UMBC

Name: Key

Math 151 Exam #2

Spring 2023

Wednesday, April 5th, 2023

[2 points] Put your name on the test above, put a check mark next to your discussion section in the following chart, and read and sign the honor pledge below:

Section 07, Sai Naga, 2 - 2:50 (MP010)		Section 09, Grace, 3 - 3:50 (MP008)	
Section 08, Grace, 2 - 2:50 (MP008)		Section 10, Sai Naga, 3 - 3:50 (MP010)	

Signature:________________________________

Question	Possible points	Score
0	2	
1	18	
2	12	
3	16	
4	16	
5	18	
6	28	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- We will **not accept** answers without justification, unless otherwise noted.
- This exam has 6 pages, and 6 questions. Please make sure that all pages are included.
- There are 110 possible points available on this exam. It will be graded out of 100 points.
- You may not use books, notes or calculators.
- You have 50 minutes to complete this exam.

Good luck!

Question 1. [18 points total]

(a) **[10 points]** Use the most appropriate technique(s) to find the derivative of $y = x^{\ln x}$.

$$\ln y = \ln x^{\ln x}$$

$$\ln y = (\ln x)(\ln x) = (\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln x + \ln x\left(\frac{1}{x}\right) = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2}{x}\ln x$$

$$\frac{dy}{dx} = y\left(\frac{2}{x}\ln x\right)$$

$$\boxed{\frac{dy}{dx} = \left(x^{\ln x}\right)\left(\frac{2}{x}\ln x\right)}$$

(b) **[8 points]** Solve for $\frac{dy}{dx}$, given the curve $2\cosh y + 4xe^y = x^3$.

$$2\sinh y \cdot \frac{dy}{dx} + 4xe^y\left(\frac{dy}{dx}\right) + 4e^y = 3x^2$$

$$\frac{dy}{dx}\left(2\sinh y + 4xe^y\right) = 3x^2 - 4e^y$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2 - 4e^y}{2\sinh y + 4xe^y}}$$

Question 2. [12 points] Find the linear approximation of the function $f(x) = \sqrt{4x+1}$ about the basepoint $a = 2$, and use it to approximate $\sqrt{9.024}$. Simplify your answer.

$f(a) = \sqrt{9} = 3$ $f'(x) = \frac{1}{2\sqrt{4x+1}}(4) = \frac{2}{\sqrt{4x+1}}$ $f'(a) = \frac{2}{\sqrt{9}} = \frac{2}{3}$

$L(x) = f(2) + f'(2)(x-2)$

$L(x) = 3 + \frac{2}{3}(x-2)$

If $\sqrt{9.024} = f(x)$
then $\sqrt{9.024} = \sqrt{4x+1}$
$9.024 = 4x+1$
$8.024 = 4x$
$x = 2.006$

$\sqrt{9.024} = f(2.006) \approx L(2.006) = 3 + \frac{2}{3}(2.006 - 2)$

$= 3 + \frac{2}{3}(0.006) = 3 + 0.004 = \boxed{3.004}$

Question 3. [16 points] Find the absolute maximum and minimum of the function $g(x) = x^3 - 6x^2 + 9x + 6$ on the interval $[-1, 2]$.

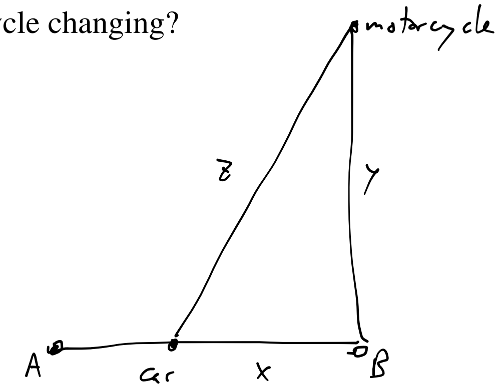
$g'(x) = 3x^2 - 12x + 9 = 0$
$x^2 - 4x + 3 = 0$
$(x-1)(x-3) = 0$
$x = 1$ or $x = 3$
Not in interval

$g(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 6 = -1 - 6 - 9 + 6 = -10$ ← abs min

$g(1) = (1)^3 - 6(1)^2 + 9(1) + 6 = 1 - 6 + 9 + 6 = 10$ ← abs max

$g(2) = (2)^3 - 6(2)^2 + 9(2) + 6 = 8 - 24 + 18 + 6 = 8$

Question 4. [16 points] Point A is 150 miles directly west of Point B. At 1pm, a car begins driving due east from Point A at a constant speed of 30 mph. Also at 1pm, a motorcycle begins driving due north from Point B at a constant speed of 60 mph. At 3pm, how is the direct distance from the car to the motorcycle changing?



$\frac{dx}{dt} = -30$ $\frac{dy}{dt} = 60$ want $\frac{dz}{dt}$

$x = 150 - 2(30) = 90$ $y = 2(60) = 120$

$x^2 + y^2 = z^2$ $90^2 + 120^2 = z^2$ $z = 150$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$2(90)(-30) + 2(120)(60) = 2(150)\frac{dz}{dt}$

$-5400 + 14400 = 300\frac{dz}{dt}$

$9000 = 300\frac{dz}{dt}$

$\boxed{\frac{dz}{dt} = 30 \text{ mph}}$

Question 5. [18 points total]

(a) **[6 points]** State the Mean Value Theorem.

Suppose f is continuous on $[a,b]$ and differentiable on (a,b).
Then there is a point c in (a,b) so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) **[12 points]** Explain why the function $f(x) = 2x^3 + x$ can have no more than one real root (i.e. there can be at most one real number x where $f(x) = 0$).

$f'(x) = 6x^2 + 1$ which is strictly greater than 0 for all x.

If f had more than one real root, say a and b with $a < b$,
then $f(a)$ and $f(b)$ would equal 0.

As a polynomial, f is continuous and differentiable everywhere.
Applying MVT to f on $[a,b]$, there would have to be a point c in (a,b) where $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$,

which is impossible since $f'(x)$ is never 0.

So f cannot have more than one root.

Question 6. [28 points total]

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f'(x)$: $-$ 0 $-$ 0 $+$ at x: 0, 3

(a) **[7 points]** Circle **True** or **False**:

True **False** The function $f(x) = x^4 - 4x^3$ has a local minimum at $x = 0$.

it's a saddle point (neither min nor max)

(b) **[7 points]** Circle **True** or **False**:

True **False** The function $f(x) = x^4 - 4x^3$ has a local minimum at $x = 3$.

For each of the following, **circle the letter of the correct answer**.

(c) **[7 points]** Suppose that Vanessa measures the side of a cubic box as 6 inches, with a possible error in measurement of 0.1 inches. What error can he expect if he uses that measurement to calculate the volume of the box?

(A) $0.3x^2$ inches^3.

(B) 3.6 inches^3.

(C) 10.8 inches^3.

(D) 21.6 inches^3.

(E) Some other answer.

$V = x^3$

$dV = 3x^2 dx$

$dV = 3(6)^2(0.1) = 10.8$

(d) **[7 points]** Suppose that $f(1) = 2$ and $f'(x) \leq 3$ for all x. Using a mean value theorem argument, what can we say about $f(5)$?

(A) $f(5) \leq -14$.

(B) $f(5) \geq -14$.

(C) $f(5) \leq -10$.

(D) $f(5) \geq -10$.

(E) $f(5) \leq 10$.

(F) $f(5) \geq 10$.

(G) $f(5) \leq 14$.

(H) $f(5) \geq 14$.

(I) We can't say anything, because the mean value theorem is not valid in these circumstances.

(J) The mean value theorem is valid, but the correct answer isn't listed.

$f'(x) \leq 3$ for all x $\Rightarrow$ f is differentiable everywhere

$\Rightarrow$ f is cts everywhere, so MVT is valid.

Applying MVT to f on $[1,5]$, there is a c in $(1,5)$

where $f'(c) = \dfrac{f(5) - f(1)}{5-1}$.

$f'(c) \leq 3$ so $\dfrac{f(5)-f(1)}{5-1} \leq 3 \Rightarrow \dfrac{f(5)-2}{4} \leq 3$

$\Rightarrow f(5) - 2 \leq 12 \Rightarrow f(5) \leq 14$