Homework #2 Solution

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Exercise 2.2

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$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 2 & 5 & 9 & 16 \\ 0 & 3 & 9 & 19 & 31 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 - R_2 \\ R_3 - 2R_2 \\ R_4 - 3R_2}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 10 & 13 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + R_3 \\ R_2 - 2R_3 \\ R_4 - 3R_3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 - R_4 \\ R_2 + 3R_4 \\ R_3 - 3R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$\rightarrow a = b = c = d = 1$

39 We need to show that the linear system can be reduced to a reduced row echelon form with identiy matrix on the left-side. Since the coefficients are variables, we should be careful not to divide by zero.

(a) $a \neq 0$

$$\left[\begin{array}{cc|c} a & b & r \\ c & d & s \end{array}\right] \xrightarrow{\substack{R_1/a \\ R_2 - cR_1}} \left[\begin{array}{cc|c} 1 & b/a & r/a \\ 0 & d - bc/a & s - cr/a \end{array}\right]$$

$$\xrightarrow{\substack{R_2/\frac{ad-bc}{a} \\ R_1 - (b/a)R_2}} \left[\begin{array}{cc|c} 1 & 0 & \frac{r}{a} - \frac{b(as-cr)}{a(ad-bc)} \\ 0 & 1 & \frac{as-cr}{ad-bc} \end{array}\right]$$

(b) $a = 0$

Note that $b \neq 0$ and $c \neq 0$ since $a = 0$ and $ad - bc \neq 0$.

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} c & d & r \\ 0 & b & s \end{array}\right]$$

$$\xrightarrow{\substack{R_1/c \\ R_2/b}} \left[\begin{array}{cc|c} 1 & d/c & r/c \\ 0 & 1 & s/b \end{array}\right]$$

$$\xrightarrow{R_1 - (d/c)R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{br-ds}{bc} \\ 0 & 1 & \frac{s}{b} \end{array}\right]$$

51 Two equations are

$$\boldsymbol{u} \cdot \boldsymbol{x} = u_1x_1 + u_2x_2 + u_3x_3 = 0$$

and

$$\boldsymbol{v} \cdot \boldsymbol{x} = v_1x_1 + v_2x_2 + v_3x_3 = 0.$$

Therefore we have a (homogeneous) linear system with its augmented matrix

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right].$$

(a) $u_1 \neq 0$ or $v_1 \neq 0$

We can assume that $u_1 = 0$ since the case $v_1 \neq 0$ can be done in the same way.

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right] \xrightarrow{\substack{R_1/u_1 \\ R_2 - v_1R_1}} \left[\begin{array}{ccc|c} 1 & u_2/u_1 & u_3/u_1 & 0 \\ 0 & v_2 - v_1u_2/u_1 & v_3 - v_1u_3/u_1 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2/\frac{u_1v_2-u_2v_1}{u_1} \\ R_1 - \frac{u_2}{u_1}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{u_3}{u_1} - \frac{u_2(u_1v_3-u_3v_1)}{u_1(u_1v_2-u_2v_1)} & 0 \\ 0 & 1 & -\frac{u_3v_1-u_1v_3}{u_1v_2-u_2v_1} & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{u_3(u_1v_2-u_2v_1)-u_2(u_1v_3-u_3v_1)}{u_1(u_1v_2-u_2v_1)} & 0 \\ 0 & 1 & -\frac{u_3v_1-u_1v_3}{u_1v_2-u_2v_1} & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{u_1(u_2v_3-u_3v_2)}{u_1(u_1v_2-u_2v_1)} & 0 \\ 0 & 1 & -\frac{u_3v_1-u_1v_3}{u_1v_2-u_2v_1} & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{u_2v_3-u_3v_2}{u_1v_2-u_2v_1} & 0 \\ 0 & 1 & -\frac{u_3v_1-u_1v_3}{u_1v_2-u_2v_1} & 0 \end{array}\right]$$

Therefore,

$$x_1 = \frac{u_2v_3 - u_3v_2}{u_1v_2 - u_2v_1}x_3 \quad \text{and} \quad x_2 = \frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1}x_3$$

hence

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} \frac{u_2v_3 - u_3v_2}{u_1v_2 - u_2v_1} x_3 \\ \frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \frac{x_3}{u_1v_2 - u_2v_1}.$$

(b) $u_1 = v_1 = 0$

Note that, in this case, either $u_2 \neq 0$ or $v_2 \neq 0$ since, if $u_2 = v_2 = 0$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and the cross product is not defined.

$$\left[\begin{array}{ccc|c} 0 & u_2 & u_3 & 0 \\ 0 & v_2 & v_3 & 0 \end{array}\right] \xrightarrow[R_2 - v_2R_1]{R_1/u_2} \left[\begin{array}{ccc|c} 0 & 1 & u_3/u_2 & 0 \\ 0 & 0 & \frac{u_2v_3 - u_3v_2}{u_2} & 0 \end{array}\right]$$

$$\xrightarrow[R_1 - \frac{u_3}{u_2}R_2]{R_2/\frac{u_2v_3 - u_3v_2}{u_2}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore the solution is

$$x_2 = x_3 = 0$$

and

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}.$$

Since $u_1 = v_1 = 0$, (t is a free parameter)

$$\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} t = \begin{bmatrix} u_2v_3 - u_3v_2 \\ 0 \\ 0 \end{bmatrix} t$$

therefore $\boldsymbol{u} \times \boldsymbol{v}$ is a multiple of $\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$.

Exercise 2.3

12 As Example 2.19, we show that any vector, say $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ can be written

as a linear combination of the tree vectors.

$$\left[\begin{array}{rrr|c} 1 & -1 & 2 & a \\ 2 & -1 & 1 & b \\ 3 & 0 & -1 & c \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left[\begin{array}{rrr|c} 1 & -1 & 2 & a \\ 0 & 1 & -3 & b-2a \\ 0 & 3 & -7 & c-3a \end{array}\right]$$

$$\xrightarrow{\substack{R_1+R_2 \\ R_3-3R_2}} \left[\begin{array}{rrr|c} 1 & 0 & -1 & b-a \\ 0 & 1 & -3 & b-2a \\ 0 & 0 & -1 & 3a-3b+c \end{array}\right]$$

$$\xrightarrow{\substack{R_3/(-1) \\ R_1+R_3 \\ R_2+3R_3}} \left[\begin{array}{rrr|c} 1 & 0 & 0 & -4a+4b-c \\ 0 & 1 & 0 & -11a+10b-3c \\ 0 & 0 & 1 & -3a+3b-c \end{array}\right]$$

Therefore

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (-4a+4b-c)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-11a+10b-3c)\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + (-3a+3b-c)\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

and

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = \mathbb{R}^3.$$

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$$\begin{aligned} \boldsymbol{u} &= 1\boldsymbol{u} + 0(\boldsymbol{u}+\boldsymbol{v}) + 0(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) \\ \boldsymbol{v} &= (-1)\boldsymbol{u} + 1(\boldsymbol{u}+\boldsymbol{v}) + 0(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) \\ \boldsymbol{w} &= 0\boldsymbol{u} + (-1)(\boldsymbol{u}+\boldsymbol{v}) + 1(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) \end{aligned}$$

43 (a) The three vectors are linearly independent if and only if the equation

$$c_1(\boldsymbol{u}+\boldsymbol{v}) + c_2(\boldsymbol{v}+\boldsymbol{w}) + c_3(\boldsymbol{u}+\boldsymbol{w}) = \boldsymbol{0}$$

is satisfied for $c_1 = c_2 = c_3 = 0$. Now

$$c_1(\boldsymbol{u}+\boldsymbol{v}) + c_2(\boldsymbol{v}+\boldsymbol{w}) + c_3(\boldsymbol{u}+\boldsymbol{w}) = (c_1+c_3)\boldsymbol{u} + (c_1+c_2)\boldsymbol{v} + (c_2+c_3)\boldsymbol{w} = \boldsymbol{0}$$

only when

$$c_1 + c_3 = 0 \quad \text{and} \quad c_1 + c_2 = 0 \quad \text{and} \quad c_2 + c_3 = 0$$

since $\boldsymbol{u}$,$\boldsymbol{v}$, and $\boldsymbol{w}$ are linearly independent. It is straightforward to show that the solution of the linear system

$$\begin{array}{ccccccc} c_1 & & & + & c_3 & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ & & c_2 & + & c_3 & = & 0 \end{array}$$

is $c_1 = c_2 = c_3 = 0$. Therefore the three vectors are linearly independent.

(b) In the same way,

$$c_1(\boldsymbol{u}-\boldsymbol{v})+c_2(\boldsymbol{v}-\boldsymbol{w})+c_3(\boldsymbol{u}-\boldsymbol{w}) = (c_1+c_3)\boldsymbol{u}+(-c_1+c_2)\boldsymbol{v}+(-c_2-c_3)\boldsymbol{w} = \boldsymbol{0}$$

if and only if the following linear system has solution.

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ -c_1 & + & c_2 & & & = & 0 \\ & & -c_2 & - & c_3 & = & 0 \end{array}$$

It is straight forward to show that the above linear system has a solution $c_1 = c_2 = c_3 = 0$ therefore the three vectors are linearly independent.

Exercise 2.4

6 Let the number of bags for house blend, special blend, and gourmet blend x, y, and z, respectively. Then

$$\begin{array}{rcrcrcrl} 300x & + & 200y & + & 100z & = & 30,000 & \text{(Colombian beans)} \\ 50x & + & 200y & + & 350z & = & 15,000 & \text{(Kenyan beans)} \\ 150x & + & 100y & + & 50z & = & 15,000 & \text{(French roast)} \end{array}$$

$$\left[\begin{array}{rrr|r} 300 & 200 & 100 & 30000 \\ 50 & 200 & 350 & 15000 \\ 150 & 100 & 50 & 15000 \end{array}\right] \xrightarrow{\substack{R_1/300 \\ R_2-50R_1 \\ R_3-150R_1}} \left[\begin{array}{rrr|r} 1 & 2/3 & 1/3 & 100 \\ 0 & 500/3 & 1000/3 & 10000 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2/(500/3) \\ R_1-(2/3)R_2}} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 60 \\ 0 & 1 & 2 & 60 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is

$$x = t + 60$$
$$y = 60 - 2t$$
$$z = t$$

The overall profit is

$$0.50x + 1.50y + 2.00z = 0.50((t+60) + 3(60-2t) + 4t) = 0.50(240 - t)$$

which is maximized when $t = 0$. Therefore the profit is maximized when

$$x = 60$$
$$y = 60$$
$$z = 0.$$

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$C_2H_2Cl_4$	$Ca(OH)_2$	C_2HCl_3	$CaCl_2$	H_2O
x_1	x_2	x_3	x_4	x_5

Then we get the linear system for each atom as follows:

$$\begin{array}{rlcrcrcrcrcr}
C: & 2x_1 & & & = & 2x_3 & & & & \\
H: & 2x_1 & + & 2x_2 & = & x_3 & & & + & 2x_5 \\
Cl: & 4x_1 & & & = & 3x_3 & + & 2x_4 & & \\
Ca: & & & x_2 & = & & & x_4 & & \\
O: & & & 2x_2 & = & & & & & x_5
\end{array}$$

which can be converted to a homogeneous lineary system

$$A\boldsymbol{x} := \begin{bmatrix} 2 & & -2 & & \\ 2 & 2 & -1 & & -2 \\ 4 & & -3 & -2 & \\ & 1 & & -1 & \\ & 2 & & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0}.$$

Using GNU Octave, we can get the reduced row echelon form of A as follows

$$\begin{bmatrix} 1 & & & & -1 \\ & 1 & & & -1/2 \\ & & 1 & & -1 \\ & & & 1 & -1/2 \\ & & & & \end{bmatrix}$$

Therefore, with $x_5 = t$ as the free parameter, the solution is

$$\boldsymbol{x} = \begin{bmatrix} t \\ 1/2t \\ t \\ 1/2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} t$$

and the simplest form of solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

16 (a) By the conservation law, we can set up the equations at each intersection as follows

intersection	flow-in		flow-out
$A:$	$10+10$	$=$	f_1+f_2
$B:$	f_1+f_3	$=$	$20+5$
$C:$	f_2+f_4	$=$	$15+10$
$D:$	$15+15$	$=$	f_3+f_4

which can be converted to the linear system

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 25 \\ 30 \end{bmatrix}.$$

A can be converted to a reduced row echelon form as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & & & 20 \\ 1 & & 1 & & 25 \\ & 1 & & 1 & 25 \\ & & 1 & 1 & 30 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & & & 20 \\ & -1 & 1 & & 5 \\ & 1 & & 1 & 25 \\ & & 1 & 1 & 30 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 - R_2 \\ R_3 - R_2}]{(-1)R_2} \left[\begin{array}{cccc|c} 1 & & 1 & & 25 \\ & 1 & -1 & & -5 \\ & & 1 & 1 & 30 \\ & & 1 & 1 & 30 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 + R_3 \\ R_4 - R_3}]{R_1 - R_3} \left[\begin{array}{cccc|c} 1 & & & -1 & -5 \\ & 1 & & 1 & 25 \\ & & 1 & 1 & 30 \\ & & & & 0 \end{array}\right].$$

Therefore, with $f_4 = t$ as the free parameters, the solution is

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} t-5 \\ 25-t \\ 30-t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -5 \\ 25 \\ 30 \\ 0 \end{bmatrix}.$$

(b) If $f_4 = 10$ then $t = 10$ therefore the solution is

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 20 \\ 10 \end{bmatrix}.$$

(c) Keeping in mind that any flow should have nonnegative value,

$$\begin{aligned} f_1 = t - 5 \geq 0 &\quad\rightarrow\quad t \geq 5 \\ f_2 = 25 - t \geq 0 &\quad\rightarrow\quad t \leq 25 \\ f_3 = 30 - t \geq 0 &\quad\rightarrow\quad t \leq 30 \\ f_4 = t \geq 0 &\quad\rightarrow\quad t \geq 0 \end{aligned}$$

Overall, the range of t is

$$5 \leq t \leq 25$$

hence the range of each flow is

$$\begin{array}{ccccc} 0 & \leq & f_1 & \leq & 20 \\ 0 & \leq & f_2 & \leq & 20 \\ 5 & \leq & f_3 & \leq & 25 \\ 5 & \leq & f_4 & \leq & 25 \end{array}$$

(d) We can easily flip the direction of each flow by allowing nonpositive values for each flow. Therefore,

$$\begin{aligned} f_1 = t - 5 \leq 0 &\quad \rightarrow \quad t \leq 5 \\ f_2 = 25 - t \leq 0 &\quad \rightarrow \quad t \geq 25 \\ f_3 = 30 - t \leq 0 &\quad \rightarrow \quad t \geq 30 \\ f_4 = t \leq 0 &\quad \rightarrow \quad t \leq 0 \end{aligned}$$

Since there is no t satisfying all the conditions, the system has no solution.

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$$\begin{aligned} &A(x-1)(x^2+x+1)(x^2+1)^3 + Bx(x^2+x+1)(x^2+1)^3 + (Cx+D)x(x-1)(x^2+1)^3 \\ &+ (Ex+F)x(x-1)(x^2+x+1)(x^2+1)^2 + (Gx+H)x(x-1)(x^2+x+1)(x^2+1) \\ &+ (Ix+J)x(x-1)(x^2+x+1) \end{aligned}$$

Exercise 2.5

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