Solution of homework #3

April 28, 2011

Excercise 3.1

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	warehouse 1	warehouse 2
doohichies	200	75
gizmos	150	100
widgets	100	125

	distribution cost per unit
warehouse 1	\$0.75
warehouse 2	\$1.00

$$\begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 225.00 \\ 212.50 \\ 200.00 \end{bmatrix}$$

	total distribution cost
doohickies	\$225.00
gizmos	\$212.50
widgets	\$200.00

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$$\begin{aligned} BA &= \left[\begin{array}{c|c|c} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{array}\right] \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \\ &= \left[\begin{array}{c|c|c} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} & -2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \end{array}\right] \\ &= \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix} \end{aligned}$$

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$$\begin{aligned} \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix} \left[\begin{array}{ccc} 1 & 0 & -2 \\ \hline -3 & 1 & 1 \\ \hline 2 & 0 & -1 \end{array}\right] &= \left[\begin{array}{c} 2\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} + 3\begin{bmatrix} -3 & 1 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \\ \hline -\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} + 6\begin{bmatrix} -3 & 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \end{array}\right] \\ &= \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix} \end{aligned}$$

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$$\left[\begin{array}{c|c|c} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & -2 \\ \hline -3 & 1 & 1 \\ \hline 2 & 0 & -1 \end{array}\right] = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \begin{bmatrix} -3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -4 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -9 & 3 & 3 \\ 3 & -1 & -1 \\ -18 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \\ 8 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix}$$

30 Let $A \in \mathbb{R}^{m\times n}$. The rows of A are linearly dependent if and only if there is a vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x}A = \mathbf{0}$.

Now, we have $\boldsymbol{x}(AB) = (\boldsymbol{x}A)B = \mathbf{0}$ therefore the rows of AB are linearly dependent.

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$$B^2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$B^4 = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$B^8 = (-I)(-I) = I$$

Therefore

$$B^{2001} = B^{8\cdot 250+1} = (B^8)^{250}B = B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Excercise 3.2

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \quad \to b = c = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \quad \to b = c = 0$$

Therefore, the condition is

$$b = c = 0.$$

36 (b)

$$(AB)^T = B^T A^T = BA$$

Therefore, AB is symmetric ($(AB)^T = AB$) if and only if $AB = BA$.

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$$(AB)^T = B^T A^T = (-B)(-A) = BA$$
Therefore, $(AB)^T = -AB$ if and only if $AB = -BA$.

43 (a) Let
$$B := \frac{1}{2}(A + A^T) \quad \text{and} \quad C := \frac{1}{2}(A - A^T).$$
Clearly, $A = B + C$. Moreover,
$$B^T = \frac{1}{2}(A^T + (A^T)^T) = B$$
therefore B is symmetric and
$$C^T = \frac{1}{2}(A^T - (A^T)^T) = -C$$
therefore C is skew-symmetric.

45 From the definition on p.139, the (j, j) entry of AB is
$$\sum_{i=1}^{n} a_{ji} b_{ij} = a_{j1} b_{1j} + a_{j2} b_{2j} + \cdots + a_{jn} b_{nj}.$$
Therefore,
$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} b_{ij} \right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ji} b_{ij} \right) \\
&\text{(Inner and outer summation variables can be switched.)} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij} a_{ji} \right) \\
&= \operatorname{tr}(BA).
\end{aligned}$$

Excercise 3.3

13 (a)
$$A^{-1} = \frac{1}{1 \cdot 6 - 2 \cdot 2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1/2 \end{bmatrix} \quad \text{(2 multiplications + 4 divisions)}$$

- $A^{-1}\boldsymbol{b}_1 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1/2 \end{bmatrix}$ (4 multiplications)

- $A^{-1}\boldsymbol{b}_2 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ (4 multiplications)
- $A^{-1}\boldsymbol{b}_3 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ (4 multiplications)

(b)

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 2 & -1 & 4 & -4 \end{bmatrix}$$

(1 division + 5 multiplications)

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 1 & -1/2 & 2 & -2 \end{bmatrix}$$

(3 divisions)

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -1/2 & 2 & -2 \end{bmatrix}$$

(1 division + 3 multiplications)

(c) For (a), 14 multiplications and 4 divisions were performed and for (b), 8 multiplications and 5 divisions were performed.

30 Clearly, no single elementary row operation of the form $R_i \leftrightarrow R_j$ or kR_i can transform A into D.

We can convert A into C by an elementary row operation as follows:

$$A \xrightarrow{R_3 \leftarrow R_3 + R_1} E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A = C.$$

Also, we can convert C into D by an elementary row operation as follows:

$$C \xrightarrow{R_2 \leftarrow R_2 - 2R_3} E_2 C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} C = D.$$

Therefore,

$$D = E_2 C = E_2(E_1 A) = (E_2 E_1)A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} A$$

But

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

is not an elementary matrix.

45 Since

$$A^2 - 2A + I = O \to I = 2A - A^2 = A(2I - A),$$

by Theorem 3.13, A is invertible and $A^{-1} = 2I - A$.

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$$\left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right]$$

- If $b = 0$, the matrix is not invertible.
- If $b \neq 0$,

$$\left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} b & 0 & c & 0 & 1 & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right]$$

  – If $a = d = 0$, the matrix is not invertible.

  – If $a \neq 0$,

$$\left[\begin{array}{ccc|ccc} b & 0 & c & 0 & 1 & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/b)R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftrightarrow R_3 - dR_2]{(1/a)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & -d/a & 0 & 1 \end{array}\right]$$

  Therefore, the matrix is not invertible.

  – If $a = 0$ and $d \neq 0$,

$$\left[\begin{array}{ccc|ccc} b & 0 & c & 0 & 1 & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/b)R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftrightarrow R_3 - aR_2]{\substack{R_2 \leftrightarrow R_3 \\ (1/d)R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 & 0 & -a/d \end{array}\right]$$

  Therefore, the matrix is not invertible.

Overall, the matrix is not invertible regardless of a, b, c and d.

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$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix}$$

$$\begin{aligned}
AP + BR &= AP - BD^{-1}CP = (A - BD^{-1}C)P = (A - BD^{-1}C)(A - BD^{-1}C)^{-1} = I \\
AQ + BS &= -APBD^{-1} + B(D^{-1} + D^{-1}CPBD^{-1}) = (-AP + I + BD^{-1}CP)BD^{-1} \\
&= (I - (A - BD^{-1}C)P)BD^{-1} = (I - P^{-1}P)BD^{-1} = O \\
CP + DR &= CP - DD^{-1}CP = CP - CP = O \\
CQ + DS &= -CPBD^{-1} + D(D^{-1} + D^{-1}CPBD^{-1}) = -CPBD^{-1} + I + CPBD^{-1} = I
\end{aligned}$$

Therefore,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I$$

Excercise 3.4

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$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - (-1)R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 6 & -6 & 7 \\ 0 & 0 & -6 & 0 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix})$$

$$\xrightarrow{R_3 \leftarrow R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -1 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix})$$

$$\xrightarrow{R_4 \leftarrow R_4 - (-2)R_3} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix})$$

Therefore,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13 Since the matrix is already in row echelon form, L is the identity matrix:

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 - (-1)R_1]{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

$$\xrightarrow{R_3 \leftrightarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

$$\xrightarrow{R_4 \leftrightarrow R_4 - (-1)R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix})$$

Therefore,

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$