Homework #5 Solution

Minho Kim

June 1, 2010

Excercise 5.1

33 (a) $A(A^T + B^T)B = AA^TB + AB^TB = (AA^T)B + A(B^TB) = B + A = A + B$.

(b) By (a),

$$\det(A+B) = \det(A(A^T+B^T)B) = \det A \det(A^T+B^T)\det B.$$

Since $(A+B)^T = A^T + B^T$ and by Theorem 4.10 (p.273),

$$\det(A^T+B^T) = \det(A+B).$$

Therefore,

$$\begin{aligned}\det(A+B) - \det(A(A^T+B^T)B) &= \det(A+B) - \det A\det(A^T+B^T)\det B \\ &= (1-\det A\det B)\det(A+B) \\ &= (1+(\det A)^2)\det(A+B) \\ &\qquad (\text{since } \det A + \det B = 0) \\ &= 0.\end{aligned}$$

Since $1 + (\det A)^2 \neq 0$, $\det(A+B) = 0$ and therefore $A+B$ is not invertible.

34 Note that $Q = Q^T$ since

$$\left[\begin{array}{c:c} x_1 & \boldsymbol{y}^T \\ \hdashline \boldsymbol{y} & I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T \end{array}\right]^T = \left[\begin{array}{c:c} x_1 & \boldsymbol{y}^T \\ \hdashline (\boldsymbol{y}^T)^T & \left(I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\right)^T \end{array}\right] = \left[\begin{array}{c:c} x_1 & \boldsymbol{y}^T \\ \hdashline \boldsymbol{y} & I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T \end{array}\right].$$

$$\begin{aligned} QQ^T = Q^2 &= \left[\begin{array}{c:c} x_1 & \boldsymbol{y}^T \\ \hdashline \boldsymbol{y} & I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T \end{array}\right]^2 \\ &= \left[\begin{array}{c:c} x_1^2 + \boldsymbol{y}^T\boldsymbol{y} & x_1\boldsymbol{y}^T + \boldsymbol{y}^T\left(I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\right) \\ \hdashline \boldsymbol{y}x_1 + \left(I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\right)\boldsymbol{y} & \boldsymbol{y}\boldsymbol{y}^T + \left(I - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\right)^2 \end{array}\right] \\ &= \left[\begin{array}{c:c} x_1^2 + \boldsymbol{y}\cdot\boldsymbol{y} & x_1\boldsymbol{y}^T + \boldsymbol{y}^T - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}^T\boldsymbol{y}\boldsymbol{y}^T \\ \hdashline x_1\boldsymbol{y} + \boldsymbol{y} - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{y} & \boldsymbol{y}\boldsymbol{y}^T + I - 2\left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T + \left(\frac{1}{1-x_1}\right)^2\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{y}\boldsymbol{y}^T \end{array}\right] \end{aligned}$$

(a) $x_1^2 + \boldsymbol{y}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot\boldsymbol{x} = 1$.

(b)

$$\begin{aligned} x_1\boldsymbol{y}^T + \boldsymbol{y}^T - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}^T\boldsymbol{y}\boldsymbol{y}^T &= (x_1+1)\boldsymbol{y}^T - \left(\frac{1}{1-x_1}\right)(\boldsymbol{y}\cdot\boldsymbol{y})\boldsymbol{y}^T \\ &= (x_1+1)\boldsymbol{y}^T - \left(\frac{1}{1-x_1}\right)(1-x_1^2)\boldsymbol{y}^T \\ &= (x_1+1)\boldsymbol{y}^T - (1+x_1^2)\boldsymbol{y}^T \\ &= \mathbf{0}^T. \end{aligned}$$

(c)

$$\begin{aligned} x_1\boldsymbol{y} + \boldsymbol{y} - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{y} &= (x_1+1)\boldsymbol{y} + - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}(\boldsymbol{y}\cdot\boldsymbol{y}) \\ &= (x_1+1)\boldsymbol{y} + - \left(\frac{1}{1-x_1}\right)\boldsymbol{y}(1-x_1^2) \\ &= (x_1+1)\boldsymbol{y} + -(1+x_1)\boldsymbol{y} = \mathbf{0}. \end{aligned}$$

(d)

$$\begin{aligned} &\boldsymbol{y}\boldsymbol{y}^T + I - 2\left(\frac{1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T + \left(\frac{1}{1-x_1}\right)^2\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{y}\boldsymbol{y}^T \\ &= I + \left(1 - 2\left(\frac{1}{1-x_1}\right)\right)\boldsymbol{y}\boldsymbol{y}^T + \left(\frac{1}{1-x_1}\right)^2\boldsymbol{y}(\boldsymbol{y}\cdot\boldsymbol{y})\boldsymbol{y}^T \\ &= I - \left(\frac{1+x_1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T + \left(\frac{1-x_1^2}{(1-x_1)^2}\right)\boldsymbol{y}\boldsymbol{y}^T \\ &= I - \left(\frac{1+x_1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T + \left(\frac{1+x_1}{1-x_1}\right)\boldsymbol{y}\boldsymbol{y}^T \\ &= I. \end{aligned}$$

Therefore, $QQ^T = \left[\begin{array}{c:c} 1 & \mathbf{0}^T \\ \hdashline \mathbf{0} & I_{n-1} \end{array}\right] = I_n$ and hence Q is orthogonal.

Excercise 5.2

14 Let a vector in $W^\perp$ be $\boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Since $\boldsymbol{x}$ should be orthogonal to $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$, we get

$$\begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3^T \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 4 & 6 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 & -3 \\ 2 & 2 & 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Gaussian elimination,

$$\begin{bmatrix} 4 & 6 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 & -3 \\ 2 & 2 & 2 & -1 & 2 \end{bmatrix} \xrightarrow{\substack{R_1/4 \\ R_2 - R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 3/2 & -1/4 & 1/4 & -1/4 \\ 0 & 1/2 & 1/4 & 3/4 & -11/4 \\ 0 & -1 & 5/2 & -3/2 & 5/2 \end{bmatrix}$$

$$\xrightarrow{\substack{2R_2 \\ R_3 + R_2 \\ R_3/3}} \begin{bmatrix} 1 & 3/2 & -1/4 & 1/4 & -1/4 \\ 0 & 1 & 1/2 & 3/2 & -11/2 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Therefore,

$$\begin{aligned} x_3 &= x_5 \\ x_2 &= -\frac{1}{2}(x_3 + 3x_4 - 11x_5) = -\frac{1}{2}(3x_4 - 10x_5) \\ x_1 &= -\frac{1}{4}(6x_2 - x_3 + x_4 - x_5) = -\frac{1}{4}(-9x_4 + 30x_5 - x_5 + x_4 - x_5) = 2x_4 - 7x_5 \end{aligned}$$

and

$$\boldsymbol{x} = \begin{bmatrix} 2x_4 - 7x_5 \\ -\frac{1}{2}(3x_4 - 10x_5) \\ x_5 \\ x_4 \\ x_5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_5.$$

Therefore,

$$W^{\perp} = \operatorname{span}\left(\begin{bmatrix} 4 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

25 In $\mathbb{R}^3$, let

- $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$,
- $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,
- $\boldsymbol{w'} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and

- $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Clearly, $\boldsymbol{w}$ and $\boldsymbol{w}'$ are orthogonal but $\boldsymbol{w}' \notin W^{\perp} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

30 (a) Let $W = \operatorname{span}(S)$. Then we can orthogonally decompose $\boldsymbol{x} \in \mathbb{R}^n$ as

$$\boldsymbol{x} = \boldsymbol{w} + \boldsymbol{w}^{\perp}$$

where $\boldsymbol{w} := \operatorname{proj}_W(\boldsymbol{x})$ and $\boldsymbol{w}^{\perp} := \operatorname{perp}_W(\boldsymbol{x})$. Also, since S is an orthonormal set, by definition on p.379,

$$\boldsymbol{w} = \sum_{i=1}^{k} (\boldsymbol{v}_i \cdot \boldsymbol{x}) \boldsymbol{v}_i.$$

Therefore,

$$\begin{aligned} \|\boldsymbol{x}\|^2 = \boldsymbol{x} \cdot \boldsymbol{x} &= \left(\boldsymbol{w} + \boldsymbol{w}^{\perp}\right) \cdot \left(\boldsymbol{w} + \boldsymbol{w}^{\perp}\right) \\ &= \boldsymbol{w}^{\perp} \cdot \boldsymbol{w}^{\perp} + \boldsymbol{w} \cdot \boldsymbol{w} \end{aligned}$$

since $\boldsymbol{w} \cdot \boldsymbol{w}^{\perp} = 0$.
Now,

$$\boldsymbol{w}\cdot\boldsymbol{w} = \left(\sum_{i=1}^{k} (\boldsymbol{v}_i \cdot \boldsymbol{w}) \boldsymbol{v}_i\right) \cdot \left(\sum_{i=1}^{k} (\boldsymbol{v}_i \cdot \boldsymbol{w}) \boldsymbol{v}_i\right) = \sum_{i=1}^{k} (\boldsymbol{v}_i \cdot \boldsymbol{x})^2 (\boldsymbol{v}_i \cdot \boldsymbol{v}_i) = \sum_{i=1}^{k} |\boldsymbol{v}_i \cdot \boldsymbol{x}|^2$$

since S is an orthonormal set. Therefore,

$$\|\boldsymbol{x}\|^2 = \|\boldsymbol{w}^{\perp}\|^2 + \sum_{i=1}^{k} |\boldsymbol{v}_i \cdot \boldsymbol{x}|^2 \geq \sum_{i=1}^{k} |\boldsymbol{v}_i \cdot \boldsymbol{x}|^2.$$

(b) In (a), the equality holds if and only if $\boldsymbol{w}^{\perp} = \mathbf{0}$, which is equivalent to satisfying $\boldsymbol{x} = \boldsymbol{w} = \operatorname{proj}_W(\boldsymbol{w})$ and hence $\boldsymbol{x} \in \operatorname{span}(S)$.

Excercise 5.3

10 (i) $\boldsymbol{v}_1 = \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

(ii)

$$\begin{aligned} \boldsymbol{v}_2 &= \boldsymbol{x}_2 - \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{x}_2}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\right) \boldsymbol{v}_1 \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

(iii)

$$\begin{aligned}
\boldsymbol{v}_3 &= \boldsymbol{x}_3 - \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{x}_3}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\right)\boldsymbol{v}_1 - \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{x}_3}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\right)\boldsymbol{v}_2 \\
&= \begin{bmatrix}1\\2\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - 0\begin{bmatrix}0\\-2\\2\\4\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}
\end{aligned}$$

Therefore, the orthogonal vectors are

$$\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\2\\4\end{bmatrix}, \text{and} \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

16 By normalizing the vectors, we get

$$Q = \begin{bmatrix}1/2 & 0 & 0\\ 1/2 & -1/\sqrt{6} & 1/\sqrt{2}\\ -1/2 & 1/\sqrt{6} & 1/\sqrt{2}\\ 1/2 & 2/\sqrt{6} & 0\end{bmatrix}$$

and since

$$R = Q^TA = \begin{bmatrix}1/2 & 1/2 & -1/2 & 1/2\\ 0 & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ -1 & 1 & 0\\ 1 & 5 & 1\end{bmatrix} = \begin{bmatrix}2 & 2 & 2\\ 0 & 2\sqrt{6} & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix},$$

therefore

$$A = \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ -1 & 1 & 0\\ 1 & 5 & 1\end{bmatrix} = QR = \begin{bmatrix}1/2 & 0 & 0\\ 1/2 & -1/\sqrt{6} & 1/\sqrt{2}\\ -1/2 & 1/\sqrt{6} & 1/\sqrt{2}\\ 1/2 & 2/\sqrt{6} & 0\end{bmatrix}\begin{bmatrix}2 & 2 & 2\\ & 2\sqrt{6} & 0\\ & & \sqrt{2}\end{bmatrix}$$

19 We don't need to apply the Gram-Schmidt process since A is already orthogonal. Since $R = Q^TA = A^TA = I$, the QR factorization results in $A = QR$ where $Q = A$ and $R = I$.

Excercise 5.4

10 (a) **Diagonalize** A **(**$A \to PDP^{-1}$**)**

The characteristic polynomial is, by the Laplace expansion theorem,

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 2-\lambda & & & 1 \\ & 1-\lambda & & \\ & & 1-\lambda & \\ 1 & & & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(-1)^{1+1}\begin{vmatrix} 2-\lambda & & 1 \\ & 1-\lambda & \\ 1 & & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)^2(-1)^{1+1}\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)^2((2-\lambda)^2-1) \\
&= (1-\lambda)^2(\lambda^2-4\lambda+3) = (\lambda-1)^3(\lambda-3)
\end{aligned}$$

(i) For $\lambda_1 = 1$,

The nullspace of $A-\lambda_1 I = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3$$

and hence

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

(ii) For $\lambda_2 = 3$,

The nullspace of $A-\lambda_2 I = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ is

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_1$$

and hence

$$E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Here, since all the eigenvectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are linearly independent, A can be diagonalizable, and

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{bmatrix}.$$

(b) **Orthogonalize and normalize the eigenvectors by Gram-Schmidt process ($P \to Q$)**

Due to Theorem 5.19, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to all the vectors in E_1. Therefore we only need to orthogonalize the vectors in E_1. But clearly, all the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are orthogonal. So we only need to normalize the vectors to get Q as follows:

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

(c) **Apply Q to orthogonally diagonalize A ($A \to QDQ^T$)**
Summing up, $A = QDQ^T$ where

$$D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{bmatrix}. \quad \text{and} \quad Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

15 By the Spectral theorem, A and B are real symmetric matrices. Clearly, the entries of AB are also real numbers. Now, $(AB)^T = B^T A^T = BA = AB$ therefore AB is also a real symmetric matrices and hence orthogonally diagonalizable.

25 We need to show that $\text{perp}_W(\boldsymbol{v}) = \boldsymbol{v} - \text{proj}_W(\boldsymbol{v})$ is orthogonal to $\boldsymbol{q}$. Now,

$$\boldsymbol{q} \cdot (\boldsymbol{v} - \text{proj}_W(\boldsymbol{v})) = \boldsymbol{q}^T(\boldsymbol{v} - \boldsymbol{q}\boldsymbol{q}^T\boldsymbol{v}) = \boldsymbol{q}^T\boldsymbol{v} - (\boldsymbol{q}^T\boldsymbol{q})\boldsymbol{q}^T\boldsymbol{v} = \boldsymbol{q}^T\boldsymbol{v} - \boldsymbol{q}^T\boldsymbol{v} = 0$$

where $\boldsymbol{q}^T\boldsymbol{q} = \boldsymbol{q} \cdot \boldsymbol{q} = 1$ since $\boldsymbol{q}$ is a unit vector.

26 (a) We need to show that for $\boldsymbol{x} \in \mathbb{R}^n$, $\text{perp}_W(\boldsymbol{x}) = \boldsymbol{x} - \text{proj}_W(\boldsymbol{x})$ is orthogonal to all the vectors in $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k\}$.

$$\begin{aligned}
\boldsymbol{q}_i \cdot \text{perp}_W(\boldsymbol{x}) &= \boldsymbol{q}_i^T(\boldsymbol{x} - \text{proj}_W(\boldsymbol{x})) \\
&= \boldsymbol{q}_i^T\left(\boldsymbol{x} - \sum_{j=1}^{k}(\boldsymbol{q}_j\boldsymbol{q}_j^T)\boldsymbol{x}\right) \\
&= \boldsymbol{q}_i^T\boldsymbol{x} - \sum_{j=1}^{k}(\boldsymbol{q}_i^T\boldsymbol{q}_j\boldsymbol{q}_j^T)\boldsymbol{x} \\
&= \boldsymbol{q}_i^T\boldsymbol{x} - \sum_{j=1}^{k}(\boldsymbol{q}_i^T\boldsymbol{q}_j)\boldsymbol{q}_j^T\boldsymbol{x} \\
&= \boldsymbol{q}_i^T\boldsymbol{x} - (\boldsymbol{q}_i^T\boldsymbol{q}_i)\boldsymbol{q}_i^T\boldsymbol{x} \\
&= 0
\end{aligned}$$

since $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k\}$ is an orthonormal set.

(b) Kepping in mind that $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k\}$ is an orthonormal set,

$$\begin{aligned}
P^T &= (\boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T)^T = (\boldsymbol{q}_1\boldsymbol{q}_1^T)^T + \cdots + (\boldsymbol{q}_k\boldsymbol{q}_k^T)^T \\
&= \boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T = P
\end{aligned}$$

and

$$\begin{aligned}
P^2 &= (\boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T)^2 = \boldsymbol{q}_1\boldsymbol{q}_1^T\boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T\boldsymbol{q}_k\boldsymbol{q}_k^T \\
&= \boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T = P
\end{aligned}$$

since

$$\boldsymbol{q}_i\boldsymbol{q}_i^T\boldsymbol{q}_j\boldsymbol{q}_j^T = \boldsymbol{q}_i(\boldsymbol{q}_i \cdot \boldsymbol{q}_j)\boldsymbol{q}_j^T = \begin{cases} \boldsymbol{q}_i\boldsymbol{q}_j^T & i = j \\ 0 & i \neq j \end{cases}$$

(c)

$$QQ^T = \begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^T \\ \vdots \\ \boldsymbol{q}_k^T \end{bmatrix} = \boldsymbol{q}_1\boldsymbol{q}_1^T + \cdots + \boldsymbol{q}_k\boldsymbol{q}_k^T = P$$

By Theorem 3.25 and Theorem 3.28(a),

$$\text{rank}(P) = \text{rank}((QQ^T)) = \text{rank}(Q^T) = \text{rank}(Q) = k$$

since the columns of Q are orthogonal therefore linearly independent.