Lecture 6

Heterogeneous Effects

Mediation Analysis and More

Causal Inference Using Graphs
August 13, 2019

Adam Glynn
Department of Political Science and QTM
Emory University

Overview

1. **Constant Effects**
2. **Interactions**
3. **Full Heterogeneity**

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2. Interactions
3. Full Heterogeneity

$$\begin{tikzcd}[column sep=huge] Z \arrow[r, "\alpha_1"] \arrow[rr, bend right, "\beta_2"'] & D \arrow[r, "\beta_1"] & Y \end{tikzcd}$$

$$(1) D_i(z) = \alpha_0 + \alpha_1 z + \nu_i$$

$$(2) Y_i(d, z) = \beta_0 + \beta_1 d + \beta_2 z + \epsilon_i$$

$$Y_i(d_i(z), z) = Y_i(z) = \beta_0 + \beta_1(\alpha_0 + \alpha_1 Z + \nu_i) + \beta_2 Z + \epsilon_i$$

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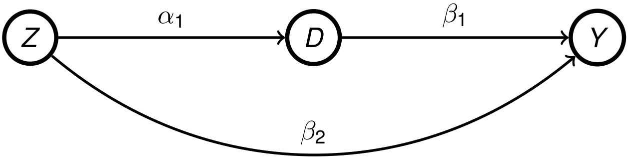

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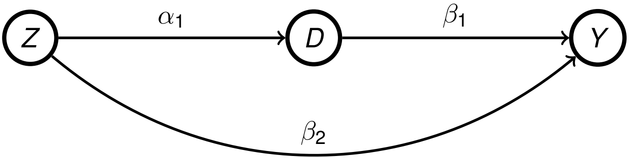

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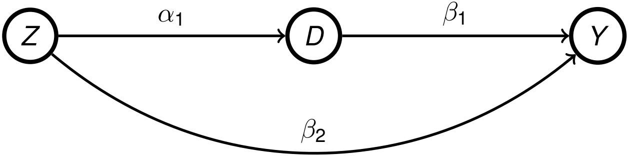

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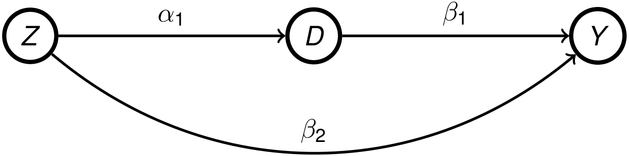

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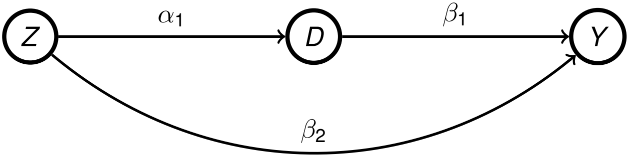
- Write some direct effects in terms of potential outcomes and coefficients.
- Write some indirect effects in terms of potential outcomes and coefficients.
- Which direct and indirect effects sum to the total effect of Z on Y?



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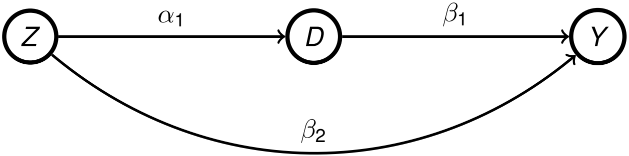
- In this model, how can we identify these direct and indirect effects?



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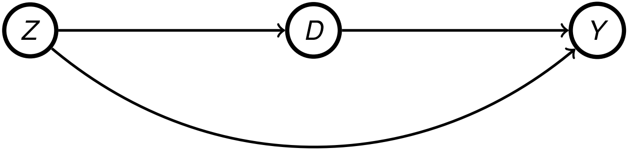
- In this model, how can we identify these direct and indirect effects?
- What happens as we complicate the model (interactions, heterogeneous effects, etc.).

1. Constant Effects
2. **Interactions**
3. Full Heterogeneity



$$(1)\, D_i(z) = \alpha_0 + \alpha_1 z + \nu_i$$

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Let's try to write the natural direct and indirect effects.

1 Constant Effects

2 Interactions

3 **Full Heterogeneity**

Natural Direct and Indirect Effects

$$\begin{aligned}
Y_1 - Y_0 &= Y_{1M_1} - Y_{0M_0} \\
&= \underbrace{Y_{1M_1} - Y_{1M_0}}_{\textit{indirect}} + \underbrace{Y_{1M_0} - Y_{0M_0}}_{\textit{direct}} \\
&= \underbrace{Y_{1M_1} - Y_{0M_1}}_{\textit{direct}} + \underbrace{Y_{0M_1} - Y_{0M_0}}_{\textit{indirect}}
\end{aligned}$$

- Vanderweele (2015) notation better in this setting
- Everything has an *i* subscript, so no need to write the subscripts.

Controlled versus Natural Direct Effects

$$
\begin{aligned}
Y_{1M_0} - Y_{0M_0} &= (Y_{10} - Y_{00}) \cdot (1 - M_0) + (Y_{11} - Y_{01}) \cdot M_0 \\
&= (Y_{10} - Y_{00}) + [(Y_{11} - Y_{01}) - (Y_{10} - Y_{00})] \cdot M_0
\end{aligned}
$$

Total minus direct effects

$$
\begin{aligned}
(Y_1 - Y_0) - (Y_{1M_0} - Y_{0M_0}) &= Y_{1M_1} - Y_{1M_0} \\
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&\quad + [(Y_{11} - Y_{01}) - (Y_{10} - Y_{00})] \cdot M_0
\end{aligned}
$$

With BTB the black sounding name gets no callback $Y_1 = 0$ while the white sounding name gets a callback $Y_0 = 1$, however without BTB both have no criminal history, and both get callbacks $Y_{10} = Y_{00} = 1$. Furthermore, with BTB the employer assumes that both have criminal histories $M_1 = M_0 = 1$, which means that $Y_{1M_1} - Y_{1M_0} = Y_{11} - Y_{11} = 0$ and there is no indirect effect. The difference between the total and controlled direct effects exists because BTB causes the employer to assume the white sounding name has a criminal history $M_0 = 1$ and white assumed criminals get a call back $Y_{01} = 1$ while black assumed criminals do not $Y_{11} = 0$.

Experimental Designs

- single experiment
- parallel experiment
- parallel encouragement
- path severing

Single Experiment Design

The single experiment design involves randomization of only A. This means that the average effects of A on Y ($E[Y_1 - Y_0]$) and A on M ($E[M_1 - M_0]$) are identified, but unfortunately, none of the other effects described above are identified by this design. This includes the joint effects and the natural direct and indirect effects. Furthermore, even bounds and sensitivity analysis may not be too informative for this design. For bounding with a binary outcome, Imai et al. 2103 shows that this design will always produce bounds that fail to rule out an average indirect effect of zero. Additionally, sensitivity analysis as in Imai and Yamamoto 2013 will also often be unable to rule out an average indirect effect of zero. In the example presented in Figure 3 of that paper, nearly all bounds contain zero for very small values of the sensitivity parameter.

Parallel Experiment Design

The parallel experiment design described in Imai et al. 2013 involves 1) randomly splitting the sample into two groups, 2) randomizing A for the first group, 3) jointly randomizing A and M for the second group. As with the single experiment design, the randomization of A in the first group identifies the average effects of A on Y ($E[Y_1 - Y_0]$) and A on M ($E[M_1 - M_0]$). The randomization of A and M for the second group identifies the average joint effects of A and M on Y. For example the average controlled direct effects ($E[Y_{10} - Y_{00}]$ and $E[Y_{11} - Y_{01}]$) are identified by this group. Furthermore, Imai et al. 2013 demonstrates that the bounds on the average indirect effect produced by this experimental design will sometimes not include an average indirect effect zero. Importantly, even when these bounds contain zero, this design will identify the difference between the total effect and the controlled direct effect. This difference can still be interpreted as a combination of the indirect effect and a mediated interaction.

Parallel Encouragement Design

Sometimes we may only be able to indirectly manipulate the mediator M through the use of an instrument. For example, we may not be able to force individuals to take aspirin or to refrain from taking aspirin, but we may be able to encourage them to take or not take aspirin. In this case, we wouldn't be able to use the parallel experiment design, but we would be able to use a parallel encouragement design where 1) the sample is randomly splitting into two groups, 2) A is randomized for the first group, 3) A and and the instrument are jointly randomized for the second group. Imai et al. 2013 shows that even with this weaker design it is possible to get bounds on the average indirect effect that do not include zero. Furthermore, it also shows that bounds will be tighter on the average complier indirect effects (where compliers are those that would always do what they are encouraged to do).

Path Severing Experiment Design

The final experiment to consider is the so-called path severing experiment (Pearl 2001). Instead of manipulating a variable, this type of experiment attempts to manipulate the causal path from A to M. In the blood pressure example, this would be a single experiment design on the redesigned drug (because the redesigned drug eliminates the causal path from the drug to aspirin intake). This type of analysis may also be referred to as implicit mediation analysis Gerber and Green 2012. If we are willing to assume that the redesigned treatment has eliminated the effect of A on M, then this experiment will point identify the natural direct effects, and in combination with the single experiment, will identify the natural indirect effect.

VanderWeele's 7 Motivations

1. scientific understanding
2. confirm or refute a theory
3. refine an intervention
4. discarding components
5. determine reason for no apparent total effect
6. not be able to intervene directly
7. bolster claim of effect

Goals and Objectives for This Afternoon:

- Review controlled direct effects and mediation analysis with constant effects.
- Constant effect CDEs and mediation analysis with interactions.
- Full heterogeneity