Topics in Computer Graphics
Chap 3: Linear Interpolation

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Linear Interpolation

Let $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$. The set of all points $\mathbf{x} \in \mathbb{E}^3$ of the form

$$\mathbf{x} = \mathbf{x}(t) = (1-t)\mathbf{a} + t\mathbf{b}, \quad t \in \mathbb{R}$$

is called the *straight line* through $\mathbf{a}$ and $\mathbf{b}$.

- For $t = 0$, $\mathbf{x}(0) = \mathbf{a}$: the line passes through $\mathbf{a}$.
- For $t = 1$, $\mathbf{x}(1) = \mathbf{b}$: the line passes through $\mathbf{b}$.
- For $0 \leqslant t \leqslant 1$, the point $\mathbf{x}$ is between $\mathbf{a}$ and $\mathbf{b}$.
- For $t < 0$ or $t > 1$, the point is outside.
- $\mathbf{x}$ is represented as a barycentric combination of two points in $\mathbb{E}^3$.
  $\rightarrow$ The three points $\mathbf{a}$,$\mathbf{x}$,$\mathbf{b}$ in $\mathbb{E}^3$ are an affine map of the three 1D points 0,t,1.
  $\rightarrow$ Linear interpolation is an affine map of the real line onto a straight line in $\mathbb{E}^3$.

Linear Interpolation (cont'd)

- Linear interpolation is affinely invariant.

$$\Phi\mathbf{x} = \Phi\left((1-t)\mathbf{a} + t\mathbf{b}\right) = (1-t)\Phi\mathbf{a} + t\Phi\mathbf{b}$$

- Can be applied to vectors as well : The vector $\vec{v} := d - c \in \mathbb{R}$ is mapped to the vector $\mathbf{l}(\vec{v})\mathbf{l}(d) - \mathbf{l}(c) \in \mathbb{R}^3$ by the linear interpolation $\mathbf{l}$.

Linear Interpolation and Barycentric Combination

- For any three conlinear points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$, the barycentric coordinates of $\mathbf{b}$ w.r.t. $\mathbf{a}$ and $\mathbf{c}$ is

$$\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}, \quad \alpha = \frac{\mathrm{vol}_1(\mathbf{b}, \mathbf{c})}{\mathrm{vol}_1(\mathbf{a}, \mathbf{c})}, \beta = \frac{\mathrm{vol}_1(\mathbf{a}, \mathbf{b})}{\mathrm{vol}_1(\mathbf{a}, \mathbf{c})}.$$

  - vol_1 : signed distance between two points
- $\mathrm{ratio}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \dfrac{\mathrm{vol}_1(\mathbf{a}, \mathbf{b})}{\mathrm{vol}_1(\mathbf{b}, \mathbf{c})} = \dfrac{\beta}{\alpha}$
- The barycentric coordinates of a point do not change under affine maps.

$$\mathrm{ratio}(\Phi\mathbf{a}, \Phi\mathbf{b}, \Phi\mathbf{c}) = \frac{\beta}{\alpha}$$

$\rightarrow$ Affine maps are ratio preserving.
$\rightarrow$ *Every map that takes straight lines to straight lines and its ratio preserving is an affine map.*

Affine Domain Transformation

The straight line

$$\mathbf{x}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in [0, 1]$ is the same as the straight line

$$\mathbf{x}(u) = \frac{b-u}{b-a}\mathbf{a} + \frac{u-a}{b-a}\mathbf{b}$$

for $u \in [a, b]$ with $t = (u-a)/(b-a)$.
$\rightarrow$ Linear interpolation is invariant under affine domain transformation.

- Affin domain transformation: An affine map of the real line onto itself.

Piecewise Linear Interpolation

- Let $\mathbf{b}_0, \ldots, \mathbf{b}_n \in \mathbb{E}^3$ form a polygon $\mathbf{B}$.
  $\rightarrow$ $\mathbf{B}$ is the *piecewise linear interpolant* $\mathcal{PL}$ to the points $\mathbf{b}_i$.
- If the points $\mathbf{b}_I$ lie on a curve $\mathbf{c}$
  $\rightarrow$ $\mathbf{B}$ is a piecewise linear interpolant to $\mathbf{c}$:

$$\mathbf{B} = \mathcal{PL}\mathbf{c}.$$

- Piecewise linear interpolation is affinely invariant.:

$$\mathcal{PL}\Phi\mathbf{c} = \Phi\mathcal{PL}\mathbf{c}.$$

- *Varoatopm diminishing property*:

$$\operatorname{cross}(\mathcal{PL}\mathbf{c}) \leqslant \operatorname{cross}\mathbf{c}.$$

Melelaos' Theorem

Let

$$\begin{aligned}\mathbf{b}[0,t] &= (1-t)\mathbf{b}_0 + t\mathbf{b}_1\\ \mathbf{b}[s,0] &= (1-s)\mathbf{b}_0 + s\mathbf{b}_1\\ \mathbf{b}[1,t] &= (1-t)\mathbf{b}_1 + t\mathbf{b}_2\\ \mathbf{b}[s,1] &= (1-s)\mathbf{b}_1 + s\mathbf{b}_2\end{aligned}$$

and

$$\begin{aligned}\mathbf{b}[s,t] &= (1-t)\mathbf{b}[s,0] + t\mathbf{b}[s,1]\\ \mathbf{b}[t,s] &= (1-s)\mathbf{b}[0,t] + s\mathbf{b}[t,1].\end{aligned}$$

Then

$$\mathbf{b}[s,t] = \mathbf{b}[t,s].$$

- Menelaus' Theorem @Wikipedia

Blossoms

A blossom is an n-variate function $\mathbf{b}[t_1, \ldots, t_n]$ from $\mathbb{R}^n$ into $\mathbb{E}^2$ or $\mathbb{E}^3$ satisfying the following three properties:

- Symmetry:

$$\mathbf{b}[t_1, \ldots, t_n] = \mathbf{b}[\pi(t_1, \ldots, t_n)]$$

  where $\pi(t_1, \ldots, t_n)$ denotes a permutation of the arguments $t_1, \ldots, t_n$.
  $\rightarrow$ The order of the arguments does not matter
  $\rightarrow$ Menelaos' theorem

- Multiaffinity

$$\mathbf{b}[(\alpha r + \beta s), *] = \alpha \mathbf{b}[r, *] + \beta \mathbf{b}[s, *], \quad \alpha + \beta = 1$$

  $\rightarrow$ Affine w.r.t. *any* argument.

- Diagonality

$$\mathbf{b}[t, \ldots, t] = \mathbf{b}[t^{<n>}]$$

  When all the n arguments are the same, it traces out a polynomial curve of degree n.

Blossoms with Vector Argument

With $\vec{h} := b - a$,

$$\mathbf{b}[\vec{h}, *] = \mathbf{b}[b - a, *] = \mathbf{b}[b, *] - \mathbf{b}[a, *]$$

$\rightarrow$ If (at least) one of the blossom arguments is a vector, then the blossom value is a vector.

Leibniz Formula

$$\mathbf{b}[(\alpha r + \beta s)^{<n>}] = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i} \mathbf{b}[r^{<i>}, s^{<n-i>}]$$

Alternative formula

$$\mathbf{b}[(\alpha r + \beta s)^{<n>}] = \sum_{\substack{i+j=n \\ i,j \geqslant 0}} \binom{n}{i, j} \alpha^i \beta^{n-i} \mathbf{b}[r^{<i>}, s^{<j>}]$$

where

$$\binom{n}{i, j} := \frac{n!}{i! j!}.$$

Barycentric Coodinates in the Plane

Considering a triangle with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{E}^2$, *any* point $\mathbf{p} \in \mathbb{E}^2$ can be represented as a barycentric combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{p} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}, \quad \text{where } u + v + w = 1.$$

- Is (u, v, w) is unique? → The coefficients $\mathbf{u} := (u, v, w)$ is the *barycentric coordinates* of $\mathbf{p}$ w.r.t. $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
- Applying the Cramer's rule,

$$u = \frac{\mathrm{area}(\mathbf{p}, \mathbf{b}, \mathbf{c})}{\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad v = \frac{\mathrm{area}(\mathbf{a}, \mathbf{p}, \mathbf{c})}{\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad w = \frac{\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{p})}{\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}.$$

  → Requires $\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$.
- Barycentric coordinates are *affinely invariant*.
- Ceva's theorem (Fig. 3.5)
- Location of $\mathbf{p}$ according to the signs of its barycentric coordinates → Fig. 3.6

Bivariate Linear Interpolation

Given three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{E}^3$, any point of the form

$$\mathbf{p} = \mathbf{p}(\mathbf{u}) = \mathbf{p}(u, v, w) = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3, \quad u + v + w = 1$$

lies in the plane spanned by $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$.

- Can be generalized to higher dimensions, e.g., a barycentric coordinates w.r.t. a tetrahedron.

Tessellations

- Bivariate piecewise linear interpolation requires triangulation of a plane.
  $\rightarrow$ related to the concept of tessellation
- Dirichlet tessellation (a.k.a. Voronoi diagram)
  "...we associate with each point $\mathbf{p}_k$ a tile T_k consisting of all points $\mathbf{p}$ that are closer to $\mathbf{p}_k$ than to any other point $\mathbf{p}_i$. The collection of all these tiles is called the Dirichlet tessellation of the given point set."

Triangulations

- A triangulation $\mathcal{T}$ of a set of 2D points $\{\mathbf{p}_i\}$ is a collection of triangles such that
  - The vertices of the triangles consist of the $\mathbf{p}_i$.
  - The interiors of any two triangles do not intersect.
  - If two triangles are not disjoint, then they share either a vertex or an edge.
- Delaunay triangulation
  - Dual of the Voronoi diagram
  - Satisfies the maxmin criterion.
  - Given a point set, is its Delaunay triangulation unique?
- Piecewise linear interpolation on a plane