Homework #6

June 17, 2011

1. If a subspace S_1 is contained in a subspace S_2 (i.e., $S_1 \subset S_2$), prove that $S_1^\perp$ contains $S_2^\perp$ (i.e., $S_2^\perp \subset S_1^\perp$). Hint: Prove that every vector $\boldsymbol{x} \in S_2^\perp$ is also contained in $S_1^\perp$ by showing that $\boldsymbol{x}$ is orthogonal to all the vectors in S_1.

Solution:

Any vector $\boldsymbol{x} \in S_2^\perp$ is orthogonal to all the vectors in S_2. Since $S_1 \subset S_2$, $\boldsymbol{x}$ is also orthogonal to all the vectors in S_1. Therefore, $\boldsymbol{x} \in S_1^\perp$, hence $S_2^\perp \subset S_1^\perp$.

2. Suppose an $n \times n$ matrix A is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the subspace spanned by which rows of A?

Solution:

Let the first column of A^{-1} be $\boldsymbol{x}$. Since $AA^{-1} = I$, $A\boldsymbol{x} = \boldsymbol{e}_1$. Therefore, $\boldsymbol{x}$ is orthogonal to all the rows of A except the first row.

3. Let $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$.
   (a) Compute the two projection matrices P_1 and P_2 onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively.
   (b) Compute P_1P_2 and P_2P_1.
   (c) Explain the result of (b) from the "transformation" point of view.

Solution:

(a)

$$P_1 = \boldsymbol{a}_1\boldsymbol{a}_1^T/\boldsymbol{a}_1^T\boldsymbol{a}_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}.$$

$$P_2 = \boldsymbol{a}_2\boldsymbol{a}_2^T/\boldsymbol{a}_2^T\boldsymbol{a}_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

(b) Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are orthogonal, $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = \boldsymbol{a}_1^T \boldsymbol{a}_2 = 0$ therefore

$$\begin{aligned} P_1 P_2 &= \frac{1}{\boldsymbol{a}_1^T \boldsymbol{a}_1 \boldsymbol{a}_2^T \boldsymbol{a}_2} (\boldsymbol{a}_1 \boldsymbol{a}_1^T)(\boldsymbol{a}_2 \boldsymbol{a}_2^T) \\ &= \frac{1}{\boldsymbol{a}_1^T \boldsymbol{a}_1 \boldsymbol{a}_2^T \boldsymbol{a}_2} \boldsymbol{a}_1 (\boldsymbol{a}_1^T \boldsymbol{a}_2) \boldsymbol{a}_2^T \\ &= \frac{1}{\boldsymbol{a}_1^T \boldsymbol{a}_1 \boldsymbol{a}_2^T \boldsymbol{a}_2} \boldsymbol{a}_1 (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2) \boldsymbol{a}_2^T \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: O \end{aligned}$$

In the same way, $P_2 P_1 = O$.

(c) $P_2 P_1$ first projects a vector onto $\boldsymbol{a}_1$ and then projects onto $\boldsymbol{a}_2$. Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are perpendicular, this always results in a zero vector, therefore $P_2 P_1 = O$.

4. Let P be a projection matrix.
   (a) Show that $(I - P)^2 = I - P$.
   (b) Let P projects onto the column space of A. Then onto which fundamental subspace of A does $I - P$ project?

Solution:

(a) $(I - P)^2 = P^2 - 2P + I = P - 2P + I = I - P$

(b) For any $\boldsymbol{x}$, $(I - P)\boldsymbol{x} = \boldsymbol{x} - P\boldsymbol{x}$. Since $P\boldsymbol{x} \in \mathrm{col}(A)$ and $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$ are orthogonal complement, $\boldsymbol{x} - P\boldsymbol{x} \in \mathrm{null}(A^T)$. Therefore, $I - P$ projects onto $\mathrm{null}(A^T)$.

5. If an $m \times m$ matrix A satisfies $A^2 = A$ and $\mathrm{rank}(A) = m$, $A = I$. Prove it.

Solution:
Since $\mathrm{rank}(A) = m$, A is invertible. Then $I = AA^{-1} = A^2 A^{-1} = A(AA^{-1}) = A$.

6. Let $P^T = P$ and $P^2 = P$. If $\boldsymbol{p}_i$ is the i-th column of P, show that $\|\boldsymbol{p}_i\|^2$ is the same as the (i, i) element of P.

Solution:

$$P^T P = P^2 = P.$$

The (i, i) element of $P^T P$ is $\boldsymbol{p}_i^T \boldsymbol{p}_i = \|\boldsymbol{p}_i\|^2$.

7. Let the matrix A have three columns and they are $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$. And let $\|\boldsymbol{a}_1\| = 1$, $\|\boldsymbol{a}_2\| = 2$, and $\|\boldsymbol{a}_3\| = 3$. Then what is $A^T A$?

Solution: There was an error in the question. They should be "three orthogonal columns".

$$A^T A = \begin{bmatrix} \boldsymbol{a}_1^T \boldsymbol{a}_1 & \boldsymbol{a}_1^T \boldsymbol{a}_2 & \boldsymbol{a}_1^T \boldsymbol{a}_3 \\ \boldsymbol{a}_2^T \boldsymbol{a}_1 & \boldsymbol{a}_2^T \boldsymbol{a}_2 & \boldsymbol{a}_2^T \boldsymbol{a}_3 \\ \boldsymbol{a}_3^T \boldsymbol{a}_1 & \boldsymbol{a}_3^T \boldsymbol{a}_2 & \boldsymbol{a}_3^T \boldsymbol{a}_3 \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{a}_1\|^2 & 0 & 0 \\ 0 & \|\boldsymbol{a}_2\|^2 & 0 \\ 0 & 0 & \|\boldsymbol{a}_3\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$