Topics in Computer Graphics
Chap 6: Bézier Curve Topics

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Degree Elevation

- What if we want to get more flexibility while keeping the shape of the current curve? $\rightarrow$ *Degree elevation* required
- Monomial form case?
- Can be used to make several curves have the same degree
- Useful for data transfer between different CAD/CAM or graphics systems

Degree Elevation: Algebraic Proof

$$
\begin{aligned}
(1-t)B_i^n(t) &= \frac{n+1-i}{n+1} B_i^{n+1}(t) \\
tB_i^n(t) &= \frac{i+1}{n+1} B_{i+1}^{n+1}(t) \\
B_i^n(t) &= \frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{i+1}{n+1} B_{i+1}^{n+1}(t)
\end{aligned}
$$

$\rightarrow$

$$
\begin{aligned}
\mathbf{x}(t) &= (1-t)\mathbf{x}(t) + t\mathbf{x}(t) \\
&= \sum_{i=0}^{n} \frac{n+1-i}{n+1} \mathbf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n} \frac{i+1}{n+1} \mathbf{b}_i B_{i+1}^{n+1}(t) \\
&= \sum_{i=0}^{n+1} \left(\frac{i}{n+1} \mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right) \mathbf{b}_i \right) B_i^{n+1}(t) \\
&=: \sum_{i=0}^{n+1} \mathbf{b}_i^{(1)} B_i^{n+1}(t)
\end{aligned}
$$

Degree Elevation: Proof Using Blossoms

$$\mathbf{b}^{(1)}[t_1, \dots, t_{n+1}] = \frac{1}{n+1} \sum_{j=1}^{n+1} \mathbf{b}[t_1, \dots, t_{n+1} | t_j]$$

- $\mathbf{b}[t_1, \dots, t_{n+1} | t_j]$: t_j is omitted from $\mathbf{b}[t_1, \dots, t_{n+1}]$.
- Why? $\rightarrow$ Definition of a blossom

$$\begin{aligned} \mathbf{b}_i^{(1)} &= \mathbf{b}^{(1)}[0^{<n+1-i>}, 1^{<i>}] \\ &= \frac{1}{n+1} \left(i\mathbf{b}_{i-1} + (n+1-i)\mathbf{b}_i \right) \end{aligned}$$

Repeated Degree Elevation

- Degree elevation: $\mathbf{P} \to \mathcal{E}\mathbf{P}$
- After r degree elevation, $\mathcal{E}^r\mathbf{P}$ has the vertices $\{\mathbf{b}_i^{(r)}\}_{i=0}^{n+r}$ where

$$\mathbf{b}_i^{(r)} = \sum_{j=0}^{n} \mathbf{b}_j \binom{n}{j} \frac{\binom{r}{i-j}}{\binom{n+r}{i}}$$

- $\lim_{r\to\infty} \mathcal{E}^r\mathbf{P} = \mathcal{B}\mathbf{P} \quad \leftarrow \lim_{i/(n+r)\to t} \frac{\binom{r}{i-j}}{\binom{r+n}{i}} = t^j(1-t)^{n-j}$
  - $\mathcal{E}^r\mathbf{P}$ converges to the Bézier curve $\mathcal{B}\mathbf{P}$.
  - But converges too slowly.

The Variation Diminishing Property

- The curve $\mathcal{B}\mathbf{P}$ has no more intersections with any plane than does the polygon $\mathbf{P}$.
- Proof
  1. Degree elevation is a piecewise linear interpolation.
  2. Piecewise linear interpolation is variation diminishing. (Sec 3.2)
     $\rightarrow$ Each $\mathcal{E}^r\mathbf{P}$ has fewer intersections than $\mathcal{E}^{r-1}\mathbf{P}$.
  3. Since $\mathcal{E}^r\mathbf{P}$ converges to the curve, the proof is done.
- A convex polygon generates a convex curve.
  How about the inverse?

Degree Reduction

- Problem: Can we write a given curve of degree $n+1$ as one of degree n?
- In general, exact degree reduction is not possible.
  $\rightarrow$ approximate method
- New problem: Given a Bézier curve with control vertices $\{\mathbf{b}_i^{(1)}\}_{i=0}^{n+1}$, can we find a Bézier curve with control vertices $\{\mathbf{b}_i\}_{i=1}^{n}$ that approximates the first curve in a "reasonable" way?

$$\begin{bmatrix} 1 & & & & & \\ * & * & & & & \\ & * & * & & & \\ & & & \ddots & & \\ & & & & * & * \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0^{(1)} \\ \vdots \\ \mathbf{b}_{n+1}^{(1)} \end{bmatrix}$$

Degree Reduction (cont'd)

- $MB = B^{(1)}, \quad M \in \mathbb{R}^{(n+2)\times(n+1)}$
  $\rightarrow$ Not solvable.
- Can be solved after converted to

$$\left(M^T M\right) B = M^T B^{(1)}$$

  - *Normal equation*
  - Guarantees that B is optimal in a least square sense.
  - Problem?
    $\mathbf{b}_0 = \mathbf{b}_0^{(1)}$ and $\mathbf{b}_n = \mathbf{b}_{n+1}^{(1)}$ are not guaranteed.

Nonparametric Curves

- Functional curves: $y = f(x)$
- How can we express a functional curve in parametric form?

$$\mathbf{b}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ f(t) \end{bmatrix}$$

- If f is a polynomial,

$$f(t) = b_0 B_0^n(t) + \cdots + b_n B_n^n(t), \quad b_i \in \mathbb{R} \text{ for } i = 0, \ldots, n.$$

- How to find the control polygon?

$$\mathbf{b}(t) = \sum_{j=0}^{n} \begin{bmatrix} j/n \\ b_j \end{bmatrix} B_j^n(t)$$

$\rightarrow$ The control polygon is given by the points $\{(j/n, b_j)\}_{j=0}^n$.
- $f(t)$ is called a *Bézier function*.
- $\{(j/n, b_j)\}_{j=0}^n$ are called *Bézier ordinates*.
- $\{j/n\}_{j=0}^n$ are called the *abscissae*.

Cross Plots

- For a 2D Bézier curve, plot each coordinate function as a Bézier function.

Integrals

- The derivative of a polynomial function $b(t)$:

$$\dot{b}(t) = n \sum_{i=0}^{n-1} \Delta b_i B_i^{n-1}(t)$$

- The indefinite integral (or antiderivative) of a polynomial $b(t)$ in Bernstein form defined by the ordinates $\{b_i\}_{i=0}^{n}$ is the polynomial $B(t)$ defined by the ordinates

$$\frac{1}{n+1}[c, c+b_0, c+b_0+b_1, \ldots, c+\sum_{i=0}^{n} b_i]$$

- $\displaystyle\int_0^1 b(t)dt = B(1) - B(0) = \frac{1}{n+1}\sum_{i=0}^{n} b_i$
- $\displaystyle\int_0^1 B_j^n(x)dx = \frac{1}{n+1}$ for any j
  - Can be obtained by setting $b_i = \delta_{i,j}$.
  - All basis functions B_j^n have the same definite integral.

The Bézier Form of a Bézier Curve

- The basis functions Bézier used:

$$\mathbf{b}^n(t) = \sum_{j=0}^{n} \mathbf{c}_j F_j^n(t)$$

- F_j^n are polynomials that obey the recursion

$$F_i^n(t) = (1-t)F_i^{n-1}(t) + tF_{i-1}^{n-1}(t)$$
$$F_0^0(t) = 1(t), \quad F_{r+1}^r(t) = 0(t), \quad F_{-1}^r(t) = 1(t)$$

- Explicit form: $F_i^n = \sum_{j=i}^{n} B_j^n$
- $F_0^n \equiv 1(t)$ for all n
  $\rightarrow$ Not a barycentric combination!
  $\rightarrow$ How can it be defined then?

The Bézier Form of a Bézier Curve (cont'd)

- $\mathbf{c}_0 = \mathbf{b}_0 \in \mathbb{E}^3$ and $\mathbf{c}_j = \Delta \mathbf{b}_{j-1} \in \mathbb{R}^3$ for $j > 0$
- $F_i^n(t)$ is NOT symmetric w.r.t. t and $1 - t$.
- The value of $\mathbf{x}(1)$ is the sum of all errors in the $\mathbf{c}_i$.

The Weierstrass Approximation Theorem

1. $\mathbf{c}(t)$: parametric curve defined over $[0, 1]$
2. For some fixed n, sample $\mathbf{c}(t)$ at parameter values i/n.
3. Define a Bézier curve $\mathbf{x}_n(t)$ by control points $\{\mathbf{c}(i/n)\}_{i=0}^{n}$:

$$\mathbf{x}_n(t) = \sum_{i=0}^{n} \mathbf{c}\left(\frac{i}{n}\right) B_i^n(t)$$

   "$\mathbf{x}(t)$ is the n^{th} degree Bernstein-Bézier approximation to $\mathbf{c}$."

4. Weierstrass approximation theorem:

$$\lim_{n\to\infty} \mathbf{x}_n(t) = \mathbf{c}(t)$$

   - Not practical (too large n required)
   - Every curve may be approximated arbitrarily closely by a polynomial curve.

Formulas for Bernstein Polynomials

- A Bernstein polynomial is defined by

$$B_i^n(t) = \begin{cases} \binom{n}{i} t^i (1-t)^{n-i} & \text{if } i \in [0, n], \\ 0 & \text{else.} \end{cases}$$

- Relationship with the power basis $\{t^i\}$

$$t^i = \sum_{j=i}^{n} \frac{\binom{j}{i}}{\binom{n}{i}} B_j^n(t)$$

and

$$B_i^n(t) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j}{i} t^j.$$

Formulas for Bernstein Polynomials (cont'd)

- Recursion:

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

- Subdivision:

$$B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t)$$

- Derivative:

$$\frac{d}{dt} B_i^n(t) = n\left[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right]$$

- Integral:

$$\int_0^t B_i^n(x)dx = \frac{1}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t)$$

$$\int_0^1 B_i^n(x)dx = \frac{1}{n+1}$$

Formulas for Bernstein Polynomials (cont'd)

- Degree elevation formulas:

$$\begin{aligned}(1-t)B_i^n(t) &= \frac{n+1-i}{n+1}B_i^{n+1}(t)\\ tB_i^n(t) &= \frac{i+1}{n+1}B_{i+1}^{n+1}(t)\\ B_i^n(t) &= \frac{n+1-i}{n+1}B_i^{n+1}(t) + \frac{i+1}{n+1}B_{i+1}^{n+1}(t)\end{aligned}$$

- Product:

$$B_i^m(u)B_j^n(u) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}}B_{i+j}^{m+n}(u)$$