Homework #5

June 6, 2011

1. A 3×3 matrix B is known to have eigenvalues 0, 1, 2.
   (a) Find $\operatorname{rank}(B)$
   (b) Find $\det\left(B^T B\right)$

2. Let
$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$
   (a) Without using Gaussian elimination, find $\operatorname{rank}(A)$.
   Hint: For any $\boldsymbol{x}$, how does $A\boldsymbol{x}$ look like?
   (b) Without using Gaussian elimination, find the eigenvalues and eigenspaces.
   Hint:
   - Try to find $\boldsymbol{x}$ such that
$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \boldsymbol{x} = \lambda \boldsymbol{x}.$$
   - What is $\operatorname{nullity}(A)$? How can we find the vectors in $\operatorname{null}(A)$ easily?

3. Let $a + b = c + d$.
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
   (a) Show that $(1, 1)$ is an eigenvector of A.
   (b) Find both eigenvalues of A.

4. Suppose A has eigenvalues 0, 3, 5 with linearly independent eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.
   (a) Give a basis for $\operatorname{null}(A)$ and a basis for $\operatorname{col}(A)$.
   Hint:
   - $\operatorname{null}(A) = E_0$.
   - Consider the linear combination $c_1\boldsymbol{v} + c_2\boldsymbol{w}$.

   (b) Show that $A\boldsymbol{x} = \boldsymbol{u}$ has no solution.
   Hint: If it did, then ( ) would be in $\operatorname{col}(A)$ and this contradicts the assumption.

5. If A has an eigenvalue $\lambda_1 = 2$ with its eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is A?

6. Let the $n \times n$ matrix A have the eigenvalues $\lambda_1, \ldots, \lambda_n$ and be diagonalizable. Find the eigenvalues of the $2n \times 2n$ block matrix
$$B = \begin{bmatrix} A & O \\ O & 2A \end{bmatrix}.$$

7. For an $n \times n$ matrix A, suppose $A^2 = A$.
   (a) Show that 0 is an eigenvalue of A and $E_0 = \text{null}(A)$.
   (b) Show that 1 is an eigenvalue of A and $E_1 = \text{col}(A)$.
   (c) Show that A is diagonalizable.
   Hint: A is diagonalizable if the sum of all the dimensions of eigenspaces (geometric multiplicities) is n.

8. Suppose that both A and B are diagonalizable by the same P:
$$A = PD_1P^{-1} \quad \text{and} \quad B = PD_2P^{-1}.$$
   Show that $AB = BA$.

9. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
   and $AB = BA$.
   (a) Show that B is diagonal.
   (b) Show that A and B have the same eigenvectors.