Mathematical Models for Engineering Problems and Differential Equations

Minho Kim

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Chapter 6: Problems Leading to Linear Differential Equations of Order Tw

L.D.E. of This Chapter

The motion of a particle whose equation of motion satisfiesa D.E. of the form

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f(t).$$

- $r = 0$ and $f(t) \equiv 0 \rightarrow \dfrac{d^2x}{dt^2} + \omega_0^2 x = 0$
  $\rightarrow$ 28A: Free undamped motion (simple harmonic motion)
- $r = 0$ and $f(t) \not\equiv 0 \rightarrow \dfrac{d^2x}{dt^2} + \omega_0^2 x = f(t)$
  $\rightarrow$ 28D: Forced undamped motion
- $r \neq$ and $f(t) \equiv 0 \rightarrow \dfrac{d^2x}{dt^2} + 2r\dfrac{dx}{dt} + \omega_0^2 x = 0$
  $\rightarrow$ 29A: Free damped motion (damped harmonic motion)
- $r \neq$ and $f(t) \not\equiv 0 \rightarrow \dfrac{d^2x}{dt^2} + 2r\dfrac{dx}{dt} + \omega_0^2 x = f(t)$
  $\rightarrow$ 29B: Forced damped motion

$\rightarrow$ Summarized in the table on p.365.

Lesson 28: Undamped Motion.

Free undamped motion (simple harmonic motion)

A particle will be said to execute "simple harmonic motion" if its equation of motion satisfies

$$\frac{d^2x}{dx^2} + \omega_0^2 x = 0$$

where ω_0 is a positive constant and $x(t)$ is the positon of the particle.

- The motion of a particle oscillating back and forth about a fixed point of equilibrium.
- Examples: displaced helical spring, a pendulum

Free undamped motion (simple harmonic motion) (cont'd)

The solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

or

$$x(t) = \sqrt{c_1^2 + c_2^2} \sin(\omega_0 t + \delta) = c \sin(\omega_0 t + \delta)$$

or

$$x(t) = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t + \delta) = c \cos(\omega_0 t + \delta).$$

Free undamped motion (simple harmonic motion) (cont'd)

Example 28.15 (p.314)

Description of the motion:

- maximum displacement?
- relation of position and velocity?
- maximum speed?

Free undamped motion (simple harmonic motion) (cont'd)

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$
$$x(t) = \sqrt{c_1^2 + c_2^2} \sin(\omega_0 t + \delta) = c \sin(\omega_0 t + \delta)$$
$$x(t) = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t + \delta) = c \cos(\omega_0 t + \delta).$$

- equilibrium position
- amplitude (c)
- phase angle (δ)
- period ($T = 2\pi/\omega_0$)
- natural (undamped) frequency (cycles per second $\nu = 1/T = \omega_0/2\pi$)
- natural (undamped) frequency (radians per second ω_0)

Harmonic Oscillators

A A particle attached to an elastic helical spring: “Hook’s law”

B A simple pendulum

Example A: A particle attached to an elastic helical spring

Hook's law

$$\mathbf{F} = -k\mathbf{x}$$

- $\mathbf{x}$ displacement of the end of the spring from its equilibrium position
- $\mathbf{F}$ restoring force exerted by the spring
- k force/spring/stiffness constant

Example A: A particle attached to an elastic helical spring (cont'd)

Fig.28.6 (p.324)

- What's the upward force of the spring?
- What's the downward force due to the weight of the particle?
- What's the relation of the forces in equilibrium?
- What's the D.E. of the position of the particle (mass m) when stretched by y?
- What is the solution?

Example B: A simple pendulum

Fig.28.7 (p.327)

- θ angle of swing
- w angular velocity
- F effective force (which moves the pendulum) due to the weight of the pendulum
- D.E. of $\theta(t)$?
- Is it the D.E. of simple harmonic motion?
- Which assumption do we need?

Forced Undamped Motion

$$m\frac{d^2y}{dt^2} + m\omega_0^2 y = f(t) \quad \rightarrow \quad \frac{d^2y}{dt^2} + \omega_0^2 y = \frac{1}{m} f(t)$$

- $f(t)$ forcing function attached to the system
- With $f(t) = mF\sin(\omega t + \beta)$, we consider two cases: (i) $\omega \neq \omega_0$ and (ii) $\omega = \omega_0$.
- ω is called ‘impressed frequency’ or ‘forcing frequency’.
- The complementary function:

$$y_c = c\sin(\omega_0 t + \delta).$$

Case 1: $\omega \neq \omega_0$

- General solution:

$$y = c\sin(\omega_0 t + \delta) + \frac{F}{\omega_0^2 - \omega^2}\sin(\omega t + \beta).$$

  $\rightarrow$ sum of two simple harmonic motions.
- Stable motion (see Fig.28.85 on p.340)
- What if $\omega \approx \omega_0$?

Case 2: $\omega = \omega_0$

- Which method to use?
- General solution:

$$y = c\sin(\omega_0 t + \delta) - \frac{F}{2\omega_0} t\cos(\omega_0 t + \beta).$$

- Unstable motion (see Fig.28.94 on p.341)
  $\rightarrow$ "undamped resonance", "undamped resonant frequency"
- See "mechanical resonance" at Wikipedia.

Lesson 29: Damped Motion.

Lesson 30: Electric Circuits. Analog Computation.

Lesson 30M: Miscellaneous Types of Problems Leading to Linear Equatic