Topics in Computer Graphics
Chap 14: Tensor Product Patches

fall, 2011

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Bilinear Interpolation

- Let $\mathbf{b}_{0,0}, \mathbf{b}_{0,1}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1} \in \mathbb{E}^3$.
- The set of all points $\mathbf{x} \in \mathbb{E}^3$ of the form

$$\mathbf{x}(u,v) = \sum_{i=0}^{1} \sum_{j=0}^{1} \mathbf{b}_{i,j} B_i^1(u) B_j^1(v)$$

  is called a *hyperbolic paraboloid* through the four points $\{\mathbf{b}_{i,j}\}$.

- In matrix form:

$$\mathbf{x}(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} \\ \mathbf{b}_{10} & \mathbf{b}_{11} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}.$$

- $\mathbf{x}(u,v)$ is linear in both u and v
  $\rightarrow \mathbf{x}$ is called the *bilinear interpolant*.

Bilinear Interpolation (cont'd)

- Can be viewed as a map of the unit square $0 \leqslant u, v \leqslant 1$ in the u, v-plane. $\rightarrow$ *domain*
- A line parallel to one of the axes in the domain corresponds to a curve in the range. (Equivalent to fixing either u or v.) $\rightarrow$ *isoparametric curve*
- Every isoparametric curve of the hyperbolic paraboloid is a straigt line.
  $\rightarrow$ *ruled surface*
- Evaluation

$$\mathbf{b}_{0,0}^{0,1} = (1-v)\mathbf{b}_{0,0} + v\mathbf{b}_{0,1}$$
$$\mathbf{b}_{1,0}^{0,1} = (1-v)\mathbf{b}_{1,0} + v\mathbf{b}_{1,1}$$
$$\mathbf{x}(u,v) = \mathbf{b}_{0,0}^{1,1}(u,v) = (1-u)\mathbf{b}_{0,0}^{0,1} + u\mathbf{b}_{1,0}^{0,1}$$

Equivalent to computing the coefficients of the isoparametric line $v = const$ first and then evaluating this parametric line at u. (or vice versa)

Bilinear Interpolation (cont'd)

- Why "hyperbolic paraboloid"?
  - The (nonparametric) surface $z = xy$ can be considered as the bilinear interpolant to the four points (Why?)

  $$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

  - The intersection of the surface and a plane parallel to the xy-plane is a hyperbola.
  - The intersection of the surface and a plane containing the z-axis is a parabola.

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The Direct de Casteljau Algorithm

- Higher degree surfaces can be obtained by repeated application of bilinear interpolation.

- Given points $\{\mathbf{b}_{i,j}\}_{i,j=0}^{n}$ and parameter values $(u, v) \in \mathbb{R}^2$, set

$$\mathbf{b}_{i,j}^{r,r} = \begin{bmatrix} 1 - u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{i,j}^{r-1,r-1} & \mathbf{b}_{i,j+1}^{r-1,r-1} \\ \mathbf{b}_{i+1,j}^{r-1,r-1} & \mathbf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1 - v \\ v \end{bmatrix},$$
$$i, j = 0, \ldots, n - r$$

and $\mathbf{b}_{i,j}^{0,0} = \mathbf{b}_{i,j}$. Then $\mathbf{b}_{0,0}^{n,n}(u, v)$ is the point with parameter values (u, v) on the *Bézier surface* $\mathbf{b}^{n,n}$.

- $\{\mathbf{b}_{i,j}\}$: Bézier points or control points

- The net of $\{\mathbf{b}_{i,j}\}$: Bézier net or control net of the surface $\mathbf{b}^{n,n}$

- What if the degrees in u and v are different?
  $(\{\mathbf{b}_{i,j}\}_{i=0,\ldots,m, j=0,\ldots,n})$

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The Tensor Product Approach

- "A surface is the locus of a curve that is moving through space and thereby changing its shape."
- Formulation
    1. Given the initial Bézier curve of degree m:

    $$\mathbf{b}^m(u) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(u)$$

    2. Let each control point $\mathbf{b}_i$ traverses a Bézier curve of degree n:

    $$\mathbf{b}_i = \mathbf{b}_i(v) = \sum_{j=0}^{n} \mathbf{b}_{i,j} B_j^n(v).$$

    3. Combining these two, we obtain the point $\mathbf{b}^{m,n}(u,v)$ on the surface $\mathbf{b}^{m,n}$ as

    $$\mathbf{b}^{m,n}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v).$$

The Tensor Product Approach (cont'd)

‣ For a fixed $\hat{v} = const$, its Bézier points are obtained by

$$\mathbf{b}_{i,0}^{0,n}(\hat{v}) = \sum_{j=0}^{n} \mathbf{b}_{ij} B_j^n(\hat{v}), \quad i = 0, \ldots, m.$$

‣ Other straight lines in the domain (i.e. diagonals) are mapped to higher-degree (generally $m + n$) curves on the patch.

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Properties

‣ Affine invariance

$$\sum_{j=0}^{n} \sum_{i=0}^{m} B_i^m(u) B_j^n(v) \equiv 1(u,v)$$

→ $\mathbf{b}^{m,n}(u,v)$ is an affine combination of the control points.

‣ Convex hull property

‣ Boundary curves of the patch $\mathbf{b}^{m,n}(u,v)$ are polynomial curves.

‣ Variation diminishing property → NOT inherited

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Degree Elevation

- To rewrite a Bézier surface of degree (m, n) as one of degree $(m + 1, n)$:

$$\mathbf{b}^{m,n}(u, v) = \sum_{j=0}^{n} \left[\sum_{i=0}^{m+1} \mathbf{b}_{i,j}^{(1,0)} B_i^{m+1}(u) \right] B_j^n(v).$$

$$\mathbf{b}_{i,j}^{(1,0)} = \frac{i}{m+1} \mathbf{b}_{i-1,j} + \left(1 - \frac{i}{m+1} \right) \mathbf{b}_{i,j}, \qquad \begin{cases} i = 0, \ldots, m+1 \\ j = 0, \ldots, n. \end{cases}$$

- as one of degree $(m + 1, n + 1)$:

$$\mathbf{b}_{i,j}^{(1,1)} = \begin{bmatrix} \frac{i}{m+1} & 1 - \frac{i}{m+1} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{i-1,j-1} & \mathbf{b}_{i-1,j} \\ \mathbf{b}_{i,j-1} & \mathbf{b}_{i,j} \end{bmatrix} \begin{bmatrix} \frac{j}{n+1} \\ 1 - \frac{j}{n+1} \end{bmatrix}$$

$$\begin{cases} i = 0, \ldots, m+1, \\ j = 0, \ldots, n+1. \end{cases}$$

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Derivatives

- Partial derivatives along u and v.
- Tangent vector of an isoparametric curve.

$$\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) = \sum_{j=0}^{n}\left[\frac{\partial}{\partial u}\sum_{i=0}^{m}\mathbf{b}_{i,j}B_i^m(u)\right]B_j^n(v)$$

$$= m\sum_{j=0}^{n}\sum_{i=0}^{m}\Delta^{(1,0)}\mathbf{b}_{i,j}B_i^{m-1}(u)B_j^n(v).$$

where $\Delta^{(1,0)}\mathbf{b}_{i,j} := \mathbf{b}_{i+1,j} - \mathbf{b}_{i,j}$.

$$\frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v) = n\sum_{i=0}^{m}\sum_{j=0}^{n}\Delta^{(0,1)}\mathbf{b}_{i,j}B_j^{n-1}(u)B_i^m(u).$$

Higher Derivatives

$$\frac{\partial^r}{\partial u^r} \mathbf{b}^{m,n}(u,v) = \frac{m!}{(m-r)!} \sum_{j=0}^{n} \sum_{i=0}^{m-r} \Delta^{(r,0)} \mathbf{b}_{i,j} B_i^{m-r}(u) B_j^n(v)$$

where $\Delta^{(r,0)} \mathbf{b}_{i,j} := \Delta^{(r-1,0)} \mathbf{b}_{i+1,j} - \Delta^{(r-1,0)} \mathbf{b}_{i,j}$ and

$$\frac{\partial^s}{\partial v^s} \mathbf{b}^{m,n}(u,v) = \frac{n!}{(n-s)!} \sum_{i=0}^{m} \sum_{j=0}^{n-s} \Delta^{(0,s)} \mathbf{b}_{i,j} B_j^{n-s}(v) B_i^m(u).$$

where $\Delta^{(0,s)} \mathbf{b}_{i,j} := \Delta^{(0,s-1)} \mathbf{b}_{i,j+1} - \Delta^{(0,s-1)} \mathbf{b}_{i,j}$.

Mixed Partial Derivatives & Cross Boundary Derivatives

- Mixed partial derivatives

$$\frac{\partial^{r+s}}{\partial u^r \partial v^s} \mathbf{b}^{m,n}(u,v) = \frac{m!n!}{(m-r)!(n-s)!} \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} \Delta^{r,s} \mathbf{b}_{i,j} B_i^{m-r}(u) B_j^{n-s}(v).$$

- Cross boundary derivatives

$$\frac{\partial^r}{\partial u^r} \mathbf{b}^{m,n}(0,v) = \frac{m!}{(m-r)!} \sum_{j=0}^{n} \Delta^{(r,0)} \mathbf{b}_{0,j} B_j^n(0).$$

$\rightarrow$ Important when we formulate condition for C^r continuity between adjacent patches.

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Blossoms

$$\mathbf{b}[u_1, \ldots, u_m | v_1, \ldots, v_n]$$

1. Compute the (curve) blossom value $\mathbf{b}_i[u_1, \ldots, u_m]$ of all rows of control points, using the same values for each row.
2. Then use those values as input to the (curve) blossom $\mathbf{b}[v_1, \ldots, v_n]$.

- $\mathbf{b}[u^{<m>} | v^{<n>}]$ is the point on the surface.
- The order of evaluation does not matter.
- Multiaffinity in both u and v.

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Curves on a Surface

1. Let $\mathbf{p} = (\mathbf{p}_u, \mathbf{p}_v)$ and $\mathbf{q} = (\mathbf{q}_u, \mathbf{q}_v)$ be u, v-coordinates of two points $\mathbf{p}$ and $\mathbf{q}$ in the domain of a tensor product patch of degree (n, n).

2. A line through $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{u}(t) = (1 - t)\mathbf{p} + t\mathbf{q}.$$

3. A point on the curve is given by

$$\mathbf{b}[((1 - t)\mathbf{p}_u + t\mathbf{q}_u)^{<n>} | ((1 - t)\mathbf{q}_v + t\mathbf{q}_v)^{<n>}].$$

4. Applying the Leibniz formula to the u-part,

$$\sum_{i+j=n} \binom{n}{i, j} (1 - t)^i t^j \mathbf{b}[\mathbf{p}_u^{<i>} \mathbf{q}_u^{<j>} | ((1 - t)\mathbf{p}_v + t\mathbf{q}_v)^{<n>}]$$

Curves on a Surface (cont'd)

5. Applying the Leibniz formula to the v-part,

$$\sum_{i+j=n} \sum_{r+s=n} \binom{n}{i,j}\binom{n}{r,s}(1-t)^i t^j (1-t)^r t^s \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<j>} | \mathbf{p}_v^{<r>}, \mathbf{q}_v^{<s>}]$$

6. Equivalently,

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i}\binom{n}{j}(1-t)^{2n-i-j} t^{i+j} \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}]$$

$$= \sum_{k=0}^{2n} \sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} B_k^{2n}(t) \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}]$$

$$= \sum_{k=0}^{2n} \left(\sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}] \right) B_k^{2n}(t)$$

$$=: \sum_{k=0}^{2n} \mathbf{c}_k B_k^{2n}(t)$$

$\rightarrow$ A curve of degree $2n$

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Normal Vectors

‣ Required for lighting computation.
‣ Can be obtained by cross product of any two tangent vectors at the given point on the surface.

$$\mathbf{b}(u,v) = \frac{\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v)}{\|\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v)\|}$$

‣ Degenerate cases?

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The Matrix Form of a Bézier patch

$$\mathbf{b}^{m,n}(u,v) = \begin{bmatrix} B_0^m(u) & \cdots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \cdots & \mathbf{b}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \cdots & \mathbf{b}_{nn} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

‣ $\{\mathbf{b}_{ij}\}$: Geometry matrix of the patch

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