Linear Algebra

Chapter 4: Eigenvalues and Eigenvectors

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Outline

Dynamical System

- For $A \in \mathbb{R}^{n \times n}$, $\lim_{k \to \infty} A^k =?$
- Try the Octave demos your selves!

1. Complete graphs (`K4.m`)
2. Petersen graph (`Petersen.m`)
3. Cyclic graphs
   3.1 Odd number of nodes (`C5.m`)
   3.2 Even number of nodes (`C6.m`)
4. Complete bipartite graphs (`K3_3.m`)

- Steady state vector

Outline

Eigenvalue Problem

- For a matrix $A \in \mathbb{R}^{n \times n}$, which vectors keep (or reverse) their directions after transformed by A? In other words, who satisfy $A\boldsymbol{x} = \lambda \boldsymbol{x}$? $\rightarrow$ "Eigenvalue problem"
- eigen- [áigən]: "own" or "charateristic of"

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a nonzero $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda \boldsymbol{x}$. Such a vector $\boldsymbol{x}$ is called an **eigenvector** of A corresponding to λ.

- Why are they important?
- Do they exist for *any* matrix?
- Is there only one eigenvector for an eigenvalue?
- Is there only one eigenvalue for an eigenvector?
- Given an eigenvalue, how can we find the corresponding eigenvectors? $\rightarrow$ Example 4.2
- How can we find eigenvalues?

Eigenspace

- Example 4.2
  $\rightarrow$ "The set of all eigenvectors corresponding to an eigenvalue λ of an $n \times n$ matrix A is just the set of *nonzero* vectors in the null space of $A - \lambda I$."

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ and is denoted by E_λ.

Geometric Interpretation of Eigenvectors

- $A\boldsymbol{x}$ and $\lambda\boldsymbol{x}$ are parallel, i.e., $\boldsymbol{x}$ is an eigenvector of A iff A transforms $\boldsymbol{x}$ into a parallel vector.
- Examples: Scaling, reflection (Ex 4.4), rotation
- Only the direction of an eigenvector matters.
  $\longrightarrow$ Only unit vectors need to be considered.

Finding Eigenvalues

- For $A \in \mathbb{R}^{n \times n}$, the eigenvectors of λ are the nonzero vectors satisfying $A\boldsymbol{x} = \lambda \boldsymbol{x}$.
  $\rightarrow A\boldsymbol{x} - \lambda I \boldsymbol{x} = \mathbf{0} \rightarrow (A - \lambda I)\boldsymbol{x} = \mathbf{0}$
  $\rightarrow$ The eigenspace E_λ is the non-trivial null space of $A - \lambda I$.
- For 2×2 matrices,
  $\rightarrow$ a matrix has a non-trivial null space iff it is non-invertible.
  $\rightarrow$ a matrix is non-invertible iff its determinant is zero.
  $\rightarrow$ λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$ (Example 4.5)
- Can be generalized to any square matrices.
  $\rightarrow$ What are the determinants for $n \times n$ matrices?

Outline

Determinants

- Notation: $\det A = |A|$
- 1×1 matrices

$$\det A = |a| = a \quad \text{(Not the absolute value)}$$

- 2×2 matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

Determinant of a 3×3 Matrix

Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Determinant of a 3×3 Matrix (cont'd)

- With A_{ij} defined as the submatrix of A obtained by deleting row i and column j,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

  - $\det A_{ij}$ is called the (i, j)-**minor** of A.

- Computed with respect to the first row.
  $\rightarrow$ Why row not column? Why the first row?
  $\rightarrow$ Can be generalized to columns or other rows (The Laplace Expansion Theorem)
- Another method (See (2) on p.264)

$$\begin{aligned}|A| =& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}\end{aligned}$$

Determinants of $n \times n$ Matrices

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

- **Cofactor expansion along the first row**:
  With (i, j)-**cofactor of** A defined as

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

  the definition becomes

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

The Laplace Expansion Theorem

Theorem 4.1: The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the **cofactor expansion along the ith row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the **cofactor expansion along the jth column**)

- Recursive computation $\rightarrow$ Not efficient (Note on p.268)
- Most useful when the matrix contains a row or column with lots of zeros. Why? (Example 4.11)

Determinants of Triangular Matrices

- The Laplace expansion theorem is particularly useful when the matrix is (upper or lower) triangular.

Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

- Why? (Example 4.12)

Properties of Determinants

Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Determinants of Elementary Matrices

- (b), (d) and (f) of the properties are related to elementary row operations.
- What are the determinatns of elementary matrices?

Theorem 4.4

Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.
b. If E results from multiplying one row of I_n by k, then $\det E = k$.
c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

- Determinants of all the elementary matrices are nonzero.

Determinants of Elementary Matrices (cont'd)

- If $B = EA$, $\det B = ?$

Lemma 4.5

Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

- How about $\det(AB)$ when A is NOT an elementary matrix? $\rightarrow$ Theorem 4.8 (p.272)

Theorem 4.6

A square matrix A is invertible if and only if $\det A \neq 0$.

Determinants and Matrix Operations

How can we write the followings in terms of $\det A$ and $\det B$?

- $\det(kA) = ?$
- $\det(A + B) = ?$
- $\det(AB) = ?$
- $\det(A^{-1}) = ?$
- $\det(A^{T}) = ?$

$\det(kA)$ and $\det(A+B)$

- $\det(kA)$

  Theorem 4.7

  If A is an $n \times n$ matrix, then

  $$\det(kA) = k^n \det A$$

  - See Theorem 4.3(d).
- $\det(A+B)$
  - $\det(A+B) = \det A + \det B? \rightarrow$ No!
  - No general forumla.

$\det(AB)$, $\det(A^{-1})$ and $\det(A^T)$

- $\det(AB)$

  Theorem 4.8

  If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

- $\det(A^{-1})$

  Theorem 4.9

  If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

- $\det(A^T)$

  Theorem 4.10

  For any square matrix A,

$$\det A = \det A^T$$

Cramer's Rule

- What is the relation between the solution of a linear system and its determinant?

Theorem 4.11: Cramer's Rule

Let A be an invertible $n \times n$ matrix and let $\boldsymbol{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\boldsymbol{x}$ os the system $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$x_i = \frac{\det\left(A_i(\boldsymbol{b})\right)}{\det A} \quad \text{for } i = 1, \cdots, n$$

- $A_i(\boldsymbol{b})$: the matrix obtained by replacing the ith column of A by $\boldsymbol{b}$
$$A_i(\boldsymbol{b}) = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{b} & \cdots & \boldsymbol{a}_n \end{bmatrix}$$
- Requires to compute determinants
$\rightarrow$ Computationally inefficient except small systems.

Adjoint

- What is the formula of the inverse of a matrix in terms of determinants?
- What is the solution of the equation $AX = I$?

$$A\boldsymbol{x}_1 = \boldsymbol{e}_1 \quad A\boldsymbol{x}_2 = \boldsymbol{e}_2 \quad \cdots \quad A\boldsymbol{x}_n = \boldsymbol{e}_n$$

Theorem 4.12

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- $\operatorname{adj} A := [C_{ji}] = [C_{ij}]^T$: the **adjoint** (or **adjugate**) of A

Outline

Finding Eigenvalues

- How to compute eigenvalues of a matrix?
- How many eigenvalues does a matrix have?
- λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

- What does $\det(A - \lambda I)$ look like?
  $\rightarrow$ Polynomial in λ of degree n
  (**Characteristic polynomial** of A)
  $\rightarrow$ At most n distinct eigenvalues

Finding Eigenvalues and Eigenvectors

Procedure

Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of A.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.
3. For each eigenvalue λ, find the null spacec of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.
4. Find a basis for each eigenspace.

- **Algebraic multiplicity** of an eigenvalue: multiplicity as a root of the characteristic equation.
- **Geometric multiplicity** of an eigenvalue λ: $\dim E_\lambda$
- Geometric multiplicity $\leq$ Algebraic multiplicity (Lemma 4.26 on p.303)

Eigenvalues of Triangular Matrices

Theorem 4.15

The eigenvalues of a triangular matrix are the entries on its main diagonal.

- Why? (See Theorem 4.2 on p.268)

Fundamental Theorem of Invertible Matrices: Ver. 3

Theorem 3.27

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every $\boldsymbol{b}$ in $\mathbb{R}^n$.
c. $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A.

Eigenvalue of A^k and A^{-1}

Theorem 4.18

Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\boldsymbol{x}$.

a. For any positive integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\boldsymbol{x}$.
b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\boldsymbol{x}$.
c. For any integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\boldsymbol{x}$.

Computing $A^k\boldsymbol{x}$

- How to compute $A^k\boldsymbol{x}$, where $\boldsymbol{x}$ is not an eigenvector of A? (Example 4.21)

Theorem 4.19

Suppose the $n \times n$ matrix A has eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. If $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors–say,

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m$$

then, for any integer k,

$$A^k\boldsymbol{x} = c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2 + \cdots + c_m\lambda_m^k\boldsymbol{v}_m$$

Theorem 4.20

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$. Then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ are linearly independent.

Outline

Why Diagonalize Matrices?

- Triangular and diagonal matrices are good.
  $\longrightarrow$ How can we relate a square matrix to a triangular or diagonal one keeping the eigenvalues?
- Gaussian elimination?
  $\longrightarrow$ Eigenvalues are not preserved.
- Diagonalization

Similar Matrices

Definition

Let A and B be $n \times n$ matrices. We say that A **is similar to** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

- Equivalent to "$A = PBP^{-1}$" or "$AP = PB$."
- "$A \sim B$" does not necessarily mean "$B \sim A$."
- P depends on A and B. Is it unique?

Theorem 4.21

Let A, B and C be $n \times n$ matrices.

a. $A \sim A$. (Reflexivity)

b. If $A \sim B$, then $B \sim A$. (Symmetry)

c. If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitivity)

- **Equivalent relation**

Similar Matrices (cont'd)

Theorem 4.22

Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$.

b. A is invertible if and only if B is invertible.

c. A and B have the same rank.

d. A and B have the same characteristic polynomial.

e. A and B have the same eigenvalues.

- The converse is not necessarily true. (See Remark on p.300.)

Diagonalization

- Good if a square matrix is similar to a diagonal matrix.
- Is it always possible?

Definition

An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D–that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Diagonalization (cont'd)

- How can we find D and P?

Theorem 4.23

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.
More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are eigenvalues of A corresponding to the eigenvectors in P in the same order.

- Is a non-invertible matrix diagonalizable? (Example 4.26)

Diagonalization (cont'd)

- How can we check if the eigenvectors are linearly independent? (Example 4.26)

Theorem 4.24

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

Theorem 4.25

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

The Diagonalization Theorem

Lemma 4.26

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

Theorem 4.27

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.

b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.

c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

- Computing A^k (Example 4.29)
  $A^k = PD^kP^{-1}$ for all $k \geq 1$.

Outline

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