Monte-Carlo methods
Tutorials 1 - Sampling

Exercise 1 (The Inverse Cumulative Distribution)**.**
For a random variable X, denote by f_X its density and $F_X(x) = \mathbb{P}\{X \leq x\}$ its cumulative distribution function. For each of these distributions, write F_X and deduce a way to sample this distributions.

1. X follows an exponential distribution : $f_X(x) = e^{-x}\mathbb{1}_{x\geq 0}$
2. X follows a Cauchy distribution: $f_X(x) = \dfrac{1}{\pi(1+x^2)}$
3. X follows a Rayleigh distribution : $f_X(x) = xe^{-x^2/2}\mathbb{1}_{x\geq 0}$
4. X follows a Pareto distribution : $f_X(x) = \dfrac{ab^a}{x^{a+1}}\mathbb{1}_{x\geq b}$

Exercise 2 (Approximating an integral)**.**
Consider the integral :

$$I = \int_0^\infty \sqrt{t}e^{-t}dt$$

Rappel : $\displaystyle\int_{-\infty}^{1.96} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 0.975$

1. Using n iid random variables under the exponential distribution, give an estimator S_n which is unbiaised and strongly consistent of I. How to sample these random variables?
2. Show that $V_n = 1 - S_n^2$ satisfies

$$V_n \xrightarrow{\mathbb{P}} \sigma^2 := \mathbb{V}\mathrm{ar}\left(\sqrt{X}\right)$$

3. Deduce a confidence interval with level 95% of I.

Exercise 3 (Mixing of distributions)**.**
Give a sampling procedure, for sampling a random variable Y with density

$$f_Y(y) = \alpha f_\sigma(y) + (1-\alpha)f_\gamma(y)$$

with $\alpha \in]0,1[$ and f_u is the density of the normal distribution $\mathcal{N}(0,u^2)$.

Exercise 4 (The rejection sampling)**.**
Show how to use the rejection sampling for sampling the law of a random variable X conditionally to $X > a$ where X is gaussian $\mathcal{N}(0,1)$ with the help of the following proposal distributions:

1. The gaussian distribution : $g(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$.
2. A shifted exponential distribution : $g_{\lambda,t}(x) = \lambda e^{-\lambda(x-t)}\mathbb{1}_{x\geq t}$ where the parameters (λ, t) should be given.

Exercise 5. Computer sessions: Illustrate all the exercises with Python programs.