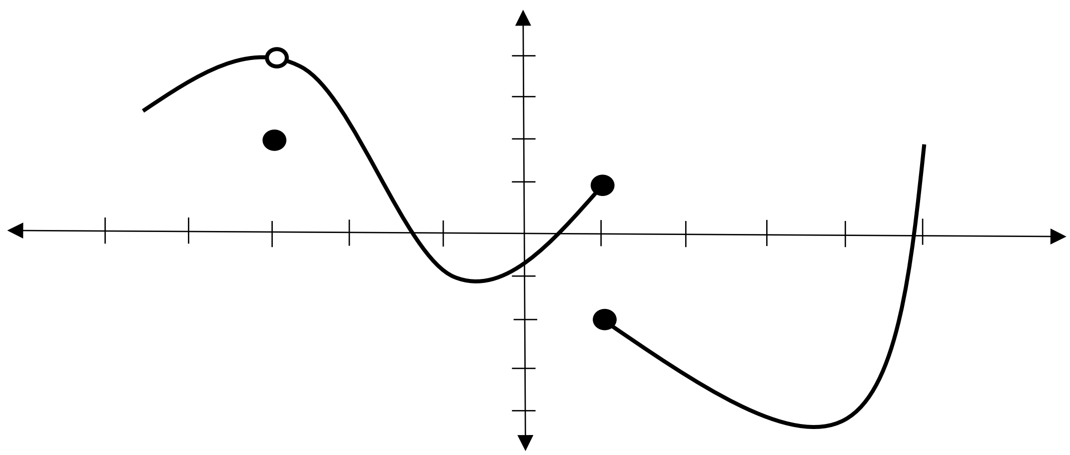
Past Quiz and Exam Questions

Quiz #1



1. Use the picture to determine the following.

(a) $\lim_{x \to -3^-} f(x)$ (b) $\lim_{x \to -3^+} f(x)$ (c) $\lim_{x \to -3} f(x)$ (d) $f(-3)$

(e) $\lim_{x \to 1^-} f(x)$ (f) $\lim_{x \to 1^+} f(x)$ (g) $\lim_{x \to 1} f(x)$ (h) $f(1)$

2. Let $f(x) = \frac{2}{2x+1}$. Evaluate $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$.

3. Write the precise definition of a limit as it applies to the expression $\lim_{x \to 2}(6x - 7) = 5$. Find the δ that you would use to prove that this definition holds for a generic given ε. You do not need to write a formal proof that this limit holds.

Quiz #2

1. Use the IVT to answer the following questions.

(a) State the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to show that the function $f(x) = x^2 + 2^x - 5$ has at least one x-intercept.

2. Use the limit definition of continuity to determine whether $f(x)$ is continuous at $x = 4$, where

$$f(x) = \begin{cases} \sqrt[3]{x+4}, & if\ x > 4 \\ 2, & if\ x < 4 \\ (x-3)^2 + 1, & if\ x = 4 \end{cases}$$

3. Evaluate the following limits.

(a) $\lim\limits_{x\to\infty} \frac{3x^2 + 28x^5 - 7x}{7x^5 + 2x^4 - 18x}$

(b) $\lim\limits_{x\to\infty} \frac{2\left(\frac{1}{3}\right)^{x-2} + 12}{\left(\frac{1}{2}\right)^{x+6} - 4}$

(c) $\lim\limits_{x\to\infty} \tan^{-1}(x)$

Quiz #3

1. Use the limit definition of the derivative (difference quotient) at a point to compute $f'(4)$, where $f(x) = 2x^3 + 6$.

Note: You may check your answer using the power rule, but it should **not** be used to obtain your answer.

2. Use the limit definition of the derivative of a function to compute the velocity function $v(t)$ of a particle, where the position function $s(t)$ of that particle is given by $s(t) = \sqrt{5t}$.

3. Find the derivative of the function $g(p) = 7p^3 - \frac{5}{p^2} + 8\sqrt[4]{p}$.

4. Differentiate the function $y = \frac{1}{e}x^e - \sqrt{2}e^x + e^\pi$.

Exam #1

Question 1. Evaluate the following limits.

(a) $\lim_{x\to-\infty} \frac{3x-4x^2+16x^3}{5x^2-8x^3}$

(b) $\lim_{x\to-1^-} \ln\left(\frac{x^5}{1+x}\right)$

Question 2.

(a) For a given position function $s(t)$, state the meaning of its three derivatives: $s'(t),\ s''(t),\ and\ s'''(t)$.

(b) The position of a F1 racecar can be modeled with the equation $s(t) = \frac{1}{2}t^4 + 2t^3 - 5t^2 + 10t$ where $s(t)$ is the position of the vehicle, in meters, and t is elapsed time, in seconds.

Find $s''(2)$ being sure to provide proper units where applicable.

Question 3. State the precise definition of a limit as it applies to the statement $\lim_{x\to3}(7-4x) = -5$. Find which δ goes with a generic ε in this definition. Then, use this definition to write the formal proof that $\lim_{x\to3}(7-4x) = -5$.

Question 4. Use the limit definition of the derivative at a point to compute $f'(2)$, where $f(x) = \frac{2}{4x-5}$.

Use your result to find an equation for the tangent line to $f(x)$ at the point where $x = 2$.

Question 5. Find the derivatives of the following functions.

(a) $k(x) = \frac{3}{\sqrt[3]{x^2}} + 5^{x^3} - e^{7x}$

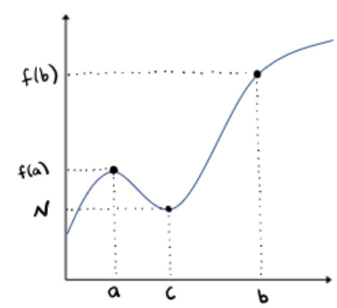
(b) $h(x) = -4\csc(\tan(x^3))$

(c) $r(\theta) = \frac{\tan\theta}{-\sin\theta}$

Question 6. True/False. Mark each statement true or false. Reasoning is not required, but it is highly suggested.

(a) If $\lim_{x\to a} f(x)$ does not exist, then $f(x)$ is not continuous at a.

(b) Examine the graph on the right. Does the Intermediate Value Theorem apply here? In other words, does the IVT guarantee that there exists $c \in (a, b)$ such that $f(c) = N$?



(c) The reason $\frac{d}{dx}e^x = e^x$ is because the unique property that $\ln e = e$.

(d) Suppose $f(4) = 3$, $f'(4) = -9$, $g(1) = 4$, and $g'(1) = 12$. Let $h(x) = f\big(g(x)\big)$. Then

Quiz #4

1. Find $\frac{dy}{dx}$ by implicit differentiation. Your final answer can/should be in terms of x and y.

$$\tan(x - y) = 2xy^3 + 1.$$

2. Find an equation for the tangent line to the asteroid equation $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 4$ at the point $(-3\sqrt{3}\,, 1)$.

3. Use implicit differentiation and to prove the formula given in class for $\frac{dy}{dx}$, where $y = \log_b x$. When you are finished, $\frac{dy}{dx}$ should be entirely in terms of x. (Hint: Rewrite the logarithm in exponential form and use implicit differentiation)

Quiz #5

1. Find the derivative of $f(x) = \frac{(3x^3-4x^2+8)^4}{(\tan^3 x)e^{5x-2x^2}}$, using the most appropriate technique(s) available.

2. Find the derivate of $g(x) = \ln[\operatorname{arccot}(x^3)]$.

3. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast if the boat approaching the dock when it is 8 m from the dock?

Quiz #6

1. Theorems from Section 4.1

a) State the Extreme Value Theorem.

b) True or False? If f is differentiable at c with $f'(c) = 0$, then f has a local maximum or minimum at c.

Extra Credit Opportunity If you said true, name the related theorem. If you said false, provide a counterexample that disproves it.

2. Find the absolute maximum and minimum of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on the interval $[-2, 3]$.

3. Use a linear approximation of the function $f(x) = \sqrt[3]{x}$ to estimate $\sqrt[3]{7.9}$

4. Use the definition of hyperbolic trig functions provided to prove the identity: $\cosh x + \sinh x = e^x$.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

5. Use the provided derivative of the hyperbolic sine function to differentiate: $y = x\, sinh^{-1}(x/3) - \sqrt{9 + x^2}$.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$$

Exam #2

Question 1.

(a) Find the derivative of $f(x) = \frac{(2x^3+5x^2-3)^3}{(sin\,x)e^{x^2+4x}}$, using the most appropriate/efficient technique(s).

(b) Solve for $\frac{dy}{dx}$ if $\sqrt[3]{y} + y * \tan x = 5x^2$.

(c) Find the derivative of $s(t) = (t^2 - 9t)^{\ln t}$

(d) Prove the identity $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Question 2. Use linear approximation or differentials for an appropriate function about an appropriate basepoint to approximate the number $(8.12)^{5/3}$. Simplify your answer.

Question 3.

(a) State Rolle's Theorem.

(b) Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

(c) **[6 points]** Use Rolle's Theorem to prove the Mean Value Theorem for a function f. [*Hint:* part (b) offers a pretty good inspiration of how to get started.]

Question 4. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 4 km from P?

Question 5. Evaluate the following limits.

(a) If $f'(x) \geq 4$ for all x, and if $f(2) = 6$, what is the smallest possible value for $g(7)$?

(b) If the side length of a cube is measured to be 18 centimeters, with a possible error in measurement of ± 0.6 centimeters, then what is the estimated error in the surface area of the cube?

Question 6. Let $f(x) = 4x^3 - 11x^2 - 14x + 13$.

(a) Find the critical numbers (x-values only) of $f(x)$ and classify each one as a local maximum, local minimum, or neither.

(b) Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-1,4]$. For the sake of time, you are provided with two of the necessary function evaluations.

Quiz #7

1. Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$

$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

Which of the following limits are indeterminate forms? For any limit that is not an indeterminate form, evaluate where possible.

a) $\lim\limits_{x\to a} \frac{f(x)}{p(x)}$

b) $\lim\limits_{x\to a} \frac{p(x)}{f(x)}$

c) $\lim\limits_{x\to a} [p(x) - q(x)]$

2. Use l'Hopital's Rule to Evaluate the following limits.

a) $\lim\limits_{x\to -2} \frac{x^3+8}{x+2}$

b) $\lim\limits_{x\to 0} \frac{e^{2x}-1}{\sin x}$

c) $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

d) $\lim\limits_{x\to 0^+} \frac{1}{x} - \frac{1}{e^x-1}$

Quiz #8

Let $f(x) = x\sqrt{x+4}$. Note that $f'(x) = \frac{3x+8}{2\sqrt{x+4}}$, and $f''(x) = \frac{3x+16}{4(x+4)^{3/2}}$

1. Find the following features of the functions using the derivatives provided above:

Domain:

X-intercept:

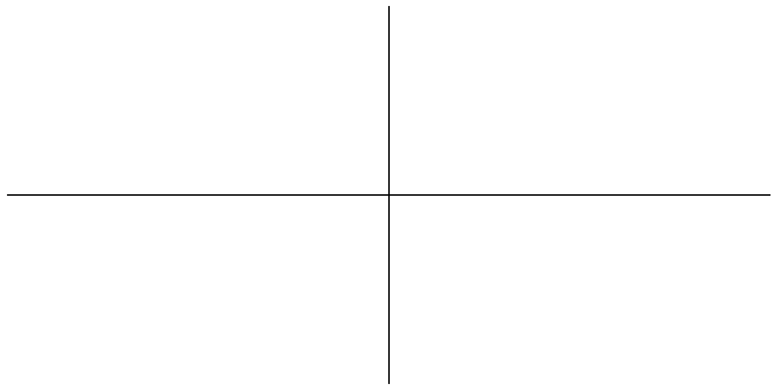
Y-intercept:

Critical Numbers:

Intervals of Increase/Decrease:

Local Maximums/Minimums: Use the approximation $\sqrt{3} = 1.7$ in your calculations.

Sketch:



2. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 4 feet wide, by cutting out a square of side length x from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw a diagram

(b) Write an expression for the volume V in terms of both x and y, where y represents the width of the square base

(c) Use the given information to write an equation for y in terms of x.

(d) Use part (c) to write the volume as a function of only x.

(e) Finish solving the problem by find the largest volume that such a box can have.

Quiz #9

1. Suppose that $f''(x) = 12x + \frac{1}{x^2}$ with $f'(1) = 6,$ and $f(1) = 0$. Find $f(x)$.

2. Let $f(x) = 3\ln x - x^2 + 2x$. Use Newton's Method to approximate a root of f. Using an initial guess of $x_1 = 1$, find x_2 and x_3 . You should simplify x_2 but do not need to simplify x_3 .

3. Find the antiderivatives of the following functions. [1 point each]

(i) $f(x) = -\csc x \cot x$

(ii) $g(x) = \frac{1}{1+x^2}$

(iii) $h(x) = \cosh x$

(iv) $k(x) = nx^{n-1}$ for $n \neq 0$

(v) $p(x) = \frac{1}{x}$

(vi) $m(x) = 2e^{2x}$

(vii) $n(x) = \sec^2 x$

Exam #3

Question 1. A ball is thrown upward from an unknown height at an unknown velocity. Six seconds later, the ball reaches its maximum height of 200 feet. Assume that acceleration due to gravity is 32 ft/s downward. **Use calculus methods** to find the velocity and height functions for the ball and determine the initial height and velocity.

Question 2.

(a) Re-write the following limit as a definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7}{n}\left[\left(3\sqrt{1+\frac{7i}{n}}\right)+\left(1+\frac{7i}{n}\right)^2\right]$$

(b) Re-write the following integral as a limit of Riemann sums.

$$\int_4^6 (2\ln x + 7\sqrt[3]{x})dx$$

Question 3. Let $f(x) = x\sqrt{2+x}$. Note that $f'(x) = \frac{4+3x}{2\sqrt{2+x}}$, and $f''(x) = \frac{8+3x}{4(2+x)^{3/2}}$

(a) Find the domain of f, as well as any intercepts, symmetries, and asymptotes. (**Hint:** Be careful with the domain! Getting it wrong will make the rest of the problem very difficult!)

(b) Determine the intervals where $f(x)$ is increasing and where it is decreasing, find any local extrema (x- and y-values) of f.

(c) Find all intervals of concavity and inflection points (again x- and y-values) of f.

(d) Draw a labelled sketch of f.

Question 4. Use the Midpoint Rule Riemann's Sum with $n = 6$ to find an approximation of the following integral. Your answer should consist entirely of numbers but does not need to be simplified.

$$\int_0^1 \sqrt{x^3 + 1}$$

Question 5. Pepsi ® is trying out a new can design. The new shape will be a rectangular prism with a square base. They don't want their customers to miss out on their product, which is far better than Coke ®, so they keep the same volume: $22\ in^3$. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Question 6. Multiple Choice. Mark the correct answer for each item.

(a) For functions such as $f(x) = x$ and $g(x) = \sqrt{r^2 - x^2}$ that have defined geometric shapes (i.e., a triangle and semi-circle, respectively), we can calculate their definite integrals by drawing a picture and calculating their area, as long as we consider the following:

(i) Areas above the x-axis are positive, those below are negative

(ii) Areas above the x-axis are negative, those below are positive

(iii) Regardless of location, area should always be positive

(iv) You cannot calculate definite integrals using geometric methods

(b) Let $f(x) = x^2 + 2x - 3$. Use Newton's Method with an initial guess of $x_1 = 2$ and produce the next two guesses, x_2 and x_3.

(c) Consider the following integrals.

$$\int_{-6}^{-1} f(x)dx = 12 \quad , \quad \int_{4}^{10} f(x)dx = -5 \quad , \quad \int_{-6}^{10} f(x)dx = 22$$

Find the following:

$$\int_{4}^{-1} f(x)dx$$

Quiz #10

1. Evaluate the following integral.

$$\frac{d}{dx}\int_{4x}^{3x^2} \frac{t+1}{\ln t}\,dt$$

2. Let $f(x) = \sin^2 x$. Use $g(x)$, defined below, to evaluate $g'(\pi)$.

$$g(x) = \int_x^8 f(t)dt$$

3. Evaluate the following integral and simplify your answer:

$$\int_2^5 (6t^2 - 4t + 1)dt$$

4. Evaluate the following integral and simplify your answer:

$$\int_1^e \left(\frac{5}{x} - \csc x \cot x\right) dx$$

Quiz #11

1. Suppose a particle moves on a straight line with velocity $v(t) = 6t^2 - 12t$ ft/s. Find the total distance travelled by the particle on the interval $0 \leq t \leq 4$.

2. Evaluating Integrals

(a) Evaluate the following integral.

$$\int 8\theta^3 \sec^2(\theta^4 + 1) \tan(\theta^4 + 1) d\theta$$

(b) Evaluate the following integral.

$$\int \frac{3e^{3t}}{e^{3t} - 12} dt$$

(c) Evaluate the following integral.

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx$$

Hypothetical Final Exam Questions

Question 1. [Evaluating limits with or without l'Hopital's Rule] Know your indeterminant forms.

Ex:

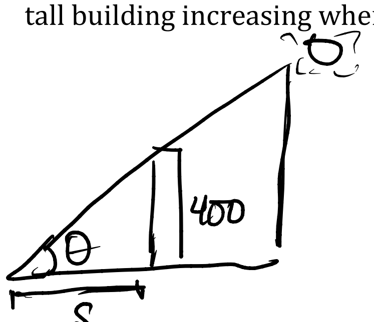
(a) $\lim_{x\to\infty} \frac{\sqrt[5]{x^5-3}}{x+7}$

(b) $\lim_{x\to\infty} (\frac{e^{1/x}}{x} + x)$

(c) $\lim_{x\to\infty} (4x)^{1/x}$

Question 2. [Related Rates Problem] Know your trigonometric identities, unit circle, and similar triangles.

Ex: The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?



$\tan\theta = \frac{400}{s} \Rightarrow \tan\frac{\pi}{6} = \frac{400}{s}$

$\frac{\sqrt{3}}{3} = \frac{400}{s}$

$s\sqrt{3} = 1200$

$s = \frac{1200}{\sqrt{3}}$

Question 3. [Optimization Problem] Know your surface area and volume equations for setups.

Ex: Phil needs a cylindrical barrel with an open top that will hold a volume of 10 ~~cubic feet.~~ Find the dimensions of such a barrel that will minimize its surface area.

$V = 10 = \pi r^2 h \Rightarrow \frac{10}{\pi r^2} = h \to \frac{10}{\pi(\sqrt{\frac{10}{\pi}})^2} = \boxed{h \to \frac{10}{\pi(\frac{10}{\pi})^{\frac{2}{3}}}}$

$SA = \pi r^2 + 2\pi r h \to \pi r^2 + \frac{20\cancel{\pi r}}{\cancel{\pi} r^{\cancel{2}}} = \pi r^2 + \frac{20}{r}$

$\frac{dSA}{dr} = 2\pi r - \frac{20}{r^2} = \quad 2\pi r = \frac{20}{r^2}$

$r^3 = \frac{10}{\pi} \neq \sqrt{\frac{10}{\pi}}$

$+ \mid -$; $\sqrt{\frac{10}{\pi}}$

$\frac{d^2SA}{dr} = 2\pi + \frac{20}{r^3} \quad < 0$

> 0 ∴ concave up = min

Question 4. [Finding Derivatives] Know all your rules, including implicit differentiation, especially the trick where you take the natural log to break apart a rational expression.

Ex: Find $\frac{dy}{dx}$

(a) $y = 4\sqrt[7]{x^5} - 3\tan^{-1}(x^3)$

$y = 4x^{\frac{5}{7}} - 3\tan^{-1}(x^3)$

$\boxed{\frac{dy}{dx} = \frac{20}{7}x^{-\frac{2}{7}} - \frac{9x^2}{1+x^6}}$

(b) $y = \frac{(x^6+9x)^2}{(x^4)e^{\tan x}}$ $= \ln y = 2\ln(x^6+9x) - 4\ln x - \tan x \cancel{\ln e}$

$\frac{1}{y}\frac{dy}{dx} = \left(\frac{2(6x^5+9)}{x^6+9x} - \frac{4}{x} - \sec^2 x\right)$

$\frac{dy}{dx} = y(\quad)$

$\frac{(x^6+9x)^2}{(x^4)e^{\tan x}}\left(\frac{2(6x^5+9)}{x^6+9x} - \frac{4}{x} - \sec^2 x\right)$

(c)

$$y = \int_1^{3x^2} te^{\sin 3t}\,dt$$

$\boxed{3x^2 e^{\sin 3(3x^2)} \cdot 6x}$

Question 5. [Evaluating Integrals] Know all your antiderivatives and the substitution rule.

Ex:

(a) $\int_1^e \left(\frac{5}{x} - \frac{4}{x^3}\right) dx$

(b) $\int \theta \cot(\theta^2)\csc^2(\theta^2)\; d\theta$

(c) $\int \frac{4x+3}{\sqrt{4x^2-6x}} dx$

Question 6. [Formal Definition of a limit] Not much more to say here. Whether you use logic notation or not is up to you, but you have to know it one way or another. I almost guarantee it'll be on the exam.

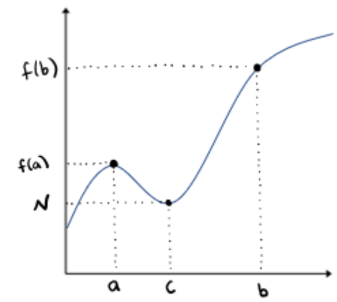
Ex: State the precise definition of a limit as it applies to the statement $\lim_{x\to1}(2x-4) = -2$. Find which δ goes with a generic ε in this definition. Then, use this definition to write the formal proof that $\lim_{x\to1}(2x-4) = -2$.

Question 7. [True/False Section] Basically everything is fair game here. Expect to see a lot of definitions and theorems. Be very careful with converse statements because they're often false.

Ex:

(a) (From Quiz 6) If f is differentiable at c with $f'(c) = 0$, then f has a local maximum or minimum at c.

(b) (From Exam 1) Examine the graph on the right. Does the Intermediate Value Theorem apply here? In other words, does the IVT guarantee that there exists $c \in (a, b)$ such that $f(c) = N$?



(c) If f is continuous at a, then f is differentiable at a.

Question 8. [Multiple Choice Section] Again, expect a little bit of everything here. The multiple choice from Exam III should give you a pretty good idea of the style of problems.

Examples of Shorter Calculations:

- Newton's Method
- Random limits: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{4}$ $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{4}$
- Random derivatives
- Random integrals/antiderivatives
- Riemann's Sums
  - Midpoint Rule
- Playing with the limits of integration to find partial sums
- Rewriting definite integrals as Riemann's Sums and vice versa
- First derivative test to find intervals of increase and decrease
- Second derivative test to find intervals of concavity
- Finding equations of tangent lines