Topics in Computer Graphics
Chap 17: Bézier Triangles

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de Casteljau Algorithm with Barycentric Coordinates: 1D Case

For $n = 3$

$$
\begin{array}{llll}
\mathbf{b}_{30} & & & \\
\mathbf{b}_{21} & \mathbf{b}_{20} & & \\
\mathbf{b}_{12} & \mathbf{b}_{11} & \mathbf{b}_{10} & \\
\mathbf{b}_{03} & \mathbf{b}_{02} & \mathbf{b}_{01} & \mathbf{b}_{00}
\end{array}
$$

With barycentric coordinates $\mathbf{u} := (u_1, u_2) \in \mathbb{Z}_+^2$ (multi-index) where $|\mathbf{u}| := u_1 + u_2 = 1$,

$$\mathbf{b}_{\mathbf{i}}(u_1, u_2) = u_1 \mathbf{b}_{\mathbf{i}+\mathbf{e}_1}(u_1, u_2) + u_2 \mathbf{b}_{\mathbf{i}+\mathbf{e}_2}(u_1, u_2)$$

- $\mathbf{e}_j$ is the j-th unit vector: $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$
- Univariate de Casteljau algorithm
- At k-th step, the subscript vector sums up to $n - k$.
- Domain is a line segment.
- $\mathbf{b}(t) = \mathbf{b}_{00}(1 - t, t)$

de Casteljau Algorithm with Barycentric Coordinates: 2D Case

For $n = 3$

$$
\begin{array}{c}
\mathbf{b}_{030} \\
\mathbf{b}_{021}\mathbf{b}_{120} \\
\mathbf{b}_{012}\mathbf{b}_{111}\mathbf{b}_{210} \\
\mathbf{b}_{003}\mathbf{b}_{102}\mathbf{b}_{201}\mathbf{b}_{300}
\end{array}
\rightarrow
\begin{array}{c}
\mathbf{b}_{020} \\
\mathbf{b}_{011}\mathbf{b}_{110} \\
\mathbf{b}_{002}\mathbf{b}_{101}\mathbf{b}_{200}
\end{array}
\rightarrow
\begin{array}{c}
\mathbf{b}_{010} \\
\mathbf{b}_{001}\mathbf{b}_{100}
\end{array}
\rightarrow
\mathbf{b}_{000}
$$

With barycentric coordinates $\mathbf{u} := (u_1, u_2, u_3) \in \mathbb{Z}_+^3$ where $|\mathbf{u}| := u_1 + u_2 + u_3 = 1$,

$$\mathbf{b}_{\mathbf{i}}(\mathbf{u}) = u_1 \mathbf{b}_{\mathbf{i}+\mathbf{e}_1}(\mathbf{u}) + u_2 \mathbf{b}_{\mathbf{i}+\mathbf{e}_2}(\mathbf{u}) + u_3 \mathbf{b}_{\mathbf{i}+\mathbf{e}_3}(\mathbf{u})$$

- $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$
- Bivariate de Casteljau algorithm
- At k-th step, the subscript vector sums up to $n - k$.
- # of control points = $\frac{1}{2}(n+1)(n+2)$
- Domain is a triangle. $\rightarrow$ A triangular patch (Bézier triangle) is generated.
- $\mathbf{b}(\mathbf{u}) = \mathbf{b}_{00}(\mathbf{u})$

Bézier Triangle: Properties

- Affine invariance
- Invariance under affine parameter transaformations
  The barycentric coordinates do not change when a triangle is transformed.
- Convex hull property
- Boundary curves

$$\mathbf{b}_{\mathbf{i}}(u_1, 0, u_3) = u_1 \mathbf{b}_{\mathbf{i}+\mathbf{e}_1} + u_3 \mathbf{b}_{\mathbf{i}+\mathbf{e}_3}, \quad u_1 + u_3 = 1$$

$\rightarrow$ Univariate de Casteljau algorithm $\rightarrow$ A Bézier curve of degree n

Outline

Blossoms with Barycentric Coordinates: 1D Case

- Let $\mathbf{x}(j)$ be the j-th element of vector $\mathbf{x}$.
- Different arguments in each level of de Casteljau algorithm: For $n = 3$, with univariate barycentric coordinates $\mathbf{u}_j$ $(\mathbf{u}_j(1) + \mathbf{u}_j(2) = 1)$,

$$\begin{array}{llll} \mathbf{b}_{30} & & & \\ \mathbf{b}_{21} & \mathbf{b}_{20}[\mathbf{u}_1] & & \\ \mathbf{b}_{12} & \mathbf{b}_{11}[\mathbf{u}_1] & \mathbf{b}_{10}[\mathbf{u}_1, \mathbf{u}_2] & \\ \mathbf{b}_{03} & \mathbf{b}_{02}[\mathbf{u}_1] & \mathbf{b}_{01}[\mathbf{u}_1, \mathbf{u}_2] & \mathbf{b}_{00}[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] \end{array}$$

- In general,

$$\begin{aligned} \mathbf{b}_{\mathbf{i}}[\mathbf{u}_1, \ldots, \mathbf{u}_{n-|\mathbf{i}|}] =& \mathbf{u}_{n-|\mathbf{i}|}(1)\mathbf{b}_{\mathbf{i}+\mathbf{e}_1}[\mathbf{u}_1, \ldots, \mathbf{u}_{n-|\mathbf{i}|-1}] \\ &+ \mathbf{u}_{n-|\mathbf{i}|}(2)\mathbf{b}_{\mathbf{i}+\mathbf{e}_2}[\mathbf{u}_1, \ldots, \mathbf{u}_{n-|\mathbf{i}|-1}] \end{aligned}$$

- The blossom is $\mathbf{b}[\mathbf{u}_1, \ldots, \mathbf{u}_n] = \mathbf{b}_{00}[\mathbf{u}_1, \ldots, \mathbf{u}_n]$

Blossoms with Barycentric Coordinates: 1D Case (cont'd)

- The Bézier curve is obtained by

$$\mathbf{b}(\mathbf{u}) = \mathbf{b}[\mathbf{u}^{<n>}]$$

- de Casteljau algorithm with blossoms

$$\begin{array}{llll}
\mathbf{b}[\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1] & & & \\
\mathbf{b}[\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2] & \mathbf{b}[\mathbf{e}_1, \mathbf{e}_1, \mathbf{u}] & & \\
\mathbf{b}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2] & \mathbf{b}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}] & \mathbf{b}[\mathbf{e}_1, \mathbf{u}, \mathbf{u}] & \\
\mathbf{b}[\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2] & \mathbf{b}[\mathbf{e}_2, \mathbf{e}_2, \mathbf{u}] & \mathbf{b}[\mathbf{e}_2, \mathbf{u}, \mathbf{u}] & \mathbf{b}[\mathbf{u}, \mathbf{u}, \mathbf{u}]
\end{array}$$

- Control points are obtained by

$$\mathbf{b}_{\mathbf{i}} = \mathbf{b}[\mathbf{e}_1^{<\mathbf{i}(1)>}, \mathbf{e}_2^{<\mathbf{i}(2)>}]$$

where $\mathbf{i} \in \mathbb{Z}_+^2$ and $|\mathbf{i}| = \mathbf{i}(1) + \mathbf{i}(2) = n$.
ex) For $n = 3$, control points are $\mathbf{b}_{30}, \mathbf{b}_{21}, \mathbf{b}_{12}, \mathbf{b}_{03}$.

Blossoms with Barycentric Coordinates: 1D Case (cont'd)

- Control points for general interval $[\mathbf{a}, \mathbf{b}]$ ($\mathbf{a}$ and $\mathbf{b}$ are barycentric coordinates and $\mathbf{i} \in \mathbb{Z}_+^2$)

$$\mathbf{c}_\mathbf{i} = \mathbf{b}[\mathbf{a}^{\mathbf{i}(1)}, \mathbf{b}^{\mathbf{i}(2)}] \quad (|\mathbf{i}| = n)$$

- Lebniz formula

$$\mathbf{b}[(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2)^{<n>}] = \sum_{|\mathbf{i}|=n} \binom{n}{\mathbf{i}} \alpha_1^{\mathbf{i}(1)} \alpha_2^{\mathbf{i}(2)} \mathbf{b}[\mathbf{u}_1^{<\mathbf{i}(1)>}, \mathbf{u}_2^{<\mathbf{i}(2)>}]$$

  - $\mathbf{u}_1$ and $\mathbf{u}_2$ are barycentric coordinates
  - $\alpha_1 + \alpha_2 = 1$
  - $\binom{n}{\mathbf{i}} := \frac{n!}{i_1! i_2!}$

Blossoms with Barycentric Coordinates: 2D Case

- Control points are

$$\mathbf{b}_{\mathbf{i}} = \mathbf{b}[\mathbf{e}_1^{<\mathbf{i}(1)>}, \mathbf{e}_2^{<\mathbf{i}(2)>}, \mathbf{e}_3^{<\mathbf{i}(3)>}], \quad |\mathbf{i}| = n$$

- New control points for general domain $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ (in barycentric coordinates):

$$\mathbf{c}_{\mathbf{i}} = \mathbf{b}[\mathbf{f}_1^{<\mathbf{i}(1)>}, \mathbf{f}_2^{<\mathbf{i}(2)>}, \mathbf{f}_3^{<\mathbf{i}(3)>}]$$

Blossoms with Barycentric Coordinates: 2D Case

- Lebniz formula

$$\mathbf{b}[(\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2)^{<n>}] = \sum_{|\mathbf{i}|=n} \binom{n}{\mathbf{i}} \alpha_1^{\mathbf{i}(1)}\alpha_2^{\mathbf{i}(2)}\mathbf{b}[\mathbf{u}_1^{<\mathbf{i}(1)>}, \mathbf{u}_2^{<\mathbf{i}(2)>}]$$

  - A line through $\mathbf{u}_1$ and $\mathbf{u}_2$ in the domain with parameter (α_1, α_2) is mapped to a curve on the Bézier triangle. (See Fig 17.3)
  - What are the control points? $\left\{\mathbf{b}[\mathbf{u}_1^{<\mathbf{i}(1)>}, \mathbf{u}_2^{<\mathbf{i}(2)>}]\right\}_{|\mathbf{i}|=n}$

- Lebniz formula

$$\begin{aligned} &\mathbf{b}[(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3)^{<n>}] \\ &= \sum_{|\mathbf{i}|=n} \binom{n}{\mathbf{i}} \alpha_1^{\mathbf{i}(1)}\alpha_2^{\mathbf{i}(2)}\alpha_3^{\mathbf{i}(3)}\mathbf{b}[\mathbf{u}_1^{<\mathbf{i}(1)>}, \mathbf{u}_2^{<\mathbf{i}(2)>}, \mathbf{u}_3^{<\mathbf{i}(3)>}] \end{aligned}$$

  - A triangle composed of vertices $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with parameter $(\alpha_1, \alpha_2, \alpha_3)$ is mapped to a Bézier triangle on the Bézier triangle.

Outline

Bernstein Polynomials with Barycentric Coordinates: 1D Case

- For univariate case barycentric coordinates

$$B_{\mathbf{i}}^{n}(\mathbf{u}) = \binom{n}{\mathbf{i}} \mathbf{u}(1)^{\mathbf{i}(1)} \mathbf{u}(2)^{\mathbf{i}(2)}, \quad \mathbf{i} \in \mathbb{Z}_{+}^{2}, |\mathbf{i}| = n$$

- Recursion

$$B_{\mathbf{i}}^{n}(\mathbf{u}) = \mathbf{u}(1) B_{\mathbf{i}-\mathbf{e}_1}^{n}(\mathbf{u}) + \mathbf{u}(2) B_{\mathbf{i}-\mathbf{e}_2}^{n}(\mathbf{u})$$

  Based on the relation

$$\binom{n}{\mathbf{i}} = \binom{n-1}{\mathbf{i}-\mathbf{e}_1} + \binom{n-1}{\mathbf{i}-\mathbf{e}_2}$$

- A Bézier curve is defined as

$$\mathbf{b}(\mathbf{u}) = \sum_{|\mathbf{i}|=n} \mathbf{b}_{\mathbf{i}} B_{\mathbf{i}}^{n}(\mathbf{u})$$

Bernstein Polynomials with Barycentric Coordinates: 2D Case

- For univariate case barycentric coordinates

$$B_{\mathbf{i}}^{n}(\mathbf{u}) = \binom{n}{\mathbf{i}} \mathbf{u}(1)^{\mathbf{i}(1)} \mathbf{u}(2)^{\mathbf{i}(2)} \mathbf{u}(3)^{\mathbf{i}(3)}, \quad \mathbf{i} \in \mathbb{Z}_{+}^{3}, |\mathbf{i}| = n$$

- Recursion

$$B_{\mathbf{i}}^{n}(\mathbf{u}) = \mathbf{u}(1) B_{\mathbf{i}-\mathbf{e}_1}^{n}(\mathbf{u}) + \mathbf{u}(2) B_{\mathbf{i}-\mathbf{e}_2}^{n}(\mathbf{u}) + \mathbf{u}(3) B_{\mathbf{i}-\mathbf{e}_3}^{n}(\mathbf{u})$$

  Based on the relation

$$\binom{n}{\mathbf{i}} = \binom{n-1}{\mathbf{i}-\mathbf{e}_1} + \binom{n-1}{\mathbf{i}-\mathbf{e}_2} + \binom{n-1}{\mathbf{i}-\mathbf{e}_3}$$

- A Bézier triangle is defined as

$$\mathbf{b}(\mathbf{u}) = \sum_{|\mathbf{i}|=n} \mathbf{b}_{\mathbf{i}} B_{\mathbf{i}}^{n}(\mathbf{u})$$

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