.

The primary role of ν_3 will of course be to match the historical basis swap spread volatility, and the primary role of ν_0 will be to bestfit the swaption volatilities. We reiterate, however, that this is a genuine joint calibration in that ν_3 will contribute to model-based swaption volatilities and $\nu_0(t)$ will contribute (mildly) to the model-based basis swap spread volatility. This being the case, the simple calibration strategy described here can be formalised as:

$$\begin{aligned} \{\nu_0^*(t), \nu_3^*\} = \underset{\{\nu_0(t), \nu_3\}}{\arg\min} \sum_i w_i (\hat{\upsilon}_3(\tau_i^{\mathrm{ex}}, \tau_i^{\mathrm{sw}}, k_i) \\ - \upsilon_3(\tau_i^{\mathrm{ex}}, \tau_i^{\mathrm{sw}}, k_i; \nu_0(t), \nu_3))^2 \\ \text{subject to } \hat{\varphi}_{0,3}(\tau^{\mathrm{bs}}) = \varphi_{0,3}(\tau^{\mathrm{bs}}; \nu_0(t), \nu_3) \qquad (28) \end{aligned}$$

where w_i are weights. One may then fit a model for 3M Ibor-versus-6M Ibor spreads with reference to their historical volatility, ie, by solving:

$$\hat{\varphi}_{3,6}(\tau^{\mathrm{bs}}) = \varphi_{3,6}(\tau^{\mathrm{bs}}; \nu_0^*(t), \nu_3^*, \nu_6)$$

for ν_6, and so on.

A worked example. We calibrate the model in (9) with a Gaussian $f_0(t, T)$ and a square-root $s_3(t, T)$. Our historical window for basis swap spreads spans October 11, 2002 through October 11, 2019. (The data used for this analysis was sourced from Bloomberg.) We present rolling volatilities for the spreads of 2Y tenor basis swaps in part (a) of figure 1. As can be seen, volatility has only been significant since 2007, and thus the historical sample for estimation begins there; the average volatility of the spread over this period was 22.60 basis points. We use a market snapshot from October 11, 2019 to collect swaption volatilities and interest rate quotes for a complete calibration set.

Our calibrated model successfully reproduces our targeted quantities. Indeed, the simulated basis swap spread volatility was 22.36bp, and market volatilities were easily fit to. Figure 1(b) presents observed normal volatilities for 10Y co-terminal swaptions along with volatilities computed off the calibrated model. The analytical approximations discussed earlier were used for calibration, and thus they agree with market volatilities precisely. The approximation, however, suffers from inaccuracy owing to the freezing of swap rate coefficients, and thus we also present 'true' model-based volatilities produced via simulation; the worst error is roughly 0.1bp.

Figure 1(c) presents simulated paths of the 2Y tenor basis swap spread. As desired, spreads generated by the model remained non-negative. Finally, in figure 1(d) we present exposure profiles (1%, mean, 99%) for a 5Y tenor basis swap with unit notional (paying the ON leg). We leave to future research a more thorough analysis of which specification, in terms of the number of factors, CEV coefficients and correlations structures, offers the best agreement with historical behaviour, and which is most effective under a backtest of CVA hedging. ■

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