Linear Algebra

Chapter 4: Eigenvalues and Eigenvectors

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Outline

Applications of Matrices (§3.7)

- Markov chain (p.228)

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots$$

  - $\mathbf{x}_k$ state vectors
  - $P = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix}$ transition matrix
  - $\mathbf{x}_k = P^k \mathbf{x}_0$
  - For an arbitrary $\mathbf{x}_0 \in \mathbb{R}^2$ $\lim_{k\to\infty} \mathbf{x}_k =?$
    1. Let $\mathbf{v}_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
    2. Then $P\mathbf{v}_1 = \mathbf{v}_1$ and $P\mathbf{v}_2 = 0.5\mathbf{v}_2$ and therefore $P^k\mathbf{v}_1 = \mathbf{v}_1$ and $P^k\mathbf{v}_2 = (0.5)^k\mathbf{v}_2$.
    3. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are l.i., any vector $\mathbf{x}_0$ can be represented as a l.c. of them: $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$
    4. $\mathbf{x}_k = P^k\mathbf{x}_0 = P^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1(P^k\mathbf{v}_1) + c_2(P^k\mathbf{v}_2) = c_1\mathbf{v}_1 + (0.5)^k c_2\mathbf{v}_2$
    5. Therefore $\lim_{k\to\infty} \mathbf{x}_k = c_1\mathbf{v}_1$
    6. Specifically, if $\mathbf{x}_0$ is a probability vector, $\lim_{k\to\infty} \mathbf{x}_k = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ regardless of $\mathbf{x}_0$.

Dynamical System

- For $A \in \mathbb{R}^{n \times n}$, $\lim_{k \to \infty} A^k =?$
- Try the Octave demos yourselves!

1. Complete graphs (`K4.m`)
2. Petersen graph (`Petersen.m`)
3. Cyclic graphs
   3.1 Odd number of nodes (`C5.m`)
   3.2 Even number of nodes (`C6.m`)
4. Complete bipartite graphs (`K3_3.m`)

- Steady state vector

Outline

Eigenvalue Problem

- For a square matrix A, are there nonzero vectors $\mathbf{x}$ such that $A\mathbf{x}$ is just a scalar multiplication of $\mathbf{x}$? In other words, which nonzero vectors satisfy $A\mathbf{x} = \lambda\mathbf{x}$? ($\lambda \in \mathbb{R}$) $\rightarrow$ "**Eigenvalue problem**"
- eigen- [áig¤n]: "own" or "charateristic of"

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a nonzero $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

- Why are they important?
- Do they exist for *any* matrix?
- Is there only one eigenvector for an eigenvalue?
- Is there only one eigenvalue for an eigenvector?
- Given an eigenvalue, how can we find the corresponding eigenvectors? $\rightarrow$ Example 4.2
- How can we find eigenvalues?

Eigenspace

- Example 4.2
  → "The set of all eigenvectors corresponding to an eigenvalue λ of an $n \times n$ matrix A is just the set of *nonzero* vectors in $\mathrm{null}(A - \lambda I)$."

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ and is denoted by E_λ.

- Is an eigenspace a subspace?
- Are all the vectors in E_λ eigenvectors of A corresponding to λ?
- $E_\lambda = \mathrm{null}(A - \lambda I) =$
  $\{$eigenvectors of A corresponding to $\lambda\} \cup \{\mathbf{0}\}$

Geometric Interpretation of Eigenvectors

- $A\mathbf{x}$ and $\lambda\mathbf{x}$ are parallel, i.e., $\mathbf{x}$ is an eigenvector of A iff T_A transforms $\mathbf{x}$ into a parallel vector.
- Examples: Scaling, reflection (Ex 4.4), rotation
- Only the direction of an eigenvector matters. (Why?)
  $\rightarrow$ Only unit vectors need to be considered. (Fig 4.7)

Finding Eigenvalues

- For $A \in \mathbb{R}^{n \times n}$, the eigenvectors of λ are the nonzero vectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$.
  $\rightarrow A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0} \rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$
  $\rightarrow$ The eigenspace E_λ is the non-trivial null space of $A - \lambda I$. (The trivial null space is $\{\mathbf{0}\}$.)
- For 2×2 matrices,
  $\rightarrow$ a matrix has a non-trivial null space iff it is non-invertible.
  $\rightarrow$ a matrix is non-invertible iff its determinant is zero.
  $\rightarrow \lambda$ is an eigenvalue of A iff $\det(A - \lambda I) = 0$ (Example 4.5)
- Can be generalized to any square matrices. (Problem?)
  $\rightarrow$ What are the determinants for $n \times n$ matrices?

Outline

Determinants

- Notation: $\det A = |A|$
- 1×1 matrices

$$\det A = |a| = a \quad \text{(Not the absolute value)}$$

- 2×2 matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

Determinant of a 3×3 Matrix

Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Determinant of a 3×3 Matrix (cont'd)

- With A_{ij} defined as the submatrix of A obtained by deleting row i and column j,

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} \\ &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

  - $\det A_{ij}$ is called the (i,j)**-minor** of A.

- Computed with respect to the first row.
  $\rightarrow$ Why row not column? Why the first row?
  $\rightarrow$ Can be generalized to columns or other rows (The Laplace Expansion Theorem)
- Another method (See (2) on p.276/p.264)

$$\begin{aligned}|A| = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}\end{aligned}$$

Determinants of $n \times n$ Matrices

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geqslant 2$. Then the **determinant** of A is the scalar

$$\begin{aligned}\det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

- **Cofactor expansion along the first row:**
  With (i, j)**-cofactor of** A defined as

  $$C_{ij} = (-1)^{i+j} \det A_{ij},$$

  the definition becomes

  $$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

The Laplace Expansion Theorem

Theorem 4.1: The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geqslant 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the **cofactor expansion along the ith row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the **cofactor expansion along the jth column**)

- Most useful when the matrix contains a row or column with lots of zeros. Why? (Example 4.11)

Determinants of Triangular Matrices

- The Laplace expansion theorem is particularly useful when the matrix is (upper or lower) triangular.

Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

- Why? (Example 4.12)

Computing Determinants

- Laplace expansion is very inefficient! (See the note below Theorem 4.2)
- The determinant of a triangular matrix can be easily found.
- We can compute the determinant of a matrix efficiently from its reduced form.
  $\rightarrow$ "How does the determinant change after elementary row operations?"

Properties of Determinants

Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Determinants of Elementary Matrices

- (b), (d) and (f) of the properties are related to elementary row operations.
- Example 4.13
- We can "mix and match" elementary row and column operations.
- What are the determinants of elementary matrices?

Theorem 4.4

Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.

b. If E results from multiplying one row of I_n by k, then $\det E = k$.

c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

- Determinants of all the elementary matrices are nonzero.

Determinants of Elementary Matrices (cont'd)

- If $B = EA$, $\det B =?$

Lemma 4.5

Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

- How about $\det(AB)$ when A is NOT an elementary matrix? $\rightarrow$ Theorem 4.8

Theorem 4.6

A square matrix A is invertible if and only if $\det A \neq 0$.

Determinants and Matrix Operations

How can we write the followings in terms of $\det A$ and $\det B$?

- $\det(kA) = ?$
- $\det(A+B) = ?$
- $\det(AB) = ?$
- $\det(A^{-1}) = ?$
- $\det(A^T) = ?$

$\det(kA)$ and $\det(A+B)$

- $\det(kA)$

  Theorem 4.7

  If A is an $n \times n$ matrix, then

  $$\det(kA) = k^n \det A$$

  - See Theorem 4.3(d).

- $\det(A+B)$
  - $\det(A+B) = \det A + \det B? \rightarrow$ No!
  - No general forumla

$\det(AB)$, $\det(A^{-1})$ and $\det(A^T)$

- $\det(AB)$

  Theorem 4.8

  If A and B are $n \times n$ matrices, then

  $$\det(AB) = (\det A)(\det B)$$

- $\det(A^{-1})$

  Theorem 4.9

  If A is invertible, then

  $$\det(A^{-1}) = \frac{1}{\det A}$$

- $\det(A^T)$

  Theorem 4.10

  For any square matrix A,

  $$\det A = \det A^T$$

Cramer's Rule and the Adjoint

- What is the relation between determinants and the solution of a linear systems? $\rightarrow$ Cramer's rule (Theorem 4.11)
- What is the relation between determinants and the inverse of a matrix? $\rightarrow$ Adjoint (Theorem 4.12)
- Notation
  $A_i(\mathbf{b})$: the matrix obtained by replacing the ith column of A by $\mathbf{b}$
  $$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix}$$

Cramer's Rule

Theorem 4.11: Cramer's Rule

Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det\left(A_i(\mathbf{b})\right)}{\det A} \quad \text{for } i = 1, \cdots, n$$

- Requires to compute determinants
  $\rightarrow$ Computationally inefficient except small systems.

Adjoint

1. What is the formula of the inverse of a matrix in terms of determinants?
2. What is the solution of the equation $AX = I$?

$$A\mathbf{x}_1 = \mathbf{e}_1 \quad A\mathbf{x}_2 = \mathbf{e}_2 \quad \cdots \quad A\mathbf{x}_n = \mathbf{e}_n$$

3. By the Cramer's rule, $x_{ij} = \dfrac{\det(A_i(\mathbf{e}_j))}{\det A}$
4. $\det(A_i(\mathbf{e}_j)) = (-1)^{j+i} \det A_{ji} = C_{ji}$ (Why?)
5. $A^{-1} = X = \dfrac{1}{\det A}[C_{ji}] = \dfrac{1}{\det A}[C_{ij}]^T = \dfrac{1}{\det A} \operatorname{adj} A$

Adjoint (cont'd)

Theorem 4.12

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- $\operatorname{adj} A := [C_{ji}] = [C_{ij}]^T$: the **adjoint** (or **adjugate**) of A
  - $C_{ij} := (-1)^{i+j} \det A_{ij}$: (i, j)-cofactor of A

Outline

Finding Eigenvalues

- How to compute eigenvalues of a matrix?
- How many eigenvalues does a matrix have?
- λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

- What does $\det(A - \lambda I)$ look like?
  $\rightarrow$ A polynomial in λ of degree n
  (**Characteristic polynomial** of A)
  $\rightarrow$ At most n distinct eigenvalues

Finding Eigenvalues and Eigenvectors

Procedure

Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of A.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.
3. For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.
4. Find a basis for each eigenspace.

- **Algebraic multiplicity** of an eigenvalue: multiplicity as a root of the characteristic equation.
- **Geometric multiplicity** of an eigenvalue λ: $\dim E_\lambda$
- What's the relation between the algebraic & geometric multiplicities? (Example 4.18 & 4.19)
  $\rightarrow$ Geometric multiplicity $\leqslant$ Algebraic multiplicity
  (Lemma 4.26 on p.303)

Eigenvalues of Triangular Matrices

- How does the characteristic equation look like if A is triangular? (See Theorem 4.2)
  $\rightarrow (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$

Theorem 4.15

The eigenvalues of a triangular matrix are the entries on its main diagonal.

What Does Eigenvalues Tell Us?

Theorem 4.16

A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A.

- Why?
- ...and there will be more (about the importance of eigenvalues).

Fundamental Theorem of Invertible Matrices: Ver. 3

Theorem 3.27

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

f. $\mathrm{rank}(A) = n$

g. $\mathrm{nullity}(A) = 0$

h. The column vectors of A are linearly independent.

i. The column vectors of A span $\mathbb{R}^n$.

j. The column vectors of A form a basis for $\mathbb{R}^n$.

k. The row vectors of A are linearly independent.

l. The row vectors of A span $\mathbb{R}^n$.

m. The row vectors of A form a basis for $\mathbb{R}^n$.

n. $\det A \neq 0$

o. 0 is not an eigenvalue of A.

Eigenvalue of A^k and A^{-1}

Theorem 4.18

Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a. For any positive integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.
b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
c. For any integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.

- Application: Computing $A^k\mathbf{x}$ where $\mathbf{x}$ is not an eigenvector of A. (Example 4.21)
  $\rightarrow$ Is this possible for any $\mathbf{x}$?

Computing $A^k\mathbf{x}$

Theorem 4.19

Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors-say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

- When does it work for any $\mathbf{x} \in \mathbb{R}^n$?

Theorem 4.20

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ are linearly independent.

Outline

Why Diagonalize Matrices?

- Triangular and diagonal matrices are good.
  $\rightarrow$ How can we relate a square matrix to a triangular or diagonal one keeping the eigenvalues?
- Gaussian elimination?
  $\rightarrow$ Eigenvalues are not preserved.
- Diagonalization

Similar Matrices

Definition

Let A and B be $n \times n$ matrices. We say that A **is similar to** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

- Equivalent to "$A = PBP^{-1}$" or "$AP = PB$."
- P depends on A and B. Is it unique?

Theorem 4.21

Let A, B and C be $n \times n$ matrices.

a. $A \sim A$. (Reflexivity)

b. If $A \sim B$, then $B \sim A$. (Symmetry)

c. If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitivity)

- **Equivalent relation**

Similar Matrices (cont'd)

Theorem 4.22

Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$.

b. A is invertible if and only if B is invertible.

c. A and B have the same rank.

d. A and B have the same characteristic polynomial.

e. A and B have the same eigenvalues.

- The converse is not necessarily true. (See Remark)
- Useful when showing two matrices are not similar. (Example 4.23)

Diagonalization

- Good if a square matrix is similar to a diagonal matrix.
- Is it always possible?
- How can we find P?

Definition

An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D-that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Diagonalization (cont'd)

- How can we find D and P?

Theorem 4.23

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.
More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are eigenvalues of A corresponding to the eigenvectors in P in the same order.

- Is a non-invertible matrix diagonalizable? (Example 4.26)

Diagonalization (cont'd)

- How can we check if the eigenvectors are linearly independent? (See the 2nd remark below Example 4.26)

Theorem 4.24

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

- We don't have to check the linear independence of eigenvectors associated with different eigenvalues.

Theorem 4.25

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

- The converse is not always true.

The Diagonalization Theorem

Lemma 4.26

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

- The sum of the algebraic multiplicities is always n. Therefore, A is diagonalizable when...?

Theorem 4.27

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.

b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.

c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

- Computing A^k (Example 4.29)
  $A^k = PD^kP^{-1}$ for all $k \geqslant 1$.

Computing $A^k\mathbf{x}$

- When $\mathbf{x}$ is an eigenvector of A. (Theorem 4.18)

$$A^k\mathbf{x} = \lambda^k\mathbf{x}$$

- When $\mathbf{x}$ is a linear combination of the eigenvectors of A. (Theorem 4.19)

$$\mathbf{x} = \sum_{j=1}^{m} c_j\mathbf{v}_j \rightarrow A^k\mathbf{x} = \sum_{j=1}^{m}(c_jA^k\mathbf{v}_j) = \sum_{j=1}^{m}(c_j\lambda_j^k\mathbf{v}_j)$$

- When A is diagonalizable. (Example 4.29)

$$A^k\mathbf{x} = (PDP^{-1})^k\mathbf{x} = PD^kP^{-1}\mathbf{x}$$

- Otherwise... Good luck!

Outline

Outline

Markov Chains

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \rightarrow \quad \mathbf{x}_k = P^k\mathbf{x}_0$$

- P: transition matrix
- All the components in each column of P add up to 1. (Why?)
- $\mathbf{x}_k$: **state vector** cf) **probability vector**
- Here, we will see...
  - Steady state vector $\mathbf{x}$: $P\mathbf{x} = \mathbf{x}$.
    $\rightarrow$ "Every Markov chain has a unique steady state vector." (§3.7)
  - $\lim_{k\rightarrow\infty} \mathbf{x}_k = \lim_{k\rightarrow\infty} P^k\mathbf{x}_0 =?$
    $\rightarrow \lim_{k\rightarrow\infty} P^k\mathbf{x}_0 = \mathbf{x}$
    :$\mathbf{x}_k$ converges to $\mathbf{x}$ regardless of $\mathbf{x}_0$.

Markov Chains (cont'd)

Theorem 4.30

If P is the $n \times n$ transition matrix of a Markov chain, then 1 is an eigenvalue of P.

- There always exists a vector $\mathbf{x}$ such that $P\mathbf{x} = \mathbf{x}$.
  $\rightarrow$ There always exists a steady state vector. But is it unique?

Theorem 4.31

Let P be an $n \times n$ transition matrix with eigenvalue λ.

a. $|\lambda| \leqslant 1$

b. If P is regular and $\lambda \neq 1$, then $|\lambda| < 1$.

- **Positive matrix**: All the entries are positive.
- **Regular matrix**: P^k is positive for some k.
- If P is regular, -1 cannot be an eigenvalue.

Markov Chains (cont'd)

Lemma 4.32

Let P is a regular $n \times n$ transition matrix. If P is diagonalizable, then the dominant eigenvalue $\lambda_1 = 1$ has algebraic multiplicity 1.

- There is only one eigenvector (and its scalar multiplications) such that $P\mathbf{x} = \mathbf{x}$.
  $\rightarrow$ the steady state vector is unique.

Markov Chains (cont'd)

Theorem 4.33

Let P be a regular $n \times n$ transition matrix. Then as $k \to \infty$, P^k approaches an $n \times n$ matrix L whose columns are identical, each equal to the same vector $\mathbf{x}$. This vector $\mathbf{x}$ is a steady state probability vector for P.

- $\lim\limits_{k\to\infty} P^k = \begin{bmatrix} \mathbf{x} & \cdots & \mathbf{x} \end{bmatrix} =: L$
  $\to \lim_{k\to\infty} P^k$ converges. But what is $\lim_{k\to\infty} P^k \mathbf{x}_0 =$?

Theorem 4.34

Let P be a regular $n \times n$ transition matrix, with $\mathbf{x}$ the steady state probability vector for P, as in Theorem 4.33. Then, for any initial probability vector $\mathbf{x}_0$, the sequence of iterates $\mathbf{x}_k$ approaches $\mathbf{x}$.

- $\lim\limits_{k\to\infty} P^k \mathbf{x}_0 = \mathbf{x}$ for any $\mathbf{x}_0$ (initial probability vector)

Population Growth

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

- Leslie matrix
- "The proportion of the population in each class is approaching a stead state." (§3.7)
  $\rightarrow$ There exists a vector such that $L\mathbf{x} = \lambda\mathbf{x}$ where $\lambda > 0$.

Theorem 4.35

Every Leslie matrix has a unique positive eigenvalue and a corresponding eigenvector with positive components.