1 Vectors

2 Systems of Linear Equations

3 Matrices

3.1 Matrix Operations

3.2 Matrix Algebra

Theorem 3.5 a. If A is a square matrix, then $A + A^T$ is a symmetric matrix.

b. For any matrix A, AA^T and A^TA are symmetric matrices.

3.3 The Inverse of a Matrix

Theorem 3.9 If A is an invertible matrix

a. then A^{-1} is invertible and $(A^{-1})^{-1} = A$

b. and c is a nonzero scalar, then cA is an invertible matrix and $(cA)^{-1} = \frac{1}{c}A^{-1}$

c. and B is an invertible matrix of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

d. then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

e. then A^n is invertible for all nonnegative integers n and $(A^n)^{-1} = (A^{-1})^n$

Theorem 3.10 Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

Theorem 3.11 Each elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

Theorem 3.13 Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

Theorem 3.14 Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

3.4 The LU Factorization

Theorem 3.15 If A is a square matrix that can be reduced to row echelon form without using any row interchanges, then A has an LU factorization.

Theorem 3.16 If A is an invertible matrix that has an LU factorization, then L and U are unique.

Theorem 3.17 If P is a permutation matrix, then $P^{-1} = P^T$.

Theorem 3.18 Every square matrix has a $P^T LU$ factorization.

3.5 Subspaces, Basis, Dimension, and Rank

Theorem 3.19 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then $\text{span}\,(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

Theorem 3.20 Let B be any matrix that is row equivalent to a matrix A. Then $\text{row}(B) = \text{row}(A)$.

Theorem 3.21 Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous linear systems $A\mathbf{x} = \mathbf{0}$. Then N is a subspace of $\mathbb{R}^n$.

Theorem 3.22 Let A be a matrix whose entries are real numbers. For any system of linear equations $A\mathbf{x} = \mathbf{b}$, exactly one of the following is true:

1. There is no solution.
2. There is a unique solution.
3. There are infinitely many solution.

Theorem 3.23 (The Basis Theorem) Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

Theorem 3.24 The row and column spaces of a matrix A have the same dimension.

Theorem 3.25 For any matrix A,

$$\text{rank}(A^T) = \text{rank}(A)$$

Theorem 3.26 (The Rank Theorem) If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Theorem 3.28 Let A be an $m \times n$ matrix. Then

a. $\text{rank}(A^T A) = \text{rank}(A)$

b. The $n \times n$ matrix $A^T A$ is invertible if and only if $\text{rank}(A) = n$.

Theorem 3.29 Let S be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a basis for S. For every vector $\mathbf{v}$ in S, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

3.6 Introduction to Linear Transformations

Theorem 3.30 Let A be an $m \times n$ matrix. Then the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad (\text{for } \mathbf{x} \text{ in } \mathbb{R}^n)$$

is a linear transformation.

Theorem 3.31 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. More specifically, $T = T_A$, where A is the $m \times n$ matrix

$$A = \left[\begin{array}{c:c:c:c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

4 Eigenvalues and Eigenvectors

4.1 Introduction to Eigenvalues and Eigenvectors

4.2 Determinants

Theorem 4.1 (The Laplace Expansion Theorem) The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geqslant 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the **cofactor expansion along the ith row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the **cofactor expansion along the jth column**)

Theorem 4.2 The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

Theorem 4.3 Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.

b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.

c. If A has two identical rows (columns), then $\det A = 0$.

d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.

e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.

f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Theorem 4.4 Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.

b. If E results from multiplying one row of I_n by k, then $\det E = k$.

c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Lemma 4.5 Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

Theorem 4.6 A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4.7 If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

Theorem 4.8 If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

Theorem 4.9 If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

Theorem 4.10 For any square matrix A,

$$\det A = \det A^T$$

Theorem 4.14 Let A be an $n \times n$ matrix and let B be obtained by interchanging any two rows (columns) of A. Then

$$\det B = -\det A$$

4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices

Theorem 4.15 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 4.16 A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A.

Theorem 4.17 (The Fundamental Theorem of Invertible Matrices: Version 3) Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

f. $\text{rank}(A) = n$

g. $\text{nullity}(A) = 0$

h. The column vectors of A are linearly independent.

i. The column vectors of A span $\mathbb{R}^n$.

j. The column vectors of A form a basis for $\mathbb{R}^n$.

k. The row vectors of A are linearly independent.

l. The row vectors of A span $\mathbb{R}^n$.

m. The row vectors of A form a basis for $\mathbb{R}^n$.

n. $\det A \neq 0$

o. 0 is not an eigenvalue of A.

Theorem 4.18 Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a. For any positive integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.

b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.

c. For any integer k, λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.

Theorem 4.19 Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors–say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

Theorem 4.20 Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ are linearly independent.

4.4 Similarity and Diagonalization

Theorem 4.21 Let A, B and C be $n \times n$ matrices.

a. $A \sim A$.

b. If $A \sim B$, then $B \sim A$.

c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Theorem 4.22 Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$.

b. A is invertible if and only if B is invertible.

c. A and B have the same rank.

d. A and B have the same characteristic polynomial.

e. A and B have the same eigenvalues.

Theorem 4.23 Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are eigenvalues of A corresponding to the eigenvectors in P in the same order.

Theorem 4.24 Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

Theorem 4.25 If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Lemma 4.26 If A is an $n \times n$ matrix. then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

Theorem 4.27 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.

b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.

c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

4.5 Iterative Methods and Computing Eigenvalues

4.6 Applications and the Perron-Frobenius Theorem

Theorem 4.30 If P is the $n \times n$ transition matrix of a Markov chain, then 1 is an eigenvalue of P.

5 Orthogonality

5.1 Orthogonality in $\mathbb{R}^n$

Theorem 5.1 If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then these vectors are linearly independent.

Theorem 5.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{w}$ be any vector in W. Then the unique scalars $c_1, \ldots, c_k$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

are given by

$$c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \quad \text{for } i = 1, \ldots, k$$

Theorem 5.3 Let $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{w}$ be any vector in W. Then

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{q}_1)\mathbf{q}_1 + \cdots + (\mathbf{w} \cdot \mathbf{q}_k)\mathbf{q}_k$$

and this representation is unique.

Theorem 5.4 The columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

Theorem 5.5 A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Theorem 5.6 Let Q be an $n \times n$ matrix. The following statements are equivalent:

a. Q is orthogonal.

b. $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x}$ in $\mathbb{R}^n$.

c. $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Theorem 5.7 If Q is an orthogonal matrix, then its rows form an orthonormal set.

Theorem 5.8 Let Q be an orthogonal matrix.

a. Q^{-1} is orthogonal.

b. $\det Q = \pm 1$.

c. If λ is an eigenvalue of Q, then $|\lambda| = 1$.

d. If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.

5.2 Orthogonal Complements and Orthogonal Projections

Theorem 5.9 Let W be a subspace in $\mathbb{R}^n$.

a. $W^\perp$ is a subspace of $\mathbb{R}^n$.

b. $(W^\perp)^\perp = W$

c. $W \cap W^\perp = \{\mathbf{0}\}$

d. If $W = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

Theorem 5.10 Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{row}(A))^\perp = \text{null}(A) \quad \text{and} \quad (\text{col}(A))^\perp = \text{null}(A^T)$$

Theorem 5.11 (The Orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Then there are unique vectors $\mathbf{w}$ in W and $\mathbf{w}^\perp$ in $W^\perp$ such that

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

Corollary 5.12 If W is a subspace of $\mathbb{R}^n$, then

$$(W^\perp)^\perp = W$$

Theorem 5.13 If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^\perp = n$$

Corollary 5.14 (The Rank Theorem) If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

5.3 The Gram-Schmidt Process and the QR Factorization

Theorem 5.15 (The Gram-Schmidt Process) Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= \text{span}\,(\mathbf{x}_1) \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, & W_2 &= \text{span}\,(\mathbf{x}_1, \mathbf{x}_2) \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, & W_3 &= \text{span}\,(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\
&\vdots \\
\mathbf{v}_k &= \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, & W_k &= \text{span}\,(\mathbf{x}_1, \dots, \mathbf{x}_k)
\end{aligned}$$

Then for each $i = 1, \dots, k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Theorem 5.16 (The QR Factorization) Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

5.4 Orthogonal Diagonalization of Symmetric Matrices

Theorem 5.17 If A is orthogonally diagonalizable, then A is symmetric.

Theorem 5.18 If A is a real symmetric matrix, then the eigenvalues of A are real.

Theorem 5.19 If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Theorem 5.20 (The Spectral Theorem) Let A be an $n \times n$ real matrix. Then A is symmetric **if and only if** it is orthogonally diagonalizable.

6 Vector Spaces

7 Distance and Approximation

7.4 The Singular Value Decomposition

Theorem 7.13 (The Singular Value Decomposition) Let A be an $m \times n$ matrix with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ of the form shown in equation (1) such that

$$A = U \Sigma V^T$$

Theorem 7.14 (The Outer Product Form of the SVD) Let A be an $m \times n$ matrix with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Let $\mathbf{u}_1, \ldots \mathbf{u}_r$ be left singular vectors and let $\mathbf{v}_1, \ldots \mathbf{v}_r$ be right singular vectors of A corresponding to these singular values. Then

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

Theorem 7.15 Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A. Let $\sigma_1, \ldots, \sigma_r$ be all the nonzero singular values of A. Then

a. The rank of A is r.

b. $\{\mathbf{u}_1, \ldots \mathbf{u}_r\}$ is an orthonormal basis for col(A).

c. $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_,\}$ is an orthonormal basis for null(A^T).

d. $\{\mathbf{v}_1, \ldots \mathbf{v}_r\}$ is an orthonormal basis for row(A).

e. $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for null(A).

Theorem 7.16 Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A with rank r. Then the image of the unit sphere in $\mathbb{R}^n$ under the matrix transformation that maps $\mathbf{x}$ to $A\mathbf{x}$ is

- the surface of an ellipsoid in $\mathbb{R}^m$ if $r = n$.
- a solid ellipsoid in $\mathbb{R}^m$ if $r < n$.