Solution for homework #2

April 20, 2012

- Excercises 2.2

13 (a) $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3-(4/3)R_1]{R_2 \leftarrow R_2-(2/3)R_1} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1/3 & -1/3 \\ 0 & -1/3 & 1/3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3+R_2]{3R_2} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\xrightarrow{\substack{R_1 \leftarrow R_1+2R_2 \\ (1/3)R_1}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

14 (a) $\begin{bmatrix} -2 & -4 & 7 \\ -3 & -6 & 10 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3+(1/2)R_1]{R_2 \leftarrow R_2-(3/2)R_1} \begin{bmatrix} -2 & -4 & 7 \\ 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3+R_2} \begin{bmatrix} -2 & -4 & 7 \\ 0 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\xrightarrow{\substack{R_1 \leftarrow R_1+14R_2 \\ (-1/2)R_1 \\ -2R_2}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

17 Applying Gauss-Jornal elimination to each matrix, we get

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2-3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1+R_2 \\ (-1/2)R_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and

$\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2-(1/3)R_1} \begin{bmatrix} 3 & -1 \\ 0 & 1/3 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1+3R_2 \\ (1/3)R_1 \\ 3R_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Since we can convert I back to B by reversing the elementary row operations, we can convert $A \to I \to B$ as follows:

(1) $R_2 \leftarrow R_2 - 3R_1$

(2) $R_1 \leftarrow R_1 + R_2$

(3) $(-1/2)R_2$ (Now A is converted to I. We need to convert I back to B.)

(4) $(1/3)R_2$

(5) $3R_1$

(6) $R_1 \leftarrow R_1 - 3R_2$

(7) $R_2 \leftarrow R_2 + (1/3)R_1$

21 $3R_2 - 2R_1$ is not an elementary row operation since it cannot be represented by any of three elementary row operations. But it can be represented by two elementary operations as follows:

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2-(2/3)R_1} \begin{bmatrix} 3 & 1 \\ 0 & 10/3 \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+2R_1} \begin{bmatrix} -1 & 3 & -2 & 4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{bmatrix} \xrightarrow{\substack{(1/3)R_2 \\ (1/2)R_3 \\ R_3+R_2}} \begin{bmatrix} -1 & 3 & -2 & 4 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

- the leading variable are x_1 and x_2 and
- the free variables are x_2 and x_4.

$$\begin{array}{rcrcrcrcr} -x_1 & + & 3x_2 & - & 2x_3 & + & 4x_4 & = & 0 \\ & & & - & x_3 & + & 2x_4 & = & -1 \end{array}$$

$$\rightarrow \begin{array}{rcl} x_3 & = & 2x_4+1 \\ x_1 & = & 3x_2-2x_3+4x_4 = 3x_2-2(2x_4+1)+4x_4 = 3x_2-2 \end{array}$$

Setting $x_2 = s$ and $x_4 = t$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s-2 \\ s \\ 2t+1 \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 3 & -2 & k^2-4 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2-k-2 \end{bmatrix}$$

The system has no solution if $k^2-k-2 = (k-2)(k+1) \neq 0$, i.e., $k \neq 2$ and $k \neq -1$. Otherwise ($k=2$ or $k=-1$) the system has infinitely many solutions since there is one free variable (z). Note that the system cannot have a unique solution due to the free variable.

44 [(a)]

The system

$$\begin{array}{rcrl} x & + & y & = 1 \\ 2x & + & 2y & = 2 \\ 3x & + & 3y & = 3 \end{array}$$

has infinitely many solutions.

– The system

$$\begin{array}{rcrl} x & + & y & = 1 \\ & & y & = 1 \\ & & 2y & = 2 \end{array}$$

has a unique solution.

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$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{array}{rcrcr} s & - & 2t & = & -4 \\ & - & 3t & = & 0 \\ s & - & t & = & -1 \end{array}$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & -2 & -4 \\ 0 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & -2 & -4 \\ 0 & -3 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3+(1/3)R_2} \begin{bmatrix} 1 & -2 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Therefore the linear system has no solution and two lines do not intersect each other.

- Excercises 2.3

4

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions are $x=3$ and $y=-1$. Therefore $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

8 We need to check if the following linear system has a solution.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}.$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{bmatrix} \xrightarrow[R_3-7R_1]{R_2-4R_1} \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system has infinitely many solutions, hence consistent, therefore $\boldsymbol{b}$ is in the span of the columns of the matrix A.

10 We need to show that, for an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ the following equality holds.

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In other words, we need to show that the following linear system

$$\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has at least one solution (consistent) regardless of x and y.
Applying Gaussian elimination,

$$\begin{bmatrix} 3 & 0 & x \\ -2 & 1 & y \end{bmatrix} \xrightarrow{R_2+(2/3)R_1} \begin{bmatrix} 3 & 0 & x \\ 0 & 1 & y+(2/3)x \end{bmatrix}$$

The soltion is

$$c_1 = x/3 \text{ and } c_2 = 2x/3 + y$$

and therefore

$$\text{span}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^2.$$

21 Let

$$\boldsymbol{w} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$$

and

$$\begin{aligned} \boldsymbol{u}_1 &= d_{11}\boldsymbol{v}_1 + d_{12}\boldsymbol{v}_2 + \cdots + d_{1m}\boldsymbol{v}_m \\ \boldsymbol{u}_2 &= d_{21}\boldsymbol{v}_1 + d_{22}\boldsymbol{v}_2 + \cdots + d_{2m}\boldsymbol{v}_m \\ &\vdots \\ \boldsymbol{u}_k &= d_{k1}\boldsymbol{v}_1 + d_{k2}\boldsymbol{v}_2 + \cdots + d_{km}\boldsymbol{v}_m. \end{aligned}$$

Then,

$$\begin{aligned} \boldsymbol{w} &= c_1(d_{11}\boldsymbol{v}_1 + \cdots + d_{1m}\boldsymbol{v}_m) + \cdots + c_k(d_{k1}\boldsymbol{v}_1 + \cdots + d_{km}\boldsymbol{v}_m) \\ &= (c_1d_{11} + \cdots + c_kd_{k1})\boldsymbol{v}_1 + (c_1d_{12} + \cdots + c_kd_{k2})\boldsymbol{v}_1 \cdots + (c_1d_{1m} + \cdots + c_kd_{km})\boldsymbol{v}_m \end{aligned}$$

therefore $\boldsymbol{w}$ is a linear combination of $\{\boldsymbol{v}_1 \ldots, \boldsymbol{v}_m\}$. In other words, if $\boldsymbol{w} \in \text{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ then $\boldsymbol{w} \in \text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ therefore $\text{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k) \subseteq \text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$.

25 It can easily be shown that the second vector is the sum of others. Therefore three vectors are linearly dependent. Therefore

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.$$

30 The vectors are linearly independent if the following homogeneous linear system has the trivial solution only.

$$\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The Gaussian elimination is easily done by reversing the order of rows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

Since the linear system has a unique solution, it should be the trivial solution and therefore the vectors are linearly independent.

35 Applying Gaussian elimination to the matrix which rows are the given vectors,

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[(R_1 \leftrightarrow R_3)]{\substack{R_1^1 \leftarrow R_3 \\ R_3^1 \leftarrow R_1}} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2^2 \leftarrow R_2^1 - R_1^1} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3^3 \leftarrow R_3^2 - R_2^2} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\mathbf{0} = R_3^3 = R_3^2 - R_2^2 = R_3^1 - (R_2^1 - R_1^1) = R_1 - (R_2 - R_3) = R_1 - R_2 + R_3$$

where

$$R_1 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, R_2 = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}, \text{ and } R_3 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}.$$

40 Applying Gaussian elimination to the matrix which rows are the given vectors,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{\substack{R_1^1 \leftarrow R_4 \\ R_2^1 \leftarrow R_3 \\ R_3^1 \leftarrow R_2 \\ R_4^1 \leftarrow R_1}} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there is no zero row at the bottom, the rows are linearly independent.

46 Let's assume that if any subset is linearly dependent. Specifically, for the linearly independent set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$, without loss of generality, assume that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ $(k \leqslant n)$ are linearly dependent. Then there are coefficients $c_1, \dots, c_k$, where at least one is nonzero, satisfying

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}.$$

Now the equality

$$d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \cdots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} + \cdots + d_n\boldsymbol{u}_n = \mathbf{0}$$

holds if

$$\begin{aligned} d_1 &= c_1 \\ d_2 &= c_2 \\ &\vdots \\ d_k &= c_k \\ d_{k+1} &= 0 \\ &\vdots \\ d_n &= 0. \end{aligned}$$

Since at least one of $c_1, \dots, c_k$ is nonzero, hehce at least one of $d_1, \dots, d_n$ is nonzero. This means that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ are linearly dependent and this contradicts the fact that they are linearly independent.

- Excercises 2.4

4 (a) Let x_1, x_2 and x_3 each denotes the number of nickels (\$0.05), dimes (\$0.1), and quarters (\$0.25), respectively. From the description we can build a linear system as follows:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 20 \\ 2x_1 & - & x_2 & & & = & 0 \\ 0.05x_1 & + & 0.10x_2 & + & 0.25x_3 & = & 3 \end{array}$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & 1 & 1 & 20 \\ 2 & -1 & 0 & 0 \\ 0.05 & 0.1 & 0.25 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ 20R_3 \\ R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 20 \\ 0 & -3 & -2 & -40 \\ 0 & 1 & 4 & 40 \end{bmatrix} \xrightarrow{R_3 + (1/3)R_2} \begin{bmatrix} 1 & 1 & 1 & 20 \\ 0 & -3 & -2 & -40 \\ 0 & 0 & 10/3 & 80/3 \end{bmatrix}$$

The solution is

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 8 \\ x_3 &= 8 \end{aligned}$$

(b) Without the second equation,

$$\begin{bmatrix} 1 & 1 & 1 & 20 \\ 0.05 & 0.1 & 0.25 & 3 \end{bmatrix} \xrightarrow{\substack{20R_3 \\ R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 20 \\ 0 & 1 & 3 & 40 \end{bmatrix}$$

With $x_3 = t$ as a free variable,

$$\begin{aligned} x_3 &= t \\ x_2 &= 40 - 3t \\ x_1 &= 20 - x_2 - x_3 = 20 - (40 - 3t) - t = 2t - 20 \end{aligned}$$

Since x_1, x_2, and $x_3 = t$ are a nonnegative integers,

$$\begin{cases} 0 \leqslant x_3 = t \\ 0 \leqslant x_2 = 40 - 3t \\ 0 \leqslant x_1 = 2t - 20 \end{cases} \rightarrow 10 \leqslant t \leqslant 13$$

Therefore the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 13 \end{bmatrix}, \right\}$$

17 (a) We can build a linear system as follows:

$$\begin{array}{ccccccccccc} f_1 & + & f_2 & & & & & & & = & 100 \\ & & f_2 & + & f_3 & - & f_4 & & & = & 150 \\ & & & & & & f_4 & + & f_5 & = & 150 \\ f_1 & & & - & f_3 & & & - & f_5 & = & -200 \end{array}$$

Applying Gauss-Jordan elimination,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & -1 & 0 & -1 & -200 \end{bmatrix} \xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & -1 & -1 & 0 & -1 & -300 \end{bmatrix} \xrightarrow[R_4+R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -50 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & -1 & -1 & -150 \end{bmatrix}$$

$$\xrightarrow{\substack{R_1-R_3 \\ R_2+R_3 \\ R_4+R_3}} \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Setting $f_3 = s$ and $f_5 = t$, the solutions are of the form

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} s + t - 200 \\ 300 - s - t \\ s \\ 150 - t \\ t \end{bmatrix}$$

(b) This means that $f_5 = t = 0$. In this case,

$$\begin{aligned} f_1 = s + t - 200 \geqslant 0 &\to s \geqslant 200 \\ f_2 = 300 - s - t \geqslant 0 &\to s \leqslant 300 \\ f_3 = s \geqslant 0 &\to s \geqslant 0 \end{aligned}$$

therefore,

$$200 \leqslant f_3 \leqslant 300.$$

1. If DB is closed, f_5 should be at least 200. But sice the flow-out on C is only 150, this is not possible. This is verified in the solution since

$$f_4 = 150 - t \to t \leqslant 150$$

but if $s = 0$ then

$$f_1 = s + t - 200 = t - 200 < 0.$$

2.

$$\begin{aligned} f_1 = s + t - 200 \geqslant 0 &\quad \to \quad s + t \geqslant 200 \\ f_2 = 300 - s - t \geqslant 0 &\quad \to \quad s + t \leqslant 300 \\ f_3 = s \geqslant 0 &\quad \to \quad s \geqslant 0 \\ f_4 = 150 - t \geqslant 0 &\quad \to \quad t \leqslant 150 \\ f_5 = t \geqslant 0 &\quad \to \quad t \geqslant 0 \end{aligned}$$

From the condisions for f_1 and f_2,

$$200 - t \leqslant s \leqslant 300 - t.$$

But since $0 \leqslant t \leqslant 150$,

$$50 \leqslant f_5 = s \leqslant 300.$$