Solution of homework #2

April 15, 2011

Excercise 2.2

16 The reverse operations are $R_i \leftrightarrow R_j$, $(1/k)R_i$, and $R_i - kR_j$,respectively.

18 For A,

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let

$$C := \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

For B,

$$\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{(1/3)R_1} \begin{bmatrix} 1 & 1/3 & -1/3 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 4 & 2 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 4 & 2 \\ 0 & 4/3 & 2/3 \end{bmatrix}$$
$$\xrightarrow{(1/2)R_2} \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 4/3 & 2/3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - (1/3)R_2} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 4/3 & 2/3 \end{bmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - (4/3)R_2} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} = C$$

Therefore, A can be converted to B by the following elementary row operations.

(a) $A \to C$

  i. $(1/2)R_1$

  ii. $R_2 \leftarrow R_2 - R_1$

iii. $R_3 \leftarrow R_3 + R_1$

iv. $R_3 \leftarrow R_3 - R_2$

(b) $C \to B$

i. $R_3 \leftarrow R_3 + (4/3)R_2$

ii. $R_1 \leftarrow R_1 + (1/3)R_2$

iii. $2R_2$

iv. $R_3 \leftarrow R_3 + 2R_1$

v. $R_2 \leftarrow R_2 + 3R_1$

vi. $3R_1$

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$$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + (1/2)R_2} \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 0 & -1/2 & 0 & -3/2 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftarrow R_4 - 4R_3} \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 0 & -1/2 & 0 & -3/2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Therefore, there is no solution.

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$$\begin{bmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - kR_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & (1-k)(k+2) & -(k+2) \end{bmatrix}$$

- If $k = 1$, the last equation become $0 = -3$ therefore there is no solution.
- If $k = -2$, the row echelon form is

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  therefore there is infinitely many solutions.

- If $k \neq 1$ and $k \neq -2$, the solution is (via back substitution)

$$\begin{aligned}
&z = 1/(k-1) \\
&y = z = 1/(k-1) \\
&x = 1 - y - kz = (k-1-1-k)/(k-1) = -2/(k-1)
\end{aligned}$$

therefore there is a unique solution.

50 The equation for the line through Q with direction vector $\boldsymbol{v}$ is

$$\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Two lines intersect when there exist t and s such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} a-1 \\ b-2 \\ c-3 \end{bmatrix}$$

Applying Gaussian elimination,

$$\begin{aligned}
\begin{bmatrix} 1 & -2 & a-1 \\ 1 & -1 & b-2 \\ -1 & 0 & c-3 \end{bmatrix} &\xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & -2 & a-1 \\ 0 & 1 & -a+b-1 \\ 0 & -2 & a+c-4 \end{bmatrix} \\
&\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{bmatrix} 1 & -2 & a-1 \\ 0 & 1 & -a+b-1 \\ 0 & 0 & -a+2b+c-6 \end{bmatrix}
\end{aligned}$$

Therefore, the solution (s and t) exist when

$$-a + 2b + c - 6 = 0.$$

In other words, all the points Q are in the plane

$$-x + 2y + z - 6 = 0.$$

Excercise 2.3

5 $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ if and only if there is a solution for the linear system

$$\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{u}.$$

Applying Gaussian elimination,

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} &\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}
\end{aligned}$$

Therefore the solution exists hence $\boldsymbol{u}$ is a linear combination of other vectors.

11 We need to show that any vector $\boldsymbol{b}$ in $\mathbb{R}^3$ is a linear combination of the three vectors. In other words, the linear system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

has a solution for any $\boldsymbol{b}$.

$$\begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & -1 & 1 & -x+z \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & -x+y+z \end{bmatrix}$$

Therefore, the system has a solution regardless of $\boldsymbol{b}$ hence

$$\mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

31 Clearly,

$$\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

therefore the vectors are linearly dependent.

42 By Theorem 2.6, n vectors in A are linearly independent if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has a trivial solution only. Since there is a unique solution we have no free variable hence by Theorem 2.2,

$$\operatorname{rank}(A) = n - \text{numer of free variables} = n.$$

Excercise 2.4

3

	small	medium	large
roses	1	2	4
daisies	3	4	8
chrysanthemums	3	6	6

We have a linear system

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 8 \\ 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 50 \\ 48 \end{bmatrix}$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & 2 & 4 & 24 \\ 3 & 4 & 8 & 50 \\ 3 & 6 & 6 & 48 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 4 & 24 \\ 0 & -2 & -4 & -22 \\ 0 & 0 & -6 & -24 \end{bmatrix}$$

Therefore,

$$\begin{aligned} z &= 4 \\ y &= (22 - 4z)/2 = 3 \\ x &= 24 - 2y - 4z = 2 \end{aligned}$$

12 Let

$$x\mathrm{HClO_4} + y\mathrm{P_4O_{10}} \to z\mathrm{H_3PO_4} + w\mathrm{Cl_2O_7}.$$

$$\begin{array}{rcccccc} \mathrm{H}: & x & & = & 3z & & \\ \mathrm{Cl}: & x & & = & & & 2w \\ \mathrm{O}: & 4x+ & 10y & = & 4z+ & & 7w \\ \mathrm{P}: & & 4y & = & z & & \end{array}$$

We have

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -2 \\ 4 & 10 & -4 & -7 \\ 0 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Gaussian elimination,

$$\begin{aligned} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 4 & 10 & -4 & -7 & 0 \\ 0 & 4 & -1 & 0 & 0 \end{bmatrix} &\xrightarrow[R_3 \leftarrow R_3 - 4R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 10 & 8 & -7 & 0 \\ 0 & 4 & -1 & 0 & 0 \end{bmatrix} \\ &\xrightarrow[R_4 \leftarrow R_4 - (2/5)R_2]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 10 & 8 & -7 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & -21/5 & 14/5 & 0 \end{bmatrix} \\ &\xrightarrow[R_4 \leftarrow R_4 + 7R_3]{5R_4} \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 10 & 8 & -7 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} z &= 2w/3 \\ y &= (-8z + 7w)/10 = w/6 \\ x &= 3z = 2w. \end{aligned}$$

To get integer solution, we set $w = 6$ hence

$$\begin{aligned} x &= 12 \\ y &= 1 \\ z &= 4 \\ w &= 6 \end{aligned}$$

18 (a)

$$\begin{aligned} A: &\qquad 200 + f_3 = 100 + f_1 \\ B: &\qquad f_1 + 150 = f_2 + f_4 \\ C: &\qquad f_2 + f_5 = 200 + 100 \\ D: &\qquad f_6 + 100 = f_3 + 200 \\ E: &\qquad f_4 + f_7 = f_6 + 100 \\ F: &\qquad 100 + 150 = f_5 + f_7 \end{aligned}$$

We get a linear system

$$\left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 100 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 150 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 250 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 250 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 50 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 \leftarrow R_1 + R_3 \\ R_2 \leftarrow R_2 + R_3 \\ R_4 \leftarrow R_4 + R_3}} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 150 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 50 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 \leftarrow R_1 - R_4 \\ R_3 \leftarrow R_3 - R_4 \\ R_5 \leftarrow R_5 + R_4}} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 \leftarrow R_2 - R_5 \\ R_4 \leftarrow R_4 - R_5 \\ R_6 \leftarrow R_6 - R_5}} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 50 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & -100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, f_6 and f_7 are free variables and

$$\begin{aligned} f_5 &= 250 - f_7 \\ f_4 &= f_6 - f_7 + 100 \\ f_3 &= f_6 - 100 \\ f_2 &= f_7 + 50 \\ f_1 &= f_6 \end{aligned}$$

(b) No, since $f_1 = f_6$. At D in Figure 2.21, if $f_6 = 150$, we get $f_3 = 50$. And at A, we have $f_1 = 150$ not 100.

(c) We get $f_7 = -f_4 + f_6 + 100 = f_6 + 100$ therefore

$$\begin{aligned}
f_1 &= f_6 \\
f_2 &= f_7 + 50 = f_6 + 150 \\
f_3 &= f_6 - 100 \\
f_4 &= 0 \\
f_5 &= 250 - f_7 = 150 - f_6
\end{aligned}$$

Since each value should be nonnegative,

$$\begin{aligned}
f_1 &= f_6 & &\rightarrow f_6 \geq 0 \\
f_2 &= f_6 + 150 & &\rightarrow f_6 \geq -150 \\
f_3 &= f_6 - 100 & &\rightarrow f_6 \geq 100 \\
f_5 &= 150 - f_6 & &\rightarrow f_6 \leq 150
\end{aligned}$$

Therefore, $100 \leq f_6 \leq 150$ and

$$\begin{aligned}
100 \leq f_1 &\leq 150 \\
250 \leq f_2 &\leq 300 \\
0 \leq f_3 &\leq 50 \\
f_4 &= 0 \\
0 \leq f_5 &\leq 50 \\
100 \leq f_6 &\leq 150
\end{aligned}$$