The model:

- First level: $X_{ij} \sim Poisson(\lambda_{ij} * E_{ij})$

where E_{ij} is the expected count of drug i and adverse event j and N_{ij} is the observed count. For simplicity in this demonstration, let's assume $E_{ij} = 1$, so

$$X_{ij} \sim Poisson(\lambda_{ij})$$

- Second level: $\lambda_{ij} \sim \, Mixture\ of\ 2\ gamma\ distributions$ with pdf:

$$\pi(\lambda; \alpha_1, \beta_1, \alpha_2, \beta_2, p) = pg(\lambda; \alpha_1, \beta_1) + (1-p)g(\lambda; \alpha_2, \beta_2), \quad \alpha_1, \beta_1, \alpha_2, \beta_2 > 0, 0 \le p \le 1$$

where $g(\lambda; \alpha_1, \beta_1)$ and $g(\lambda; \alpha_2, \beta_2)$ are the probability density functions of the Gamma Distribution with shape parameters α_1, α_2 and scale parameters β_1, β_2, and p is the weight of the first distribution. $\alpha_1, \beta_1, \alpha_2, \beta_2, p$ are estimated from the data (Empirical Bayes).

We already know that the marginal distribution of X follows a mixture of 2 negative binomial distributions. We also know that $\lambda | X \sim \pi(\lambda; \alpha_1 + X, \beta_1 + 1, \alpha_2 + X, \beta_2 + 1, p)$. But let's ignore these truths and use Gibbs Sampling and MH algorithm to simulate X and λ and compare with the true distributions.

1. Gibbs Sampling steps:

- Choose arbitrary X_1 and λ_1
- For each t = 2, 3, 4, etc.:
  - Draw X_t from $Poisson(\lambda_{t-1})$
  - Draw λ from posterior distribution of λ given X_t:

$$\lambda | X_{t-1} \sim \frac{\pi(\lambda; \alpha_1, \beta_1, \alpha_2, \beta_2, p) * \frac{e^{-\lambda}\lambda^X}{X!}}{\sum_{\lambda=1}^{\infty} \pi(\lambda; \alpha_1, \beta_1, \alpha_2, \beta_2, p) * \frac{e^{-\lambda}\lambda^X}{X!}}$$

We numerically estimate the Cumulative Distribution function and then use inversion method to draw λ

This was done for 4 sequences with 4 initial points $(X, \lambda) = (1,1), (1,10), (10,1), (10,1)$ and we use Potential Scale Reduction to determine convergence.

2. Metropolis – Hastings algorithm steps:

I tried different proposal functions such as Binomial, Geometric, discrete Uniform for X and Gamma, continuous Uniform for λ. It turned out discrete Uniform for X and continuous Uniform for λ works the best.

Steps:

- Choose arbitrary X_1 and λ_1
- For each t = 2, 3, 4, etc.:
    - Draw λ^* from $Continuous\ Uniform(0, 30)$, draw $X^*\ from\ Discrete\ Uniform(0, 30)$
    - Calculate the density ratio:

$$\alpha = \frac{\pi(\lambda^*; \alpha_1, \beta_1, \alpha_2, \beta_2, p) * \frac{e^{-\lambda^*}\lambda^{X^*}}{X^*!}}{\pi(\lambda_{t-1}; \alpha_1, \beta_1, \alpha_2, \beta_2, p) * \frac{e^{-\lambda_{t-1}}\lambda^{X_{t-1}}}{X_{t-1}!}}$$

    - Accept/Reject proposal: set $X_t = X^*\ and\ \lambda_t = \lambda^*$ with probability $\min(\alpha, 1)$, otherwise $X_t = X_{t-1}\ and\ \lambda_t = \lambda_{t-1}$
- This was done for 4 sequences with 4 initial points $(X, \lambda) = (1,1), (1,10), (10,1), (10,1)$ and we use Potential Scale Reduction to determine convergence.