Monte-Carlo methods
TD2 - Exact and approximate Sampling

Exercice 1. Let $k, \ell \in \mathbb{N}$ (k may be different from ℓ). For two random vectors, U, V with dimension k and ℓ respectively, we define the covariance matrix of (U, V) as the following $k \times \ell$-matrix

$$\Sigma_{U,V} = \mathbb{E}(UV^T) - \mathbb{E}(U)\{\mathbb{E}(V)\}^T = \mathbb{E}([U - \mathbb{E}(U)][V - \mathbb{E}(V)]^T).$$

By abuse of notation, we write $\Sigma_U = \Sigma_{U,U}$.

1. Set $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$. Show that

$$\Sigma_Z = \begin{pmatrix} \Sigma_X & \Sigma_{X,Y} \\ \Sigma_{Y,X} & \Sigma_Y \end{pmatrix} \quad \text{and} \quad \Sigma_{X,Y} = \Sigma_{Y,X}^T$$

2. If $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ is a Gaussian vector, show that X and Y are independent if and only if $\Sigma_{X,Y} = 0$

Exercice 2 (The rejection algorithm for a Gaussian distribution)**.**
Let $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ be the density of a centered standard gaussian r.v. and let $g(x) = \alpha e^{-|x|}$.

1. Find α such that g is a density and calculate its cumulative distribution function G.
2. How can we draw a r.v. according to the density g ?
3. Let ε be such that $\mathbb{P}\{\varepsilon = -1\} = \mathbb{P}\{\varepsilon = 1\} = 1/2$ and Z be sampled according to an exponential distribution with parameter 1 independent from ε. We set $Y = \varepsilon Z$. Show that Y follows a distribution of density g and deduce a way to sample according to g.
4. Use the rejection algorithm to draw a r.v. according to $\mathcal{N}(0, 1)$.

Exercice 3 (Estimation of the Black-Scholes formula)**.**
Using n iid r.v. according to the distribution $\mathcal{N}(0, 1)$, give a confidence interval of level 95% of the quantity :

$$C = \mathbb{E}\left[e^{-rT}(S_T - K)^+\right]$$

where S follows the Black and Scholes model : $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma B_t}$ and (B_t) is a Brownian motion.