Lecture 1
Graphical Models for Dependence Relationships
An Introduction

Causal Inference Using Graphs
August 6, 2019

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Goals and Objectives for This Course:

- Introduce causal graphs
- Show how causal graphs help us think about standard techniques
- Show how causal graphs help us think about not yet standard techniques

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Day by Day Plan

1. Nonparametrics
2. Sketching with strong assumptions
3. Weakening the assumptions
4. Non-standard stuff

How I Use Graphs for Causal Inference

Two Possible Metaphors:

1. Draw with graphs, ink with potential outcomes
2. Sketch with graphs, paint with potential outcomes

How I Use Graphs for Causal Inference

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Many of the mistakes I see in seminar could be avoided if the author(s) sketched a bit before starting to paint.

Acknowledgements

Many of the slides from lectures 1 and 2 today were written by Kevin Quinn.

Goals and Objectives for This Morning:

- Introduce graphical notation and terminology
- Build intuition about properties of probabilistic systems represented as directed graphs
- Provide some motivating examples

Overview

1. **Why Study Graphical Models?**

2. **Definitions and Terminology**
   - Basic Definitions
   - Graph Drawing Conventions
   - Some Additional Definitions
   - Familial Relations Specific to Directed Graphs

3. **Directed Acyclic Graphs (DAGs)**
   - Using DAGs to Represent Complicated Joint Distributions
   - Observational Equivalence
   - Conditional Independence and *d*-Separation
   - Faithfulness

What Application Areas Make Use of Graphical Models?

Graphical models are widely used in a variety of application areas:

- Social network analysis
- Causal inference
- Pattern recognition and machine learning
- Information retrieval
- Document summarization
- Multivariate analysis
- ...

Some Questions that are Easy to Answer Using Graphical Models

- Are X and Y marginally independent?

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- What's the full conditional distribution of $[\theta_1 | y, \theta_2]$?
- What will happen to Y if a variable X is set to x by outside intervention?
- Which background variables need to be adjusted for in order to get a consistent estimate of the causal effect of X on Y?

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- express complex mathematical systems with an equivalent visual representation
- read off non-trivial mathematical properties of the system directly from the graph
- simplify complicated computations by taking account of the structure of the graph
- present substantive assumptions in a transparent manner

Basic Definitions

Definition (Directed Graph)

A directed graph $\mathcal{G}$ is a pair $\langle \mathcal{V}, \mathcal{E} \rangle$ where $\mathcal{V}$ is a finite set of vertices (a.k.a. nodes) and $\mathcal{E}$ is the set of directed edges (a.k.a. directed arcs or directed links).

Each directed edge in $\mathcal{E}$ is an ordered pair of distinct vertices from $\mathcal{V} \times \mathcal{V}$.

A directed edge $(V_i, V_j) \in \mathcal{E}$ is also denoted $V_i \rightarrow V_j$.

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In this class, we will think of each $V \in \mathcal{V}$ as being a (possibly non-scalar) random variable and each directed edge $(V_i, V_j) \in \mathcal{E}$ as some relationship between V_i and V_j.

This will be made more precise below.

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Note the difference between "(un)observed" and "(un)observable".

Example

Example (Drawing a Simple Directed Graph)

Consider the simple directed graph with vertices $\{X, Y, Z\}$ and edges $\{X \to Y,\ Z \to X,\ Z \to Y\}$ with X and Y observable and Z unobservable.

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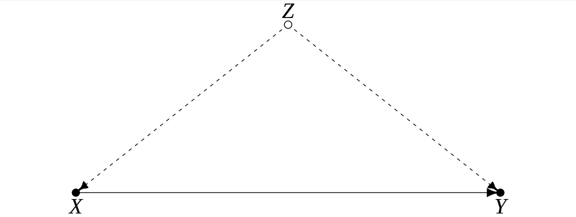
We would express this graph visually as:



Some Additional Definitions

Definition (Path)

A path from V_i to V_j in a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a sequence of distinct nodes $V_i = X_0, \ldots, X_n = V_j$ such that $(X_{k-1}, X_k) \in \mathcal{E}$ or $(X_k, X_{k-1}) \in \mathcal{E}$ for each $k = 1, \ldots, n$. We write $V_i \sim V_j$ to denote a path from V_i to V_j.

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Note that the direction of the edges does not matter and that a path can't visit the same node more than once.

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Example

A path from V_1 to V_3 exists in each of the following four graphs.

$$V_1 \rightarrow V_2 \rightarrow V_3$$
$$V_1 \leftarrow V_2 \rightarrow V_3$$
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A directed path from V_i to V_j in a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a sequence of distinct nodes $V_i = X_0, \ldots, X_n = V_j$ such that $(X_{k-1}, X_k) \in \mathcal{E}$ and $(X_{k-1}, X_k) \notin \mathcal{E}$ for each $k = 1, \ldots, n$. We write $V_i \rightsquigarrow V_j$ to denote a directed path from V_i to V_j.

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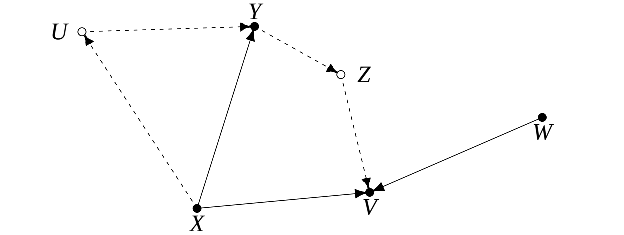
Put more informally, a directed path from V_i to a V_j is a path from V_i to a V_j in which all the edges on the path have arrows pointing toward V_j.

Example

Example (Paths and Directed Paths)

Consider the graph depicted below.



Here there are 3 paths from X to Z:
1) $(X \rightarrow U \rightarrow Y \rightarrow Z)$,
2) $(X \rightarrow Y \rightarrow Z)$, and
3) $(X \rightarrow V \leftarrow Z)$.

Of these, only $(X \rightarrow U \rightarrow Y \rightarrow Z)$ and $(X \rightarrow Y \rightarrow Z)$ are directed paths from X to Z.

Some Additional Definitions

Definition (Back-door Path)

A back-door path from V_i to V_j in a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a sequence of distinct nodes $V_i = X_0, \ldots, X_n = V_j$ such that $(X_1, X_0) \in \mathcal{E}$ and $(X_{k-1}, X_k) \in \mathcal{E}$ or $(X_k, X_{k-1}) \in \mathcal{E}$ for each $k = 2, \ldots, n$. We write $V_i \sim V_j$ to denote a path from V_i to V_j.

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Note that an arrow must point into V_1 for a back-door path, but otherwise the direction of the edges does not matter.

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Note that an arrow must point into V_1 for a back-door path, but otherwise the direction of the edges does not matter.

Example

A back-door path from V_1 to V_3 exists in the second and fourth of the following four graphs.

$$V_1 \rightarrow V_2 \rightarrow V_3$$
$$V_1 \leftarrow V_2 \rightarrow V_3$$
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$$V_1 \leftarrow V_2 \leftarrow V_3$$

Familial Relations Specific to Directed Graphs

Definition (Parents)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of parents of a node $V \in \mathcal{V}$ is defined to be:

$$\text{pa}(V) = \{Z \in \mathcal{V} : (Z, V) \in \mathcal{E}\}$$

The set of parents of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

$$\text{pa}(\mathcal{W}) = \bigcup_{V \in \mathcal{W}} \text{pa}(V)$$

Put more informally, the parents of V are all nodes from which there is a directed edge to V.

Familial Relations Specific to Directed Graphs

Definition (Ancestors)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of ancestors of a node $V \in \mathcal{V}$ is defined to be:

$$\text{an}(V) = \{Z \in \mathcal{V} : Z \rightsquigarrow V\}$$

The set of ancestors of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

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In words, the set of ancestors of V consists of the vertices from which there is a directed path to V.

Familial Relations Specific to Directed Graphs

Definition (Children)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of children of a node $V \in \mathcal{V}$ is defined to be:

$$\text{ch}(V) = \{Z \in \mathcal{V} : (V, Z) \in \mathcal{E}\}$$

The set of children of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

$$\text{ch}(\mathcal{W}) = \bigcup_{V \in \mathcal{W}} \text{ch}(V)$$

Put more informally, the children of V are all nodes to which there is a directed edge from V.

Familial Relations Specific to Directed Graphs

Definition (Descendents)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of descendents of a node $V \in \mathcal{V}$ is defined to be:

$$\mathrm{de}(V) = \{Z \in \mathcal{V} : V \rightsquigarrow Z\}$$

The set of descendents of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

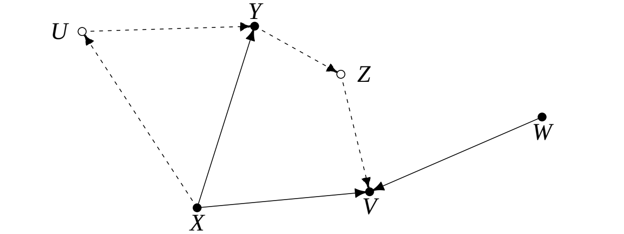
$$\mathrm{de}(\mathcal{W}) = \bigcup_{V \in \mathcal{W}} \mathrm{de}(V)$$

In words, the set of descendents of V consists of the vertices to which there exists a directed path from V.

Example

Example (Familial Relations)

Consider the graph depicted below.



Here:
$\text{pa}(Z) = \{Y\}$, $\text{ch}(Z) = \{V\}$, $\text{an}(Z) = \{U, X, Y\}$, $\text{de}(Z) = \{V\}$

and

$\text{pa}(X) = \emptyset$, $\text{ch}(X) = \{U, V, Y\}$, $\text{an}(X) = \emptyset$,
$\text{de}(X) = \{U, V, Y, Z\}$

Directed Acycylic Graphs (DAGs)

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- Relatedly, they can help one think about ways to achieve dimension reduction / data summarization
- They are natural representation of systems evolving in time
- They can be given a causal interpretation

Using DAGs to Represent Complicated Joint Distributions

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$$p(x, y, z) = p(z|x, y)p(y|x)p(x)$$

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Note that there are five other such factorizations that are possible.

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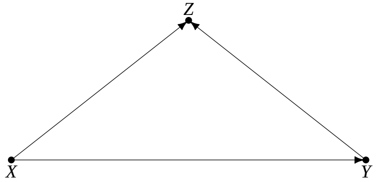
For the 3 variable example above his gives rise to:



Note the relationship to $p(x, y, z) = p(z|x, y)p(y|x)p(x)$

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It is easy to show, given our rules for forming graphs, that for a DAG with n vertices $V_1, \ldots, V_n$ one can write

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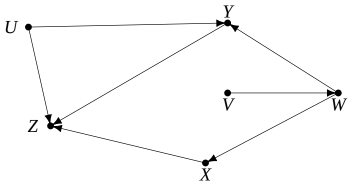
This formula allows us to look at a graph and then write down the factorization that the graph implies.

Using DAGs to Represent Complicated Joint Distributions

Let's look at a more complicated example that starts with the following graph.



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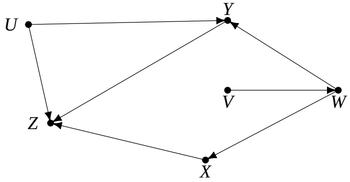

This is consistent with the factorization

$$p(u, v, w, x, y, z) = p(z|u, x, y)p(y|u, w)p(x|w)p(w|v)p(u)p(v)$$

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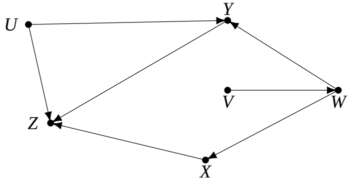


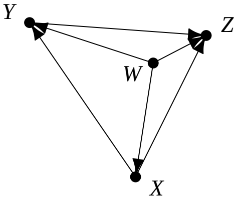
This is consistent with the factorization

$$p(u, v, w, x, y, z) = p(z|u, x, y)p(y|u, w)p(x|w)p(w|v)p(u)p(v)$$

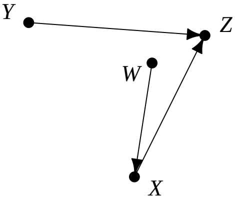
Note that the graph embodies a lot of conditional independence assumptions.

Using DAGs to Represent Complicated Joint Distributions

Question: Which graph below embodies more assumptions?



(a)



(b)

Using DAGs to Represent Complicated Joint Distributions

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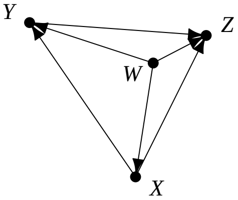

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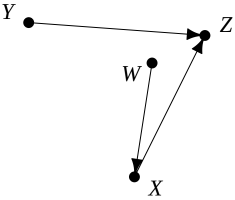

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Answer: More assumptions are implied by (b).

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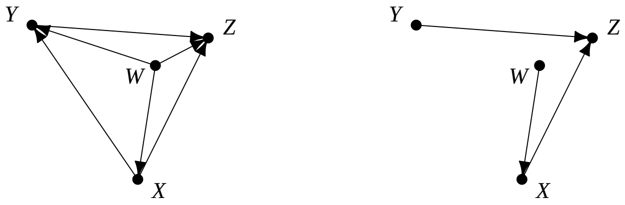

(a) (b)

Answer: More assumptions are implied by (b).

Missing edges correspond to conditional independence assumptions.

Using DAGs to Represent Complicated Joint Distributions

Returning to our three variable example, note that the factorization $p(x, y, z) = p(z|x, y)p(y|x)p(x)$ suggests a simple means (called the method of composition) of obtaining a sample from the joint distribution of (X, Y, Z):

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4. Return (x, y, z) as a draw from the joint distribution of (X, Y, Z)

Using DAGs to Represent Complicated Joint Distributions

Note that there is a graphical version of the method of composition that can be used to sample from the joint distribution of the variables governed by a DAG $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$.

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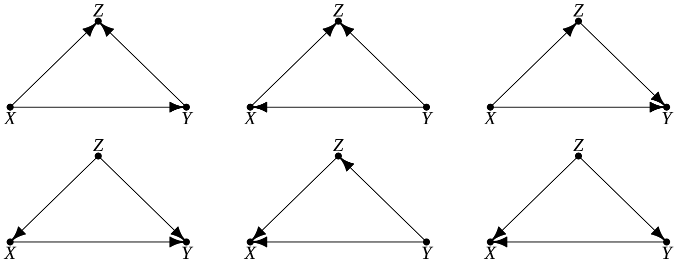
4. Return $(v_1, \ldots, v_n)$

Observational Equivalence

As an aside, note that all of the following DAGs are consistent with any choice of $p(x, y, z)$.



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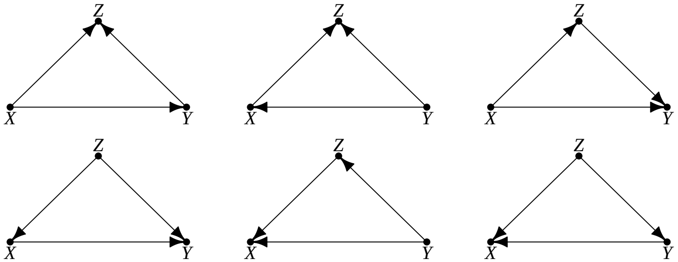

There is actually a general result here.

Goals and Objectives
Why Study Graphical Models?
Definitions and Terminology
Basic Definitions
Graph Drawing Conventions
Some Additional Definitions
Familial Relations Specific to Directed Graphs
Directed Acyclic Graphs (DAGs)
Using DAGs to Represent Complicated Joint Distributions
Observational Equivalence
Conditional Independence and *d*-Separation
Faithfulness

Observational Equivalence

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Two DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be observationally equivalent if every probability distribution that is compatible with $\mathcal{G}_1$ is also compatible with $\mathcal{G}_2$.

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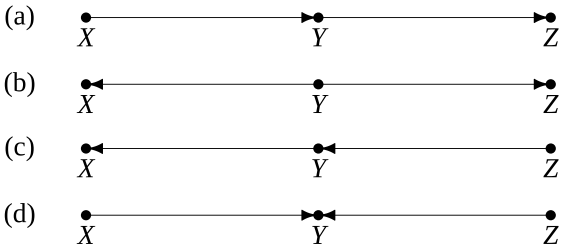
Theorem (Observational Equivalence (Verma and Pearl))

Two DAGs are observationally equivalent if and only if they have the same skeletons and the same sets of v-structures.

Example

Example (Observational Equivalence)

Consider the four graphs below.



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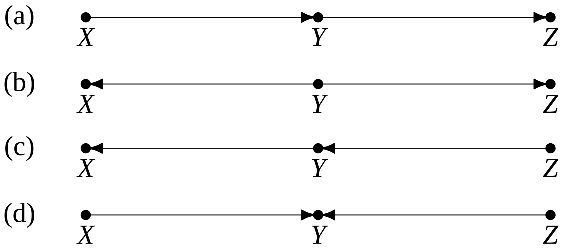

Here we see that graphs (a), (b), and (c) are all observationally equivalent. *Graph (d) is not observationally equivalent to any of the other graphs.*

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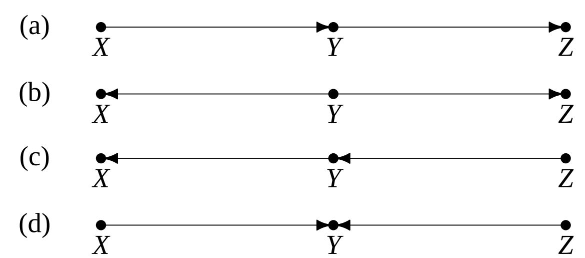

Here we see that graphs (a), (b), and (c) are all observationally equivalent. *Graph (d) is not observationally equivalent to any of the other graphs.*

Note that graphs (a), (b), and (c) imply very different causal relationships.

Observational Equivalence

Some things to think about:

- A DAG implies a particular factorization but an unfactorized joint distribution implies a wide range of DAGs.
- Because a DAG gives a recipe for generating data from a joint distribution it is tempting to think of a DAG (by itself) in causal terms. This is a mistake because of the point immediately above.
- Interpreting a DAG causally can be reasonable—we'll talk about this extensively in later today—but it does require additional assumptions.

Conditional Independence

Definition (Conditional Independence)

Let $\mathcal{V} = \{V_1, \ldots, V_n\}$ be a finite set of random variables and let X, Y, and Z denote three subsets of $\mathcal{V}$. If

$$p(x|y,z) = p(x|z) \quad \text{whenever} \quad p(y,z) > 0$$

we say that X is conditionally independent of Y given Z and write

$$[X \perp\!\!\!\perp Y | Z]$$

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It is easy to see that $[X \perp\!\!\!\perp Y | Z]$ also implies that

$$p(x,y|z) = p(x|z)p(y|z)$$

Conditional Independence

The following are some properties that hold for conditionally independent random variables.

Property (Symmetry)

$$[X \perp\!\!\!\perp Y | Z] \implies [Y \perp\!\!\!\perp X | Z]$$

Property (Decomposition)

$$[X \perp\!\!\!\perp (Y, W) | Z] \implies [X \perp\!\!\!\perp Y | Z]$$

Property (Weak Union)

$$[X \perp\!\!\!\perp (Y, W) | Z] \implies [X \perp\!\!\!\perp Y | (Z, W)]$$

Property (Contraction)

$$[X \perp\!\!\!\perp Y | Z] \ \& \ [X \perp\!\!\!\perp W | (Z, Y)] \implies [X \perp\!\!\!\perp (Y, W) | Z]$$

The Importance of Conditional Independence

Conditional independence is fundamental to almost all areas of probability, statistics, and data analysis.

- causal inference
- Bayesian inference
- data summarization
- Markov chains
- sufficiency
- ancillarity
- identification
- etc.

See Dawid (1979). “Conditional Independence in Statistical Theory” *JRSS B*. for additional information.

d-Separation

Definition (d-Separation)

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a DAG and X, Y, and Z be disjoint subsets of $\mathcal{V}$.

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X is said to be *d*-separated from Y by Z in $\mathcal{G}$ (written $[X \perp\!\!\!\perp Y | Z]_{\mathcal{G}}$) if and only if Z blocks every path from a vertex in X to a vertex in Y.

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A path p is said to be blocked by a set of vertices Z if and only if at least one of the following conditions hold:

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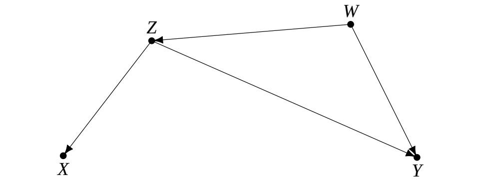
If X is *not* *d*-separated from Y by Z we say that X is *d*-connected to Y by Z.

Example

Example (d-Separation)

Consider the graph below.



Which sets of variables (if any) d-separate X from Y?

Example

Let's draw the paths between X and Y and check to see if various conditioning sets block these paths.

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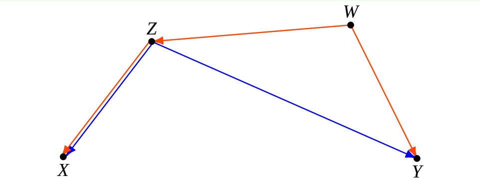

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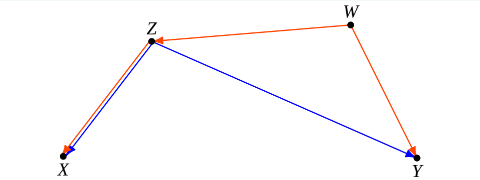

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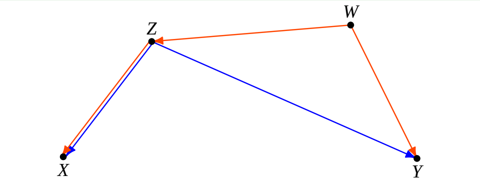

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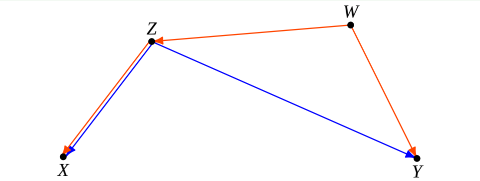

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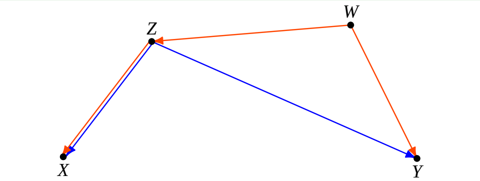

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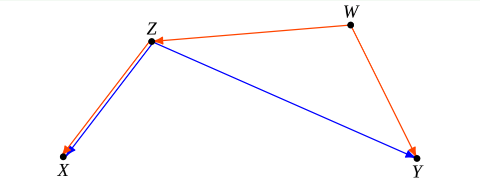

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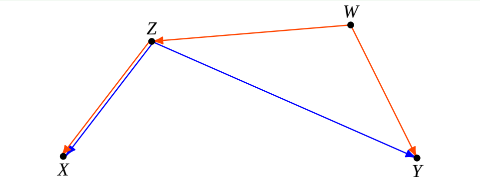

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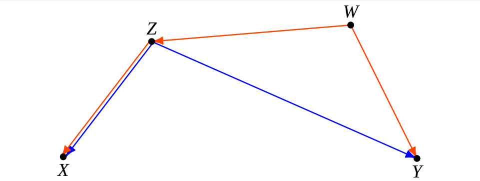

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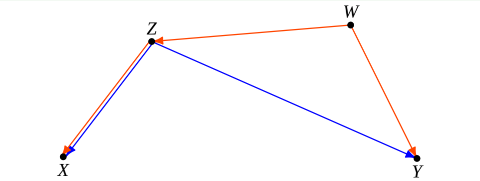

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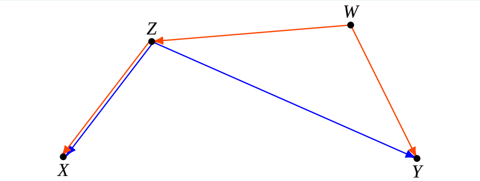

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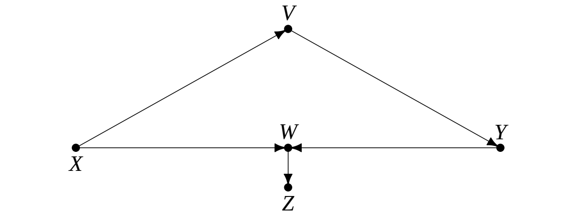

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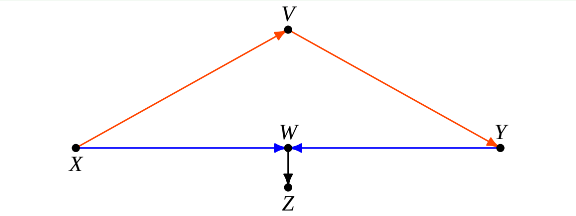

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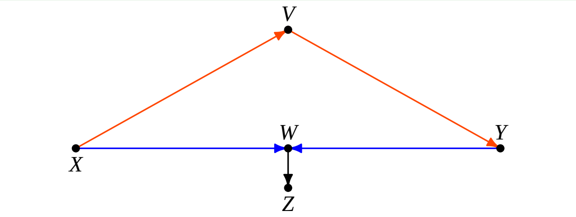

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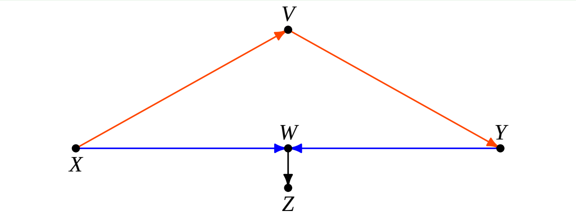

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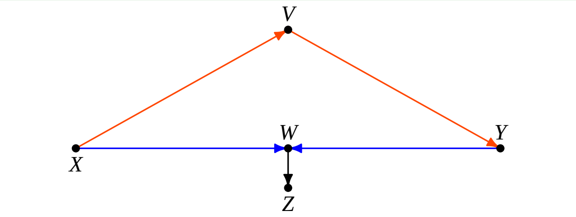

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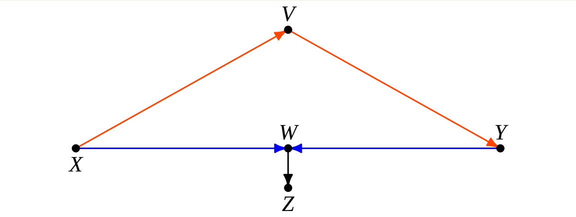

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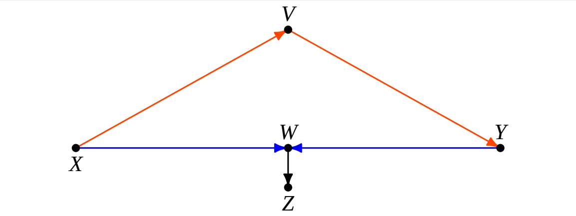

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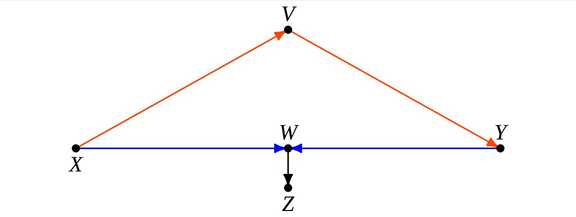

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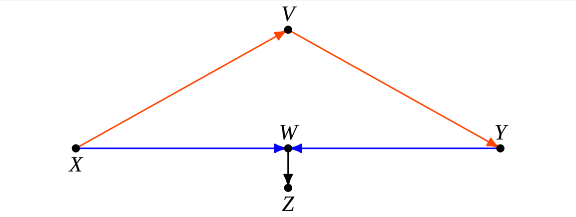

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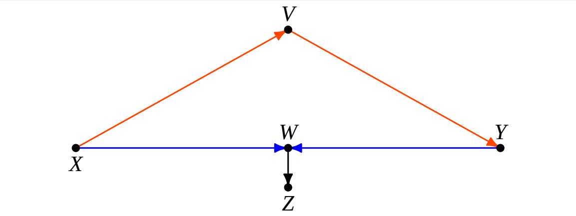

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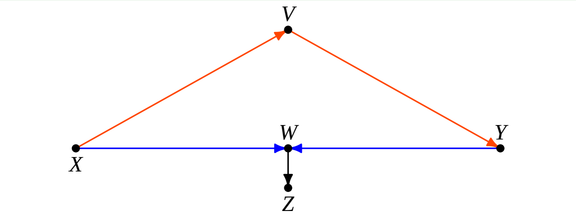

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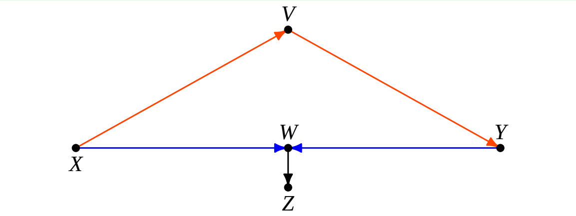

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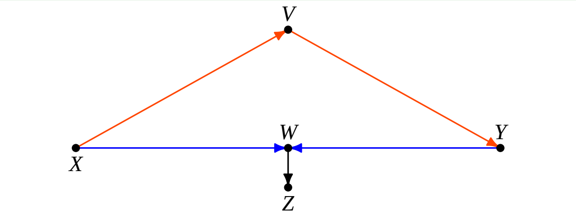

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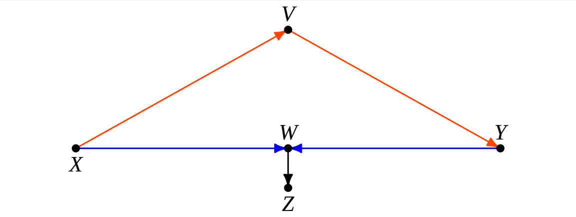

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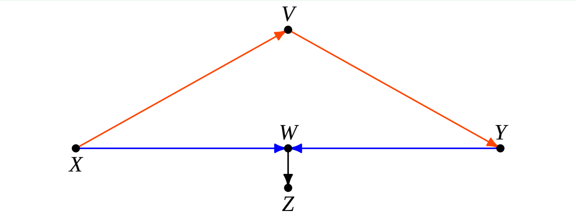

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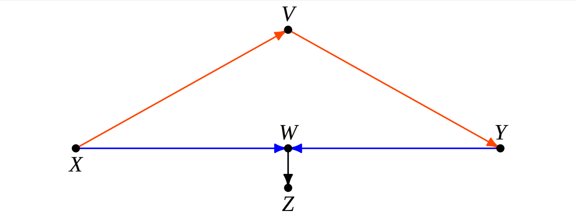

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Probabilistic Implications of *d*-Separation

Theorem (Probabilistic Implications of *d*-Separation (Verma and Pearl))

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a DAG and X, Y, and Z be disjoint subsets of $\mathcal{V}$.

If Z d-separates X from Y in $\mathcal{G}$, then X is conditionally independent of Y given Z in every distribution compatible with $\mathcal{G}$.

Conversely, if X and Y are not d-separated by Z in $\mathcal{G}$, then X and Y are conditionally dependent given Z in at least one distribution compatible with $\mathcal{G}$.

Example

Example (*d*-Separation Implies Conditional Independence)

Consider a joint density that factorizes as:

$$p(v, w, x, y, z) = p(x|v, z)p(v|w)p(w|y, z)p(y)p(z)$$

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Is $[X \perp\!\!\!\perp Y | W]$ in this distribution?
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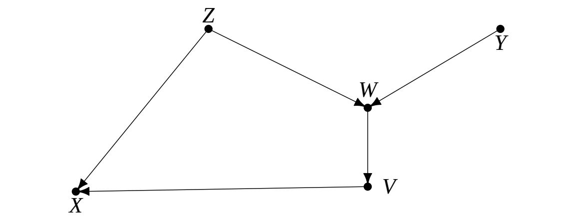
Difficult to say without a lot of tedious calculations.

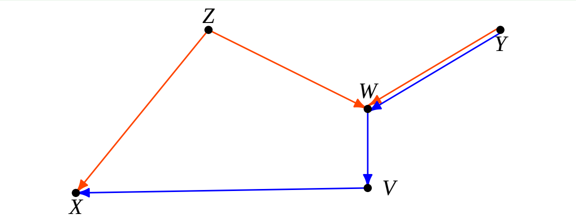
Example

Unless we draw the associated graph:



Example

And highlight the paths from X to Y



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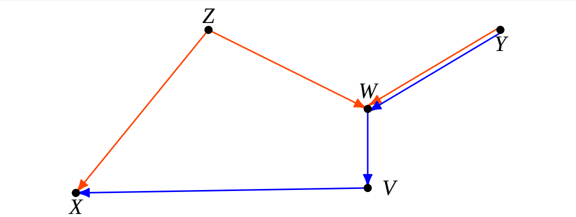
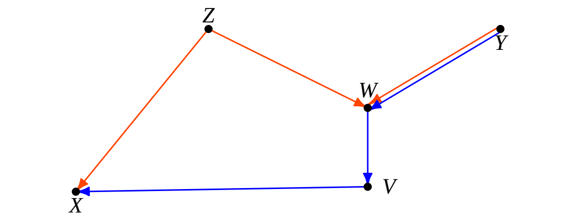


Recall that a path p is said to be blocked by a set of vertices U if and only if at least one of the following conditions hold:

1. p contains a chain $a \to b \to c$ or a fork $a \leftarrow b \to c$ where the node b is in U
2. p contains a collider $a \to b \leftarrow c$ where b is *not* in U and no descendent of b is in U

Example



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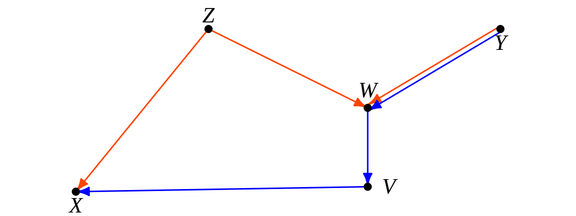

This implies that:
$[X \not\perp\!\!\!\perp Y | V]_{\mathcal{G}}$

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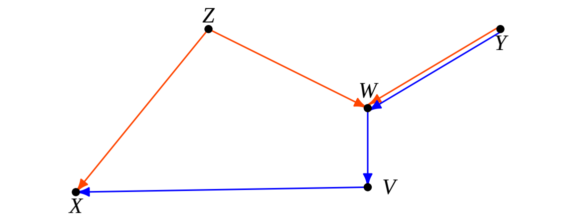

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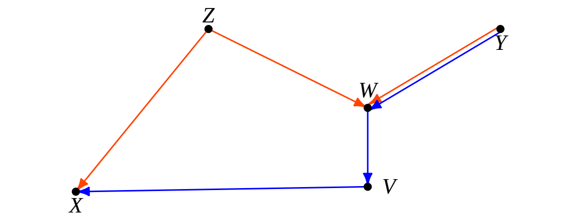

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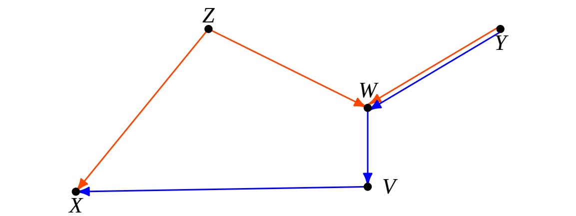

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and
$[X \perp\!\!\!\perp Y | (V, Z)]_{\mathcal{G}}$

Example

Example (d-Connection Does Not Imply Conditional Dependence in *All* Distributions Compatible with $\mathcal{G}$)

Consider the following system:

$$X = U_1$$

$$\begin{aligned} Z &= \alpha X + U_2 \\ &= \alpha U_1 + U_2 \end{aligned}$$

$$\begin{aligned} Y &= \gamma X + \beta Z + U_3 \\ &= \gamma U_1 + \beta(\alpha U_1 + U_2) + U_3 \\ &= (\gamma + \alpha\beta) U_1 + \beta U_2 + U_3 \end{aligned}$$

where it is assumed that $(U_1, U_2, U_3)' \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.

Example

We can write this in matrix notation as:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ (\gamma + \alpha\beta) & \beta & 1 \\ \alpha & 1 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

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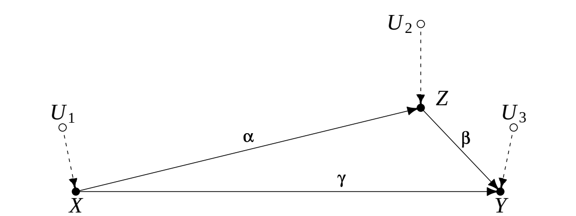

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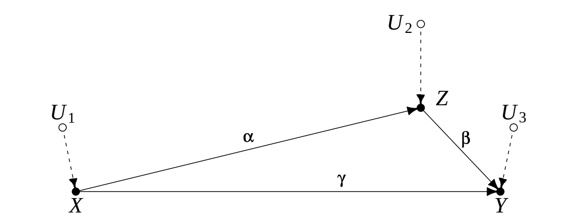

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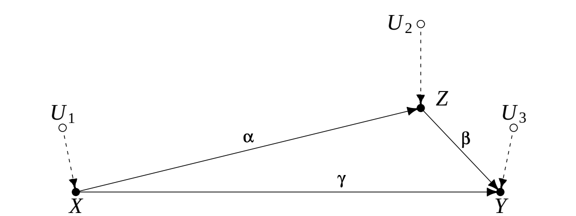

Here we see that Z does not d-separate X from Y– in other words, X and Y are d-connected given Z.

Are they always conditionally dependent given Z?

Example

Because X, Y, and Z are linear functions of Gaussian random variables we know that their joint distribution is also Gaussian with variance-covariance matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ (\gamma + \alpha\beta) & \beta & 1 \\ \alpha & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (\gamma + \alpha\beta) & \alpha \\ 0 & \beta & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (\gamma + \alpha\beta) & \alpha \\ (\gamma + \alpha\beta) & [(\gamma + \alpha\beta)^2 + \beta^2 + 1] & [\alpha(\gamma + \alpha\beta) + \beta] \\ \alpha & [\alpha(\gamma + \alpha\beta) + \beta] & (\alpha^2 + 1) \end{pmatrix}$$

Example

Now that we know the joint distribution of X, Y, and Z we can use standard results from multivariate normal theory to calculate the conditional distribution of $(X, Y)|Z$. We know that this is also Gaussian with variance-covariance matrix $\mathbf{\Sigma}_{XY|Z}$:

$$\mathbf{\Sigma}_{XY|Z} = \begin{pmatrix} 1 & (\gamma + \alpha\beta) \\ (\gamma + \alpha\beta) & [(\gamma + \alpha\beta)^2 + \beta^2 + 1] \end{pmatrix} - \frac{1}{\alpha^2 + 1} \begin{pmatrix} \alpha \\ [\alpha(\gamma + \alpha\beta) + \beta] \end{pmatrix} \begin{pmatrix} \alpha & [\alpha(\gamma + \alpha\beta) + \beta] \end{pmatrix}$$

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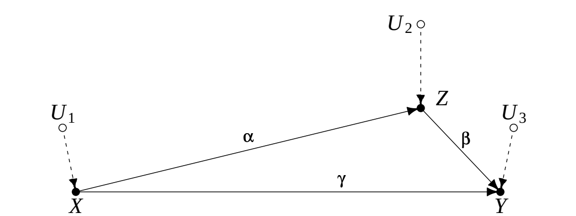
After multiplying this out and simplifying we see that

$$\mathrm{Cov}(X, Y|Z) = \frac{\gamma}{\alpha^2 + 1}$$

Thus, when $\gamma = 0$ $\mathrm{Cov}(X, Y|Z) = 0$ and because $p(x, y|z)$ is Gaussian we know that $[X \perp\!\!\!\perp Y|Z]$ when $\gamma = 0$ — even though $[X \not\!\perp\!\!\!\perp Y|Z]_{\mathcal{G}}$.

Example

It turns out that because of the linear Gaussian nature of this system we can reach the same result by looking at the edge coefficients on the graph.



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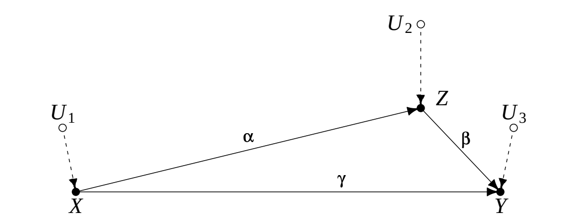

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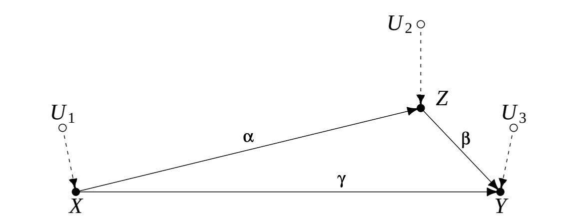

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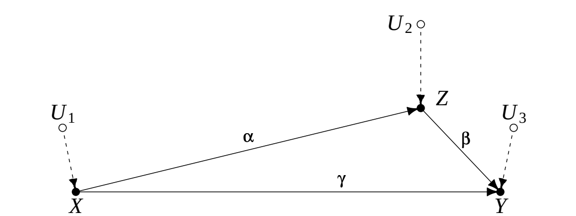

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Standard results from the linear model tell us that $\text{Cov}(X, Y|Z)$ will be 0 when $\gamma = 0$.
Because the joint distribution of (X, Y, Z) is Gaussian we know that $\text{Cov}(X, Y|Z) = 0 \implies [X \perp\!\!\!\perp Y|Z]$.

Faithfulness

Conditional dependence does not follow from d-connection in the example immediately above because the graph is not faithful to the distribution of (X, Y, Z).

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Violations of faithulness tend to be knife-edge situations.

Goals and Objectives for This Morning:

- Introduce graphical notation and terminology
- Build intuition about properties of probabilistic systems represented as directed graphs
- Provide some motivating examples

This afternoon we will demonstrate what this implies for estimation techniques the rely on selection-on-observables/unconfoundedness assumptions (regression, matching, IPW, doubly robust, etc.)