Mathematical Models for Engineering Problems and Differential Equations

Minho Kim

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Introductory remarks

1. Only very special types of 1st order DE possess solutions which can be expressed in terms of the elementary functions.
2. The "look" of a DE doesn't tell you how easy (or difficult) it is to solve it.
3. Implicit solutions usually are not practical.
4. The solution you find may be *extraneous*. Therefore you should always verify that it does in fact satisfy the given DE.
5. The examples in the textbook are only for illustration and not from real applications.

Table of contents

Lesson 6: Meaning of the Differential of a Function. Separable Differenti

Increment (Δy) and differential (dy)

Let $y = f(x)$ define y as a function of x.

- Increament: "How much y increases from x to $x + \Delta x$?"
  $\Delta y = (\Delta f)(x, \Delta x) = f(x + \Delta x) - f(x)$
- Differential: Approximates Δy using $f'(x)$.
  $dy = (df)(x, \Delta x) = f'(x)\Delta x$

$\rightarrow (dy)(x, \Delta x) = f'(x)(d\hat{x})(x, \Delta x)$
$\rightarrow dy = f'(x)dx$

Type #1: $f(x)dx + g(y)dy = 0$

A 1-parameter family of solutions of the DE with **separable variables**,

$$f(x)dx + g(y)dy = 0,$$

is

$$\int f(x)dx + \int g(y)dy = C.$$

Lesson 7: First Order Differential Equation with Homogeneous Coefficien

Homogeneous function

Definition

The function $f(x, y)$ is n-**th order homogeneous** if it can be written as

$$f(x, y) = x^n g(u), u = y/x$$

or

$$f(x, y) = y^n h(u), u = x/y$$

Alternatively, $f(x, y)$ is homogeneous of order n if

$$f(tx, ty) = t^n f(x, y).$$

Type #2: Homogeneous $P(x,y)dx + Q(x,y)dy = 0$

The 1st order DE with n-th order homogeneous coefficients,

$$P(x,y)dx + Q(x,y)dy = 0,$$

can be converted to the equation with separable variables (Type #1)

$$\frac{dx}{x} + \frac{g_2(u)}{g_1(u) + ug_2(u)}du = 0, x \neq 0, g_1(u) + ug_2(u) \neq 0,$$

where $P(x,y) = x^n g_1(u)$, $Q(x,y) = x^n g_2(u)$, and $u = y/x$.
Alternatively, we can convert by substituting $u = x/y$.

Lesson 8: Differential Equations with Linear Coefficients

Type #3: $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$

1. Two lines are not parallel.
   The DE with linear coefficients
   $$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$
   can be converted to the equation with homogeneous coefficients
   $$(a_1\bar{x} + b_1\bar{y})d\bar{x} + (a_2\bar{x} + b_2\bar{y})d\bar{y} = 0$$
   where $\bar{x} = x - h$, $\bar{y} = y - k$, and (h, k) is the unique intersection point of the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.
2. Two lines are parallel or coincide.

Lesson 9: Exact Differential Equations

Type #4: Exact $P(x, y)dx + Q(x, y)dy = 0$

A DE

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact** if there exists a function $f(x, y)$ such that

$$P(x, y) = \frac{\partial f(x, y)}{\partial x}$$

and

$$Q(x, y) = \frac{\partial f(x, y)}{\partial y}.$$

A 1-parameter family of solutions of this equation is

$$f(x, y) = c.$$

Type #4: Exact $P(x, y)dx + Q(x, y)dy = 0$ (cont'd)

- Necessary and sufficient condition for exactness:

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}.$$

- The 1-parameter solution:

$$f(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy = c$$

or

$$f(x, y) = \int_{y_0}^{y} Q(x, y)dy + \int_{x_0}^{x} P(x, y_0)dx = c.$$

Lesson 10: Recognizing Exact Differential Equations. Integrating Factors.

Type #5: Integrating factors

Definition

A multiplying factor which will convert an inexact DE into an exact one is called an **integrating factor**.

1. $h = h(x)$
2. $h = h(y)$
3. $h = h(u), u = xy$
4. $h = h(u), u = x/y$
5. $h = h(u), u = y/x$

Lesson 11: The Linear Differential Equation of the First Order. Bernoulli

Type #6: $\frac{dy}{dx} + P(x)y = Q(x)$

The integrating factor of the **linear differential equation of the 1st order**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$e^{\int P(x)dx}.$$

Type #7: Bernoulli equation

The Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

can be converted to Type #6 by multiplying $(1-n)y^{-n}$.

Lesson 12: Miscellaneous Methods of Solving a First Order Differential E