# Review

## 1 $A\boldsymbol{x} = \boldsymbol{b}$ $\left(A \in \mathbb{R}^{m\times n}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m\right)$

- Gaussian elimination: Why does it work?
- Three cases

(i) no solution

$\Leftrightarrow$ no point where all the hyperplanes do intersect
$\Leftrightarrow$ $\boldsymbol{b}$ cannot be represented as a linear combination of the column vectors of $A$
$\Leftrightarrow$ $\boldsymbol{b} \notin \operatorname{span}$ (columns of $A$)
$\Leftrightarrow$ $\boldsymbol{b} \notin \operatorname{col}(A)$
$\Leftrightarrow$ There is no vector in $\mathbb{R}^n$ that is transformed to $\boldsymbol{b}$ by the linear transformation $T_A$
$\Rightarrow$ The columns of $A$ are either linearly dependent ( $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ ) of linearly independent ( $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ ).
$\Rightarrow$ $\dim \operatorname{col}(A) = \operatorname{rank}(A) < m$ (otherwise $\operatorname{col}(A) = \mathbb{R}^m$ and the linear system always have a solution)
$\Rightarrow$ If $m = n$, $\det(A) = 0$ ($A$ is not invertible)
$\Rightarrow$ There can be any number of free variables. (including 0)

(ii) a unique solution

$\Leftrightarrow$ all the hyperplanes intersect at one point
$\Rightarrow$ $\boldsymbol{b} \in \operatorname{col}(A)$
$\Rightarrow$ no free variable (converse is not always true: e.g. $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ )
$\Rightarrow$ $\operatorname{null}(A) = \{\mathbf{0}\}$ (otherwise we can get another solution by adding a nonzero vector in $\operatorname{null}(A)$)
$\Rightarrow$ $\operatorname{rank}(A) = n$
$\Rightarrow$ columns of $A$ are linearly independent
$\Rightarrow$ $n <= m$ (otherwise columns of $A$ are linearly dependent)
$\Leftrightarrow$ There is a unique way to represent $\boldsymbol{b}$ as a linear combination of the columns of $A$
$\Leftrightarrow$ There is only one vector $\boldsymbol{x} \in \mathbb{R}^n$ that is transformed to $\boldsymbol{b}$ by the linear transformation $T_A$
$\Leftrightarrow$ $A$ maps $\mathbb{R}^n$ to the subspace $\operatorname{col}(A) \subset \mathbb{R}^m$ 1-to-1
$\Leftrightarrow$ $\operatorname{col}(A)$ is an $n$-dimensional hyperplane in $\mathbb{R}^m$ through the origin
$\Rightarrow$ If $m = n$, $\det(A) \neq 0$ ($A$ is invertible)

(iii) infinitely many solutions

$\Leftrightarrow$ all the hyperplanes meet infinitely many points (lines or planes...)
$\Rightarrow$ $\boldsymbol{b} \in \operatorname{col}(A)$
$\Rightarrow$ columns of $A$ are linearly dependent
$\Rightarrow$ $\operatorname{nullity}(A) > 0$
$\Rightarrow$ one or more free variables
$\Leftrightarrow$ There are more than one ways to represent $\boldsymbol{b}$ as a linear combination of the columns of $A$.
$\Leftrightarrow$ There are infinitely many vectors in $\mathbb{R}^n$ that are transformed to $\boldsymbol{b}$ by the linear transformation $T_A$
$\Leftrightarrow$ $A$ maps $\mathbb{R}^n$ to $\operatorname{col}(A)$ many-to-1.
$\Rightarrow$ If $m = n$, $\det(A) = 0$ ($A$ is not invertible)

## 2 $A\boldsymbol{x} = \mathbf{0}$ $(A \in \mathbb{R}^{m\times n}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m)$

- Two cases

  (i) a unique solution ($\mathbf{0}$)

  $\Leftrightarrow$ all the hyperplanes meet only at the origin
  $\Leftrightarrow$ $\text{null}(A) = \{\mathbf{0}\}$
  $\Leftrightarrow$ $\text{nullity}(A) = 0$
  $\Leftrightarrow$ no free variable
  $\Leftrightarrow$ columns of $A$ are linearly independent
  $\Leftrightarrow$ $\text{col}(A)$ is an $n$-dimensional subspace in $\mathbb{R}^m$
  $\Leftrightarrow$ $\dim \text{col}(A) = \text{rank}(A) = n$
  $\Leftrightarrow$ $A$ maps $\mathbb{R}^n$ to $\text{col}(A)$ 1-to-1
  $\Leftrightarrow$ If $m = n$, $\det A \neq 0$

  (ii) infinitely many solutions

  $\Leftrightarrow$ all the hyperplanes meet infinitely many points (lines, planes...)
  $\Leftrightarrow$ The set of solution vectors forms a nontrivial subspace $\text{null}(A) \subset \mathbb{R}^n$
  $\Leftrightarrow$ $\text{null}(A)$ is non-trivial.
  $\Leftrightarrow$ $\dim \text{null}(A) = \text{nullity}(A) > 0$
  $\Rightarrow$ columns of $A$ are linearly dependent
  $\Leftrightarrow$ $A$ maps $\mathbb{R}^n$ to $\text{col}(A)$ many-to-1
  $\Leftrightarrow$ There is a non-zero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$
  $\Leftrightarrow$ If $m = n$, $\det(A) = 0$

## 3 $A = LU$ $(A, L, U \in \mathbb{R}^{n\times n})$

- How does it work?

  (a) Gaussian elimination: $E_k \cdots E_2 E_1 A = U$
  (b) $A = (E_k \cdots E_1)^{-1} U = (E_1^{-1} \cdots E_k^{-1}) U$ (elementray matrices $E_1, \ldots, E_k$ are invertible)
  (c) $A = LU$ where $L = E_1^{-1} \cdots E_k^{-1}$ ($E_1^{-1}, \ldots, E_k^{-1}$ are unit lower triangular matrices)

## 4 $A$ is invertible $(A \in \mathbb{R}^{n\times n})$

- How to find $A^{-1}$? Gauss-Jordan elimination: $[A|I] \to [I|A^{-1}]$
- Why does it work? $E_k \cdots E_1 A = I \Leftrightarrow E_k \cdots E_1 I = A^{-1}$
- "The fundamental theorem of invertible matrices"

## 5 $\text{col}(A), \text{row}(A), \text{null}(A)$ $(A \in \mathbb{R}^{m\times n})$

- $\boldsymbol{x} \in \text{col}(A)$ $(\boldsymbol{x} \in \mathbb{R}^m)$

  $\Leftrightarrow$ There is a column vector $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{x} = A\boldsymbol{y}$
  $\Leftrightarrow$ $\boldsymbol{x}$ can be represented as a linear combination of the columns of $A$
  $\Leftrightarrow$ The linear system $A\boldsymbol{y} = \boldsymbol{x}$ is consistent.

- $\boldsymbol{x} \in \text{row}(A)$ $(\boldsymbol{x} \in \mathbb{R}^n)$

  $\Leftrightarrow$ There is a row vector $\boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{x} = \boldsymbol{y}A$
  $\Leftrightarrow$ $\boldsymbol{x}$ can be represented as a linear combination of the rows of $A$
  $\Leftrightarrow$ The linear system $(A^T)\boldsymbol{y} = \boldsymbol{x}^T$ is consistent

- $\boldsymbol{x} \in \text{null}(A)$ $(\boldsymbol{x} \in \mathbb{R}^n)$

  $\Leftrightarrow$ $A\boldsymbol{x} = \mathbf{0}$

# 6 $A\boldsymbol{x} = \lambda\boldsymbol{x}$ $(A \in \mathbb{R}^{n\times n}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \neq \boldsymbol{0}, \lambda \in \mathbb{R})$

$\Leftrightarrow$ $\lambda$ is an eigenvalue of $A$

$\Leftrightarrow$ $\boldsymbol{x}$ is an eigenvector of $A$ associated with the eigenvalue $\lambda$

$\Leftrightarrow$ Geometric meaning: $\boldsymbol{x}$ does not change its direction by the linear transformation $T_A$ (including becoming the zero vector)

$\Leftrightarrow$ $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

$\Leftrightarrow$ $\boldsymbol{x} \in \mathrm{null}(A - \lambda I)$

$\Leftrightarrow$ $A - \lambda I$ is not invertible.

$\Leftrightarrow$ $\det(A - \lambda I) = 0$

$\Leftrightarrow$ $\lambda$ is a solution of the characteristic equation of $A$

$\Leftrightarrow$ $\mathrm{nullity}(A - \lambda I) > 0$

# 7 $P^{-1}AP = D$ $(A, P, D \in \mathbb{R}^{n\times n}$, $D$ is a diagonal matrix)

$\Leftrightarrow$ $A \sim D$

$\Rightarrow$ $A^k = P^{-1}D^kP$

$\Leftrightarrow$ "The diagonalization theorem"

# 8 Computing $A^k\boldsymbol{x}$ $(A \in \mathbb{R}^{n\times n}, \boldsymbol{x} \in \mathbb{R}^n)$

(i) Is $A$ diagonalizable?
$\rightarrow A^k\boldsymbol{x} = PD^kP^{-1}\boldsymbol{x}$

(ii) Is $\boldsymbol{x}$ a linear combination of eigenvectors of $A$? ($\boldsymbol{x} = \sum_{j=1}^{m} c_j\boldsymbol{v}_j$)
$\rightarrow A^k\boldsymbol{x} = \sum_{j=1}^{m} c_j\lambda^k\boldsymbol{v}_j^k$