Generating Functions

If you take $f(x) = 1/(1-x-x^2)$ and expand it as a power series by long division, say, then you get $f(x) = 1/(1-x-x^2) = 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\cdots$. It certainly seems as though the coefficient of x^n in the power series is the n-th Fibonacci number. This is not difficult to prove directly, or using techniques discussed below. You can think of the process of long division as *generating* the coefficients, so you might want to call $1/(1-x-x^2)$ the generating function for the Fibonacci numbers. The actual definition of generating function is a bit more general. Since the closed form and the power series represent the same function (within the circle of convergence), we will regard either one as being *the* generating function.

Definition of generating function. The *generating function* for the sequence $a_0, a_1, \ldots$ is defined to be the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

That is, the generating function for the sequence $a_0, a_1, \ldots$ is the function whose power series representation has a_n as the coefficient of x^n. We'll call $a_0, a_1, \ldots$ the *sequence generated* by $f(x)$. We will not be concerned with matters of convergence, and instead treat these as *formal* power series. Perhaps "symbolic" would be a better word than formal.

When determining the sequence generated by a generating function, you will want to get a formula for the n-the term (that is, for the coefficient of x^n), rather than just computing numerical values for the first few coefficients.

Useful facts.
- If $x \neq 1$ then $1+x+x^2+\cdots+x^r = \frac{x^{r+1}-1}{x-1}$ (although it is often best not to do anything with sums of only a few terms).
- $\frac{1}{1-x} = 1+x+x^2+\cdots$.

Both of these facts can be proved by letting S be the sum, calculating $xS - S$, and doing a little bit of algebra.

The second fact above says that $\frac{1}{1-x}$ is the generating function for the sequence $1, 1, 1, 1, \ldots$. It also lets you determine the sequence generated by many other functions. For example:
- $\frac{1}{1-ax} = 1 + a + a^2x^2 + a^3x^3 + \cdots = \sum_{n=0}^{\infty} a^n x^n$. To see this, substitute $y = ax$ into the series expansion of $\frac{1}{1-y}$. Thus $\frac{1}{1-ax}$ is the generating function for the sequence $a^0, a^1, a^2, \ldots$. The coefficient of x^n is a^n.

- $\frac{1}{1+ax} = 1 - a + a^2x^2 - a^3x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n a^n x^n$. To see this, substitute $y = (-a)x$ into the series expansion of $\frac{1}{1-y}$. Thus $\frac{1}{1+ax}$ is the generating function for the sequence $a^0, -a^1, a^2, \ldots$. The coefficient of x^n is $(-a)^n = (-1)^n x^n$.

More useful facts.

- $\left(\frac{1}{1-x}\right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$.
- $\left(\frac{1}{1-ax}\right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} a^n x^n$.
- $\left(\frac{1}{1+ax}\right)^k = \sum_{n=0}^{\infty}(-1)^n \binom{n+k-1}{n} a^n x^n$.

To see the first of these, consider the LHS, $(1 + x + x^2 + \cdots)^k$, multiplied out, but not simplified, Each term is a product of k (possibly different) powers of x (remembering that $x^0 = 1$). By the rules of exponents, $x^{n_1}x^{n_2}\cdots x^{n_k} = x^{n_1+n^2+\cdots+n_k}$. Thus, after simplifying, there is a term x^n for every way of expressing n as a sum of k numbers each of which is one of $0, 1, 2, \ldots$. That is, the number of terms x^n equals the number of integer solutions to $n_1 + n^2 + \cdots + n_k = n,\ 0 \leq n_i,\ \forall i$. We know that this number is $\binom{n+k-1}{k-1}$ (count using "bars and stars").

To see the second fact, let $y = ax$ in $\left(\frac{1}{1-y}\right)^k$. (This is the same idea as before.) To see the third fact, let let $y = -ax$ in $\left(\frac{1}{1-y}\right)^k$.

Multiplying a generating function by a constant multiplies every coefficient by that constant. For example, $\frac{3}{1+5x} = 3\frac{1}{1+5x} = 3(1 - 5x + 5^2x^2 - 5^3x^3 + \cdots) = 3 - 3\cdot 5x + 3\cdot 5^2x^2 - 3\cdot 5^3x^3 + \cdots = \sum_{n=0}^{\infty} 3\cdot(-1)^n 5^n x^n$. The coefficient of x^n is $3\cdot(-1)^n 5^n$.

Multiplying a generating function by x^k "shifts" the coefficients by k. This has the effect of introducing k zeros at the start of the sequence generated. For example,

$$\begin{aligned}
\left(\frac{x}{1+7x}\right)^4 &= x^4\left(\frac{1}{1+7x}\right)^4 \\
&= x^4 \sum_{n=0}^{\infty}(-1)^n \binom{n+4-1}{n} 7^n x^n \\
&= \sum_{n=0}^{\infty}(-1)^n \binom{n+4-1}{3} 7^n x^{n+4} \\
&= \sum_{t=4}^{\infty}(-1)^{t-4} \binom{t-1}{3} 7^{t-4} x^t.
\end{aligned}$$

(Let $t = n + 4$ in the second last sum. If $n = 0$ then $t = 4$, so that gives the lower limit for the sum. If $n \to \infty$, so does t, which gives the upper limit. And $n = t - 4$.) Since $(-1)^{t-4} = (-1)^t$, the last sum equals: $\sum_{t=4}^{\infty}(-1)^t \binom{t-1}{3} 7^{t-4} x^t$. The coefficient of x^t is zero if $t < 4$, and $(-1)^t\binom{t-1}{3}7^{t-4}$ if $t \geq 4$.

If you add two generating functions together, the coefficient of x^n in the sum is what you would expect, the sum of the coefficients of x^n in the summands. For example, adding the generating functions from the above two paragraphs gives:

$\frac{3}{1+5x} + \left(\frac{x}{1+7x}\right)^4$
$$= \sum_{n=0}^{\infty} 3(-1)^n 5^n x^n + \sum_{n=4}^{\infty} (-1)^n \binom{n-1}{3} 7^{n-4} x^n$$
$$= 3 - 3(5^1)x + 3(5^2)x^2 - 3(5^3)x^3 + \sum_{n=4}^{\infty} \left(3(-1)^n 5^n + (-1)^n \binom{n-1}{3} 7^{n-4}\right) x^n$$
$$= 3 - 3(5^1)x + 3(5^2)x^2 - 3(5^3)x^3 + \sum_{n=4}^{\infty} (-1)^n \left(3 \cdot 5^n + \binom{n-1}{3} 7^{n-4}\right) x^n.$$
Thus, if $n \leq 3$, the coefficient of x^n is $3(-1)^n 5^n$, and if $n \geq 4$ it is $(-1)^n \left(3 \cdot 5^n + \binom{n-1}{3} 7^{n-4}\right)$.

The above describes most of the basic tools you need. *When trying to determine what sequence is generated by some generating function, your goal will be to write it as a sum of known generating functions, some of which may be multiplied by constants, or constants times some power of x.* Once you've done this, you can use the techniques above to determine the sequence. Most of the time the known generating functions are among those described above. Occasionally you may need to directly compute the product of two generating functions:

Cauchy Product.
$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$
To see this, carry out the multiplication and simplify.

The last tool you will need allows you to write a product of (known) generating functions as a sum of known generating functions, some of which may be multiplied by constants, or constants times some power of x. It is the method of **partial fractions**, which you should have learned while studying Calculus. We will do a couple of examples in the course of this section, but won't give a comprehensive treatment of the method. For more info, look in (almost) any Calculus textbook.

Example 1. What sequence is generated by $\frac{2+x}{1-x-8x^2+12x^3}$?

The first task is to factor the denominator. We want to write it as a product of terms of the form $(1 - ax)$. Since the constant term is 1, if the a's are all integers then they are among the divisors of the coefficient of the highest power of x, and their negatives. (This is similar to the hints for finding integer roots of polynomials discussed in the section on solving recurrences. To see it, imagine such a factorisation multiplied out and notice how the coefficient of the highest power of x arises.) Here, we find that $1 - x - 8x^2 + 12x^3 = (1 + 3x)(1 - 2x)^2$. Thus, we want to use partial fractions to write $\frac{2+x}{(1+3x)(1-2x)^2}$ as a sum of “known” generating functions. These arise from the factors in the denominator. Using partial fractions

$$\frac{2+x}{(1+3x)(1-2x)^2} = \frac{A}{1+3x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$
$$= \frac{A(1-2x)^2+B(1+3x)(1-2x)+C(1+3x)}{(1+3x)(1-2x)^2}$$

Thus, $2 + x = A(1-2x)^2 + B(1+3x)(1-2x) + C(1+3x)$. Expanding the RHS and equating like powers of x on the LHS and RHS gives

$$A + B + C = 2, \quad -4A + B + 3C = 1, \quad 4A - 6B = 0$$

The solution is $A = 3/5,\ B = 2/5,\ C = 1$. Therefore,

$$\frac{2+x}{(1+3x)(1-2x)^2} = (3/5)\frac{1}{1+3x} + (2/5)\frac{1}{1-2x} + \left(\frac{1}{1-2x}\right)^2.$$

The coefficient of x^n on the RHS is $(3/5)(-1)^n3^n + (2/5)2^n + \binom{n+1}{n}2^n = (3/5)(-1)^n3^n + (2/5)2^n + (n+1)2^n$. Thus the sequence generated is $(3/5)(-1)^n3^n + (2/5)2^n + (n+1)2^n$.

Deriving generating functions from recurrences. If you are given a sequence defined by a recurrence relation and initial conditions, you can use these to get a generating function for the sequence. Having done that, you can then apply the facts and methods above to get a formula for the n-th term of the sequence. This is **another method of solving recurrences**. We'll illustrate the method first, and then try to give a description of it below. It would be wise to work through the example twice: once before reading the description of the method, and once after. That way you should be able to recognize (and understand) the major steps.

Example 2. Find the generating function for the sequence a_n defined by $a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3},\ n \geq 3$, with initial conditions $a_0 = 2$, $a_1 = 3$, and $a_3 = 19$.

Let $g(x) = \sum_{n=0}^{\infty} a_nx^n$
$= 2 + 3x + 19x^2 + \sum_{n=3}^{\infty} a_nx^n$
$= 2 + 3x + 19x^2 + \sum_{n=3}^{\infty}(a_{n-1} + 8a_{n-2} - 12a_{n-3})x^n$
$= 2 + 3x + 19x^2 + \sum_{n=3}^{\infty} a_{n-1}x^n + \sum_{n=3}^{\infty} 8a_{n-2}x^n - \sum_{n=3}^{\infty} 12a_{n-3}x^n$
$= 2 + 3x + 19x^2 + x\sum_{n=3}^{\infty} a_{n-1}x^{n-1} + 8x^2\sum_{n=3}^{\infty} a_{n-2}x^{n-2} - 12x^3\sum_{n=3}^{\infty} a_{n-3}x^{n-3}$
$= 2 + 3x + 19x^2 + x\sum_{k=2}^{\infty} a_kx^k + 8x^2\sum_{k=1}^{\infty} a_kx^k - 12x^3\sum_{k=0}^{\infty} a_kx^k$
$= 2+3x+19x^2+\left(x\sum_{k=0}^{\infty} a_kx^k\right) - x(2+3x) + 8x^2\left(\sum_{k=0}^{\infty} a_kx^k\right) - 8x^2(2) - 12x^3\sum_{k=0}^{\infty} a_kx^k$
$= 2 + x + xg(x) + 8x^2g(x) - 12x^3g(x)$.

Therefore $g(x) - xg(x) - 8x^2g(x) - 12x^xg(x) = 2+x$. That is, $g(x)(1-x-8x^2-12x^3) = 2+x$, so $g(x) = \frac{2+x}{1-x-8x^2-12x^3}$ is the generating function. You can now apply the method in Example 1 (since this *is* the generating function from Example 1) to find that $a_n = (3/5)(-1)^n3^n + (2/5)2^n + (n+1)2^n$.

The main idea is to let $g(x)$ be the generating function, say $g(x) = \sum_{n=0}^{\infty} a_n x^n$, and then use the recurrence, initial conditions, manipulation of the sum(s), and algebra to convert the RHS into an expression which is a sum of terms some of which involve $g(x)$ itself. The following steps are usually involved (and in this order):

(1) Split the terms corresponding to the initial values from the sum on the RHS. The sum should then run from the first integer for which the recursive definition takes effect.

(2) Substitute the recursive definition of a_n into the summation.

(3) Split the sum into several sums, and in each one factor out the constant times a high enough power of x so that the subscript on the coefficient and the exponent of x are equal.

(4) Make a change of variable in each summation so each is the sum of $a_n x^n$ from some value to infinity. One of the sums should start at 0, another at one, and so on, until finally one starts at the subscript of the "last" initial value (assuming consecutive initial values are given). Also, the sum that starts at 0 should be multiplied by a constant (maybe 1), the one that starts at 1 should be multiplied by a constant (maybe 1) times x, and in general each sum that starts at t should be multiplied by a constant (maybe 1) times x^t.

(5) By adding the "missing" (first few) terms, and subtracting them off outside the sum (don't forget to multiply by whatever is in front of the summation!), convert each sum so that it goes from 0 to infinity. That is, do algebra so that each sum involving a term of the recurrence equals $g(x)$. If the recurrence is non-homogeneous, you may need to do the same sort of thing to convert the sum(s) arising from the non-homogeneous term(s) into known sums.

(6) Collect all terms involving $g(x)$ on the LHS, factor $g(x)$ out (if possible), then divide (or do whatever algebra is needed), to obtain a closed form for the generating function.

You might have noticed that the characteristic equation for the recurrence in Example 2 is $x^3 - x^2 - 8x + 12$, while the denominator for the generating function is $1 - x - 8x^2 - 12x^3$. It is always true that *if the characteristic equation for a LHRRWCC is* $a_0 x^k + a_1 x^{k-1} + \cdots + a_{k-1}$, *then the denominator for (some form of) the generating function is* $a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^k$. This follows from the general method described above.

Once you have the generating function, you can factor the denominator and (hopefully) use partial fractions to write it as a sum of multiples of known generating functions. By finding the coefficient of x^n in each summand, you get a formula for a_n.

Example 3. Solve $a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n,\ n \geq 2$ with initial conditions $a_0 = 0$

and $a_1 = 6$.

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$
$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x_n$
$= 0 + 6x + \sum_{n=2}^{\infty}(-3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n)x^n$
$= 6x - 3x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 10x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} + 3 \sum_{n=2}^{\infty} 2^n x^n$
$= 6x - 3x \sum_{k=1}^{\infty} a_k x^k + 10x^2 \sum_{k=0}^{\infty} a_k x^k + 3 \sum_{n=2}^{\infty} 2^n x^n$
$= 6x - 3x\Big(\sum_{k=1}^{\infty} a_k x^k\Big) - (-3x)(0) + 10x^2 \sum_{k=0}^{\infty} a_k x^k + 3 \sum_{n=2}^{\infty} 2^n x^n$
$= 6x - 3xg(x) + 10x^2 g(x) + 3\Big(\sum_{n=0}^{\infty} 2^n x^n\Big) - 3(1 + 2x)$
Therefore, $g(x)(1 + 3x - 10x^2) = -3 + \frac{3}{1-2x}$, so $g(x) = \frac{-3}{1+3x-10x^2} + \frac{3}{(1-2x)(1+3x-10x^2)}$
$= -3\frac{1}{(1+5x)(1-2x)} + 3\frac{1}{(1-2x)^2(1+5x)} = 3\frac{1-(1-2x)}{(1-2x)^2(1+5x)} = \frac{6x}{(1-2x)^2(1+5x)}$.
We need to expand the RHS using partial fractions.
$\frac{6x}{(1-2x)^2(1+5x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x} = \frac{A(1-2x)(1+5x)+B(1+5x)+C(1-2x)^2}{(1-2x)^2(1+5x)}$.
Thus $6x = A(1 - 2x)(1 + 5x) + B(1 + 5x) + C(1 - 2x)^2$. Expanding the RHS and then equating coefficients of like powers of x yields:

$$A + B + C = 0, \quad 3A + 5B - 4C = 6, \quad -10A + 4C = 0$$

The solution is $A = -12/49$, $B = 6/7$, $C = -30/49$. Thus, $g(x) = (-12/49)\frac{1}{1-2x} + (6/7)\frac{1}{(1-2x)^2} + (-30/49)\frac{1}{1+5x}$, so $a_n = (-12/49)2^n + (6/7)\binom{n+1}{1}2^n - (30/49)(-1)^n 5^n = (-12/49)2^n + (6/7)(n+1)2^n + (30/49)(-1)^{n+1}5^n$.

Using generating functions to solve counting problems. In showing that $(1 + x + x^2 + \cdots)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$ we argued that the coefficient of x^n equals the number of ways of writing n as a sum of k integers, each of which occurs as an exponent in one of the factors on the LHS. Stated slightly differently, $\left(\frac{1}{1-x}\right)^k$ is the generating function for the number of ways to write n as a sum of k non-negative integers.

Continuing this line of reasoning, the generating function for the number of ways to write n as a sum of five odd integers would be $(x + x^3 + x^5 + \cdots)^5 = [x(1 + x^2 + x^4 + \cdots)]^5 = \left(\frac{x}{1-x^2}\right)^5$. (Think about what happens with the exponents when the LHS is multiplied out.) Similarly, the generating function for the number of ways to write n as a sum of two odd integers and an even integer would be $(x + x^3 + x^5 + \cdots)^2(1 + x^2 + x^4 + \cdots) = \frac{x^2}{(1-x)^3}$.

In general, suppose you want the generating function for the number of ways to write n as an (ordered) sum of k terms, some of which may be restricted (as in being odd, for example). Then the generating function will be a product of k factors each of which is a sum of powers of x. The exponents of x in the first factor will be the possibilities for the first term in sum (that adds to n), the exponents of x in the second factor will be the possibilities for the second term in the sum, and so on until, finally, the exponents in the k-th term are the possibilities for the k-th term in the sum.

Remember that $x^0 = 1$, so a factor has a 1 if and only if zero is one of the possibilities for the corresponding term in the sum. Similarly for $x^1 = x$, and so on.

Example 4. Determine the generating function for the number of ways to distribute a large number of identical candies to four children so that the first two children receive an odd number of candies, the third child receives at least three candies, and the fourth child receives at most two candies.

Suppose the number of candies distributed is n. Then we are looking for the generating function for the number of ways to write n as a sum of four integers, the first two of which are odd, the third of which is at least 3, and the fourth of which is at most 2. Following the reasoning from above, the generating function is a product of four factors, and the exponents of x in each factor give the possible values for the corresponding term in the sum. Thus, the generating function is $(x + x^3 + x^5 + \cdots)^2(x^3 + x^4 + \cdots)(1 + x + x^2)$ $= \left(\frac{x}{1-x^2}\right)^2 \frac{x^3}{1-x}(1 + x + x^2)$.

Once you have the generating function for the number of ways to do something, you can apply the methods you know (*i.e.* write the generating function as a sum of known generating functions) to determine the cooefficient of x^n, and hence the number of ways.

Example 4 continued. (This gets a touch ugly at the end, but it is a good illustration of the methods, so hang with it.) Use the generating function to determine the number of ways.

The generating function is
$g(x) = \left(\frac{x}{1-x^2}\right)^2 \frac{x^3}{1-x}(1+x+x^2) = (x^2+x^3+x^4)\left(\frac{1}{(1-x^2)^2(1-x)}\right) = (x^2+x^3+x^4)\left(\frac{1}{(1+x)^2(1-x)^3}\right)$.
An exercise in partial fractions shows that:
$\frac{1}{(1+x)^2(1-x)^3} = (1/8)\left(\frac{1}{1-x}\right)^2 + (3/16)\left(\frac{1}{1-x}\right) + (1/4)\left(\frac{1}{1+x}\right)^3 + (1/4)\left(\frac{1}{1+x}\right)^2 + (3/16)\left(\frac{1}{1+x}\right)$.
Since these are all known generating functions, the coefficient of x^n in *this* expression (which is not $g(x)$) is $(1/8)+(3/16)(n+1)+(1/4)(-1)^n\binom{n+3-1}{3-1}+(1/4)(-1)^n(n+1)+(3/16)(-1)^n$.
We need the coefficient of x^n in $g(x)$, which equals $\frac{1}{(1+x)^2(1-x)^3}$ multiplied by $(x^2+x^3+x^4)$.
Thus, the coefficient of x^n in $g(x)$ is the sum of the coefficient of x^{n-2} in $\frac{1}{(1+x)^2(1-x)^3}$, the coefficient of x^{n-3} in $\frac{1}{(1+x)^2(1-x)^3}$, and the coefficient of x^{n-4} in $\frac{1}{(1+x)^2(1-x)^3}$. This equals $a + b + c$, where

$a = (1/8) + (3/16)(n-1) + (1/4)(-1)^{n-2}\binom{n}{2} + (1/4)(-1)^{n-2}(n-1) + (3/16)(-1)^{n-2}$,
$b = (1/8) + (3/16)(n-2) + (1/4)(-1)^{n-2}\binom{n-1}{2} + (1/4)(-1)^{n-3}(n-2) + (3/16)(-1)^{n-3}$,
$c = (1/8) + (3/16)(n-3) + (1/4)(-1)^{n-4}\binom{n-2}{2} + (1/4)(-1)^{n-4}(n-3) + (3/16)(-1)^{n-4}$.
No doubt this expression can be simplified.

Example 5. Use generating functions to find b_n, the number of ways that $n \geq 0$ identical candies can be distributed among 4 children and 1 adult so that each child receives an odd number of candies, and the adult receives 1 or 2 candies.

Following the method outlined above, the generating function is

$$g(x) = (x + x^3 + x^5 + \cdots)^4(x + x^2) = \left(\frac{x}{1-x^2}\right)^4(x + x^2)$$
$$= (x^5 + x^6)\sum_{k=0}^{\infty}\binom{k+4-1}{3}x^{2k}$$
$$= \sum_{k=0}^{\infty}\binom{k+3}{3}x^{2k+5} + \sum_{k=0}^{\infty}\binom{k+3}{3}x^{2k+6}$$

The first sum contains all terms with odd exponents, and the second sum contains all terms with even exponents. Thus, if $n \leq 4, b_n = 0$. If $n \geq 5$ and odd, $b_n = \binom{n+1/2}{3}$ (let $n = 2k+5$ in the first sum). If $n \geq 6$ and even, $b_n = \binom{n/2}{3}$ (let $n = 2k+6$ in the second sum). These cases can all be described by the single expression $b_n = \binom{\lceil n/2 \rceil}{3}$.

Example 6. In a certain game it is possible to score 1, 2, or 4 points on each turn. Find the generating functions for the number of ways to score n points in a game in which there are at least two turns where 4 points are scored.

Here we are looking for the number of ways to write n as an ordered sum of three terms. The first term represents the number of points obtained from turns where 1 point was scored. The second term represents the number of points obtained from turns where 2 points were scored (and thus is a multiple of 2). The third term represents the number of points obtained from turns where 4 points were scored (and thus is a multiple of 4, and at least 8). Thus, the generating function will be a product of three factors, where the exponents in each factor correspond to the possibilities just discussed. Therefore,

$$g(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(x^8 + x^{12} + x^{16} + \cdots) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{x^8}{1-x^4}.$$

If you want an exercise in partial fractions, continue with Example 6 and determine the number of ways by writing $g(x)$ as a sum of known generating functions and finding the coefficient of x^n. This involves a 6×6 linear system (unless you makle an astute observation that is eluding me right now).