Linear Algebra

Chapter 1: Vectors

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Introduction to Linear Algebra

What is *algebra*(대수학 : 代數學)?

- "Algebra is the branch of mathematics concerning the study of the rules of **operations** and **relations**, and the constructions and concepts arising from them, including **terms**, **polynomials**, **equations** and **algebraic structures**..." (from Wikipedia)
- Elementary algebra
  - operations(연산자): $+$, $-$, $\times$,...
  - relations(관계): $>$, $<$, $=$, $\leqslant$,...
  - variables/unknowns(미지수): x, y,...
  - terms(항): '$3x$' and '$4y$' in "$3x + 4y$"
  - polynomials(다항식): $ax^2 + by + cz$
  - functions
  - algebraic structures(대수적 구조): $\mathbb{Z}$(정수) under addition & multiplication

Introduction to Linear Algebra (cont'd)

What is *linear algebra*?

- "Linear algebra is a branch of mathematics that studies **vector spaces**, also called **linear spaces**, along with **linear functions** that input one vector and output another... (from Wikipedia)
- Vector space is an algebraic structure. → operations?

What You've Already Learned in High School...

- 수학관련 교과목: 수학 (고1과정), 수학의활용, 수학I, 미적분과통계기본, 수학II, 적분과통계, 기하와벡터
- "7차교육과정" (한국교육과정평가원)
- "대한민국의 고등학교 수학 교과목" (Wikipedia korea)

Related topics:

- 수학 (고1과정)
  - 수와 연산 (elementary algebra: 기초대수학)
- 수학I
  - 행렬 (matrices) 과 그래프
- 기하와 벡터
  - 일차변환 (linear transformations) 과 행렬 (matrices)
  - 공간도형 (three-dimensional geometries) 과 공간좌표 (three-dimensional coordinates)
  - 벡터 (vectors)

Review on Vectors

Let's refresh your memory...

- Definition
- Vector *representation* using coordinates
- Algebra of vectors
- Dot product (점곱) or scalar product (스칼라곱)
  NOTE: Dot(scalar) product *is an* inner product (내적), but *not vice versa*.
- Length of a vector

→ Watch "Vectors" video on YouTube

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...Enjoy with your friends!

Outline

Vectors

- Direction + magnitude
- **Displacement** from **initial point** (**tail**) to **terminal point (head)**
- Notation: $\overrightarrow{AB}$ or $\mathbf{v}$
- Position vectors: 1-to-1 correspondence between **points** on the plane and (2D) **vectors**
- Points $\neq$ vectors
- **Representation** $\rightarrow$ Why?
  - **Coordinate system** (e.g., Cartesian, polar)
  - Components
  - **Ordered** pair of real numbers
  - Notation:

    Row vector $[x, y]$ or column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$

    cf) Notation for points: $P = (x, y)$
- Zero vector ($\mathbf{0}$)
- (Real) vector space $\mathbb{R}^2$: "the set of all vectors with two components"

Vector Arithmetic

Let $\mathbf{u} := [u_1, u_2]$ and $\mathbf{v} := [v_1, v_2]$.
$(\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $u_1, u_2, v_1, v_2 \in \mathbb{R})$

- Addition: $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$
  - 'Head-to-tail rule' or 'parallelogram rule'
- Scalar multiplication: $c\mathbf{u} = c[v_1, v_2] = [cv_1, cv_2]$
  - **Scalar**: constant
- Subtraction: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = [u_1 - v_1, u_2 - v_2]$

Higher Dimensional Vectors

Vectors in $\mathbb{R}^n$

- **Ordered n-tuples** of real numbers
- $\mathbf{v} = [v_1, \cdots, v_n] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$
- Geometric method doesn't work anymore
  $\rightarrow$ We need algebraic methods using n-tuples.

Theorem 1.1: Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity)

f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity)

g. $c(d\mathbf{u}) = (cd)\mathbf{u}$

h. $1\mathbf{u} = \mathbf{u}$

$\rightarrow$ Try to prove them yourselves!

Proof of "$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$" of Theorem 1.1

Any vector can be *uniquely represented* by a coordinate. So let

$$\mathbf{u} := [u_1, u_2, \ldots, u_n] \text{ and } \mathbf{v} := [v_1, v_2, \ldots, v_n].$$

Then,

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n] \\
&= [u_1 + v_1, \ldots, u_n + v_n] \\
&\qquad \text{(by the definition of vector addition)} \\
&= [v_1 + u_1, \ldots, v_n + u_n] \\
&\qquad \text{(by the commutativity of addition of real numbers)} \\
&= [v_1, \ldots, v_n] + [u_1, \ldots, u_n] \\
&\qquad \text{((by the definition of vector addition)} \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

Linear Combination

Definition

A vector $\mathbf{v}$ is a **linear combination** of **vectors** $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ if there are scalars $c_1, c_2, \cdots, c_k$ such that

$$\mathbf{v} = \sum_{j=1}^{k} c_j \mathbf{v}_j = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k.$$

The scalars $c_1, c_2, \cdots, c_k$ are called the **coefficients** of the linear combination.

Linear Combination (cont'd)

- Do such scalars $c_1, c_2, \cdots, c_k$ always exist for any $\mathbf{v}$ and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$?
- How many vectors do we need to represent any vector in $\mathbb{R}^2$ as a linear combination?
- Is it enough just to have sufficiently many vectors to represent any vector in $\mathbb{R}^2$ as a linear combination?
- How is the situation different between $\mathbb{R}$ and $\mathbb{R}^2$?
- **Coordinate axes** and **coordinate grid**
- More in Chapter 6

Binary Vectors and Modular Arithmetic

- Binary arithmetic

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

- Ternary arithmetic

$$\begin{array}{c|ccc} + & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} \cdot & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{array}$$

Outline

Dot Product

Can we multiply a vector with another vector?
$\rightarrow$ **Dot (scalar) product** and **cross product**

Definition

If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then the **dot product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + \cdots + u_n v_n \in \mathbb{R}.$$

- $\mathbf{u} \cdot \mathbf{v} \in \mathbb{R}$ $\rightarrow$a.k.a. **scalar product**
- The dot product is one of the more general **inner products.** (Chap. 7)

Properties of Dot Product

Theorem 1.2

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity)

b. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity)

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geqslant 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ iff (if and only if) $\mathbf{u} = \mathbf{0}$.

$\rightarrow$ Try to prove them yourselves!

- $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) =?$

Length

Definition

The **length** (or **norm**) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the non-negative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2} \rightarrow \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

Theorem 1.3

Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

a. $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$

b. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

Unit Vectors

- Vectors with length 1
- **Normalization**: to make the length of a vector 1
  $\mathbf{v} \to (1/\|\mathbf{v}\|)\mathbf{v}$
- **Standard unit vectors**
  Examples:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

More on Length

Theorem 1.4: The Cauchy-Schwarz inequality

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leqslant \|\mathbf{u}\|\|\mathbf{v}\|$$

- Try Excercises 71 & 72 (p.31) (65 & 66 (p.27) for 2nd ed.)
- Is "$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$" true?

Theorem 1.5: The triangle inequality

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$$

Distance

- How can we *define* the distance of two vectors?

Definition

The **distance** $d(\mathbf{u}, \mathbf{v})$ between (position) vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is *defined* by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Angles

- Geometric view in $\mathbb{R}^2$ and $\mathbb{R}^3$
  - What is the relationship of $\mathbf{u}$, $\mathbf{v}$, and θ?
  - How should we define θ? See Fig 1.32 on p.24 (Fig 1.29 on p.21 for 2nd ed.)
  - What is the relationship of $\mathbf{u}$, $\mathbf{v}$, and $\cos\theta$? $\leftarrow$ "the law of cosines"
- How can we *define* the angle beween two vectors in $\mathbb{R}^n$?

Definition: Angle

For nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Orthogonality

- How can we generalize the concept of "perpendicularity" to $\mathbb{R}^n$?

Definition: Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other ($\mathbf{u} \perp \mathbf{v}$) if $\mathbf{u} \cdot \mathbf{v} = 0$.

- What if one vector is $\mathbf{0}$? $\rightarrow$ The zero vector is orthogonal to every vector!

Theorem 1.6: Pythagoras' Theorem

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

iff $\mathbf{u} \perp \mathbf{v}$.

- More in Chapter 5

Projections

- How to find the distance from a point to a line?
- How can we represent $\mathbf{p}$ using $\mathbf{u}$ and $\mathbf{v}$ in Fig 1.37 on p.27 (Fig 1.34 on p.24 for 2nd ed.)?

Definition

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, the the **projection of $\mathbf{v}$ onto $\mathbf{u}$** is the vector $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ defined by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}$$

- $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is a vector. $(\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) \in \mathbb{R}^n) \rightarrow$ length & direction?
- Is "$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathrm{proj}_{\mathbf{v}}(\mathbf{u})$" true?
- Parallel to $\mathbf{u}$
- What if $\mathbf{v}$ is the zero vector?
- What if the angle is obtuse?
- What if $\mathbf{u}$ is a unit vector?

Cartesian Coordinate System

- Standard unit vectors $\mathbf{e}_1, \cdots, \mathbf{e}_n \in \mathbb{R}^n$
- Any (point) vector $\mathbf{v} \in \mathbb{R}^n$ can be represented as a **linear combination** of the standard unit vectors:

$$\mathbf{v} = \sum_{j=1}^{n} \mathrm{proj}_{\mathbf{e}_j}(\mathbf{v}) = \sum_{j=1}^{n} (\mathbf{v} \cdot \mathbf{e}_j)\mathbf{e}_j = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{v} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

- Example:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The Cartesian coordinate of a (point) vector $\mathbf{v}$ is

$$\mathbf{v} = \begin{bmatrix} \mathbf{v} \cdot \mathbf{e}_1 \\ \vdots \\ \mathbf{v} \cdot \mathbf{e}_n \end{bmatrix}.$$

- More on §5.1

Outline

Line Equations

Think geometrically!!!

- "slope-intercept form" $y = mx + k$ (lines on the plane)
  - m: slope
  - k: y-intercept
- $ax + by = c$ (lines on the plane)
- $\mathbf{n} \cdot \mathbf{x} = 0$ (**normal form** of lines passing through the origin)
  - $\mathbf{x}$: position vectors on the line
  - $\mathbf{n}$: **normal vector**
- $\mathbf{x} = t\mathbf{d}$ (**vector form** of lines passing through the origin)
  - $\mathbf{d}$: direction vector

Line Equations (cont'd)

- What if a line does not pass through the origin?
- (Fig 1.58 on p.36 (Fig 1.55 on p.33 for 2nd ed.)) For all points $\mathbf{x}$ on the line ℓ, $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{n}$.

Definition

The **normal form of the equation of a line** ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \text{ or } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ.
The **general form of the equation of** ℓ is

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax + by = c$$

where $\mathbf{n}$ is a normal vector for ℓ.

- What is the geometric meaning of c in the equation $\mathbf{n} \cdot \mathbf{x} = c$ when $\|\mathbf{n}\| = 1$?

Line Equations (cont'd)

Definition

The **vector form of the equation of a line** ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{d} \neq \mathbf{0}$ is a direction vector for ℓ.
The equation corresponding to the components of the vector form of the equation are called **parametric equations** of ℓ.

- What is the parametric equation of the line passing through two points P and Q? Let $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$.

Plane Equations

- How can we generalize the general form of the line equation to $\mathbb{R}^3$? Does it still represent a line?

Definition

The **normal form of the equation of a plane** $\mathscr{P}$ in $\mathbb{R}^3$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \text{ or } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on $\mathscr{P}$ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for $\mathscr{P}$.
The **general form of the equation of** $\mathscr{P}$ is

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz = d$$

where $\mathbf{n}$ is a normal vector for $\mathscr{P}$.

- Hyperplanes

Plane Equations (cont'd)

- Fig 1.61 on p.39 (Fig 1.58 on p.36 for 2nd ed.)
- Any vector in $\mathbb{R}^2$ is a linear combination of two non-zero vectors in $\mathbb{R}^2$ if they are NOT parallel each other. $\rightarrow$ Can be generalized to vectors parallel to a plane in $\mathbb{R}^3$!

Definition

The **vector form of the equation of a plane** $\mathscr{P}$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where $\mathbf{p}$ is a point on $\mathscr{P}$ and $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for $\mathscr{P}$ ($\mathbf{u}$ and $\mathbf{v}$ are non-zero and parallel to $\mathscr{P}$, but not parallel to each other).
The equations corresponding to the components of the vector form of the equation are called **parametric equations** of $\mathscr{P}$.

- What is the parametric equation of the plane passing through three points P, Q and R? Let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$ and $\mathbf{r} = \overrightarrow{OR}$.

Lines and Planes: Summary

	normal form	vector form
lines in $\mathbb{R}^2$	$(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}=0$ "$\mathbf{x}-\mathbf{p}$ is orthogonal to $\mathbf{n}$"	$\mathbf{x}-\mathbf{p}=t\mathbf{d}$ "$\mathbf{x}-\mathbf{p}$ is parallel to $\mathbf{d}$" "$\mathbf{x}-\mathbf{p}$ is a l.c. of $\mathbf{d}$"
lines in $\mathbb{R}^3$	$(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}_1=0$ $(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}_2=0$ "$\mathbf{x}-\mathbf{p}$ is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$"	$\mathbf{x}-\mathbf{p}=t\mathbf{d}$ "$\mathbf{x}-\mathbf{p}$ is parallel to $\mathbf{d}$"
planes in $\mathbb{R}^3$	$(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}=0$ "$\mathbf{x}-\mathbf{p}$ is orthogonal to $\mathbf{n}$"	$\mathbf{x}-\mathbf{p}=s\mathbf{u}+t\mathbf{v}$ "$\mathbf{x}-\mathbf{p}$ is a l.c. of $\mathbf{u}$ and $\mathbf{v}$"

* l.c.: linear combination

Distance between a Point and a Hyperplane

Distance between a point and a line in 2D

- General form of line equation of ℓ: $ax + by = c$
- Point $B = (x_0, y_0)$
- Proof: Let $X = (p, q)$ be the shortest point on ℓ to B.
  1. The vector $X - B$ is parallel to the vector (a, b). (Why?) $\rightarrow$ $\begin{bmatrix} p-x_0 \\ q-y_0 \end{bmatrix} = k \begin{bmatrix} a \\ b \end{bmatrix}$
  2. X satisfies the line equation. $\rightarrow ap + bq = c \rightarrow$ $a(x_0 + ka) + b(y_0 + kb) = c$
  3. $\|X - B\|$ is the distance. $\rightarrow$ $\|X - B\| = \sqrt{(p - x_0)^2 + (q - y_0)^2} = |k|\sqrt{a^2 + b^2}$

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

Distance between a point and a plane in 3D

- General form of plane equation of $\mathscr{P}$: $ax + by + cz = d$
- Point $B = (x_0, y_0, z_0)$

$$d(B, \mathscr{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product

Definition

The **cross product** of

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is the vector $\mathbf{u} \times \mathbf{v}$ defined by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

- Defined only for the vectors in $\mathbb{R}^3$.
- Show that $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{u}$ and $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{v}$
- $(\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 =?$
- $\|\mathbf{u} \times \mathbf{v}\| =?$
- What is the geometric meaning of $\|\mathbf{u} \times \mathbf{v}\|$?

Scalar Triple Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then **the scalar triple product** of the three vectors is defined as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$
- $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$
- Geometric meaning: (signed) volume of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Outline

Code Vectors

- Example 1.40 (1.37 for 2nd ed.) – UPS (Universal Product Code)
  - $\in \mathbb{Z}_{10}^{12}$: 10-ary vector of length 12
  - check vector $\mathbf{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$
  - Detects all single errors and most adjacent transposition errors.
- Example 1.41 (1.38 for 2nd ed.) – ISBN-10 (International Standard Book Number)
  - $\in \mathbb{Z}_{11}^{10}$ (X denotes 10)
  - check vector $\mathbf{c} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$
  - Detects all single errors and adjacent transposition errors.