# Linear Algebra
## Chapter 4: Eigenvalues and Eigenvectors

University of Seoul
School of Computer Science
Minho Kim

# Table of contents

# Outline

# Dynamical System

- For $A \in \mathbb{R}^{n \times n}$, $\lim\limits_{k \to \infty} A^k =$?
- Try the Octave demos yourselves!

1. Complete graphs (`K4.m`)
2. Petersen graph (`Petersen.m`)
3. Cyclic graphs
   3.1 Odd number of nodes (`C5.m`)
   3.2 Even number of nodes (`C6.m`)
4. Complete bipartite graphs (`K3_3.m`)

- Steady state vector

# Outline

# Eigenvalue Problem

- For a square matrix $A$, are there nonzero vectors $x$ such that $Ax$ is just a scalar multiplication of $x$? In other words, which nonzero vectors satisfy $Ax = \lambda x$? ($\lambda \in \mathbb{R}$) → "**Eigenvalue problem**"
- eigen- [áigǝn]: "own" or "charateristic of"

### Definition

Let $A$ be an $n \times n$ matrix. A scalar $\lambda$ is called an **eigenvalue** of $A$ if there is a nonzero $x$ such that $Ax = \lambda x$. Such a vector $x$ is called an **eigenvector** of $A$ corresponding to $\lambda$.

- Why are they important?
- Do they exist for *any* matrix?
- Is there only one eigenvector for an eigenvalue?
- Is there only one eigenvalue for an eigenvector?
- Given an eigenvalue, how can we find the corresponding eigenvectors? → Example 4.2
- How can we find eigenvalues?

# Eigenspace

‣ Example 4.2
→ "The set of all eigenvectors corresponding to an eigenvalue $\lambda$ of an $n \times n$ matrix $A$ is just the set of *nonzero* vectors in $\text{null}(A - \lambda I)$."

## Definition

Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. The collection of all eigenvectors corresponding to $\lambda$, together with the zero vector, is called the **eigenspace** of $\lambda$ and is denoted by $E_\lambda$.

# Geometric Interpretation of Eigenvectors

- $A\boldsymbol{x}$ and $\lambda\boldsymbol{x}$ are parallel, i.e., $\boldsymbol{x}$ is an eigenvector of $A$ iff $T_A$ transforms $\boldsymbol{x}$ into a parallel vector.
- Examples: Scaling, reflection (Ex 4.4), rotation
- Only the direction of an eigenvector matters. (Why?)
  $\rightarrow$ Only unit vectors need to be considered. (Fig 4.7)

# Finding Eigenvalues

- For $A \in \mathbb{R}^{n \times n}$, the eigenvectors of $\lambda$ are the nonzero vectors satisfying $A\boldsymbol{x} = \lambda\boldsymbol{x}$.
  $\rightarrow A\boldsymbol{x} - \lambda I\boldsymbol{x} = \boldsymbol{0} \rightarrow (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$
  $\rightarrow$ The eigenspace $E_\lambda$ is the non-trivial null space of $A - \lambda I$.

- For $2 \times 2$ matrices,
  $\rightarrow$ a matrix has a non-trivial null space iff it is non-invertible.
  $\rightarrow$ a matrix is non-invertible iff its determinant is zero.
  $\rightarrow \lambda$ is an eigenvalue of $A$ iff $\det(A - \lambda I) = 0$ (Example 4.5)

- Can be generalized to any square matrices.
  $\rightarrow$ What are the determinants for $n \times n$ matrices?

# Outline

# Determinants

- Notation: $\det A = |A|$
- $1 \times 1$ matrices

$$\det A = |a| = a \quad \text{(Not the absolute value)}$$

- $2 \times 2$ matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- $3 \times 3$ matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

# Determinant of a $3 \times 3$ Matrix
Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of $A$ is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

‣ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

‣ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

‣ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

# Determinant of a $3 \times 3$ Matrix (cont'd)

‣ With $A_{ij}$ defined as the submatrix of $A$ obtained by deleting row $i$ and column $j$,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$
$$= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

   ‣ $\det A_{ij}$ is called the $(i, j)$-**minor** of $A$.

‣ Computed with respect to the first row.
   $\rightarrow$ Why row not column? Why the first row?
   $\rightarrow$ Can be generalized to columns or other rows (The Laplace Expansion Theorem)

‣ Another method (See (2) on p.264)

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

# Determinants of $n \times n$ Matrices

### Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geqslant 2$. Then the **determinant** of $A$ is the scalar

$$\det A = |A| = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

‣ **Cofactor expansion along the first row**:
With $(i, j)$-**cofactor of** $A$ defined as

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

the definition becomes

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

# The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geqslant 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the **cofactor expansion along the $i$th row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the **cofactor expansion along the $j$th column**)

▸ Most useful when the matrix contains a row or column with lots of zeros. Why? (Example 4.11)

# Determinants of Triangular Matrices

‣ The Laplace expansion theorem is particularly useful when the matrix is (upper or lower) triangular.

### Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

‣ Why? (Example 4.12)

# Computing Determinants

- Laplace expansion is very inefficient! (p.268)
- The determinant of a triangular matrix can be easily found.
- We can compute the determinant of a matrix efficiently from its reduced form.
  $\rightarrow$ "How does the determinant change after elementary row operations?"

# Properties of Determinants

### Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

a. If $A$ has a zero row (column), then $\det A = 0$.

b. If $B$ is obtained by interchanging two rows (columns) of $A$, then $\det B = -\det A$.

c. If $A$ has two identical rows (columns), then $\det A = 0$.

d. If $B$ is obtained by multiplying a row (column) of $A$ by $k$, then $\det B = k \det A$.

e. If $A$, $B$, and $C$ are identical except that the $i$th row (column) of $C$ is the sum of the $i$th rows (columns) of $A$ and $B$, then $\det C = \det A + \det B$.

f. If $B$ is obtained by adding a multiple of one row (column) of $A$ to another row (column), then $\det B = \det A$.

# Determinants of Elementary Matrices

- ‣ (b), (d) and (f) of the properties are related to elementary row operations.
- ‣ What are the determinants of elementary matrices?

### Theorem 4.4

Let $E$ be an $n \times n$ elementary matrix.

a. If $E$ results from interchanging two rows of $I_n$, then $\det E = -1$.

b. If $E$ results from multiplying one row of $I_n$ by $k$, then $\det E = k$.

c. If $E$ results from adding a multiple of one row of $I_n$ to another row, then $\det E = 1$.

- ‣ Determinants of all the elementary matrices are nonzero.

# Determinants of Elementary Matrices (cont'd)

- If $B = EA$, $\det B =\,$?

### Lemma 4.5

Let $B$ be an $n \times n$ matrix and let $E$ be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

- How about $\det(AB)$ when $A$ is NOT an elementary matrix? $\rightarrow$ Theorem 4.8 (p.272)

### Theorem 4.6

A square matrix $A$ is invertible if and only if $\det A \neq 0$.

# Determinants and Matrix Operations

How can we write the followings in terms of $\det A$ and $\det B$?

- $\det(kA) = ?$
- $\det(A + B) = ?$
- $\det(AB) = ?$
- $\det(A^{-1}) = ?$
- $\det(A^T) = ?$

# $\det (kA)$ and $\det (A + B)$

- $\det (kA)$

  Theorem 4.7

  If $A$ is an $n \times n$ matrix, then

  $$\det (kA) = k^n \det A$$

    - See Theorem 4.3(d).
- $\det (A + B)$
    - $\det (A + B) = \det A + \det B$? $\rightarrow$ No!
    - No general forumla. (p.271)

# $\det(AB)$, $\det(A^{-1})$ and $\det(A^T)$

- $\det(AB)$

  **Theorem 4.8**

  If $A$ and $B$ are $n \times n$ matrices, then

  $$\det(AB) = (\det A)(\det B)$$

- $\det(A^{-1})$

  **Theorem 4.9**

  If $A$ is invertible, then

  $$\det(A^{-1}) = \frac{1}{\det A}$$

- $\det(A^T)$

  **Theorem 4.10**

  For any square matrix $A$,

  $$\det A = \det A^T$$

# Cramer's Rule and the Adjoint

- What is the relation between determinants and the solution of a linear systems? $\rightarrow$ Cramer's rule (Theorem 4.11)
- What is the relation between determinants and the inverse of a matrix? $\rightarrow$ Adjoint (Theorem 4.12)
- Notation

  $A_i(\boldsymbol{b})$: the matrix obtained by replacing the $i$th column of $A$ by $\boldsymbol{b}$

  $$A_i(\boldsymbol{b}) = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{b} & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

# Cramer's Rule

### Theorem 4.11: Cramer's Rule

Let $A$ be an invertible $n \times n$ matrix and let $b$ be a vector in $\mathbb{R}^n$.
Then the unique solution $x$ of the system $Ax = b$ is given by

$$x_i = \frac{\det\left(A_i(b)\right)}{\det A} \quad \text{for } i = 1, \cdots, n$$

‣ Requires to compute determinants
  → Computationally inefficient except small systems.

# Adjoint

1. What is the formula of the inverse of a matrix in terms of determinants?

2. What is the solution of the equation $AX = I$?

$$Ax_1 = e_1 \quad Ax_2 = e_2 \quad \cdots \quad Ax_n = e_n$$

3. By the Cramer's rule, $x_{ij} = \dfrac{\det(A_i(e_j))}{\det A}$

4. $\det(A_i(e_j)) = (-1)^{j+i} \det A_{ji} = C_{ji}$ (Why?)

5. $A^{-1} = X = \dfrac{1}{\det A}[C_{ji}] = \dfrac{1}{\det A}[C_{ij}]^T = \dfrac{1}{\det A} \operatorname{adj} A$

# Adjoint (cont'd)

Theorem 4.12

Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- $\operatorname{adj} A := [C_{ji}] = [C_{ij}]^T$: the **adjoint** (or **adjugate**) of $A$
  - $C_{ij} := (-1)^{i+j} \det A_{ij}$: $(i, j)$-cofactor of $A$

# Outline

# Finding Eigenvalues

- How to compute eigenvalues of a matrix?
- How many eigenvalues does a matrix have?
- $\lambda$ is an eigenvalue of $A$ iff $\det(A - \lambda I) = 0$.

The eigenvalues of a square matrix $A$ are precisely the solutions $\lambda$ of the equation

$$\det(A - \lambda I) = 0$$

- What does $\det(A - \lambda I)$ look like?
  $\to$ A polynomial in $\lambda$ of degree $n$
  (**Characteristic polynomial** of $A$)
  $\to$ At most $n$ distinct eigenvalues

# Finding Eigenvalues and Eigenvectors
## Procedure

Let $A$ be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of $A$.
2. Find the eigenvalues of $A$ by solving the characteristic equation $\det(A - \lambda I) = 0$ for $\lambda$.
3. For each eigenvalue $\lambda$, find the null space of the matrix $A - \lambda I$. This is the eigenspace $E_\lambda$, the nonzero vectors of which are the eigenvectors of $A$ corresponding to $\lambda$.
4. Find a basis for each eigenspace.

- **Algebraic multiplicity** of an eigenvalue: multiplicity as a root of the characteristic equation.
- **Geometric multiplicity** of an eigenvalue $\lambda$: $\dim E_\lambda$
- What's the relation between the algebraic & geometric multiplicities? (Example 4.18 & 4.19)
  $\rightarrow$ Geometric multiplicity $\leqslant$ Algebraic multiplicity
  (Lemma 4.26 on p.303)

# Eigenvalues of Triangular Matrices

- How does the characteristic equation look like if $A$ is triangular? (See Theorem 4.2 on p.268)
  $\rightarrow (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$

### Theorem 4.15

The eigenvalues of a triangular matrix are the entries on its main diagonal.

# What Does Eigenvalues Tell Us?

### Theorem 4.16

A square matrix $A$ is invertible if and only if $0$ is *not* an eigenvalue of $A$.

- ‣ Why?
- ‣ ...and there will be more (about the importance of eigenvalues).

# Fundamental Theorem of Invertible Matrices: Ver. 3

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

a. $A$ is invertible.

b. $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every $\boldsymbol{b}$ in $\mathbb{R}^n$.

c. $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

d. The reduced row echelon form of $A$ is $I_n$.

e. $A$ is a product of elementary matrices.

f. $\operatorname{rank}(A) = n$

g. $\operatorname{nullity}(A) = 0$

h. The column vectors of $A$ are linearly independent.

i. The column vectors of $A$ span $\mathbb{R}^n$.

j. The column vectors of $A$ form a basis for $\mathbb{R}^n$.

k. The row vectors of $A$ are linearly independent.

l. The row vectors of $A$ span $\mathbb{R}^n$.

m. The row vectors of $A$ form a basis for $\mathbb{R}^n$.

n. $\det A \neq 0$

o. $0$ is not an eigenvalue of $A$.

# Eigenvalue of $A^k$ and $A^{-1}$

### Theorem 4.18

Let $A$ be a square matrix with eigenvalue $\lambda$ and corresponding eigenvector $x$.

a. For any positive integer $k$, $\lambda^k$ is an eigenvalue of $A^k$ with corresponding eigenvector $x$.

b. If $A$ is invertible, then $1/\lambda$ is an eigenvalue of $A^{-1}$ with corresponding eigenvector $x$.

c. For any integer $k$, $\lambda^k$ is an eigenvalue of $A^k$ with corresponding eigenvector $x$.

- Application: Computing $A^k x$ where $x$ is not an eigenvector of $A$. (Example 4.21)
  $\rightarrow$ Is this possible for any $x$?

# Computing $A^k \boldsymbol{x}$

Suppose the $n \times n$ matrix $A$ has eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. If $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors–say,

$$\boldsymbol{x} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_m \boldsymbol{v}_m$$

then, for any integer $k$,

$$A^k \boldsymbol{x} = c_1 \lambda_1^k \boldsymbol{v}_1 + c_2 \lambda_2^k \boldsymbol{v}_2 + \cdots + c_m \lambda_m^k \boldsymbol{v}_m$$

- When does it work for any $\boldsymbol{x}$?

## Theorem 4.20

Let $A$ be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of $A$ with corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$. Then $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ are linearly independent.

# Outline

# Why Diagonalize Matrices?

- Triangular and diagonal matrices are good.
  $\rightarrow$ How can we relate a square matrix to a triangular or diagonal one keeping the eigenvalues?
- Gaussian elimination?
  $\rightarrow$ Eigenvalues are not preserved.
- Diagonalization

# Similar Matrices

### Definition

Let $A$ and $B$ be $n \times n$ matrices. We say that $A$ **is similar to** $B$ if there is an invertible $n \times n$ matrix $P$ such that $P^{-1}AP = B$. If $A$ is similar to $B$, we write $A \sim B$.

- Equivalent to "$A = PBP^{-1}$" or "$AP = PB$."
- $P$ depends on $A$ and $B$. Is it unique?

### Theorem 4.21

Let $A$, $B$ and $C$ be $n \times n$ matrices.

a. $A \sim A$. (Reflexivity)

b. If $A \sim B$, then $B \sim A$. (Symmetry)

c. If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitivity)

- **Equivalent relation**

# Similar Matrices (cont'd)

### Theorem 4.22

Let $A$ and $B$ be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$.

b. $A$ is invertible if and only if $B$ is invertible.

c. $A$ and $B$ have the same rank.

d. $A$ and $B$ have the same characteristic polynomial.

e. $A$ and $B$ have the same eigenvalues.

> ‣ The converse is not necessarily true. (See Remark on p.300.)
> ‣ Useful when showing two matrices are not similar. (How?)

# Diagonalization

- Good if a square matrix is similar to a diagonal matrix.
- Is it always possible?
- How can we find $P$?

### Definition

An $n \times n$ matrix $A$ is **diagonalizable** if there is a diagonal matrix $D$ such that $A$ is similar to $D$–that is, if there is an invertible $n \times n$ matrix $P$ such that $P^{-1}AP = D$.

# Diagonalization (cont'd)

‣ How can we find $D$ and $P$?

### Theorem 4.23

Let $A$ be an $n \times n$ matrix. Then $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

More precisely, there exist an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$ if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$ and the diagonal entries of $D$ are eigenvalues of $A$ corresponding to the eigenvectors in $P$ in the same order.

‣ Is a non-invertible matrix diagonalizable? (Example 4.26)

# Diagonalization (cont'd)

‣ How can we check if the eigenvectors are linearly independent? (See the 2nd remark below Example 4.26)

## Theorem 4.24

Let $A$ be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of $A$. If $\mathcal{B}_i$ is a basis for the eigenspace $E_{\lambda_i}$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

‣ We don't have to check the linear independence of eigenvectors associated with different eigenvalues.

## Theorem 4.25

If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues, then $A$ is diagonalizable.

‣ The converse is not always true.

# The Diagonalization Theorem

### Lemma 4.26

If $A$ is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

‣ The sum of the algebraic multiplicities is always $n$.
  Therefore, $A$ is diagonalizable when...?

### Theorem 4.27

Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_k$. The following statements are equivalent:

a. $A$ is diagonalizable.

b. The union $\mathcal{B}$ of the bases of the eigenspaces of $A$ (as in Theorem 4.24) contains $n$ vectors.

c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

‣ Computing $A^k$ (Example 4.29)
  $A^k = PD^kP^{-1}$ for all $k \geqslant 1$.

# Computing $A^k\boldsymbol{x}$

‣ When $\boldsymbol{x}$ is an eigenvector of $A$. (Theorem 4.18)

$$A^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$$

‣ When $\boldsymbol{x}$ is a linear combination of the eigenvectors of $A$. (Theorem 4.19)

$$\boldsymbol{x} = \sum_{j=1}^{m} c_j\boldsymbol{v}_j \rightarrow A^k\boldsymbol{x} = \sum_{j=1}^{m} (c_j A^k\boldsymbol{v}_j) = \sum_{j=1}^{m} (c_j \lambda_j^k\boldsymbol{v}_j)$$

‣ When $A$ is diagonalizable. (Example 4.29)

$$A^k\boldsymbol{x} = (PDP^{-1})^k\boldsymbol{x} = PD^kP^{-1}\boldsymbol{x}$$

‣ Otherwise... Good luck!

# Outline

# Outline

# Markov Chains

$$\boldsymbol{x}_{k+1} = P\boldsymbol{x}_k \quad \rightarrow \quad \boldsymbol{x}_k = P^k\boldsymbol{x}_0$$

- $P$: transition matrix
- All the components in each column of $P$ add up to $1$. (Why?)
- $\boldsymbol{x}_k$: **state vector** cf) **probability vector**
- Here, we will see...
  - Steady state vector $\boldsymbol{x}$: $P\boldsymbol{x} = \boldsymbol{x}$.
    $\rightarrow$ "Every Markov chain has a unique steady state vector." (p.231)
  - $\lim_{k\rightarrow\infty} \boldsymbol{x}_k = \lim_{k\rightarrow\infty} P^k\boldsymbol{x}_0 =?$
    $\rightarrow \lim_{k\rightarrow\infty} P^k\boldsymbol{x}_0 = \boldsymbol{x}$
    : $\boldsymbol{x}_k$ converges to $\boldsymbol{x}$ regardless of $\boldsymbol{x}_0$.

# Markov Chains (cont'd)

### Theorem 4.30

If $P$ is the $n \times n$ transition matrix of a Markov chain, then $1$ is an eigenvalue of $P$.

- There always exists a vector $x$ such that $Px = x$.
  $\rightarrow$ There always exists a steady state vector. But is it unique?

### Theorem 4.31

Let $P$ be an $n \times n$ transition matrix with eigenvalue $\lambda$.

a. $|\lambda| \leqslant 1$

b. If $P$ is regular and $\lambda \neq 1$, then $|\lambda| < 1$.

- **Positive matrix**: All the entries are positive.
- **Regular matrix**: $P^k$ is positive for some $k$.
- If $P$ is regular, $-1$ cannot be an eigenvalue.

# Markov Chains (cont'd)

### Lemma 4.32

Let $P$ is a regular $n \times n$ transition matrix. If $P$ is diagonalizable, then the dominant eigenvalue $\lambda_1 = 1$ has algebraic multiplicity $1$.

- There is only one eigenvector (and its scalar multiplications) such that $P\boldsymbol{x} = \boldsymbol{x}$.
  $\rightarrow$ the steady state vector is unique.

# Markov Chains (cont'd)

### Theorem 4.33

Let $P$ be a regular $n \times n$ transition matrix. Then as $k \to \infty$, $P^k$ approaches an $n \times n$ matrix $L$ whose columns are identical, each equal to the same vector $x$. This vector $x$ is a steady state probability vector for $P$.

- $\lim_{k \to \infty} P^k = \begin{bmatrix} x & \cdots & x \end{bmatrix} =: L$
  $\to \lim_{k \to \infty} P^k$ converges. But what is $\lim_{k \to \infty} P^k x_0 = ?$

### Theorem 4.34

Let $P$ be a regular $n \times n$ transition matrix, with $x$ the steady state probability vector for $P$, as in Theorem 4.33. Then, for any initial probability vector $x_0$, the sequence of iterates $x_k$ approaches $x$.

- $\lim_{k \to \infty} P^k x_0 = x$ for any $x_0$ (initial probability vector)

# Population Growth

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

- Leslie matrix
- "The proportion of the population in each class is approaching a stead state." (p.235)
  $\rightarrow$ There exists a vector such that $L\boldsymbol{x} = \lambda\boldsymbol{x}$ where $\lambda > 0$.

## Theorem 4.35

Every Leslie matrix has a unique positive eigenvalue and a corresponding eigenvector with positive components.