Solution of homework #1

April 5, 2011

Excercise 1.2

17 (a) The norm is for vectors while $\boldsymbol{u} \cdot \boldsymbol{v}$ is a scale value.

(b) We cannot add a scalar ($\boldsymbol{u} \cdot \boldsymbol{v}$) with a vector ($\boldsymbol{w}$).

(c) We cannot take a dot product of a vector ($\boldsymbol{u}$) and a scalar ($\boldsymbol{v} \cdot \boldsymbol{w}$).

(d) We cannot take a dot product of a scalar (c) and a vector ($\boldsymbol{u} + \boldsymbol{w}$).

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$$\boldsymbol{u} \cdot \boldsymbol{v} = 2(k+1) + 3(k-1) = 5k - 1 = 0 \quad \to k = 1/5$$

46 (a) If we take the squares of both sides,

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\boldsymbol{u} \cdot \boldsymbol{v} \\ &= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta\end{aligned}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. On the other hands,

$$(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

Therefore, since $\|\boldsymbol{u}+\boldsymbol{v}\| \geq 0$ and $\|\boldsymbol{u}\|+\|\boldsymbol{v}\| \geq 0$, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ when $\cos\theta = 1$, in other words, ~~when $\theta = 0$.~~ when $\boldsymbol{u} = k\boldsymbol{v}$, $k \geq 0$.

(b) Since $\|\boldsymbol{u}+\boldsymbol{v}\| \geq 0$, it should be satisfied that $\|\boldsymbol{u}\| \geq \|\boldsymbol{v}\|$.
If we take the squares of both sides, from the above,

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta.$$

On the other hands,

$$(\|\boldsymbol{u}\| - \|\boldsymbol{v}\|)^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

Therefore, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\| - \|\boldsymbol{v}\|$ when

- $\|\boldsymbol{u}\| \geq \|\boldsymbol{v}\|$ and
- $\cos\theta = -1$, in other words, ~~$\theta = \pi$.~~ $\boldsymbol{u} = -k\boldsymbol{v}$, $k \geq 0$.

50 In Theorem 1.5 (p.19)

$$\|\boldsymbol{u}' + \boldsymbol{v}'\| \leq \|\boldsymbol{u}'\| + \|\boldsymbol{v}'\|,$$

let

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v} \quad \text{and} \quad \boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{w}.$$

Then the theorem becomes

$$\|(\boldsymbol{u}-\boldsymbol{v})+(\boldsymbol{v}-\boldsymbol{w})\| = \|\boldsymbol{u}-\boldsymbol{w}\| = d(\boldsymbol{u}, \boldsymbol{w}) \leq \|\boldsymbol{u}-\boldsymbol{v}\|+\|\boldsymbol{v}-\boldsymbol{w}\| = d(\boldsymbol{u}, \boldsymbol{v})+d(\boldsymbol{v}, \boldsymbol{w}).$$

53 In Theorem 1.5 (p.19)

$$\|\boldsymbol{u}' + \boldsymbol{v}'\| \leq \|\boldsymbol{u}'\| + \|\boldsymbol{v}'\| \quad \rightarrow \quad \|\boldsymbol{u}'\| \geq \|\boldsymbol{u}' + \boldsymbol{v}'\| - \|\boldsymbol{v}'\|$$

let

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v} \quad \text{and} \quad \boldsymbol{v}' = \boldsymbol{v}.$$

Then the theorem becomes

$$\|\boldsymbol{u} - \boldsymbol{v}\| \geq \|\boldsymbol{u}\| - \|\boldsymbol{v}\|.$$

59 (a)

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = \|\boldsymbol{u}\|^2 - \|\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)(\|\boldsymbol{u}\| - \|\boldsymbol{v}\|).$$

Since $\|\boldsymbol{u}\|, \|\boldsymbol{v}\| \geq 0$, $(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = 0$ if and only if $\|\boldsymbol{u}\| = \|\boldsymbol{v}\|$.

64 (a)

$$\begin{aligned}
\mathrm{proj}_{\boldsymbol{u}}\left(\mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v}\right)\right) &= \mathrm{proj}_{\boldsymbol{u}}\left(\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u}\right) = \left(\frac{\boldsymbol{u} \cdot \left(\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u}\right)}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} \\
&= \left(\frac{(\boldsymbol{u} \cdot \boldsymbol{u})\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} = \mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v}\right).
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v} - \mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v}\right)\right) &= \mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u}\right) = \left(\frac{\boldsymbol{u} \cdot \left(\boldsymbol{v} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u}\right)}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} \\
&= \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{u})\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right)\boldsymbol{u} = \boldsymbol{0}.
\end{aligned}$$

(c) (a) Once projected, the same projection doesn't change it anymore.
(b) The vector $\boldsymbol{v} - \mathrm{proj}_{\boldsymbol{u}}\left(\boldsymbol{v}\right)$ is orthogonal to $\boldsymbol{u}$.

Excercise 1.3

13 In the definition of the *vector form of the equation of a plane* on p.36, replace as follows:

$$\boldsymbol{p} = P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u} = Q - P = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v} = R - P = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.$$

Then the equation becomes

$$P + s(Q-P) + t(R-P) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.$$

15 (a)

- Parametric equation
  Pick any two points on the line: for example, $(0, -1)$ and $(1, 2)$. In the definition (p.33)

$$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d},$$

  let

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

  Then the equation becomes

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

- Vector form

$$y = 3x - 1 \to 3x - y = 1 \to \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 3a - b = 1.$$

  Therefore, we can pick any a and b satisfying $3a - b = 1$. Let $a = 0$ and $b = -1$ then the equation becomes

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = 0.$$

(b)

- Parametric equation
  In the same way, let's pick $(1, 1)$ and $(3, -2)$. Then

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

- Vector form
  In the same way,

$$3x + 2y = 5 \to \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 3a + 2b = 5.$$

Let $a = b = 1$. Then the equation becomes

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0.$$

23 The direction vector of the parametric quation is

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

Therefore, the equation is

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}. + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

24 Since the normal vector of the plane is

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix},$$

the plane we're looking for should have the same normal vector. Therefore, the equation becomes

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = 0.$$

Since the point $(0, -2, 5)$ is on this plane, we have

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -a \\ -2-b \\ 5-c \end{bmatrix} = -6a + 2 + b + 10 - 2c = 0.$$

Since any a,b, and c are acceptable as long as they satisfy this condition, let $a = 0$, $b = 0$ and $c = 6$. Then the equation becomes

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \right) = 0.$$

46 If we replace x, y, and z in the plane equation, we get

$$4t - (1 + 2t) - (2 + 3t) = -t - 3 = 6 \rightarrow t = -9.$$

Therefore, they intersect at $(-9, -17, -25)$. Since the normal vector of the plane is $\boldsymbol{u} := \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$ and the direction vector of the line is $\boldsymbol{v} := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, by the definition on p.21,

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{-1}{\sqrt{18}\sqrt{14}} = \frac{-1}{6\sqrt{7}} \to \theta = \cos^{-1}\left(-\frac{1}{6\sqrt{7}}\right).$$

Assuming $0 \leq \theta \leq \pi$, Since this is the angle between the line and the normal vector of the plane, the angle between the line and the plane is

$$\theta - \pi/2 = \theta = \cos^{-1}\left(-\frac{1}{6\sqrt{7}}\right) - \pi/2.$$