Homework #4

May 30, 2011

1. For which sides (find a condition on b_1, b_2, b_3) are these systems solvable?

   (a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution:

(a)

$$\left[\begin{array}{rrr|l} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

Therefore, the linear system is solvable when $b_2 - 2b_1 = 0$ and $b_3 + b_1 = 0$, in other words, $b_2 = 2b_1$ and $b_3 = -b_1$.

(b)

$$\left[\begin{array}{rr|l} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{array}\right]$$

Therefore, the linear system is solvable when $b_3 = -b_1$.

2. Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = \boldsymbol{c}$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{c}$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{c}$ are in the column space $\operatorname{col}(A)$, then $\boldsymbol{b} + \boldsymbol{c}$ is in $\operatorname{col}(A)$.

Solution: If we add two linear systems, we get

$$A(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{b} + \boldsymbol{c}$$

therefore, $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$.

3. True or false (with a counterexample if false):

   (a) The vectors $\boldsymbol{b}$ that are not in the column space $\operatorname{col}(A)$ form a subspace.

   (b) If $\operatorname{col}(A)$ contains only the zero vector, then A is the zero matrix.

   (c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A.

Solution:

(a) False, since any subspace should have zero vector. (Any matrix can be a counterexample.)

(b) True, since if A contains any nonzero column vector, that vector should be in $\mathrm{col}(A)$.

(c) True. The proof is as follows.

- $\mathrm{col}(2A) \subset \mathrm{col}(A)$.
  Let $\boldsymbol{x} \in \mathrm{col}(2A)$, i.e., $\boldsymbol{x} = (2A)\boldsymbol{y}$ with some $\boldsymbol{y}$. Then $\boldsymbol{x} = A(2\boldsymbol{y})$ therefore $\boldsymbol{x} \in \mathrm{col}(A)$.
- $\mathrm{col}(A) \subset \mathrm{col}(2A)$.
  Let $\boldsymbol{x} \in \mathrm{col}(A)$, i.e., $\boldsymbol{x} = A\boldsymbol{y}$ with some $\boldsymbol{y}$. Then $\boldsymbol{x} = (2A)(\boldsymbol{y}/2)$ therefore $\boldsymbol{x} \in \mathrm{col}(2A)$.

(d) False. For $A = I$, $\mathrm{col}(A) = \mathbb{R}^n$, but $\mathrm{col}(A - I) = \mathrm{col}(O) = \{\mathbf{0}\}$.

4. Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Solution:

- Since the matrix has five columns and four pivots, only one column has no pivot.

  Suppose the column 5 has a pivot. Let the k-th entry of the column 5 has the leading (nonzero) entry. Obviously, the k-th entry of the column 1 & 3 are all zeros. (Otherwise it cannot be a leading entry, by definition.) Therefore, the column 5 cannot have a pivot.

- Let the matrix A. Then

$$A \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \text{column } 1 + \text{column } 2 + \text{column } 3 = \mathbf{0}$$

  therefore $(1, 0, 1, 0, 1)$ is a special solution.

  (The special solution here means a nonzero solution of the homogeneous linear system. Since this is not explained, this part won't be graded.)

- $\mathrm{null}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$

5. Why does no 3 by 3 matrix have a nullspace that equals its column space?

Solution: Let A be a 3×3 matrix. By rank theorem,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 3.$$

Since $\operatorname{rank}(A)$ and $\operatorname{nullity}(A)$ are all positive integers, $\operatorname{rank}(A)$ cannot be the same as $\operatorname{nullity}(A)$ therefore $\operatorname{col}(A) \neq \operatorname{null}(A)$.

6. If the nullspace of A consists of all multiples of $\boldsymbol{x} = (2, 1, 0, 1)$, how many pivots appear in U (row echelon form)? What is R (reduced row echelon form)?

Solution: Let the last entry of $\boldsymbol{x}$ be the free variable. Then,

$$x_1 = 2x_4, \quad x_2 = x_4, \quad x_3 = 0, \text{ and } x_4 = t.$$

In other words, the homogeneous linear system

$$A\boldsymbol{x} = \mathbf{0}$$

can be reduced to the reduced row echelon form (R)

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, there are three pivots in U.

7. Find a basis for each of the three subspace associated with

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Let

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } U := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Row space

$$\operatorname{row}(A) = \operatorname{row}(U) = \operatorname{span}\left(\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}\right).$$

- Column space

  Since the basis of $\operatorname{col}(U)$ is composed of its 2nd and 4th columns,

$$\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\right).$$

- Null space

  If we further reduce U,

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  Therefore, the solution of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is

$$\begin{cases} x_2 + 2x_3 - 2x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \to \boldsymbol{x} = \begin{bmatrix} x_1 \\ -2x_3 + 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

  Therefore,

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right).$$

8. What are the dimensions of the three subspaces for A, B, and C if I is the 3 by 3 identity matrix and O is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & O \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ O^T & O^T \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} O \end{bmatrix}.$$

Solution:

- $\text{row}(A) = \text{span}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}\right)$

  Therefore,

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A) = 3 \text{ and } \dim(\text{null}(A)) = 5 - \text{rank}(A) = 2.$$

- $\text{row}(B)\,\text{span}\left(\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}\right)$

  Therefore,

$$\dim(\text{row}(B)) = \dim(\text{col}(B)) = 3 \text{ and } \text{null}(B) = 6 - 3 = 3.$$

- $\text{row}(C) = \{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$

  Therefore,

$$\dim(\text{row}(C)) = \dim(\text{col}(C)) = 0 \text{ and } \text{nullity}(C) = 2 - 0 = 2.$$

9. A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution.*

   (a) What are all inequalities ($<$ or $\leqslant$) that must be true between m, n, and r?

   (b) How do you know that $A^T\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

Solution:

(a) Obviously, r cannot be larger than m and n. If $r = m$, $\text{rank}(A) = r = \dim(\text{col}(A))$ therefore $\text{col}(A) = \mathbb{R}^r = \mathbb{R}^m$. In other words, for any $\boldsymbol{b} \in \mathbb{R}^m$ there exists a linear combination of the columns of A such that $A\boldsymbol{x} = \boldsymbol{b}$. Therefore, r should be strictly smaller than m: $r < m$.

(b) By the rank theorem,

$$\text{rank}(A^T) + \text{nullity}(A^T) = r + \text{nullity}(A^T) = m.$$

Since $r < m$ by (a), $\text{nullity}(A^T) > 0$ therefore there is a non-trivial solution for $A^T\boldsymbol{y} = \mathbf{0}$.

10. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

Solution: Let

$$B := \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \text{ and } C := \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

- $\text{row}(A)$

  We first show that $\text{row}(A) = \text{row}(C)$.

  - $\text{row}(A) \subset \text{row}(C)$

    For any $\boldsymbol{y} \in \text{row}(A)$, there exists $\boldsymbol{x} \in \mathbb{R}^3$ such that $\boldsymbol{x}A = \boldsymbol{y}$. Therefore,

    $$\boldsymbol{y} = \boldsymbol{x}A = \boldsymbol{x}(BC) = (\boldsymbol{x}B)C$$

    hence $\boldsymbol{y} \in \text{row}(C)$.

  - $\text{row}(C) \subset \text{row}(A)$

    For any $\boldsymbol{y} \in \text{row}(C)$, there exists $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{x}C = \boldsymbol{y}$. On the other hand, $\boldsymbol{z}B = \boldsymbol{x}$ always has a solution since $\text{row}(B) = \mathbb{R}^2$. Therefore,

    $$\boldsymbol{y} = \boldsymbol{x}C = (\boldsymbol{z}B)C = \boldsymbol{z}(BC) = \boldsymbol{z}A$$

    hence $\boldsymbol{y} \in \text{row}(A)$.

  Now, to find the basis of $\text{row}(C)$,

  $$\begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

  Therefore,

  $$\text{row}(A) = \text{row}(C) = \text{span}\left(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\right).$$

- $\text{col}(A)$

  We first show that $\text{col}(A) = \text{col}(B)$.

- $\mathrm{col}(A) \subset \mathrm{col}(B)$
  For any $\boldsymbol{y} \in \mathrm{col}(A)$, there exists $\boldsymbol{x} \in \mathbb{R}^3$ such that $A\boldsymbol{x} = \boldsymbol{y}$. Therefore,
  $$\boldsymbol{y} = A\boldsymbol{x} = BC\boldsymbol{x} = B(C\boldsymbol{x})$$
  hence $\boldsymbol{y} \in \mathrm{col}(B)$.
- $\mathrm{col}(B) \subset \mathrm{col}(A)$
  For any $\boldsymbol{y} \in \mathrm{col}(B)$, there exists $\boldsymbol{x} \in \mathbb{R}^2$ such that $B\boldsymbol{x} = \boldsymbol{y}$. On the other hand, $C\boldsymbol{z} = \boldsymbol{x}$ always has a solution since $\mathrm{col}(C) = \mathbb{R}^2$. Therefore,
  $$\boldsymbol{y} = B\boldsymbol{x} = B(C\boldsymbol{z}) = (BC)\boldsymbol{z} = A\boldsymbol{z}$$
  hence $\boldsymbol{y} \in \mathrm{col}(A)$.

Now, to find the basis of $\mathrm{col}(B)$,

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore,

$$\mathrm{col}(A) = \mathrm{col}(B) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \right).$$

A is not invertible since, while A is a 3×3 matrix, $\mathrm{rank}(A) = \mathrm{rank}(B) < 3$.

11. (Fill (a),(b), and (c) below.)
    If $AB = C$, the rows of C are combinations of the rows of (a). So the rank of C is not greater than the rank of (b). Since $B^T A^T = C^T$, the rank of C is also not greater than the rank of (c).

Solution:

(a) B

(b) B

(c) A

12. Which of these transformations satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ and which satisfy $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$?
    (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$
    (b) $T(\boldsymbol{v}) = v_1 + v_2 + v_3$
    (c) $T(\boldsymbol{v}) = (v_1, 2v_2, 3v_3)$
    (d) $T(\boldsymbol{v}) =$ largest component of $\boldsymbol{v}$

Solution:

(a) Satisfies neither.

- For $\boldsymbol{v} = (3,4)$ and $\boldsymbol{w} = (1,0)$,

$$T(\boldsymbol{v}+\boldsymbol{w}) = T((4,4)) = (4,4)/4\sqrt{2} = (1,1)/\sqrt{2} \neq T(\boldsymbol{v})+T(\boldsymbol{w}) = (3,4)/5+(1,0) = (8/5,4/5)$$

- For $\boldsymbol{v} = (1,1)$ and $c = 2$,

$$T(c\boldsymbol{v}) = (2,2)/2\sqrt{2} = (1,1)/\sqrt{2} \neq cT(\boldsymbol{v}) = 2(1,1)/\sqrt{2} = \sqrt{2}(1,1)$$

(b) $T(\boldsymbol{v}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{v}$: Satisfies both.

(c) $T(\boldsymbol{v}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{v}$: Satisfies both.

(d) Satisfies neither.

- For $\boldsymbol{v} = (3,2,0)$ and $\boldsymbol{w} = (0,2,3)$,

$$T(\boldsymbol{v}+\boldsymbol{w}) = 4 \neq T(\boldsymbol{v}) + T(\boldsymbol{w}) = 3+3 = 6$$

- For $\boldsymbol{v} = (1,2)$ and $c = -1$,

$$cT(\boldsymbol{v}) = -2 \neq T(c\boldsymbol{v}) = -1$$

13. Suppose T is reflection across the x axis and S is reflection across the y axis. The domain V is the xy plane. If $\boldsymbol{v} = (x,y)$ what is $S(T(\boldsymbol{v}))$? Find a simpler description of the product ST.

Solution:

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } [S] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore

$$S(T(\boldsymbol{v})) = [ST]\boldsymbol{v} = [S][T]\boldsymbol{v} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

Therefore, ST is a reflection with respect to the origin.

14. Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $\boldsymbol{v} = (2,1)$ then $T(\boldsymbol{v}) = (1,2)$. Find $S(T(\boldsymbol{v}))$ and $T(S(\boldsymbol{v}))$. This shows that generall $ST \neq TS$.

Solution:

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } [S] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore

$$S(T(\boldsymbol{v})) = [ST]\boldsymbol{v} = [S][T]\boldsymbol{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

and

$$T(S(\boldsymbol{v})) = [TS]\boldsymbol{v} = [T][S]\boldsymbol{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

15. Let ℓ_1 and ℓ_2 are lines through the origin. And let T_1 and T_2 are reflections across the line ℓ_1 and ℓ_2, respectively. Show that the product T_1T_2 is a rotation. (Hint: See problem 26 on p.222 of our textbook.)

Solution: Let θ be the angle with respect to the x-axis and R_θ be the rotation with respect to the origin by θ. Then, the reflection across the line with its angle θ, T_θ, is defined by

$$\begin{aligned}
[T_\theta] &= [R_\theta]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}[R_{-\theta}]\\
&= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\\
&= \begin{bmatrix}\cos^2\theta-\sin^2\theta & 2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta\end{bmatrix}\\
&= \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}.
\end{aligned}$$

Now let θ_1 and θ_2 be the angles of ℓ_1 and ℓ_2, respectively. Then, by the trigonometric identities (`http://en.wikipedia.org/wiki/List_of_trigonometric_identities`),

$$\begin{aligned}
[T_1T_2] &= [T_{\theta_1}T_{\theta_2}] = [T_{\theta_1}][T_{\theta_2}]\\
&= \begin{bmatrix}\cos 2\theta_1 & \sin 2\theta_1\\ \sin 2\theta_1 & -\cos 2\theta_1\end{bmatrix}\begin{bmatrix}\cos 2\theta_2 & \sin 2\theta_2\\ \sin 2\theta_2 & -\cos 2\theta_2\end{bmatrix}\\
&= \begin{bmatrix}\cos 2\theta_1\cos 2\theta_2+\sin 2\theta_1\sin 2\theta_2 & \cos 2\theta_1\sin 2\theta_2-\sin 2\theta_1\cos 2\theta_2\\ \sin 2\theta_1\cos 2\theta_2-\cos 2\theta_1\sin 2\theta_2 & \sin 2\theta_1\sin 2\theta_2+\cos 2\theta_1\cos 2\theta_2\end{bmatrix}\\
&= \begin{bmatrix}\cos(\theta_1-\theta_2) & -\sin 2(\theta_1-\theta_2)\\ \sin 2(\theta_1-\theta_2) & \cos(\theta_1-\theta_2)\end{bmatrix}\\
&= [R_{\theta_1-\theta_2}].
\end{aligned}$$