Topics in Computer Graphics
Chap 2: Introductory Material

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Affine Space

- *Coordinate-free* or *coordinate-independent* methods
  $\rightarrow$ "affine geometry"
- Distinction between points and vectors
  - *Points* are elements of 3D Euclidean (or point) space $\mathbb{E}^3$.
    $\rightarrow$ A.k.a. "affine space"
  - *Vectors* are elements of 3D linear (or vector) space $\mathbb{R}^3$.
- Operations
  - Vector + vector $\in \mathbb{R}^3$
  - Point + vector $\in \mathbb{E}^3$
  - Point + point not allowed

Barycenteric Combinations

- A.k.a. *affine combinations*
- In general, a linear combination of points

$$\sum_{j=0}^{n} \alpha_j \mathbf{b}_j, \quad \mathbf{b}_j \in \mathbb{E}^3$$

  is not allowed. (Why?)
- But allowed/defined when $\sum_{j=0}^{n} = 1$.

$$\sum_{j=0}^{n} \alpha_j \mathbf{b}_j, \quad \mathbf{b}_j \in \mathbb{E}^3, \sum_{j=0}^{n} \alpha_j = 1.$$

  (Why?)

$$\sum_{j=0}^{n} \alpha_j \mathbf{b}_j = \mathbf{b}_0 + \sum_{j=1}^{n} \alpha_j (\mathbf{b}_j - \mathbf{b}_0)$$

  - $\mathbf{b}_0 \in \mathbb{E}^3$ and $\mathbf{b}_j - \mathbf{b}_0 \in \mathbb{R}^3$
- Examples: centroid of a triangle, midpoint of a line, etc.

Convex Combinations

$$\sum_{j=0}^{n} \alpha_j \mathbf{b}_j, \quad \mathbf{b}_j \in \mathbb{E}^3, \sum_{j=0}^{n} \alpha_j = 1, \alpha_j \geqslant 0 \; \forall j.$$

- A convex combination of points is always inside of the *convex hull* of those points.
- For any two points in the set, the straight line connecting them is also contained in the set.
- Affine maps preserve convexity.

Other Combinations

- What if the sum of coefficients is 0?
  For $\mathbf{p}_j \in \mathbb{E}^3$,
  $$\sum_{j=0}^{n} \sigma_j \mathbf{p}_j \in \mathbb{R}^3.$$

- For any form $\mathbf{a} = \sum \beta_j \mathbf{b}_j$, if $\mathbf{a}$ is supposed to be a point, we must be able to split the sum into three groups:
  $$\mathbf{a} = \sum_{\sum \beta_j = 1} \beta_j \mathbf{b}_j + \sum_{\sum \beta_j = 0} \beta_j \mathbf{b}_j + \sum_{\text{remaining } \beta\text{s}} \beta_j \mathbf{b}_j$$
  - $\mathbf{b}_j$s in $\sum_{\sum \beta_j = 1} \beta_j \mathbf{b}_j$ are points (mandotary)
  - $\mathbf{b}_j$s in $\sum_{\sum \beta_j = 0} \beta_j \mathbf{b}_j$ are either points or vectors (optional)
  - $\mathbf{b}_j$s in $\sum_{\text{remaining } \beta\text{s}} \beta_j \mathbf{b}_j$ are vectors (optional)

Affine Maps

Definition

A map Φ that maps $\mathbb{E}^3$ into itself is called an affine map if it leaves barycentric combinations invariant.

- A.k.a. affine transformation
- If

  $$\mathbf{x} = \sum \alpha_j \mathbf{a}_j, \quad \sum \alpha_j = 1, \mathbf{x}, \mathbf{a}_j \in \mathbb{E}^3,$$

  and Φ is an affine map, then also

  $$\Phi\mathbf{x} = \Phi\left(\sum \alpha_j \mathbf{a}_j\right) = \sum \alpha_j \Phi\mathbf{a}_j, \quad \Phi\mathbf{x}, \Phi\mathbf{a}_j \in \mathbb{E}^3.$$

- Example: The midpoint of two points will be mapped to the midpoint of the affine image of the points.

Affine Maps (cont'd)

Any affine map is of the form

$$\Phi\mathbf{x} = A\mathbf{x} + \mathbf{v}, \quad A \in \mathbb{R}^{3\times 3}, \mathbf{v} \in \mathbb{R}^3.$$

- Proof: Show that the form preserves a barycentric combination.
- The inverse is true as well: Every map of the form above represents an affine map.

Affine Maps (cont'd)

- Examples: The identity, translation, scaling, rotation, shear, parallel projection
- What is the different from the linear transformations? $\rightarrow$ "translation" added
- Euclidean maps (a.k.a. rigid body motions)
  - Characterized by orthonormal matrices A ($A^T A = I$)
  - Leaves lengths and angles unchanged
  - Rotations or translations.
- Affine maps can be composed.
- Every affine map can be composed of translations, rotations, shears, and scalings.
- *Rank* of A: dimension of the image
- An affine map from $\mathbb{E}^2$ ($\mathbb{E}^3$) to $\mathbb{E}^2$ ($\mathbb{E}^3$) is uniquely determined by a nondegenerate triangle (tetrahedron) and its image.
- Affine maps of vectors $\rightarrow$ Same as the linear map A:

$$\Phi(\mathbf{w}) = A\mathbf{w}, \quad \mathbf{w} \in \mathbb{R}^3.$$

Norm Ellipse

1. An ellipse with center at the origin is given by a quadratic form $\mathbf{x}^T A\mathbf{x} = 1$. where A is a symmetric matrix with two nonnegative eigenvalues. (Why?)
2. We're given a 2D point set $\mathbf{p}_1, \ldots, \mathbf{p}_L$ whose centroid is located at the origin.: $\sum_{j=1}^{L} \mathbf{p}_j = \mathbf{0}$.
3. If a point $\mathbf{p}_i$ *were* on the ellipse defined by A, then all points would satisfy ${\mathbf{p}_i}^T A\mathbf{p}_i = 1, \quad i = 1, \ldots, L.$
4. Define $\mathbf{P} := [\mathbf{p}_1 \quad \ldots \quad \mathbf{p}_L] \in \mathbb{R}^{2\times L}$.
5. Then $\mathbf{P}^T A\mathbf{P} = I \in \mathbb{R}^{L\times L}$
6. $\mathbf{P}\mathbf{P}^T A\mathbf{P}\mathbf{P}^T = \mathbf{P}\mathbf{P}^T$
7. Defining $B := \mathbf{P}\mathbf{P}^T \in \mathbb{R}^{2\times 2}$ and assuming it is invertible, $A = B^{-1}$.

Norm Ellipse (cont'd)

8. An ellipse is uniquely defined by the points in an affinely invariant way. → "norm ellipse"
   - The axes of the ellipse defined by A represent the distribution of the points.
   - The axes are given by the eigenvectors of A.
   - The lengths of the axes are determined by the corresponding eigenvalues.
   - Application: image registration

Function Spaces

- Example #1: $C[a,b]$: the set of all real-valued continous functions defined over the interval $[a,b]$ of the real axis
  - By defining
    $$(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t),$$
    $C[a,b]$ forms a *linear space* over the reals.
  - $f_1, \ldots, f_n \in C[a,b]$ are *linearly independent* if $\sum c_i f_i = 0$ for all $t \in [a,b]$ implies $c_1 = \cdots = c_n = 0$.
- Example #2: $C^k[a,b]$: the set of all real-valued functions defined over $[a,b]$ that are k-times continuously differentiable.

Function Spaces (cont'd)

- Example #3: $\mathcal{P}^n$: the set of all polynomials of degree n.
  - The dimension of $\mathcal{P}^n$ is $n+1$. (Why?)
  - A basis of $\mathcal{P}^n$ is the *monomials* $\{1, t, t^2, \ldots, t^n\}$. (Why?)
- Example #4: Piecewise linear functions
  - Forms a linear function space.
  - Basis: *hat functions* $H_i(t)$
    $\rightarrow$ any pecewise linear function f with $f(t_j) = f_j$ can always be written as

$$f(t) = \sum_{j=0}^{n} f_j H_j(t).$$

- *Linear operators*
  - Assigns a function $\mathcal{A}f$ to a given function f
    $\mathcal{A} : C[a,b] \rightarrow C[a,b]$
  - $\mathcal{A}(\alpha f + \beta g) = \alpha \mathcal{A} f + \beta \mathcal{A} g, \quad \alpha, \beta \in \mathbb{R}$.
  - Example: detivative operator