Investigation of the effect of using stochastic and local volatility when pricing barrier options

Mert Camlibel and Johan Lundgren

December 13, 2010

Abstract

The main purpose of this thesis is to price barrier options using local and stochastic volatility and compare the results to option prices retrieved using Merton, Reiner and Rubinstein's analytical expressions for barrier options. For the stochastic volatility we use the SABR model and to implement the local volatility we use Tikhonov regularization and the Crank-Nicholson scheme. The models are calibrated to European call options based on the S&P 500 index from October 1995. Our results show that the barrier option prices under local and stochastic volatility coincide. However, we find that these prices differ significantly from the analytical prices for barrier options with barrier levels far away from the spot price.

Acknowledgements

We would like to thank Magnus Lundin, Erik Svensson and Stefan Toll at Handelsbanken and Robert Thorén at Algorithmica AB for guidance during this thesis. We are also thankful for Professor Boualem Djehiche at KTH who was our tutor.

Contents

1 Introduction **5**

2 Theory **7**
2.1 Notation 7
2.2 Local volatility 7
2.3 Stochastic volatility 9
2.4 Monte Carlo simulation 10
2.5 Barrier options 10

3 Data **13**

4 Calibration **15**
4.1 Dupire's formula 15
4.2 Local volatility - Tikhonov regularization 15
4.3 Local volatility - the Crank-Nicholson scheme 18
4.4 Stochastic volatility - SABR 19

5 Calibration results **21**
5.1 Local volatility 21
5.2 Stochastic volatility 27

6 Pricing barrier options **35**
6.1 Preliminary conditions 35
6.2 Barrier option prices - tables 36
6.3 Barrier option prices - graphs 52

7 Analysis and Discussion **54**

8 Conclusions **55**

9 References **56**

1 Introduction

In the Black & Scholes formula it is assumed that the volatility is constant. However, using historical option prices one can uniquely solve for the implied volatility. The implied volatility can then be used in the Black & Scholes formula to retrieve the market prices. In practice, options with the same underlying asset but different strikes/maturities require different implied volatilities. This is inconsistent since the implied volatility should not depend on the specifications of the contract. Thus the implied volatility is a function of the strike price. The plot of the implied volatility against the strike price is often refered to as volatility smile or market skew.

A way to handle these skews is to use local volatility models developed by Dupire. In local volatility models it is assumed that the volatility depends on the current stock price and time. The idea is to calibrate the local volatility model to market prices of liquid European options. In other words the local volatility function is varied until the theoretical prices match the actual market prices of the option. When the calibration is finished, the model is fit to correctly reproduce all the market prices of options for all strikes and maturities. Thus the local volatility model provides a method of pricing options in the presence of market skews.

One can also introduce a stochastic process for the volatility itself. In that case the volatility process is neither constant (as in Black & Scholes) nor deterministic (as in local volatility models). These models are called stochastic volatility models. One particular stochastic volatility model is the SABR model (derived by Hagan) where the asset price and the volatility are correlated. The SABR model is handy because there exists a closed form approximation for the implied volatility. The main reason for choosing the SABR model in this thesis is due to the fact that it is widely used in the financial industry.

The advantage of using local and stochastic volatility is that it provides us with additional information compared to using the implied volatility. The implied volatility skew tells us information about the distribution of the underlying asset at maturity. This is satisfactory when working with vanilla options but to price path dependent options (i.e. barrier options) it is also neccessary to know whether the path of the underlying asset has hit the option barrier.

Both local and stochastic volatility approaches have their pitfalls. The dynamic behavior of skews predicted by local volatility models is the opposite of the behavior observed in the market: when the price of the underlying asset decreases the model suggests that the smile shifts to higher prices and vice versa. Stochastic volatility on the other hand, introduces another source of randomness (i.e. the volatility of the volatility) which leads to an incomplete market meaning that hedging using the underlying asset and the risk free interest rate is not sufficient.

The purpose of this study is to evaluate the effect of using stochastic volatility (SV) and local volatility (LV) versus using constant volatility (i.e. using the implied volatility (IV) surface) when pricing barrier options. The SV model to be used is the SABR model and for the LV we will use Dupire's local volatility formula (using Tikhonov regularization and the Crank-Nicholson method). To simulate the prices we will use Monte Carlo simulation. To price the barrier options with constant volatility we will use Merton, Reiner and Rubinstein's developed formulas for pricing standard barrier options.

To start with we will calibrate our SV model (SABR) to market data (S&P 500 european call options from October 1995). Then we will recreate the market prices under SV through Monte Carlo simulations. Lastly we will price barrier options under SV based on the same market data but for different barrier levels. The LV calibration will be similiar to the SV calibration. First we calibrate our LV model to market data, then we recreate market prices under LV through Monte Carlo simulations. Finally we will price barrier options under LV based on the same market data but for different barrier levels.
After both calibrations are done we will calculate analytical barrier option prices based on the same market data as above for different barrier levels (the implied volatility surface will be used as input for the volatility). Even though we calibrate the SV and LV models for a variety of strike prices, we find it more appealing to focus on one single strike price (while varying the maturities and barrier levels) when pricing the actual barrier option prices.
The last step will be to compare the analytical barrier option prices with SV and LV barrier option prices. All implementation is done using Matlab.

2 Theory

2.1 Notation

Throughout the paper following notation will be used

S - Spot price
F - Forward price
K - Strike price
Hu - Upper barrier level
Hd - Lower barrier level
R - Rebate
T - Time to maturity
c - Price of an European call option
r - Risk free interest rate
D - Dividend
b - Cost of carry rate (i.e. r-D)
$N(x)$ - Cumulative normal distribution function
N - Number of simulations
nT - Number of time steps

2.2 Local volatility

Unlike the traditional Black & Scholes model, where the volatility is assumed to be constant, it is assumed that under the local volatility model the volatility depends on the underlying stock price and the time. The developer of local volatility was Bruno Dupire and the main idea with the model is to calibrate it to fit the market prices of liquid European options so that the model fits the actual market prices for all maturities and strikes for the option. When the calibration is done it is possible to reproduce all the option prices in the market for every maturity and strike. It can then be used to price path dependent options.

The new dynamics under local volatility then becomes

$$\frac{dS}{S} = rdt + \sigma_L(S,t)dW_t \tag{2.1}$$

The unique local volatility function is the solution to Dupire's equation

$$\frac{\partial C}{\partial T} = \frac{\sigma_L^2 K^2}{2}\frac{\delta^2 C}{\delta K^2} + (r_t - D_t)\left(C - K\frac{\delta C}{\delta K}\right) \tag{2.2}$$

where D_t is the dividend yield and C is the European option price $C(S_0, K, T)$ (see [1]).

Proof (from [2]).

Suppose the stock price diffuses with a risk-neutral drift $\mu_t = r_t - D_t$ and local volatility $\sigma_L(S,t)$ according to the equation

$$\frac{dS}{S} = \mu_t dt + \sigma_L(S_t,t)dW. \tag{2.3}$$

The undiscounted risk-neutral value $C(S_0,K,T)$ of a European option with strike K and expiration T is given by

$$C(S_0,K,T) = \int_K^\infty \varphi(S_T,T;S_0)(S_T - K)dS_T, \tag{2.4}$$

where $\varphi(S_T,T;S_0)$ is the probability density of the final spot at time T. It evolves according to the Flokker-Planck equation

$$\frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_L^2 S_T^2 \varphi) - \frac{\varphi}{\partial S_T}(\mu S_T \varphi) = \frac{\partial \varphi}{\partial T}. \tag{2.5}$$

Differentiating (2.4) with respect to K gives

$$\frac{\partial C}{\partial K} = -\int_K^\infty \varphi(S_T,T;S_0)dS_T, \tag{2.6}$$

$$\frac{\partial^2 C}{\partial K^2} = \varphi(K,T;S_0). \tag{2.7}$$

Now, differentiating (2.4) with respect to time T gives

$$\frac{\partial C}{\partial T} = \int_K^\infty \left\{ \frac{\partial}{\partial T}\varphi(S_T,T;S_0) \right\} (S_T - K)dS_T. \tag{2.8}$$

By using (2.5) in (2.8) we get

$$\frac{\partial C}{\partial T} = \int_K^\infty \left\{ \frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma_L^2 S_T^2 \varphi\right\} - \frac{\partial}{\partial S_T}(\mu S_T \varphi) \right\} (S_T - K)dS_T. \tag{2.9}$$

Integrating the first term of (2.9) by parts, $\int_a^b fg'\,dx = [fg]_a^b - \int_a^b f'g\,dx$, gives

$$\int_K^\infty \left\{ \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_L^2 S_T^2 \varphi) \right\} (S_T - K)dS_T =$$
$$\left[\frac{1}{2}\frac{\partial}{\partial S_T}(\sigma_L^2 S_T \varphi)(S_T - K) \right]_K^\infty - \int_K^\infty \frac{1}{2}\frac{\partial}{\partial S_T}(\sigma_L^2 S_T^2 \varphi)dS_T. \tag{2.10}$$

Using that $\lim_{K->\infty} S_T = 0$ the term in brackets vanishes and the second term becomes $\frac{\sigma_L^2 K^2}{2}\varphi$. Integrating the second term of (2.9) by parts yields

$$
\begin{aligned}
&-\int_K^\infty \left\{ \frac{\partial}{\partial S_T}(\mu S_T \varphi) \right\} (S_T - K) dS_T = \\
&-\left[\mu S_T \varphi (S_T - K)\right]_K^\infty + \int_K^\infty (\mu S_T \varphi) dS_T. \qquad (2.11)
\end{aligned}
$$

In the same way as before, the term in brackets vanishes. This leads to the following equation

$$
\frac{\partial C}{\partial T} = \frac{\sigma_L^2 K^2}{2} \varphi + \int_K^\infty \mu S_T \varphi dS_T. \qquad (2.12)
$$

The second term of (2.12) may be written as

$$
\int_K^\infty dS_T \mu S_T \varphi = \mu \left[\int_K^\infty \varphi (S_T - K) dS_T + K \int_K^\infty \varphi \; dS_T \right]. \qquad (2.13)
$$

Note that the first term of the right hand side above is exactly the undiscounted option value from (2.4). By using (2.6) to the second term of (2.13) and (2.7) to the first term of (2.12) we finally get

$$
\frac{\partial C}{\partial T} = \frac{\sigma_L^2 K^2}{2} \frac{\partial^2 C}{\partial K^2} + \mu(T) \left(C - K \frac{\partial C}{\partial K} \right). \qquad (2.14)
$$

2.3 Stochastic volatility

Stochastic volatility is another approach that can be used to resolve the shortcoming of the Black & Scholes model with constant volatility. The stochastic volatility model assumes that the volatility of the underlying is a stochastic processes instead of constant.
One particular stochastic model is the SABR model (Stochastic Alpha Beta Rho) developed by Hagan et al. (see [3]). Here the dynamics are given by

$$
\begin{aligned}
dF_t &= \sigma_t F_t^\beta dW_1(t), \qquad & F_t(0) &= f, \qquad (2.15) \\
d\alpha_t &= \nu \alpha_t dW_2(t), \qquad & \alpha_t(0) &= \alpha_0, \qquad (2.16)
\end{aligned}
$$

where F_t is the forward measure, α is the volatility, ν is the volatility of the volatility and W_1 and W_2 are Brownian motions, where the two processes are correletad by

$$
d \langle W_1, W_2 \rangle_t = \rho dt \qquad (2.17)
$$

The forward value is $F_t = Se^{rt}$ with r as the interest rate. Note that α_0, β, ρ and ν are all constants. There are many other stochastic models but the SABR model has the advantage that it is easy to calibrate. The disadvantage of the SABR model is that it is not relatively accurate for longer maturities since the model does not have the volatility mean reversion property, which means that the process strives to a long term mean value. See [3] for further details.

2.4 Monte Carlo simulation

A common method used to price options is Monte Carlo simulation. All options have an underlying asset and the idea in Monte Carlo simulation is to generate paths that the underlying asset price can take. This is done many times and the value of the option equals the discounted value of the mean of all the payoffs. The advantage of Monte Carlo simulation is that it is very flexible and easy to implement, even for path dependent options (i.e. barrier options). The disadvantage is that it is computationally intensive. One needs to run a large number of simulations in order to get good estimates (see [1, 4]).

To use Monte Carlo simulation under local volatility we simulate

$$S(t+\Delta t) = S(t) + r(t)S(t)\Delta t + \sigma_L(S,t)S(t)\sqrt{\Delta t}N(0,1) \tag{2.18}$$

To simulate the corresponding path under stochastic volatility (SABR) we simulate the spot price path according to

$$S(t+\Delta t) = S(t) + r(t)S(t)\Delta t + \alpha(t)(D(t,T))^{1-\beta}(S(t)^{\beta})\epsilon_1(t)\sqrt{\Delta t} \tag{2.19}$$

$$\alpha(t+\Delta t) = \alpha(t) + \nu\alpha(t)(\rho\epsilon_1(t) + \epsilon_2(t)\sqrt{1-\rho^2})\sqrt{\Delta t}. \tag{2.20}$$

where ϵ_1 and ϵ_2 are independent random samples from a standard normal distribution (independent of each other and independent with respect to the time).

2.5 Barrier options

A barrier option is a typical exotic option where the payoff depends on whether the price of the underlying asset reaches a certain level during a certain period of time. One example of a barrier option could be an option that knocks in if and only if the underlying asset price reaches a certain upper level. This is called an up and in option. There can also be multiple barrier levels.

The standard barrier options that will be used in this thesis are

- Up and In (S<H)
- Up and Out (S<H)
- Down and In (S>H)
- Down and Out (S>H)

One method to price standard barrier options is to use the formulas derived by Merton (1973) and Reiner and Rubinstein (1991a) (see [5] for further details). We have that

$$
\begin{aligned}
A &= \phi S e^{(b-r)T} N(\phi x_1) - \phi K e^{-rT} N(\phi x_1 - \phi\sigma\sqrt{T}) \\
B &= \phi S e^{(b-r)T} N(\phi x_2) - \phi K e^{-rT} N(\phi x_2 - \phi\sigma\sqrt{T}) \\
C &= \phi S e^{(b-r)T} (H/S)^{2(\mu+1)} N(\eta y_1) - \phi K e^{-rT} (H/S)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T}) \\
D &= \phi S_0 e^{(b-r)T} (H/S)^{2(\mu+1)} N(\eta y_2) - \phi K e^{-rT} (H/S)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T}) \\
E &= R e^{-rT} [N(\eta x_2 - \eta\sigma\sqrt{T}) - (H/S)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})] \\
F &= R[(H/S)^{\mu+\lambda} N(\eta z) + (H/S)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})]
\end{aligned}
$$

and

$$
\begin{aligned}
x_1 &= \frac{ln(S/K)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \\
x_2 &= \frac{ln(S/H)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \\
y_1 &= \frac{ln(H^2/SK)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \\
y_2 &= \frac{ln(H/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \\
z &= \frac{ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
\mu &= \frac{b - \sigma^2/2}{\sigma^2} \\
\lambda &= \sqrt{\mu^2 + \frac{2r}{\sigma^2}}
\end{aligned}
$$

The price of the different types of barrier options mentioned earlier are determined by

Down and in call option

$$\begin{aligned} c_{di}(K > H) &= C + E & \eta = 1, \phi = 1 \\ c_{di}(K < H) &= A - B + D + E & \eta = 1, \phi = 1 \end{aligned}$$

Down and out call option

$$\begin{aligned} c_{do}(K > H) &= A - C + F & \eta = 1, \phi = 1 \\ c_{do}(K < H) &= B - D + F & \eta = 1, \phi = 1 \end{aligned}$$

Up and in call option

$$\begin{aligned} c_{ui}(K > H) &= A + E & \eta = 1, \phi = -1 \\ c_{ui}(K < H) &= B - C + D + E & \eta = 1, \phi = -1 \end{aligned}$$

Up and out call option

$$\begin{aligned} c_{uo}(K > H) &= F & \eta = -1, \phi = 1 \\ c_{uo}(K < H) &= A - B + C - D + F & \eta = -1, \phi = 1 \end{aligned}$$

3 Data

The models are calibrated to liquid European call options with the S&P 500 index from October 1995 as underlying asset. We have 8 maturities and for each maturitiy we have the same 10 strike prices, i.e. we have 80 option prices along with 80 implied volatilities. Also we assume zero dividends and a risk free interest rate $r = 0.06$ (see [1]).
The maturity vector is then defined as

$$T = [T_1, T_2, ..., T_8]$$

and the strike vector as

$$K = [K_1, K_2, ..., K_{10}]$$

The option prices are summarized in the table below

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.96	102.47	111.63	119.81	121.79	137.44	152.30	179.56
531.00	65.36	74.94	85.38	94.46	96.57	113.46	129.20	157.77
560.50	37.49	49.19	60.88	70.90	73.26	90.73	107.13	136.95
590.00	14.42	27.28	39.38	49.26	51.62	69.62	86.35	117.07
619.50	2.59	10.87	21.19	31.15	33.34	50.63	67.17	98.28
649.00	0.17	3.17	8.60	16.22	18.22	34.26	50.26	80.77
678.50	0.05	0.65	3.04	7.45	8.68	22.12	36.68	65.47
708.00	0.02	0.30	1.05	3.10	3.71	12.90	24.63	51.45
767.00	0.00	0.04	0.14	0.38	0.49	3.71	10.80	31.10
826.00	0.00	0.01	0.05	0.10	0.12	0.76	3.95	18.25

Table 3.1: Table of option prices for European call options based on S&P 500 index from October 1995

By inverting the Black & Scholes model we can calculate the implied volatilities. See table below.

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	0.19	0.18	0.17	0.17	0.17	0.17	0.17	0.17
531.00	0.17	0.16	0.16	0.16	0.16	0.16	0.16	0.16
560.50	0.13	0.14	0.14	0.15	0.15	0.15	0.15	0.16
590.00	0.11	0.13	0.13	0.14	0.14	0.14	0.15	0.15
619.50	0.10	0.11	0.12	0.13	0.13	0.13	0.14	0.14
649.00	0.10	0.10	0.10	0.11	0.12	0.12	0.13	0.14
678.50	0.12	0.10	0.10	0.11	0.11	0.12	0.13	0.13
708.00	0.14	0.11	0.10	0.10	0.10	0.11	0.12	0.13
767.00	0.17	0.13	0.11	0.10	0.10	0.11	0.12	0.12
826.00	0.20	0.15	0.12	0.11	0.11	0.10	0.11	0.12

Table 3.2: Table of market implied volatilities prices corresponding to European call options based on S&P 500 index from October 1995

4 Calibration

4.1 Dupire's formula

One way to obtain the volatility function is to look at the Fokker-Planch equation associated with the diffusion process

$$dS_t/S_t = \mu dt + \sigma dW_t, \qquad t > t_0, \quad S(t_0) = S_0 \tag{4.1}$$

The equation above states that we can write the option price

$$u = u(t_0, S_0; T, K)$$

as a function of K (strike price) and T (maturity) that satisfies the following differential equation

$$\frac{\partial u}{\partial T} = \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 u}{\partial K^2} - rK\frac{\partial u}{\partial K}, \qquad T > t_0, \quad K > 0, \tag{4.2}$$

$$u(t_0, S_0; t_0, K) = max(S_0 - K, 0), \quad K \geq 0,$$

with boundary values

$$u(t_0, S_0; T, 0) = S_0, \quad \lim_{K\to\infty} u(t_0, S_0; T, K) = 0$$

From (4.2) we can solve for the volatility

$$\sigma(T,K) = \sqrt{\frac{2(\frac{\partial u}{\partial T} + rK\frac{\partial u}{\partial K})}{K^2\frac{\partial^2 u}{\partial K^2}}} \tag{4.3}$$

Although (4.3) seems simple it is very sensitive to input data. First of all, we only have a discrete sample of strikes and prefixed maturity dates so we will need to do some sort of numerical differentiation to evaluate the volatility. Another problem is that the Brownian motion in (4.1) is just a model of true market dynamics which means that (4.3) will at best be an approximation. Infact, true option prices are usually monotonically increasing in T (and decreasing and convex in K) which implies that the denominator of (4.3) is usually positive but this may easily not be the case when using real data (see [7] for further details).

4.2 Local volatility - Tikhonov regularization

To obtain a smooth solution we will rewrite (4.2) as a linear system of equations to determine the unknown volatilities and use Tikhonov reularization with a discrete smoothing term. We will use natural cubic splines to interpolate between strike prices. The reason we are using natural cubic splines is because they are known to be continuous, smooth and twice differentiable everywhere in the interval (which is required to calculate the volatilities).

Denote the maturities as $T_1, ..., T_m$ and for each maturity we have corresponding strike prices $K_1, ..., K_n$. To evaluate the volatilities in (4.3) we need to differentiate these discrete data once and twice with respect to K but also once with respect to T. To do this we will follow Reinsch's algorithm which basically means that for each maturity $T_i, i = 1, ..., m$, we derive a smoothing natural cubic spline u_i such that

$$\sum_K \|u_i(K) - u(t_0, S_0; T_i, K)\|^2 + \lambda \|u_i''(K)\|^2 \tag{4.4}$$

is minimized.

Given an interval $a < t_1 < ... < t_n < b$, g is a cubic spline if

1. On each interval $(a, t_1), ..., (t_n, b)$, g is a cubic polynomial
2. The polynomial pieces fit together at knots (points t_i) such that g, g' and g'' are continous at each t_i and hence on the whole interval [a,b]

Natural cubic splines can be defined as

$$g(t) = d_i(t - t_i)^3 + c_i(t - t_i)^2 + b_i(t - t_i) + a_i, \quad t_i \leq t \leq t_i + 1,$$

if

$$g''(a) = g''(b) = g'''(a) = g'''(b) = 0.$$

The penalized sum of squares is defined as

$$S(g) = \sum_{i=1}^{n} (Y_i - g(t_i))^2 + \lambda \int_a^b (g''(t))^2 dt$$

where Y_i are the option prices, t_i are the strike prices and λ is the penalty parameter.

The Reinsch algorithm for spline smoothing can be defined in the following way (see [6]).

1. Evaluate $Q'Y$
2. Determine LDL = R+λQ'Q
3. Solve for γ in LDL'γ=Y'Y
4. Finally determine g by evaluating g = Y-λQγ

where γ is the second derivative of the cubic splines.

With $h_i = t_{i+1} - t_i$, i = 1,...,n, Q and R are defined as

$$Q = \begin{bmatrix} h_1^{-1} & 0 & \cdots & 0 \\ h_1^{-1} - h_2^{-1} & h_2^{-1} & \cdots & 0 \\ h_2^{-1} & h_2^{-1} - h_3^{-1} & \cdots & 0 \\ 0 & h_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{n-1}^{-}1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} \frac{1}{3}(h_1 + h_2) & \frac{1}{6}h_2 & \cdots & 0 \\ \frac{1}{6}h_2 & \frac{1}{3}(h_2 + h_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{3}(h_{n-2} + h_{n-1}) \end{bmatrix}$$

respectively.

Differentation with respect to T is more troublesome because of potential large gaps between maturities. This means that we do not have an issue with stability but at the same time the numerical differentation cannot be particularly accurate. Therefore we use an approximation, i.e. centered differences at $T_2, ..., T_{m-1}$ and one-sided differences at the end-points T_1 and T_m. Since the maturities are not equispaced the approximation is defined as

$$\frac{\partial u}{\partial T}(K_i, T_j) \approx \frac{1}{\tau_j + \tau_{j+1}}\left(\frac{\tau_j}{\tau_{j+1}} u_{j+1}(K_i) + \left(\frac{\tau_{j+1}}{\tau_j} - \frac{\tau_j}{\tau_{j+1}}\right) u_j(K_i) - \frac{\tau_{j+1}}{\tau_j} u_{j-1}(K_i)\right)$$

with $\tau_j = T_j - T_{j-1}, \quad j = 2, ..., m-1$.

The idea is to write the elements in (4.3) in matrix form. Therefore we define $\mathbf{D}$ as a diagonal matrix with the denominator of (4.3) as its elements, z as the square of the local volatility (i.e. σ_L^2) and $\mathbf{b}$ as the corresponding vector of the enumerator of (4.3). In other words we want to solve

$$\mathbf{D}z \approx \mathbf{b} \tag{4.5}$$

but as mentioned before, this could lead to complex volatilites (i.e. negative denominator) so we need to apply regularization. Then the problem transforms to the following minimization problem

$$\|\mathbf{D}z - \mathbf{b}\|_2^2 + \alpha\|\mathbf{L}z\|_2^2 \tag{4.6}$$

where $\alpha > 0$ is the penalty parameter and L is a discrete gradient operator defined as

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_m \otimes \mathbf{L}_K \\ \mathbf{L}_T \otimes \mathbf{I}_n \end{bmatrix}$$

where

$$\mathbf{L}_K = \begin{bmatrix} -1 & 1 & 0 & \cdots & & \\ 0 & -1 & 1 & \cdots & & \\ 0 & 0 & -1 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{L}_T = \begin{bmatrix} -\frac{1}{\tau_1} & \frac{1}{\tau_1} & 0 & \cdots & & \\ 0 & -\frac{1}{\tau_2} & \frac{1}{\tau_2} & \cdots & & \\ 0 & 0 & -1 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & -\frac{1}{\tau_{m-1}} & \frac{1}{\tau_{m-1}} \end{bmatrix}$$

and the symbol $\otimes$ denotes the Kronecker product of two matrices.

Note that $\mathbf{L}_K \in R^{n-1,n}, \mathbf{L}_T \in R^{m-1,m}$.

Minimizing (4.6) with respect to z we get the following linear equation system

$$(\mathbf{D}^2 + \alpha \mathbf{L}'\mathbf{L})z = \mathbf{Db} \tag{4.7}$$

Solving for z gives

$$z = (\mathbf{D^2} + \alpha \mathbf{L'L})^{-1}\mathbf{Db} \tag{4.8}$$

where the inverse always exists (see [7]).

4.3 Local volatility - the Crank-Nicholson scheme

An alternative approach to using the cubic splines method described above is that we use the Crank-Nicholson method for solving the numerical problem. This is a finite difference method used for numerically solving the heat equation and similar differential equations. First, we use the ordinary data to obtain the partial derivative with respect to strike and maturity as well as the second partial derivative with respect to strikes. This is done by calculating the forward and backward Euler and then the Crank-Nicholson equation follows as the sum of the forward and backward Euler equations divided by two.
Let the partial differential equation be

$$\frac{\partial x}{\partial t} = F(x, k, t, \frac{\partial x}{\partial k}, \frac{\partial^2 x}{\partial k^2})$$

then letting $x(i\Delta k, n\Delta t) = x_i^n$, the forward Euler equation is given by

$$\frac{x_i^{n+1} - x_i^n}{\Delta t} = F_i^n(x, k, t, \frac{\partial x}{\partial k}, \frac{\partial^2 x}{\partial k^2})$$

the backward Euler equation is given by

$$\frac{x_i^{n+1} - x_i^n}{\Delta t} = F_i^{n+1}(x, k, t, \frac{\partial x}{\partial k}, \frac{\partial^2 x}{\partial k^2})$$

and the Crank-Nicholson equation is then

$$\frac{x_i^{n+1} - x_i^n}{\Delta t} = \frac{1}{2}(F_i^{n+1}(x, k, t, \frac{\partial x}{\partial k}, \frac{\partial^2 x}{\partial k^2}) + F_i^n(x, k, t, \frac{\partial x}{\partial k}, \frac{\partial^2 x}{\partial k^2}))$$

By using these equations we can calculate the derivatives of the implied volatility with respect to maturity and strike and the second derivative of the implied volatility with respect to strikes. These are then used in the following equation

$$\sigma^2(K,T) = \frac{2\Sigma\frac{\partial\Sigma}{\partial T}T + \Sigma^2 + 2\Sigma rKT\frac{\partial\Sigma}{\partial K}}{(1 + Kd_1\frac{\partial\Sigma}{\partial K}\sqrt{T})^2 + K^2T\Sigma(\frac{\partial^2\Sigma}{\partial K^2} - d_1(\frac{\partial\Sigma}{\partial K})^2\sqrt{T})} \tag{4.9}$$

to obtain the local variance. Here, Σ is the implied volatility and

$$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\Sigma^2)T}{\Sigma\sqrt{T}}.$$

Next, the negative values are removed by the use of linear interpolation. Then, by taking the square root of the local variance we obtain the local volatility, which is then interpolated to get the local volatility surface (see [8]).

4.4 Stochastic volatility - SABR

The expression for the implied volatility under the SABR model is given by

$\sigma(F, K) =$

$$\frac{\alpha}{(FK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}log^2F/K + \frac{(1-\beta)^4}{1920}log^4F/K + \dots\right\}}\left(\frac{z}{x(z)}\right) \bullet$$
$$\bullet\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(FK)^{1-\beta}/2} + \frac{2-3p^2}{24}\nu^2\right]t + O(t^2)\right\} \tag{4.10}$$

where

$$z = \frac{\nu}{\alpha}(FK)^{(1-\beta)/2}ln(\frac{F}{K}),$$

and

$$x(z) = ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\},$$

where (4.10) has four parameters α, β, ρ and ν. It is well established that given the price of a European call option, one can solve for the implied volatility in the Black & Scholes formula. Holding everything else fixed while varying the strike price the implied volatility can be calculated as a function of the strike price. Plotting this gives us the implied volatility curve, also known as the smile curve.

In the SABR model the implied volatility is given by (4.10). The idea is to calibrate (4.10) with respect to the four parameters so we can compare it to the implied volatility from market data. Calibrating the model basically means choosing these parameter values in such a way that the two volatility curves lie as close to each other as possible, i.e. minimizing the mean squared errors (see [9]). Note that the model only has a limited number of parameters so if the data consists of more strike prices than parameteres available then we cannot hope for a perfect fit.

Setting the strike price K in (4.10) equal to the forward price, we get an expression for the at-the-money volatility

$$\sigma_{ATM}(f,f) = \frac{\alpha}{f(1-\beta)}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta)}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+O(t_{ex}^2)\right\},$$

and taking the logarithm gives us

$$ln\sigma_{ATM} \approx ln\alpha - (1-\beta)lnf$$

where we can use linear regression on a time series of logs of at-the-money volatilites and logarithms of forward rates to estimate β. Alternatively we can choose β from a priori view of the futures price distribution with

- $\beta = 0$ corresponds to a normal distribution
- $\beta = 0.5$ corresponds to a non-central χ^2-distribution
- $\beta = 1$ corresponds to a lognormal distribution

It is worth mentioning that although the choice of β can affect the Greeks it does not have a huge effect on the volatility curve. A final way is to include β in the optimization problem which leads to

$$min_{\alpha,\beta,\rho,\nu}\sum_{K,T}\|\sigma_{SABR}(K,T)-\sigma^*(K,T)\|_2 \tag{4.11}$$

where $\|\cdot\|_2$ is the L_2 norm which is defined as $\|a\|_2 = \sqrt{\Sigma_i a_i^2}$, where a is a vector.

5 Calibration results

5.1 Local volatility

Implied volatility surface

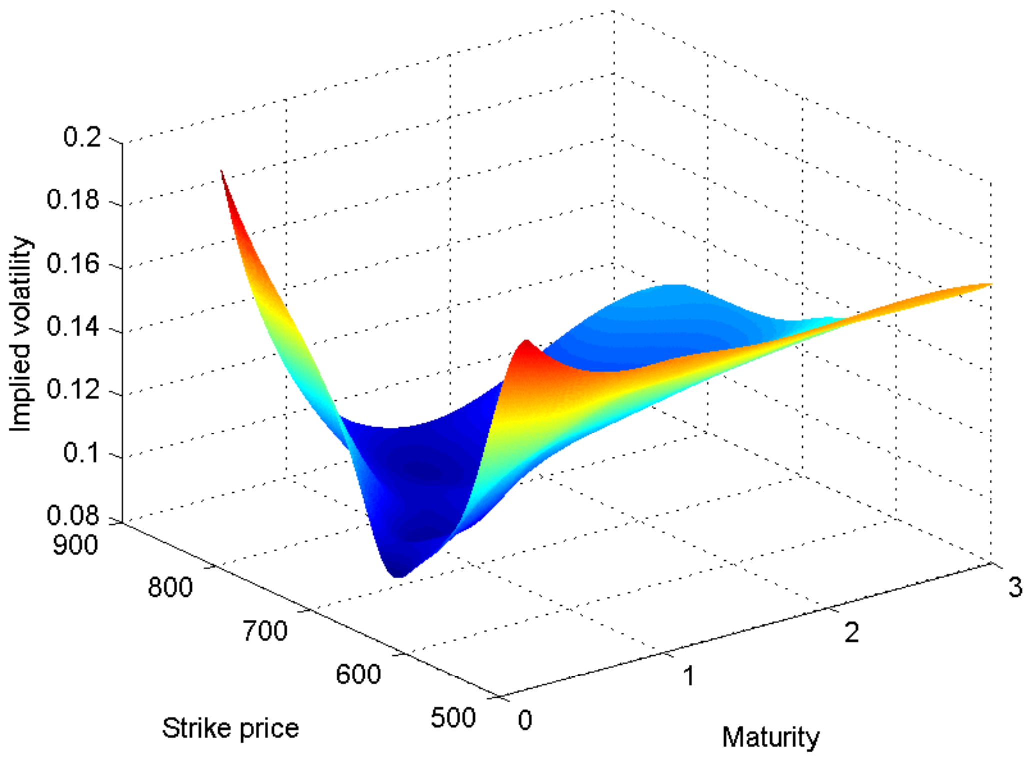


Figure 5.1: Market implied volatility surface

Local volatility surface - Tikhonov regularization

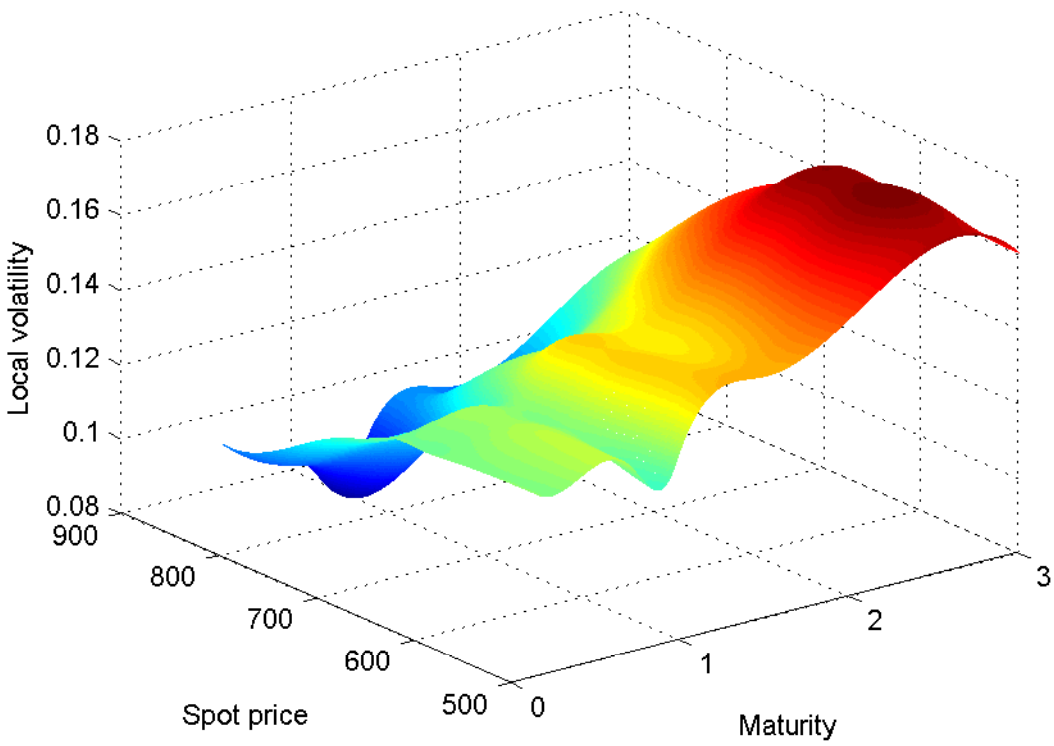


Figure 5.2: Local volatility surface - Tikhonov regularization

Local volatility surface - the Crank-Nicholson scheme

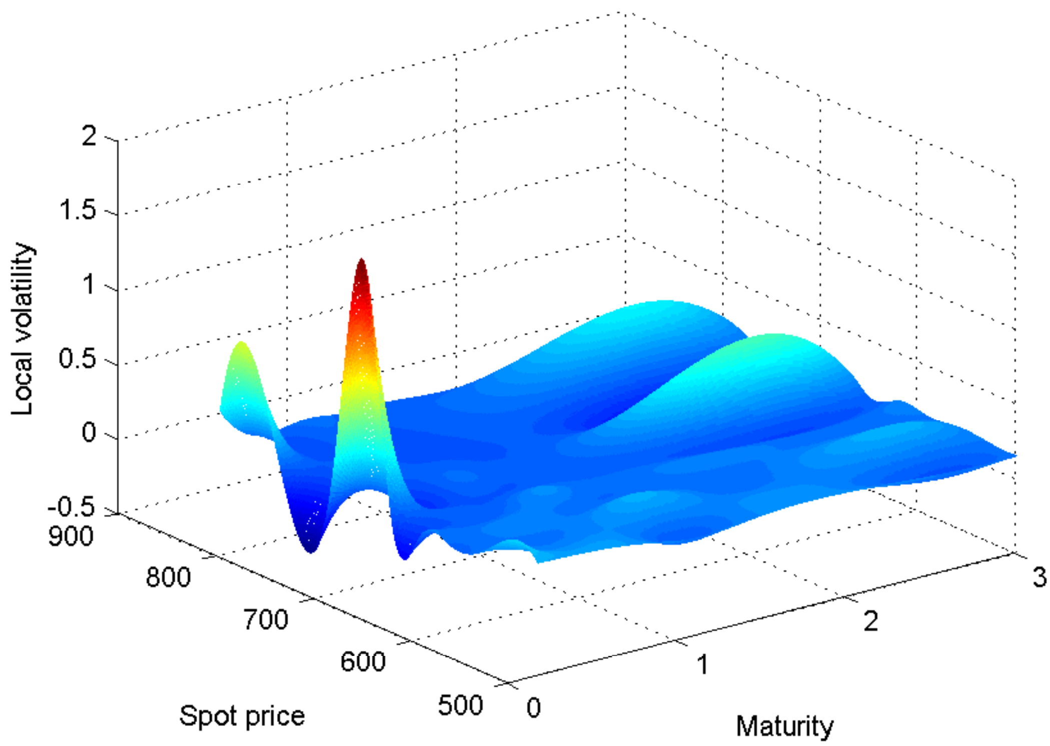


Figure 5.3: Local volatility surface - the Crank-Nicholson scheme

Market prices

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.96	102.47	111.63	119.81	121.79	137.44	152.30	179.56
531.00	65.36	74.94	85.38	94.46	96.57	113.46	129.20	157.77
560.50	37.49	49.19	60.88	70.90	73.26	90.73	107.13	136.95
590.00	14.42	27.28	39.38	49.26	51.62	69.62	86.35	117.07
619.50	2.59	10.87	21.19	31.15	33.34	50.63	67.17	98.28
649.00	0.17	3.17	8.60	16.22	18.22	34.26	50.26	80.77
678.50	0.05	0.65	3.04	7.45	8.68	22.12	36.68	65.47
708.00	0.02	0.30	1.05	3.10	3.71	12.90	24.63	51.45
767.00	0.00	0.04	0.14	0.38	0.49	3.71	10.80	31.10
826.00	0.00	0.01	0.05	0.10	0.12	0.76	3.95	18.25

Table 5.1: Table of option prices for European call options based on S&P 500 index from October 1995

Monte Carlo prices - Tikhonov regularization

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.73	101.31	109.44	117.13	118.92	133.52	148.57	175.64
531.00	64.69	73.53	82.86	91.31	93.28	109.22	125.31	154.08
560.50	37.19	48.17	58.73	67.81	69.92	86.84	103.66	133.74
590.00	15.75	27.59	38.48	47.62	49.78	66.97	83.98	114.81
619.50	4.39	13.47	23.06	31.45	33.48	49.98	66.55	97.40
649.00	0.75	5.53	12.53	19.39	21.12	35.98	51.49	81.54
678.50	0.08	1.85	6.04	11.00	12.34	24.82	38.76	67.27
708.00	0.00	0.50	2.56	5.66	6.58	16.29	28.24	54.57
767.00	0.00	0.02	0.29	1.03	1.34	5.85	13.28	33.90
826.00	0.00	0.00	0.02	0.11	0.17	1.62	5.25	19.40

Table 5.2: Table of option prices for European call options based on S&P 500 index from October 1995 using Monte Carlo simulation - Tikhonov regularization (N = 100 000, nT = 100)

Monte Carlo prices - the Crank-Nicholson scheme

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.97	102.21	111.31	119.24	120.80	136.33	150.79	177.58
531.00	65.22	74.80	85.07	94.07	95.92	112.63	127.75	156.12
560.50	37.28	49.22	60.62	70.60	72.80	90.27	105.82	135.59
590.00	15.23	27.50	39.14	49.12	51.38	69.36	85.36	115.98
619.50	2.98	10.37	20.81	31.12	33.29	50.36	66.68	97.44
649.00	0.03	2.07	7.96	15.98	18.02	33.90	50.60	80.17
678.50	0.02	0.56	2.07	7.03	8.27	21.51	37.29	64.28
708.00	0.01	0.22	0.69	2.86	3.54	12.20	24.47	48.89
767.00	0.00	0.04	0.23	0.46	0.58	3.48	8.76	26.79
826.00	0.00	0.02	0.09	0.20	0.24	1.05	4.66	18.00

Table 5.3: Table of option prices for European call options based on S&P 500 index from October 1995 using Monte Carlo simulation - the Crank-Nicholson scheme (N = 100 000, nT = 100)

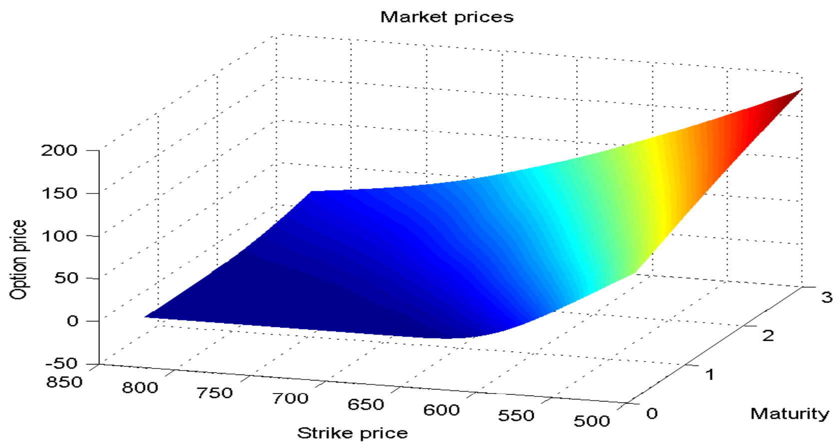


Figure 5.4: Plot of market prices.

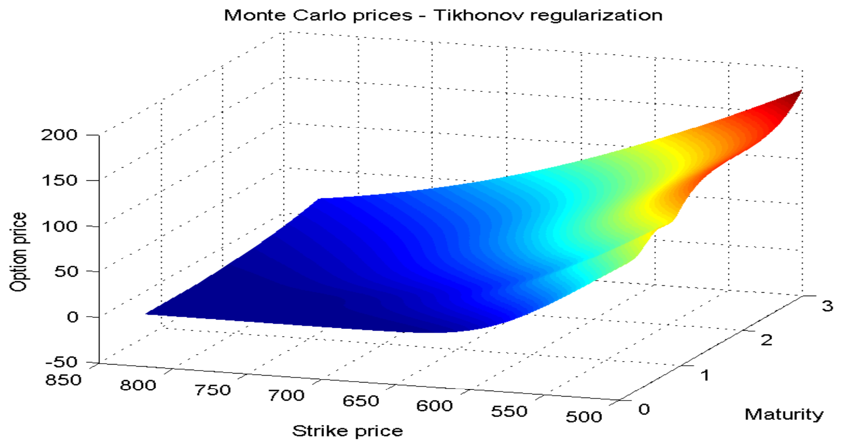


Figure 5.5: Plot of Monte Carlo prices - Tikhonov regularization.

Figure 5.6: Plot of Monte Carlo prices - the Crank-Nicholson scheme.

5.2 Stochastic volatility

Results from the SABR calibration are shown below

C/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
α	3.00	0.01	0.01	0.01	0.01	0.10	0.13	0.01
β	1.44	0.99	0.81	0.69	0.66	0.52	0.47	0.48
ν	0.48	1.38	1.39	1.39	1.39	1.04	1.00	1.39
ρ	-0.40	-0.59	-0.71	-0.73	-0.75	-0.63	-0.54	-0.44

Table 5.4: Table of calibrated α, β, ν and ρ values for each maturity.

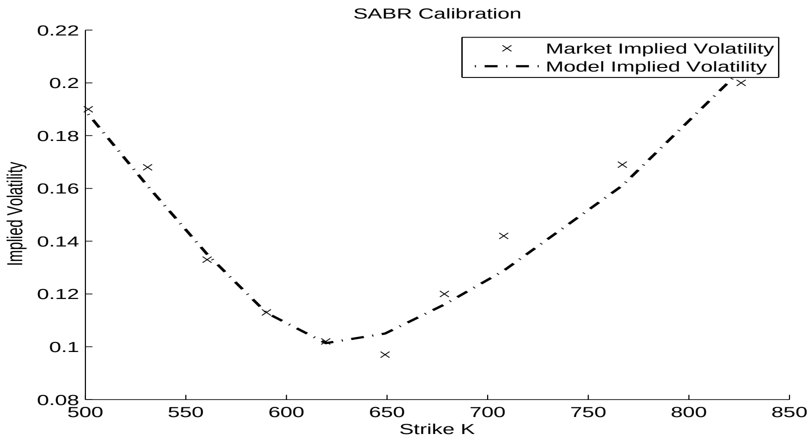


Figure 5.7: Calibration corresponding to T_1 = 0.18

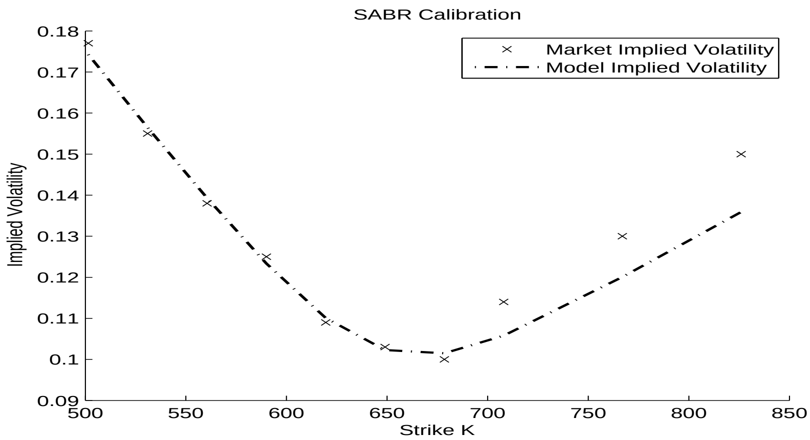


Figure 5.8: Calibration corresponding to T_2 *= 0.43*

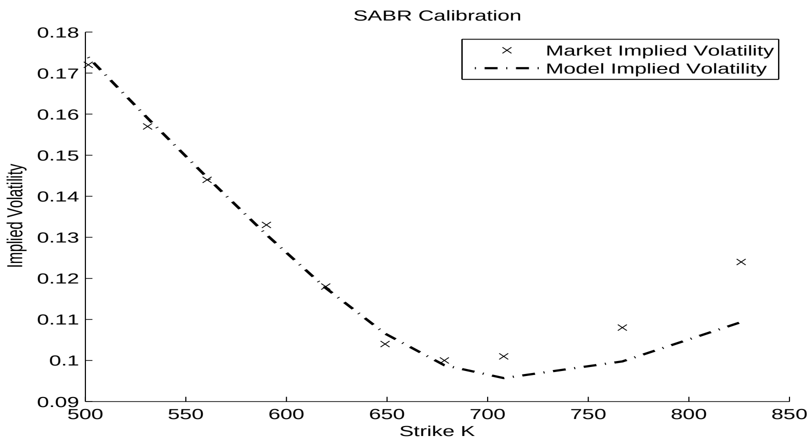


Figure 5.9: Calibration corresponding to T_3 *= 0.70*

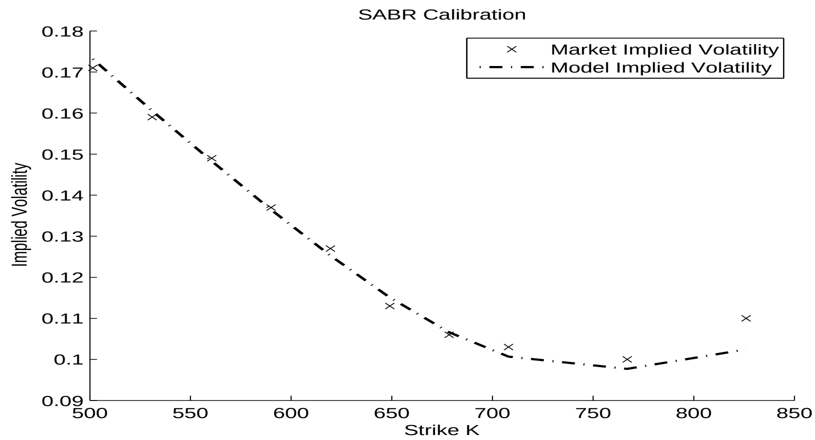


Figure 5.10: Calibration corresponding to T_4 ***= 0.94***

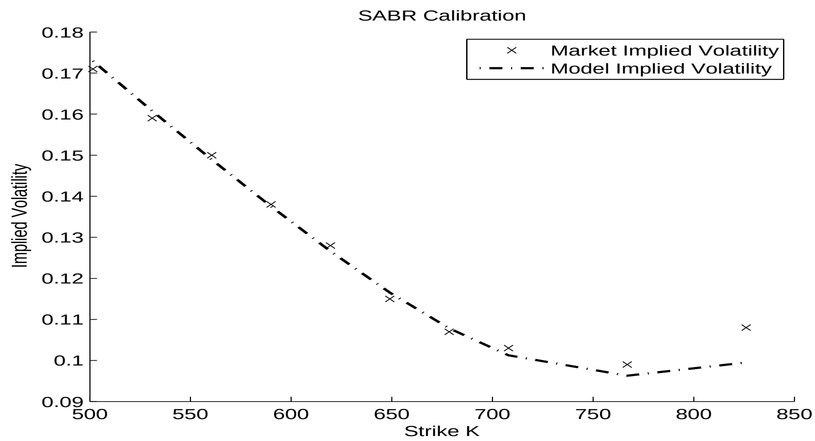


Figure 5.11: Calibration corresponding to T_5 ***= 1.00***

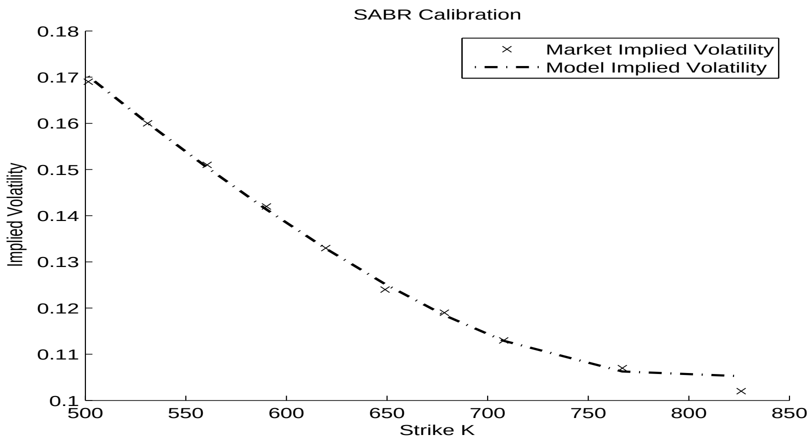


Figure 5.12: Calibration corresponding to T_6 ***= 1.50***

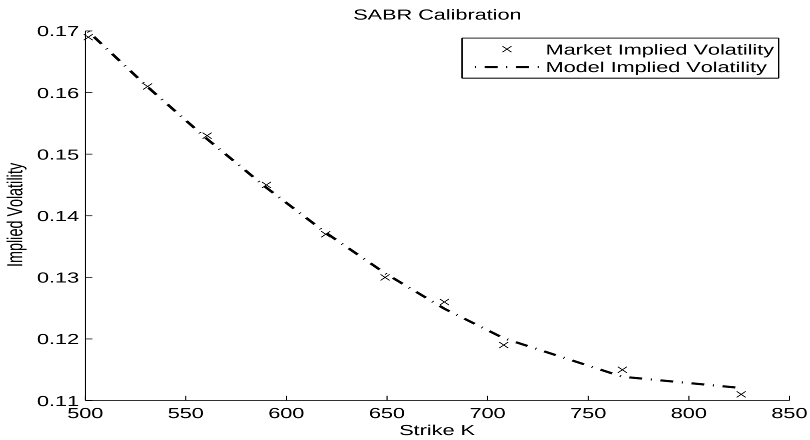


Figure 5.13: Calibration corresponding to T_7 ***= 2.00***

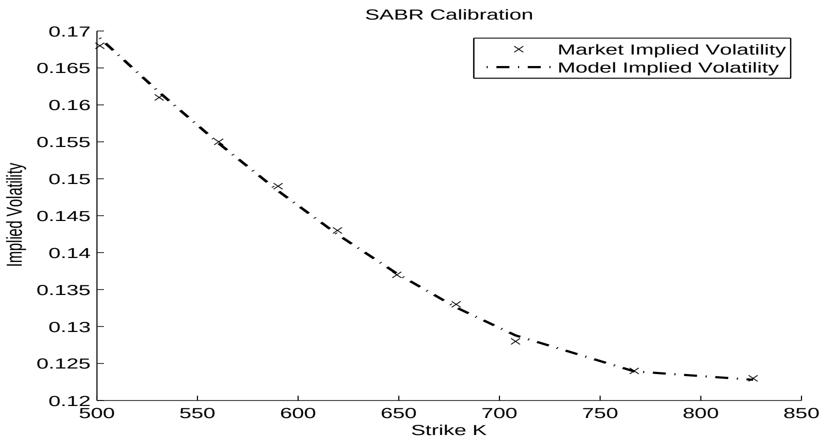


Figure 5.14: Calibration corresponding to T_8 = 3.00

The retrieved parameter values are then used to replicate the vanilla options in table 5.5 by using

1. Black-76
2. Monte Carlo simulation

Market prices

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.96	102.47	111.63	119.81	121.79	137.44	152.30	179.56
531.00	65.36	74.94	85.38	94.46	96.57	113.46	129.20	157.77
560.50	37.49	49.19	60.88	70.90	73.26	90.73	107.13	136.95
590.00	14.42	27.28	39.38	49.26	51.62	69.62	86.35	117.07
619.50	2.59	10.87	21.19	31.15	33.34	50.63	67.17	98.28
649.00	0.17	3.17	8.60	16.22	18.22	34.26	50.26	80.77
678.50	0.05	0.65	3.04	7.45	8.68	22.12	36.68	65.47
708.00	0.02	0.30	1.05	3.10	3.71	12.90	24.63	51.45
767.00	0.00	0.04	0.14	0.38	0.49	3.71	10.80	31.10
826.00	0.00	0.01	0.05	0.10	0.12	0.76	3.95	18.25

Table 5.5: Table of option prices for European call options based on S&P 500 index from October 1995

Black-76 prices

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	93.95	102.37	111.75	120.00	121.97	137.58	152.43	179.73
531.00	65.20	75.03	85.58	94.65	96.80	113.47	129.19	157.91
560.50	37.60	49.32	60.96	70.78	73.08	90.61	107.01	136.89
590.00	14.41	27.04	38.96	49.15	51.52	69.47	86.23	116.88
619.50	2.55	11.03	21.12	30.75	33.02	50.60	67.25	98.11
649.00	0.28	3.10	9.02	16.67	18.57	34.60	50.48	80.83
678.50	0.04	0.71	2.92	7.55	8.83	21.97	36.31	65.31
708.00	0.01	0.17	0.79	2.85	3.50	12.88	24.98	51.78
767.00	0.00	0.02	0.07	0.33	0.40	3.61	10.53	31.08
826.00	0.00	0.00	0.01	0.05	0.06	0.92	4.10	18.19

Table 5.6: Table of option prices for European call options based on S&P 500 index from October 1995 using Black-76.

Monte Carlo prices

K/T	0.18	0.43	0.70	0.94	1.00	1.50	2.00	3.00
501.50	94.01	102.37	111.54	119.77	121.76	137.08	151.56	179.50
531.00	65.22	74.99	85.37	94.44	96.61	112.96	128.31	157.79
560.50	37.57	49.28	60.80	70.63	72.97	90.11	106.16	136.94
590.00	14.38	27.08	38.91	49.11	51.53	69.02	85.46	117.15
619.50	2.55	11.18	21.21	30.88	33.18	50.28	66.60	98.65
649.00	0.26	3.24	9.22	16.95	18.86	34.45	49.98	81.66
678.50	0.03	0.76	3.09	7.90	9.17	21.98	35.98	66.43
708.00	0.00	0.19	0.88	3.13	3.77	13.02	24.82	53.14
767.00	0.00	0.02	0.07	0.41	0.48	3.79	10.58	32.66
826.00	0.00	0.00	0.01	0.07	0.07	0.99	4.18	19.62

Table 5.7: Table of option prices for European call options based on S&P 500 index from October 1995 using Monte Carlo simulation (N = 100 000, nT = 100)

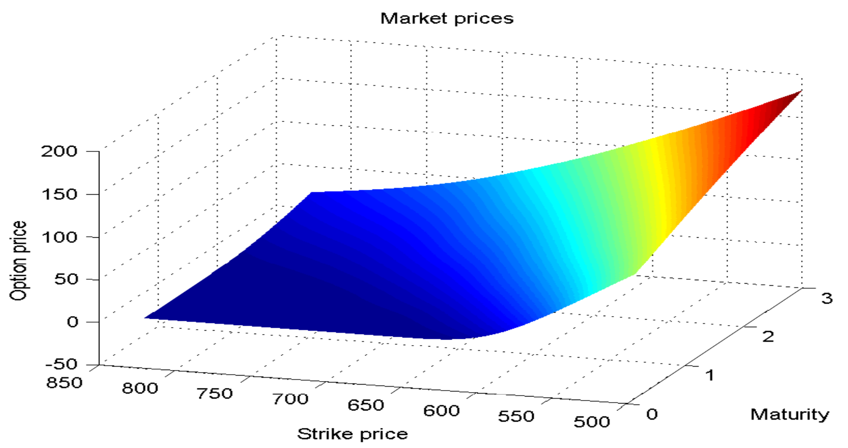


Figure 5.15: Plot of Monte Carlo prices - Market prices.

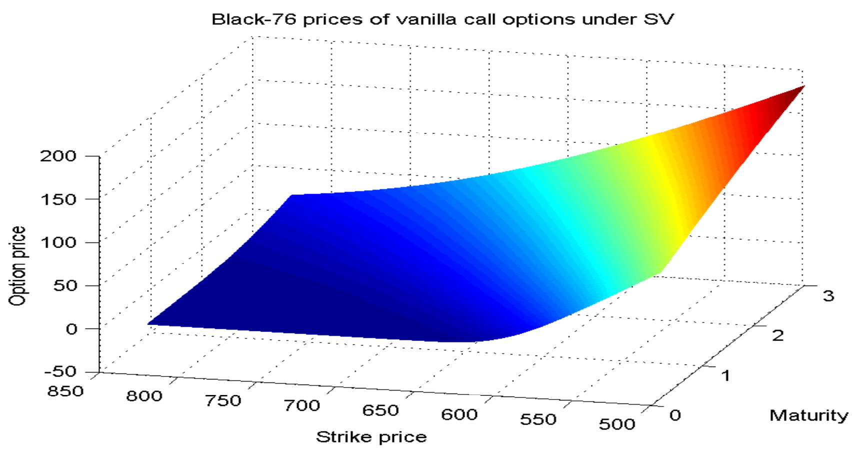


Figure 5.16: Plot of Monte Carlo prices - Black-76 prices.

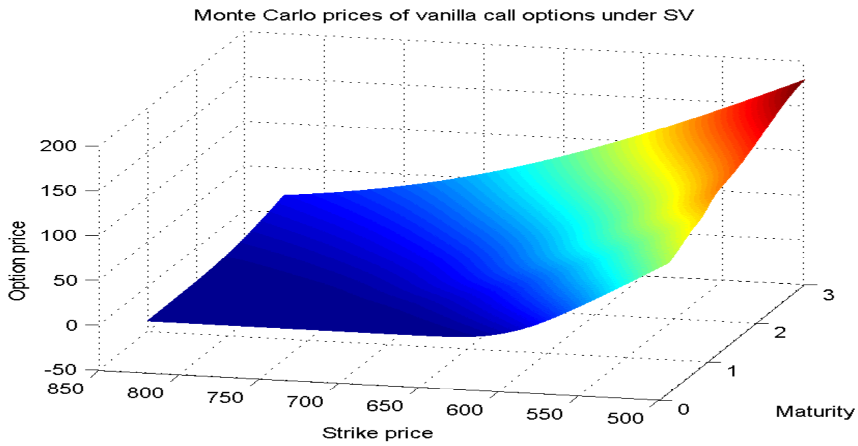


Figure 5.17: Plot of Monte Carlo prices - SABR prices.

6 Pricing barrier options

Looking at tables 5.1-5.3 we can see that we got a better fit using the Crank-Nicholson scheme while the Tikhonov regularization method did not yield an excellent fit. The reason for this could easily be the choices of parameter values, see λ in (4.4) and α in (4.6). It is very difficult to choose the right parameter values mainly because one has to run a new Monte Carlo simulation for each pair (λ,α). Since we are running the Monte Carlo simulations $100,000$ times with 100 time steps finding the optimal parameter values would be very time-consuming (in this thesis we found that the values $\lambda = 2,000$ and $\alpha = 5,000,000$ gave the best results). Hence it makes sense to compute barrier option prices using the Crank-Nicholson scheme because it resulted in a better calibration against the market data than Tikhonov regularization.

6.1 Preliminary conditions

When pricing barrier options we will use the at-the-money strike, i.e. S = K = 590 and vary the maturity and barrier levels. The maturities will be the same T vector as before while the up and down barrier levels will be defined as

- Hd = S[0.5:0.05:0.95]
- Hu = S[1.05:0.05:1.05]

and we will run $10,000$ simulations with 100 times steps. The analytical expressions used here are the ones mentioned in section 2.4.
Broadie, Glasserman and Kou suggests that one should adjust the barrier level when the price of the underlying is observed discretely. Thus when pricing barrier options using the analytical expression we must replace the continuous barrier level with a discrete barrier level [10]. The barrier level for down-options then becomes

$$Hd = He^{-0.5826\sigma\sqrt{nT}}$$

and the barrier level for up-options

$$Hu = He^{0.5826\sigma\sqrt{nT}}$$

6.2 Barrier option prices - tables

Barrier call option prices corresponding to $T_1 = 0.18$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,00
383,50	0,00	0,00	0,00
413,00	0,00	0,00	0,00
442,50	0,00	0,00	0,00
472,00	0,00	0,00	0,00
501,50	0,00	0,00	0,00
531,00	0,00	0,00	0,02
560,50	0,12	0,18	0,48

Table 6.1: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	14,42	14,94	14,30
324,50	14,42	15,25	14,38
354,00	14,42	15,36	14,10
383,50	14,42	15,32	14,34
413,00	14,42	15,35	14,41
442,50	14,42	15,01	14,24
472,00	14,42	15,08	14,28
501,50	14,42	15,08	14,29
531,00	14,42	14,94	14,51
560,50	14,30	14,77	13,72

Table 6.2: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	11,55	13,36	9,89
649,00	3,50	0,26	2,14
678,50	0,42	0,03	0,29
708,00	0,02	0,03	0,08
737,50	0,00	0,00	0,01
767,00	0,00	0,00	0,00
796,50	0,00	0,00	0,00
826,00	0,00	0,00	0,00
855,50	0,00	0,00	0,00
885,00	0,00	0,00	0,00

Table 6.3: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	2,87	1,58	4,55
649,00	10,92	14,99	12,25
678,50	14,00	15,33	13,82
708,00	14,40	15,30	14,29
737,50	14,42	15,35	14,37
767,00	14,42	15,01	14,24
796,50	14,42	15,08	14,28
826,00	14,42	15,08	14,30
855,50	14,42	14,94	14,54
885,00	14,42	14,95	14,20

Table 6.4: Up and Out call option prices

Barrier call option prices corresponding to $T_2 = 0.43$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,00
383,50	0,00	0,00	0,00
413,00	0,00	0,00	0,00
442,50	0,00	0,00	0,00
472,00	0,00	0,00	0,02
501,50	0,00	0,04	0,10
531,00	0,06	0,31	0,44
560,50	2,28	2,90	3,13

Table 6.5: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	27,28	27,69	26,68
324,50	27,28	27,12	27,49
354,00	27,28	27,38	26,44
383,50	27,28	27,79	26,52
413,00	27,28	27,48	26,76
442,50	27,28	27,64	27,38
472,00	27,28	27,10	27,60
501,50	27,28	27,72	26,63
531,00	27,22	26,74	26,51
560,50	25,00	24,69	24,32

Table 6.6: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	26,33	26,52	25,27
649,00	20,24	12,88	14,74
678,50	11,35	2,80	5,10
708,00	4,81	1,47	1,55
737,50	1,61	0,71	0,59
767,00	0,44	0,37	0,13
796,50	0,10	0,14	0,04
826,00	0,02	0,01	0,03
855,50	0,00	0,04	0,05
885,00	0,00	0,04	0,00

Table 6.7: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,95	1,17	1,83
649,00	7,04	14,24	12,30
678,50	15,93	24,58	21,23
708,00	22,47	26,32	24,91
737,50	25,67	26,77	26,38
767,00	26,84	27,27	27,28
796,50	27,18	26,96	27,53
826,00	27,26	27,74	26,69
855,50	27,28	27,01	26,89
885,00	27,28	27,55	27,45

Table 6.8: Up and Out call option prices

Barrier call option prices corresponding to $T_3 = 0.70$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,00
383,50	0,00	0,00	0,00
413,00	0,00	0,00	0,00
442,50	0,00	0,01	0,04
472,00	0,00	0,05	0,10
501,50	0,02	0,21	0,27
531,00	0,63	1,24	1,56
560,50	6,55	7,01	7,52

Table 6.9: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	39,38	39,48	40,26
324,50	39,38	39,02	39,16
354,00	39,38	39,91	38,63
383,50	39,38	38,54	38,80
413,00	39,38	39,07	38,63
442,50	39,38	39,48	39,16
472,00	39,38	39,34	38,98
501,50	39,36	39,01	38,63
531,00	38,75	38,13	37,03
560,50	32,84	31,72	30,99

Table 6.10: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	38,93	39,10	37,55
649,00	35,34	31,19	31,37
678,50	27,69	14,86	16,60
708,00	18,65	2,97	6,98
737,50	11,00	1,78	2,04
767,00	5,79	1,31	0,84
796,50	2,76	0,74	0,11
826,00	1,21	0,46	0,17
855,50	0,49	0,45	0,06
885,00	0,19	0,21	0,03

Table 6.11: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,46	0,38	0,77
649,00	4,05	7,83	8,12
678,50	11,69	25,05	21,52
708,00	20,73	35,57	31,59
737,50	28,38	37,28	36,34
767,00	33,60	38,18	38,53
796,50	36,62	38,65	38,74
826,00	38,18	38,76	38,80
855,50	38,89	38,92	38,52
885,00	39,20	38,52	38,51

Table 6.12: Up and Out call option prices

Barrier call option prices corresponding to $T_4 = 0.94$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,00
383,50	0,00	0,00	0,00
413,00	0,00	0,00	0,04
442,50	0,00	0,02	0,08
472,00	0,01	0,14	0,20
501,50	0,15	0,64	1,08
531,00	1,76	2,68	3,34
560,50	10,91	10,64	11,53

Table 6.13: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	49,26	48,80	48,84
324,50	49,26	49,14	48,49
354,00	49,26	49,63	48,99
383,50	49,26	49,42	48,65
413,00	49,26	49,06	49,39
442,50	49,26	48,97	48,43
472,00	49,25	49,27	49,21
501,50	49,11	48,70	48,66
531,00	47,50	45,71	46,37
560,50	38,35	38,95	36,99

Table 6.14: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	48,96	48,60	48,11
649,00	46,52	46,80	43,91
678,50	40,51	32,48	32,84
708,00	31,94	18,73	19,99
737,50	22,95	8,09	8,53
767,00	15,21	2,80	3,46
796,50	9,39	1,88	1,23
826,00	5,45	1,22	0,70
855,50	3,00	0,70	0,33
885,00	1,57	0,62	0,13

Table 6.15: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,30	0,20	0,53
649,00	2,74	2,34	5,00
678,50	8,75	17,15	15,73
708,00	17,32	30,69	30,32
737,50	26,30	40,98	41,02
767,00	34,05	46,19	44,96
796,50	39,87	47,53	48,21
826,00	43,81	48,12	48,98
855,50	46,26	47,70	49,48
885,00	47,68	48,97	48,46

Table 6.16: Up and Out call option prices

Barrier call option prices corresponding to $T_5 = 1.00$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,00
383,50	0,00	0,00	0,01
413,00	0,00	0,02	0,04
442,50	0,00	0,03	0,07
472,00	0,01	0,17	0,30
501,50	0,21	0,73	0,79
531,00	2,13	2,73	3,43
560,50	12,05	11,94	12,66

Table 6.17: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	51,62	52,02	51,36
324,50	51,62	51,78	51,55
354,00	51,62	51,91	52,37
383,50	51,62	51,66	52,25
413,00	51,62	51,70	51,91
442,50	51,62	51,60	50,87
472,00	51,61	51,61	50,97
501,50	51,41	51,59	50,76
531,00	49,49	48,19	47,36
560,50	39,57	39,53	39,13

Table 6.18: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	51,35	51,74	50,84
649,00	49,11	49,70	46,94
678,50	43,46	36,40	37,30
708,00	35,15	21,49	21,58
737,50	26,09	10,44	10,51
767,00	17,96	3,94	4,73
796,50	11,57	1,95	1,79
826,00	7,04	1,18	0,63
855,50	4,08	1,04	0,30
885,00	2,26	0,67	0,11

Table 6.19: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,27	0,28	0,43
649,00	2,51	2,08	4,47
678,50	8,15	15,51	15,65
708,00	16,47	30,17	30,02
737,50	25,52	41,27	40,74
767,00	33,66	47,69	46,42
796,50	40,04	49,83	49,46
826,00	44,57	51,15	50,77
855,50	47,54	49,88	50,47
885,00	49,36	50,80	51,62

Table 6.20: Up and Out call option prices

Barrier call option prices corresponding to $T_6 = 1.50$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,00
324,50	0,00	0,00	0,00
354,00	0,00	0,00	0,03
383,50	0,00	0,00	0,02
413,00	0,00	0,01	0,09
442,50	0,01	0,14	0,30
472,00	0,13	0,48	0,67
501,50	1,10	1,85	2,34
531,00	5,81	6,29	7,08
560,50	21,17	19,66	19,62

Table 6.21: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	69,62	69,33	69,14
324,50	69,62	69,27	68,48
354,00	69,62	69,34	68,58
383,50	69,62	69,36	69,50
413,00	69,62	67,84	70,25
442,50	69,61	68,74	70,04
472,00	69,49	67,99	69,09
501,50	68,52	67,91	67,47
531,00	63,82	62,85	63,61
560,50	48,45	49,70	49,66

Table 6.22: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	69,47	69,22	68,29
649,00	68,19	67,73	66,94
678,50	64,58	62,01	60,63
708,00	58,35	51,29	48,49
737,50	50,18	33,45	33,83
767,00	41,19	21,20	22,01
796,50	32,42	12,78	12,69
826,00	24,58	7,92	7,69
855,50	18,04	4,55	4,61
885,00	12,85	2,57	2,26

Table 6.23: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,15	0,11	0,27
649,00	1,43	1,54	2,49
678,50	5,04	7,33	9,37
708,00	11,27	18,07	21,01
737,50	19,44	34,40	34,86
767,00	28,43	47,68	47,14
796,50	37,20	55,69	56,08
826,00	45,04	61,84	61,92
855,50	51,59	64,59	65,75
885,00	56,77	66,78	67,11

Table 6.24: Up and Out call option prices

Barrier call option prices corresponding to $T_7 = \mathbf{2.00}$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,01
324,50	0,00	0,00	0,06
354,00	0,00	0,00	0,08
383,50	0,00	0,01	0,18
413,00	0,01	0,08	0,39
442,50	0,07	0,34	0,70
472,00	0,53	1,29	1,52
501,50	2,79	3,92	3,97
531,00	10,48	10,66	11,04
560,50	30,39	26,27	26,92

Table 6.25: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	86,35	85,83	85,16
324,50	86,35	85,07	85,62
354,00	86,35	85,31	85,48
383,50	86,35	85,98	86,73
413,00	86,34	84,64	85,74
442,50	86,28	84,38	85,14
472,00	85,81	84,58	83,24
501,50	83,56	80,43	80,41
531,00	75,86	74,75	73,80
560,50	55,96	58,39	57,53

Table 6.26: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	86,25	85,76	84,90
649,00	85,41	84,11	83,58
678,50	82,95	80,92	79,12
708,00	78,36	78,06	70,70
737,50	71,74	62,87	58,15
767,00	63,66	35,80	47,53
796,50	54,86	26,99	35,10
826,00	46,03	23,59	23,94
855,50	37,71	19,81	15,79
885,00	30,24	13,24	10,41

Table 6.27: Up and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,10	0,07	0,20
649,00	0,94	0,96	1,72
678,50	3,40	4,39	6,34
708,00	7,99	7,93	14,75
737,50	14,61	21,85	25,73
767,00	22,68	48,92	39,60
796,50	31,48	58,88	51,28
826,00	40,31	60,77	60,77
855,50	48,63	65,61	69,52
885,00	56,10	71,42	73,65

Table 6.28: Up and Out call option prices

Barrier call option prices corresponding to $T_8 = 3.00$

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	0,00	0,00	0,12
324,50	0,00	0,00	0,13
354,00	0,00	0,01	0,34
383,50	0,01	0,07	0,34
413,00	0,10	0,49	0,92
442,50	0,56	1,30	2,20
472,00	2,38	3,94	4,43
501,50	7,86	8,18	10,12
531,00	21,14	19,86	19,88
560,50	48,26	41,71	42,54

Table 6.29: Down and In call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
295,00	117,07	117,49	118,31
324,50	117,07	115,71	116,33
354,00	117,06	116,19	117,17
383,50	117,05	116,25	117,76
413,00	116,97	115,24	115,06
442,50	116,51	114,05	114,90
472,00	114,69	110,99	112,92
501,50	109,21	108,07	108,37
531,00	95,93	95,67	97,45
560,50	68,81	75,04	74,59

Table 6.30: Down and Out call option prices

Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	117,01	117,46	116,21
649,00	116,57	115,36	115,67
678,50	115,22	113,82	115,92
708,00	112,53	111,30	109,81
737,50	108,28	105,28	100,59
767,00	102,54	89,51	88,90
796,50	95,54	69,17	76,93
826,00	87,66	60,66	66,53
855,50	79,27	53,60	54,03
885,00	70,73	49,43	46,98

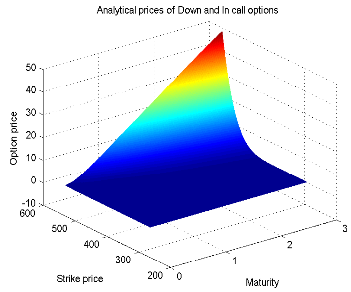
Table 6.31: Up and In call option prices

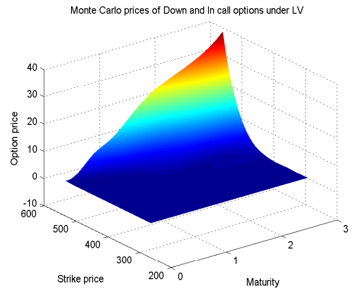
Barrier/Model	Analytical IV	Monte Carlo LV	Monte Carlo SV
619,50	0,06	0,03	0,13
649,00	0,50	0,36	1,06
678,50	1,84	2,37	3,53
708,00	4,54	5,02	9,00
737,50	8,78	10,44	16,40
767,00	14,53	25,84	26,48
796,50	21,52	45,76	37,35
826,00	29,41	55,59	50,50
855,50	37,80	61,93	60,64
885,00	46,33	67,32	69,54

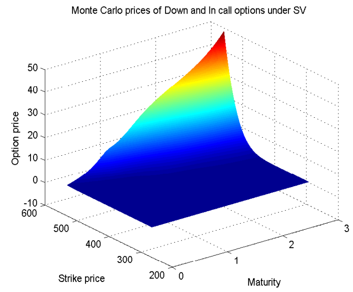
Table 6.32: Up and Out call option prices

6.3 Barrier option prices - graphs

Graphs of Down and In prices

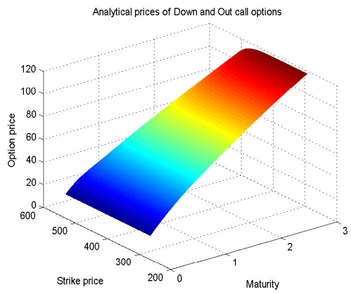


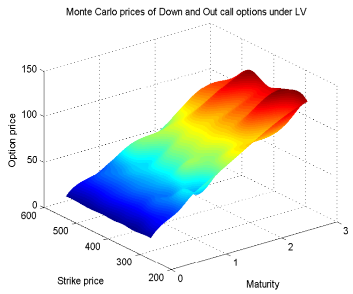




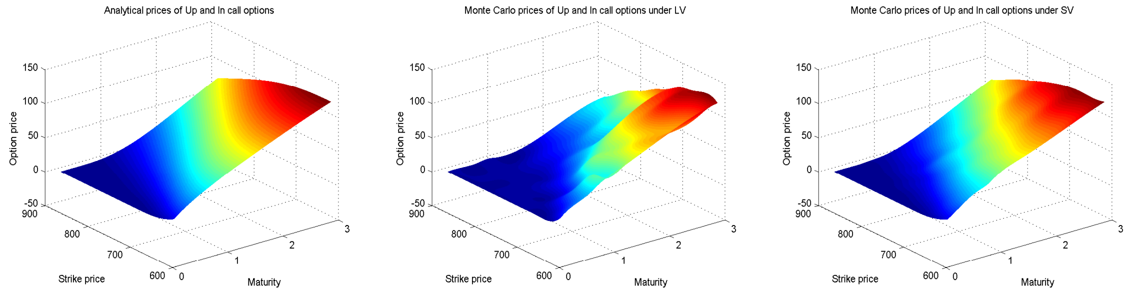
c

Graphs of Down and Out prices



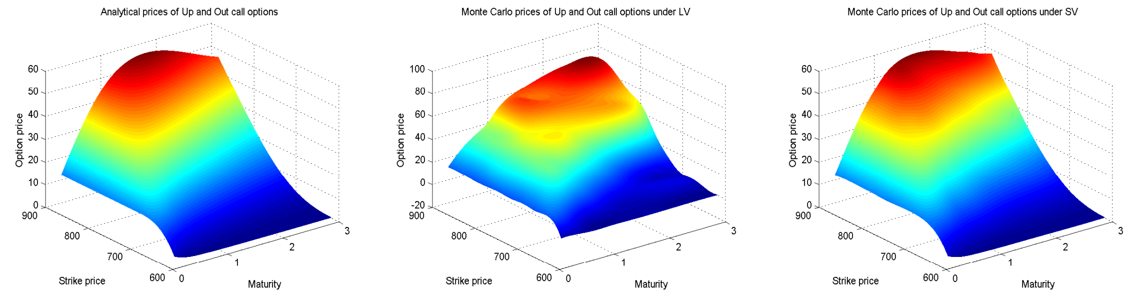


Graphs of Up and In prices



c

Graphs of Up and Out prices



7 Analysis and Discussion

The reader should look at section 6.2, tables 6.1-6.32 for a breakdown of the calculated barrier option prices for each model. The first column is the prices retrieved from the analytical expression with the use of the implied volatility. The other two columns show that we did not observe particularly large differences between the SV and LV models regarding retrieved barrier option prices. This is quite satisfactory but also expected because both models were calibrated against the same market data. However, note that the LV and SV prices differ somewhat a lot for option prices very close to zero (leading to very sensitive relative price differences).
Comparing the SV and LV prices to the analytical prices we do however note large price differences. The first observation is that the prices of down and in options for low barrier levels are significantly higher using LV and SV compared to the analytical prices. It is well documented that the volatility is higher in declining markets than in an up-going market and the analytical formulas fail to capture this effect. This means that the volatility is underestimated in the analytical case, which ultimately leads to an underestimation of the option price. Under SV and LV the corresponding volatility will then be higher, i.e. the probability that the stock price reaches the down barrier level is higher than in the analytical case for low barrier levels. Hence, we will get a higher option price in the SV and LV case.
Similarly, one would expect that there is an equal reverse price difference in the down and out case. However, since the down and out prices are much larger than the down and in prices the differences are not as clear. The reason for this is because the differences in prices for down and in options corresponds to very large relative differences which translates into very small relative differences for the down and out options. The trend is particularly visable for longer maturities.
Next we note that for large barrier levels, the up and in barrier option prices under SV and LV are significantly smaller than the corresponding analytical prices (for all maturities). A plausible explanation is the fact that the volatility is overestimated when using the analytical expressions compared to SV and LV in up-going markets. Higher volatility means that there is a higher probability of the stock price hitting the up-barrier, which means that the option price should be larger in the analytical case. In other words, the probability that the option stays worthless is larger in the SV and LV case.
The corresponding price difference between SV/LV and analytical prices should be visible when looking at the up and out option prices. As above we assume that under SV and LV the volatility is underestimated in up-going markets, compared to the analytical case. Hence the probability of the option becoming worthless is underestimated which means that the price should be higher under SV and LV compared to the analytical case. The results show that this is the case.

8 Conclusions

The purpose of this thesis was to price barrier options using stochastic and local volatility models and compare the results to option prices retrieved using Merton, Reiner and Rubinstein's analytical expressions for barrier options. Our results show that the barrier option prices under stochastic and local volatility coincide but differ significantly from the analytical prices for barrier levels far away from the current spot price.
Furthermore, we noted that the volatility is underestimated in declining markets when pricing down and in options with the analytical expressions. In other words the down and in option prices under SV/LV are significantly larger than the corresponding analytical prices. The reverse differences are noted for the down and out case (but much smaller relative difference).
Finally we concluded that for large barrier levels, the up and in barrier option prices under SV and LV are significantly smaller than the corresponding analytical prices. An explanation was preposed that the volatility might be overestimated using the analytical expressions compared to SV/LV. This results in a higher risk that the option becomes worthless hence a lower analytical price compared to the SV and LV prices.
There are many ways to extend the research. For example, one could start by pricing more complicated barrier options (i.e. double, partial-time, look, discrete, soft, first-then etc.). Then there are also many different stochastic volatility models one could look at, one example is the Heston model. For the local volatility we used Tikhonov regularization (which was very noisy and yielded not that satisfactory results) and the Crank-Nicholson scheme but there are many other methods available (i.e. PDE, parameterization methods etc.).

9 References

[1] Nilsson, E. Calibration of the Volatility Surface. Masters Thesis, Royal Insititute of Technology, Stockholm, 2008

[2] Gatheral, J. The volatility surface: A Practitioner's Guide, John Wiley & Sons, Inc., 2006

[3] Hagan, P. S. Managing Smile Risk, Wilmott Magazine, 2002

[4] Vlaming, G. Pricing Options with the SABR Model. Master Thesis, University of Utrecht, Utrecht, 2008, pp. 7-8

[5] Haug, E. G. Option Pricing Formulas, second edition. McGraw-Hill, 2007, pp. 152-153

[6] Wu, T. Introduction to Smoothing Splines, 2004.

[7] Hanke, M. & Rösler, E. Computation of Local Volatilities from Regularized Dupire Equations, Fachbereich Mathematik, Johannes Gutenberg-Universität, 2004, pp. 1-9

[8] Majmin, L. Local and Stochastic Volatility Models: An Investigation into the Pricing of Exotic Equity Options, Master Thesis, University of the Witwatersrand, Johannesburg, 2005

[9] Blix, M. Essays in Mathematical Finance Modeling the Futures Price, Dissertation, 2004, pp. 45-50

[10] Hull, John C. Options, Futures, and Other Derivatives, 6th edition. Pearson International, 2006, pp. 535