Homework #4

May 23, 2011

1. For which sides (find a condition on b_1, b_2, b_3) are these systems solvable?

   (a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

2. Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = \boldsymbol{c}$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{c}$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{c}$ are in the column space $\mathrm{col}(A)$, then $\boldsymbol{b} + \boldsymbol{c}$ is in $\mathrm{col}(A)$.

3. True or false (with a counterexample if false):

   (a) The vectors $\boldsymbol{b}$ that are not in the column space $\mathrm{col}(A)$ form a subspace.

   (b) If $\mathrm{col}(A)$ contains only the zero vector, then A is the zero matrix.

   (c) The column space of $2A$ equals the column space of A.

   (d) The column space of $A - I$ equals the column space of A.

4. Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

5. Why does no 3 by 3 matrix have a nullspace that equals its column space?

6. If the nullspace of A consists of all multiples of $\boldsymbol{x} = (2, 1, 0, 1)$, how many pivots appear in U (row echelon form)? What is R (reduced row echelon form)?

7. Find a basis for each of the three subspace associated with

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8. What are the dimensions of the three subspaces for A, B, and C if I is the 3 by 3 identity matrix and O is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & O \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ O^T & O^T \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} O \end{bmatrix}.$$

9. A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *non solution.*

   (a) What are all inequalities ($<$ or $\leqslant$) that must be true between m, n, and r?

   (b) How do you know that $A^T\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

10. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

11. (Fill (a),(b), and (c) below.)

If $AB = C$, the rows of C are combinations of the rows of (a). So the rank of C is not greater than the rank of (b). Since $B^T A^T = C^T$, the rank of C is also not greater than the rank of (c).

12. Which of these transformations satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ and which satisfy $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$?

    (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$

    (b) $T(\boldsymbol{v}) = v_1 + v_2 + v_3$

    (c) $T(\boldsymbol{v}) = (v_1, 2v_2, 3v_3)$

    (d) $T(\boldsymbol{v}) =$ largest component of $\boldsymbol{v}$

13. Suppose T is reflection across the x axis and S is reflection across the y axis. The domain V is the xy plane. If $\boldsymbol{v} = (x, y)$ what is $S(T(\boldsymbol{v}))$? Find a simpler description of the product ST.

14. Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $\boldsymbol{v} = (2, 1)$ then $T(\boldsymbol{v}) = (1, 2)$. Find $S(T(\boldsymbol{v}))$ and $T(S(\boldsymbol{v}))$. This shows that generall $ST \neq TS$.

15. Let ℓ_1 and ℓ_2 are lines through the origin. And let T_1 and T_2 are reflections across the line ℓ_1 and ℓ_2, respectively. Show that the product $T_1 T_2$ is a rotation. (Hint: See problem 26 on p.222 of our textbook.)