Lecture 1

Graphical Models for Dependence Relationships

An Introduction

Causal Inference Using Graphs
August 6, 2019

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Acknowledgements

Many of the slides from lectures 1 and 2 today were written by Kevin Quinn.

Goals and Objectives for This Course:

- Introduce causal graphs
- Show how causal graphs help us think about standard techniques
- Show how causal graphs help us think about not yet standard techniques

How I Use Graphs for Causal Inference

Two Possible Metaphors:

1. Draw with graphs, ink with potential outcomes
2. Sketch with graphs, paint with potential outcomes

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Many of the mistakes I see in seminar could be avoided if the author(s) sketched a bit before starting to paint.

Goals and Objectives for This Morning:

- Introduce graphical notation and terminology
- Build intuition about properties of probabilistic systems represented as directed graphs
- Provide some motivating examples

Overview

1. **Why Study Graphical Models?**

2. **Definitions and Terminology**
   - Basic Definitions
   - Graph Drawing Conventions
   - Some Additional Definitions
   - Familial Relations Specific to Directed Graphs

3. **Directed Acyclic Graphs (DAGs)**
   - Using DAGs to Represent Complicated Joint Distributions
   - Observational Equivalence
   - Conditional Independence and *d*-Separation
   - Faithfulness

What Application Areas Make Use of Graphical Models?

Graphical models are widely used in a variety of application areas:

- Social network analysis
- Causal inference
- Pattern recognition and machine learning
- Information retrieval
- Document summarization
- Multivariate analysis
- ...

Some Questions that are Easy to Answer Using Graphical Models

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- What's the full conditional distribution of $[\theta_1 | y, \theta_2]$?
- What will happen to Y if a variable X is set to x by outside intervention?
- Which background variables need to be adjusted for in order to get a consistent estimate of the causal effect of X on Y?

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- read off non-trivial mathematical properties of the system directly from the graph
- simplify complicated computations by taking account of the structure of the graph
- present substantive assumptions in a transparent manner

Basic Definitions

Definition (Directed Graph)

A directed graph $\mathcal{G}$ is a pair $\langle \mathcal{V}, \mathcal{E} \rangle$ where $\mathcal{V}$ is a finite set of vertices (a.k.a. nodes) and $\mathcal{E}$ is the set of directed edges (a.k.a. directed arcs or directed links).

Each directed edge in $\mathcal{E}$ is an ordered pair of distinct vertices from $\mathcal{V} \times \mathcal{V}$.

A directed edge $(V_i, V_j) \in \mathcal{E}$ is also denoted $V_i \rightarrow V_j$.

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Note that this definition does not allow edges from a node to itself or multiple copies of edges.

In this class, we will think of each $V \in \mathcal{V}$ as being a (possibly non-scalar) random variable and each directed edge $(V_i, V_j) \in \mathcal{E}$ as some relationship between V_i and V_j.

This will be made more precise below.

Graph Drawing Conventions

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Note the difference between "(un)observed" and "(un)observable".

Example

Example (Drawing a Simple Directed Graph)

Consider the simple directed graph with vertices $\{X, Y, Z\}$ and edges $\{X \rightarrow Y,\ Z \rightarrow X,\ Z \rightarrow Y\}$ with X and Y observable and Z unobservable.

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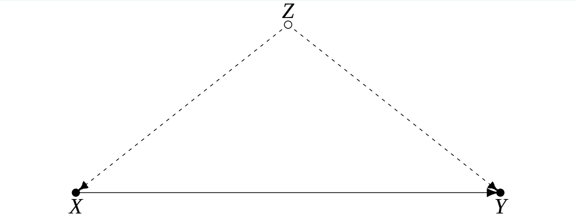
Consider the simple directed graph with vertices $\{X, Y, Z\}$ and edges $\{X \to Y,\ Z \to X,\ Z \to Y\}$ with X and Y observable and Z unobservable.

We would express this graph visually as:



Some Additional Definitions

Definition (Path)

A path from V_i to V_j in a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a sequence of distinct nodes $V_i = X_0, \ldots, X_n = V_j$ such that $(X_{k-1}, X_k) \in \mathcal{E}$ or $(X_k, X_{k-1}) \in \mathcal{E}$ for each $k = 1, \ldots, n$. We write $V_i \sim V_j$ to denote a path from V_i to V_j.

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Example

A path from V_1 to V_3 exists in each of the following four graphs.

$$V_1 \rightarrow V_2 \rightarrow V_3$$
$$V_1 \leftarrow V_2 \rightarrow V_3$$
$$V_1 \rightarrow V_2 \leftarrow V_3$$
$$V_1 \leftarrow V_2 \leftarrow V_3$$

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A directed path from V_i to V_j in a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a sequence of distinct nodes $V_i = X_0, \ldots, X_n = V_j$ such that $(X_{k-1}, X_k) \in \mathcal{E}$ and $(X_{k-1}, X_k) \notin \mathcal{E}$ for each $k = 1, \ldots, n$. We write $V_i \rightsquigarrow V_j$ to denote a directed path from V_i to V_j.

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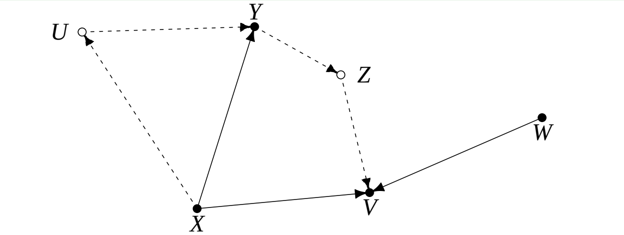
Put more informally, a directed path from V_i to a V_j is a path from V_i to a V_j in which all the edges on the path have arrows pointing toward V_j.

Example

Example (Paths and Directed Paths)

Consider the graph depicted below.



Here there are 3 paths from X to Z:
1) $(X \to U \to Y \to Z)$,
2) $(X \to Y \to Z)$, and
3) $(X \to V \leftarrow Z)$.

Of these, only $(X \to U \to Y \to Z)$ and $(X \to Y \to Z)$ are directed paths from X to Z.

Familial Relations Specific to Directed Graphs

Definition (Parents)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of parents of a node $V \in \mathcal{V}$ is defined to be:

$$\mathrm{pa}(V) = \{Z \in \mathcal{V} : (Z, V) \in \mathcal{E}\}$$

The set of parents of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

$$\mathrm{pa}(\mathcal{W}) = \bigcup_{V \in \mathcal{W}} \mathrm{pa}(V)$$

Put more informally, the parents of V are all nodes from which there is a directed edge to V.

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Definition (Ancestors)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of ancestors of a node $V \in \mathcal{V}$ is defined to be:

$$\mathrm{an}(V) = \{Z \in \mathcal{V} : Z \rightsquigarrow V\}$$

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In words, the set of ancestors of V consists of the vertices from which there is a directed path to V.

Familial Relations Specific to Directed Graphs

Definition (Children)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of children of a node $V \in \mathcal{V}$ is defined to be:

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The set of children of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

$$\text{ch}(\mathcal{W}) = \bigcup_{V \in \mathcal{W}} \text{ch}(V)$$

Put more informally, the children of V are all nodes to which there is a directed edge from V.

Familial Relations Specific to Directed Graphs

Definition (Descendents)

In a directed graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ the set of descendents of a node $V \in \mathcal{V}$ is defined to be:

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The set of descendents of a set of nodes $\mathcal{W} \subseteq \mathcal{V}$ is defined to be:

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In words, the set of descendents of V consists of the vertices to which there exists a directed path from V.

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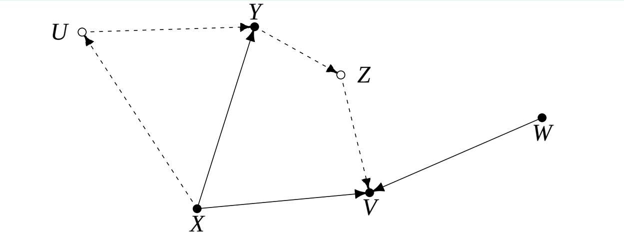
Example (Familial Relations)

Consider the graph depicted below.



Here:
$\mathrm{pa}(Z) = \{Y\}$, $\mathrm{ch}(Z) = \{V\}$, $\mathrm{an}(Z) = \{U, X, Y\}$, $\mathrm{de}(Z) = \{V\}$

and

$\mathrm{pa}(X) = \emptyset$, $\mathrm{ch}(X) = \{U, V, Y\}$, $\mathrm{an}(X) = \emptyset$,
$\mathrm{de}(X) = \{U, V, Y, Z\}$

Directed Acycylic Graphs (DAGs)

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- Relatedly, they can help one think about ways to achieve dimension reduction / data summarization
- They are natural representation of systems evolving in time
- They can be given a causal interpretation

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Note that there are five other such factorizations that are possible.

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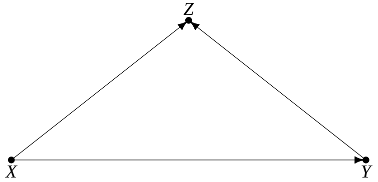
For the 3 variable example above his gives rise to:



Note the relationship to $p(x, y, z) = p(z|x, y)p(y|x)p(x)$

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It is easy to show, given our rules for forming graphs, that for a DAG with n vertices $V_1, \ldots, V_n$ one can write

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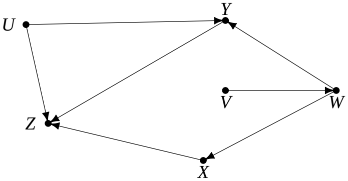
This formula allows us to look at a graph and then write down the factorization that the graph implies.

Using DAGs to Represent Complicated Joint Distributions

Let's look at a more complicated example that starts with the following graph.



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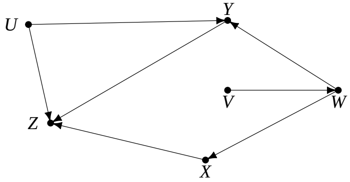

This is consistent with the factorization

$$p(u, v, w, x, y, z) = p(z|u, x, y)p(y|u, w)p(x|w)p(w|v)p(u)p(v)$$

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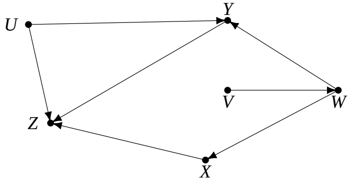

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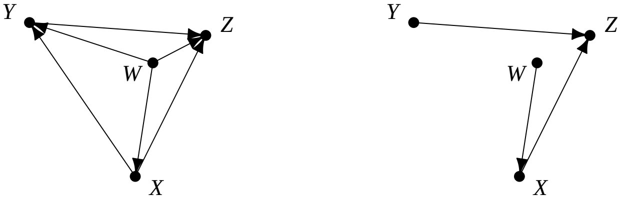
$$p(u, v, w, x, y, z) = p(z|u, x, y)p(y|u, w)p(x|w)p(w|v)p(u)p(v)$$

Note that the graph embodies a lot of conditional independence assumptions.

Using DAGs to Represent Complicated Joint Distributions

Question: Which graph below embodies more assumptions?



(a) (b)

Using DAGs to Represent Complicated Joint Distributions

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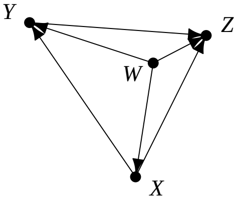
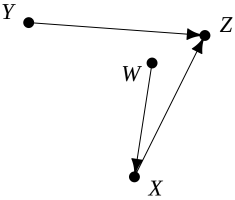


(a)



(b)

Answer: More assumptions are implied by (b).

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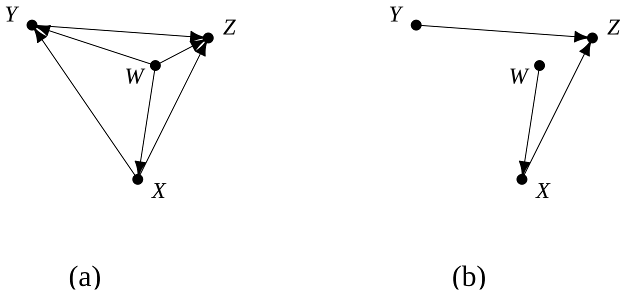

Answer: More assumptions are implied by (b).

Missing edges correspond to conditional independence assumptions.

Using DAGs to Represent Complicated Joint Distributions

Returning to our three variable example, note that the factorization $p(x, y, z) = p(z|x, y)p(y|x)p(x)$ suggests a simple means (called the method of composition) of obtaining a sample from the joint distribution of (X, Y, Z):

1. Sample x from the marginal distribution of X

Using DAGs to Represent Complicated Joint Distributions

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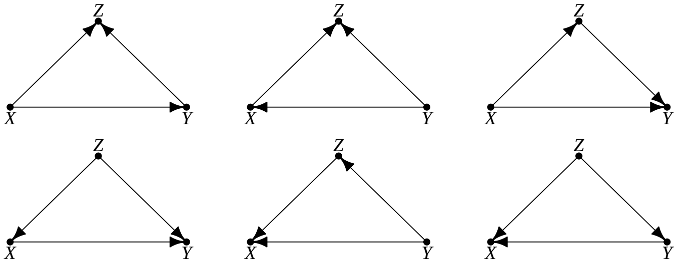
Observational Equivalence

As an aside, note that **all** of the following DAGs are consistent with any choice of $p(x, y, z)$.



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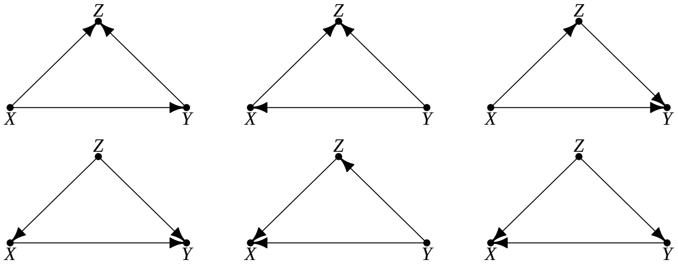

There is actually a general result here.

Observational Equivalence

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Two DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be observationally equivalent if every probability distribution that is compatible with $\mathcal{G}_1$ is also compatible with $\mathcal{G}_2$.

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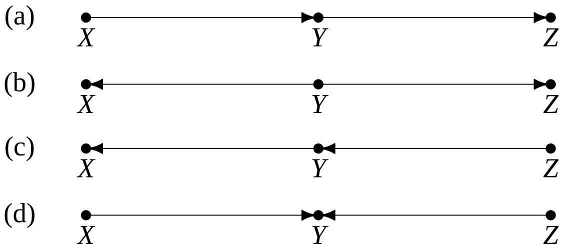
Theorem (Observational Equivalence (Verma and Pearl))

Two DAGs are observationally equivalent if and only if they have the same skeletons and the same sets of v-structures.

Example

Example (Observational Equivalence)

Consider the four graphs below.



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Consider the four graphs below.

(a) $X \rightarrow Y \rightarrow Z$

(b) $X \leftarrow Y \rightarrow Z$

(c) $X \leftarrow Y \leftarrow Z$

(d) $X \rightarrow Y \leftarrow Z$

Here we see that graphs (a), (b), and (c) are all observationally equivalent. *Graph (d) is not observationally equivalent to any of the other graphs.*

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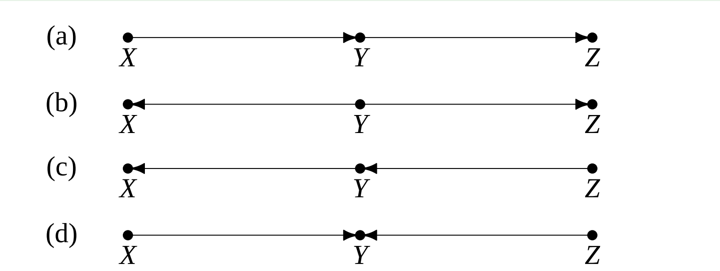

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Note that graphs (a), (b), and (c) imply very different causal relationships.

Observational Equivalence

Some things to think about:

- A DAG implies a particular factorization but an unfactorized joint distribution implies a wide range of DAGs.
- Because a DAG gives a recipe for generating data from a joint distribution it is tempting to think of a DAG (by itself) in causal terms. This is a mistake because of the point immediately above.
- Interpreting a DAG causally can be reasonable—we'll talk about this extensively in later weeks—but it does require additional assumptions.

Conditional Independence

Definition (Conditional Independence)

Let $\mathcal{V} = \{V_1, \ldots, V_n\}$ be a finite set of random variables and let X, Y, and Z denote three subsets of $\mathcal{V}$. If

$$p(x|y,z) = p(x|z) \quad \text{whenever} \quad p(y,z) > 0$$

we say that X is conditionally independent of Y given Z and write

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It is easy to see that $[X \perp\!\!\!\perp Y|Z]$ also implies that

$$p(x,y|z) = p(x|z)p(y|z)$$

Conditional Independence

The following are some properties that hold for conditionally independent random variables.

Property (Symmetry)

$$[X \perp\!\!\!\perp Y | Z] \implies [Y \perp\!\!\!\perp X | Z]$$

Property (Decomposition)

$$[X \perp\!\!\!\perp (Y, W) | Z] \implies [X \perp\!\!\!\perp Y | Z]$$

Property (Weak Union)

$$[X \perp\!\!\!\perp (Y, W) | Z] \implies [X \perp\!\!\!\perp Y | (Z, W)]$$

Property (Contraction)

$$[X \perp\!\!\!\perp Y | Z] \;\&\; [X \perp\!\!\!\perp W | (Z, Y)] \implies [X \perp\!\!\!\perp (Y, W) | Z]$$

The Importance of Conditional Independence

Conditional independence is fundamental to almost all areas of probability, statistics, and data analysis.

- causal inference
- Bayesian inference
- data summarization
- Markov chains
- sufficiency
- ancillarity
- identification
- etc.

See Dawid (1979). “Conditional Independence in Statistical Theory” *JRSS B*. for additional information.

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Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a DAG and X, Y, and Z be disjoint subsets of $\mathcal{V}$.

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A path p is said to be blocked by a set of vertices Z if and only if at least one of the following conditions hold:

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If X is *not* d-separated from Y by Z we say that X is d-connected to Y by Z.

Example

Example (d-Separation)

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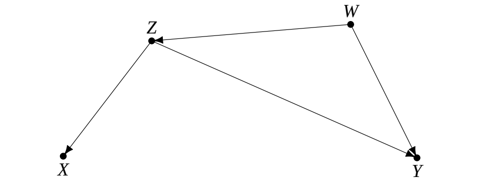

Which sets of variables (if any) d-separate X from Y?

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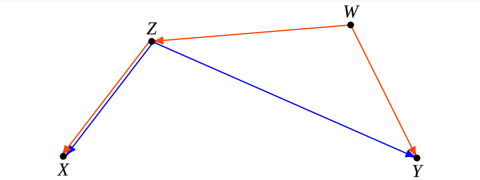

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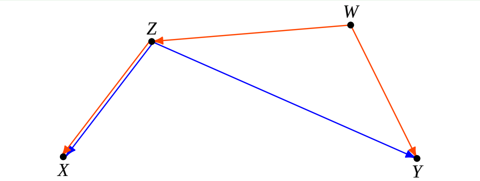

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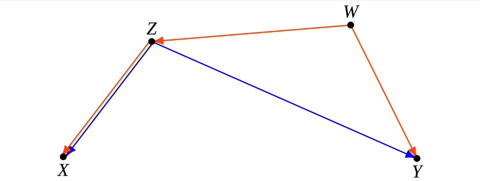

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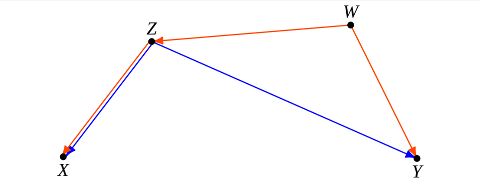

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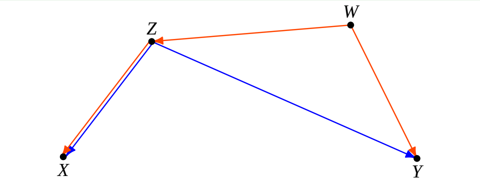

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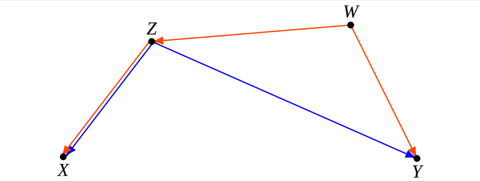

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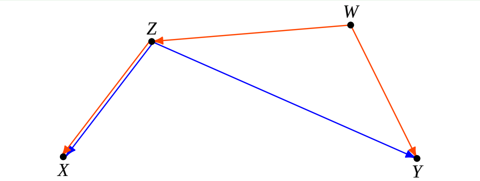

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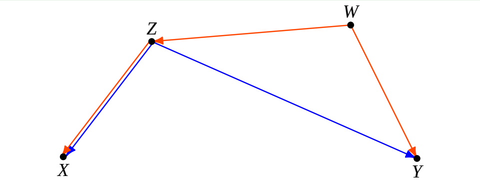

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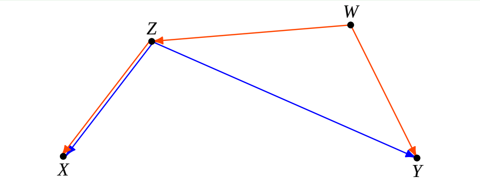

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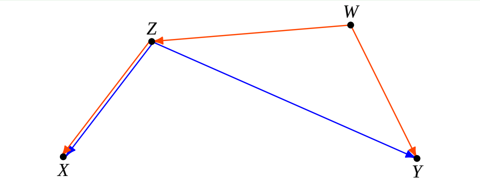

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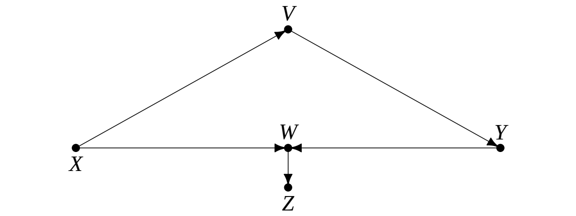

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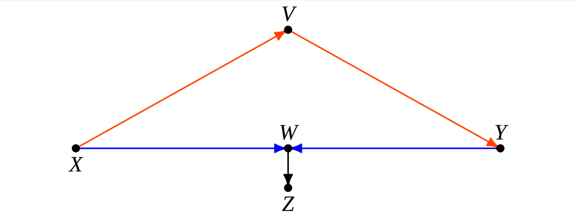

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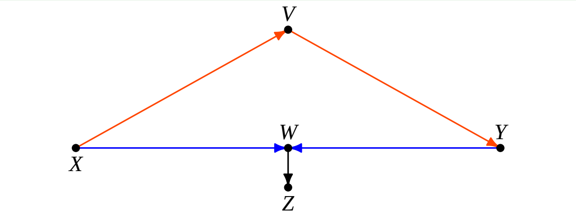

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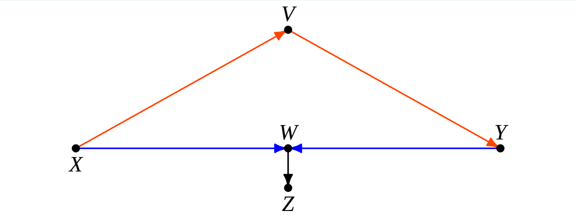

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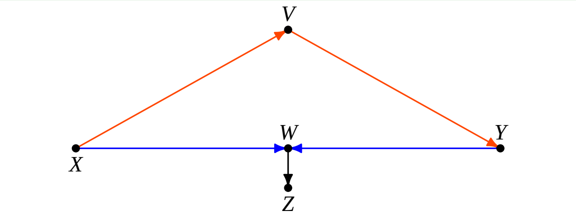

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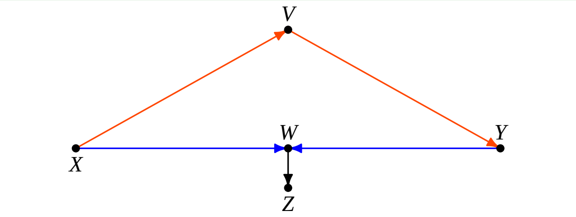

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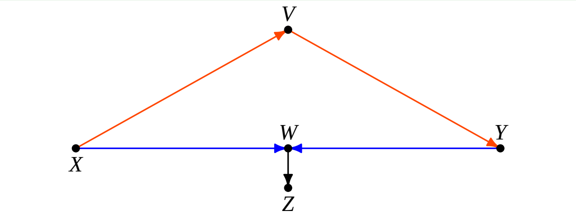

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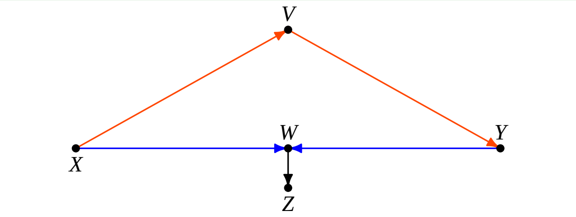

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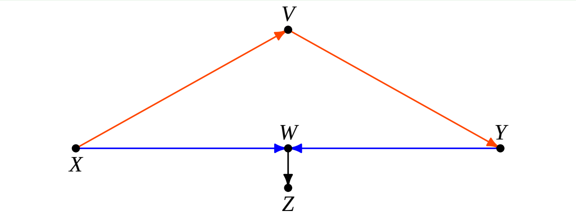

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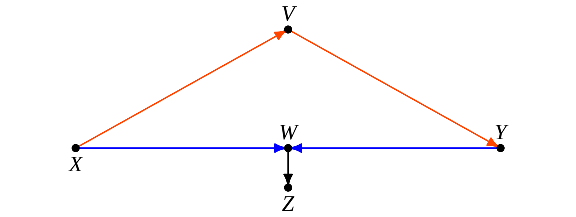

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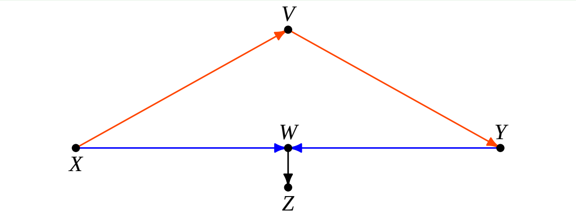

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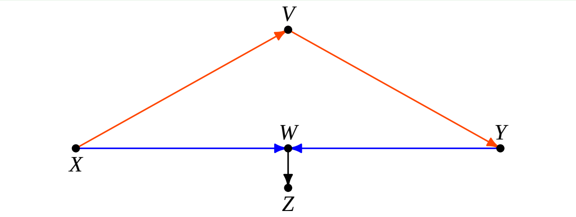

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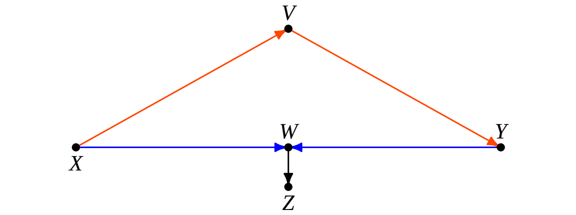

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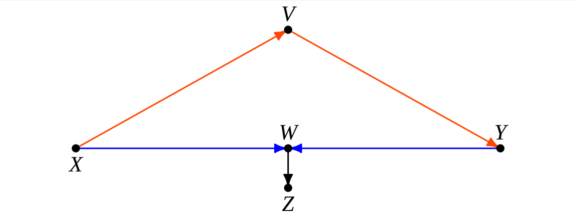

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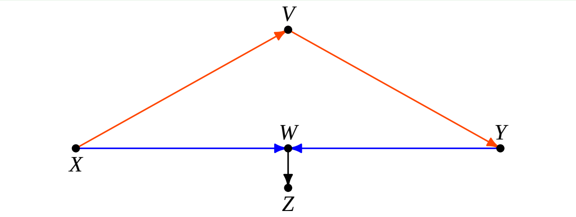

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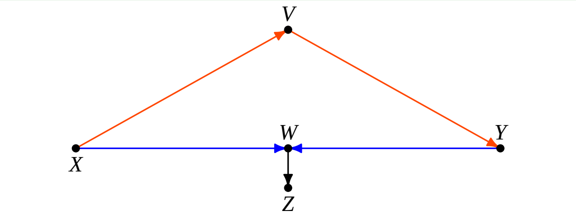

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Probabilistic Implications of *d*-Separation

Theorem (Probabilistic Implications of *d*-Separation (Verma and Pearl))

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a DAG and X, Y, and Z be disjoint subsets of $\mathcal{V}$.

If Z d-separates X from Y in $\mathcal{G}$, then X is conditionally independent of Y given Z in every distribution compatible with $\mathcal{G}$.

Conversely, if X and Y are not d-separated by Z in $\mathcal{G}$, then X and Y are conditionally dependent given Z in at least one distribution compatible with $\mathcal{G}$.

Example

Example (d-Separation Implies Conditional Independence)

Consider a joint density that factorizes as:

$$p(v, w, x, y, z) = p(x|v, z)p(v|w)p(w|y, z)p(y)p(z)$$

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Questions:
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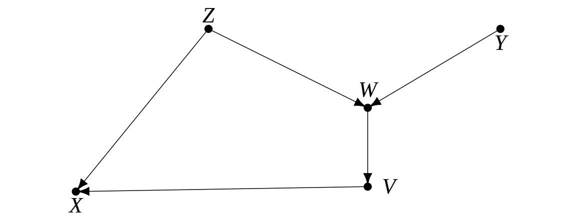
Difficult to say without a lot of tedious calculations.

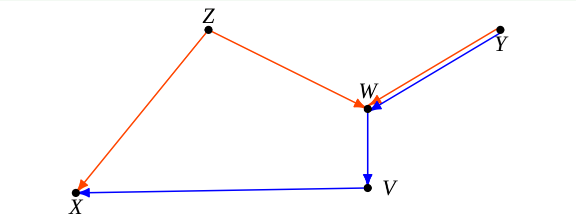
Example

Unless we draw the associated graph:



Example

And highlight the paths from X to Y



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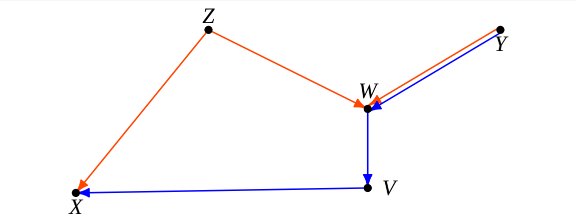
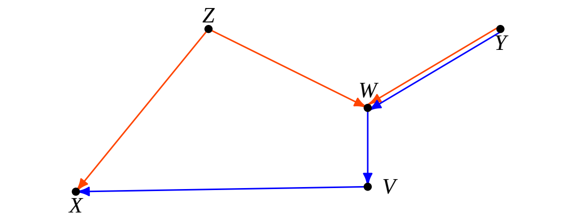


Recall that a path p is said to be blocked by a set of vertices U if and only if at least one of the following conditions hold:

1. p contains a chain $a \to b \to c$ or a fork $a \leftarrow b \to c$ where the node b is in U
2. p contains a collider $a \to b \leftarrow c$ where b is *not* in U and no descendent of b is in U

Example



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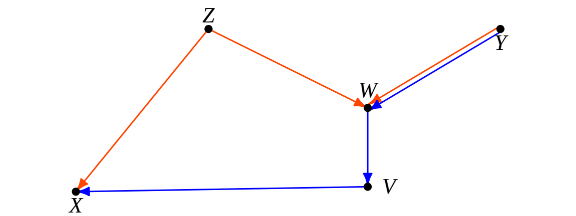

This implies that:
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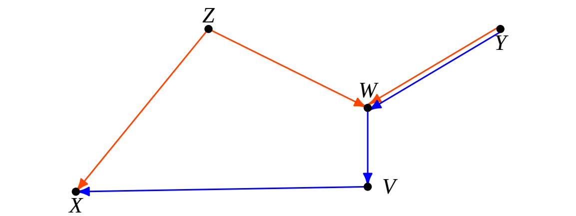

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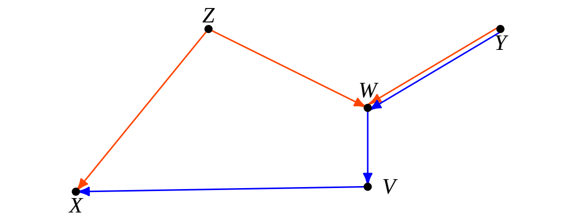

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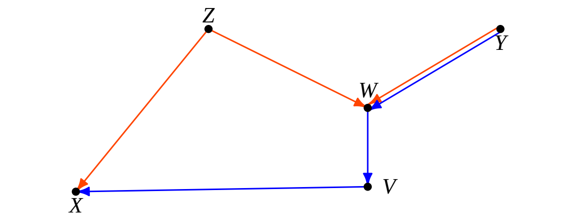

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Example

Example (d-Connection Does Not Imply Conditional Dependence in *All* Distributions Compatible with $\mathcal{G}$)

Consider the following system:

$$X = U_1$$

$$\begin{aligned} Z &= \alpha X + U_2 \\ &= \alpha U_1 + U_2 \end{aligned}$$

$$\begin{aligned} Y &= \gamma X + \beta Z + U_3 \\ &= \gamma U_1 + \beta(\alpha U_1 + U_2) + U_3 \\ &= (\gamma + \alpha\beta) U_1 + \beta U_2 + U_3 \end{aligned}$$

where it is assumed that $(U_1, U_2, U_3)' \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.

Example

We can write this in matrix notation as:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ (\gamma + \alpha\beta) & \beta & 1 \\ \alpha & 1 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

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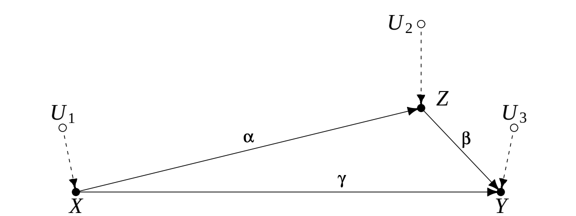

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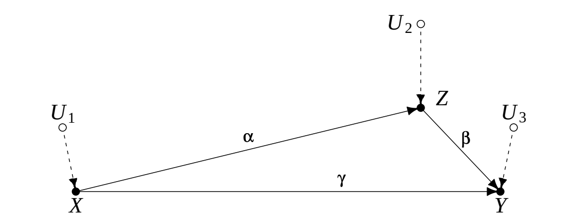

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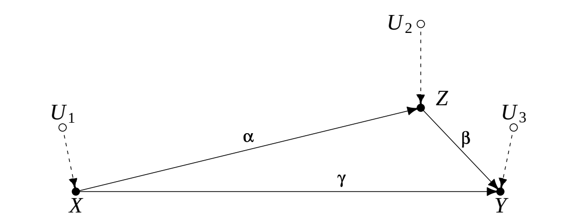

Here we see that Z does not d-separate X from Y– in other words, X and Y are d-connected given Z.

Are they always conditionally dependent given Z?

Example

Because X, Y, and Z are linear functions of Gaussian random variables we know that their joint distribution is also Gaussian with variance-covariance matrix $\mathbf{\Sigma}$:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ (\gamma + \alpha\beta) & \beta & 1 \\ \alpha & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (\gamma + \alpha\beta) & \alpha \\ 0 & \beta & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (\gamma + \alpha\beta) & \alpha \\ (\gamma + \alpha\beta) & \left[(\gamma + \alpha\beta)^2 + \beta^2 + 1\right] & [\alpha(\gamma + \alpha\beta) + \beta] \\ \alpha & [\alpha(\gamma + \alpha\beta) + \beta] & (\alpha^2 + 1) \end{pmatrix}$$

Example

Now that we know the joint distribution of X, Y, and Z we can use standard results from multivariate normal theory to calculate the conditional distribution of $(X, Y)|Z$. We know that this is also Gaussian with variance-covariance matrix $\mathbf{\Sigma}_{XY|Z}$:

$$\mathbf{\Sigma}_{XY|Z} = \begin{pmatrix} 1 & (\gamma + \alpha\beta) \\ (\gamma + \alpha\beta) & \left[(\gamma + \alpha\beta)^2 + \beta^2 + 1\right] \end{pmatrix} - \frac{1}{\alpha^2 + 1} \begin{pmatrix} \alpha \\ [\alpha(\gamma + \alpha\beta) + \beta] \end{pmatrix} \begin{pmatrix} \alpha & [\alpha(\gamma + \alpha\beta) + \beta] \end{pmatrix}$$

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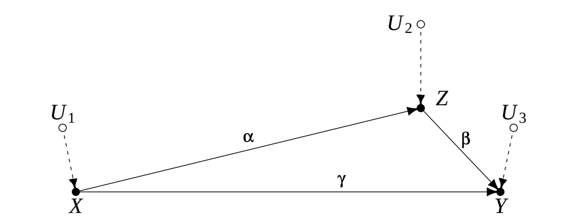
After multiplying this out and simplifying we see that

$$\mathrm{Cov}(X, Y|Z) = \frac{\gamma}{\alpha^2 + 1}$$

Thus, when $\gamma = 0$ $\mathrm{Cov}(X, Y|Z) = 0$ and because $p(x, y|z)$ is Gaussian we know that $[X \perp\!\!\!\perp Y|Z]$ when $\gamma = 0$ — even though $[X \not\perp\!\!\!\perp Y|Z]_{\mathcal{G}}$.

Example

It turns out that because of the linear Gaussian nature of this system we can reach the same result by looking at the edge coefficients on the graph.



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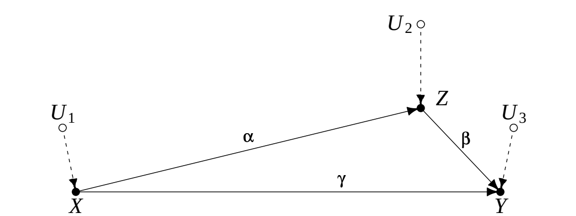

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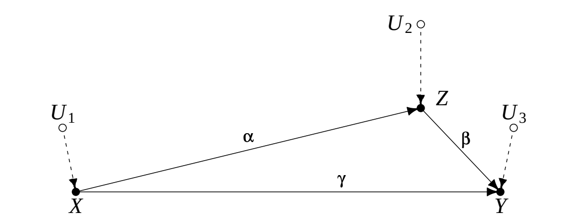

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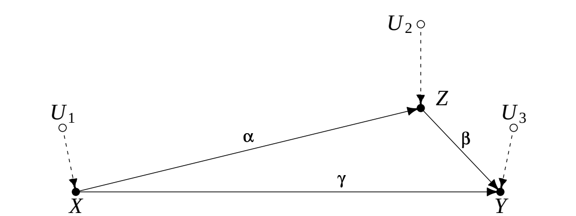

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Standard results from the linear model tell us that $\text{Cov}(X, Y|Z)$ will be 0 when $\gamma = 0$.
Because the joint distribution of (X, Y, Z) is Gaussian we know that $\text{Cov}(X, Y|Z) = 0 \implies [X \perp\!\!\!\perp Y|Z]$.

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