Homework #1 Solution

Minho Kim

March 23, 2010

Exercise 1.2

58 (a) We need to prove both "if" and "only if" statements.

(i) "if" statement: $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal $\Rightarrow \|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}-\boldsymbol{v}\|$
Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, $\boldsymbol{u}\cdot\boldsymbol{v} = 0$. Firstly,

$$\|\boldsymbol{u}+\boldsymbol{v}\| = \sqrt{(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v})} = \sqrt{\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2+2\boldsymbol{u}\cdot\boldsymbol{v}} = \sqrt{\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2}.$$

Secondly,

$$\|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})} = \sqrt{\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2-2\boldsymbol{u}\cdot\boldsymbol{v}} = \sqrt{\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2}.$$

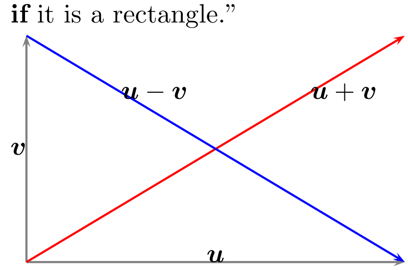
Therefore, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}-\boldsymbol{v}\|$.

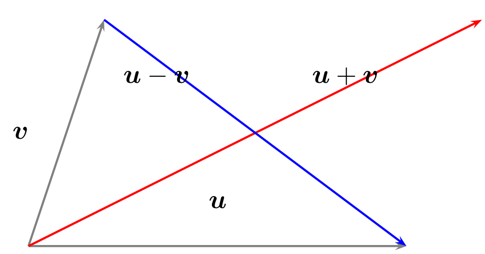
(ii) "only if" statement: $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}-\boldsymbol{v}\| \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal
By squaring both sides, we get

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= \|\boldsymbol{u}-\boldsymbol{v}\|^2\\
(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) &= (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})\\
\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2+2\boldsymbol{u}\cdot\boldsymbol{v} &= \|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2-2\boldsymbol{u}\cdot\boldsymbol{v}\\
4\boldsymbol{u}\cdot\boldsymbol{v} &= 0
\end{aligned}$$

therefore $\boldsymbol{u}\cdot\boldsymbol{v} = 0$ hence $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

(b) From the pictures below, we can conclude that
"Two diagonals of a parallelogram have the same lengths **if and only if** it is a rectangle."





65 (a)

$$\begin{aligned}
&(u_1^2+u_2^2)(v_1^2+v_2^2)-(u_1v_1+u_2v_2)^2\\
&=[(u_1v_1)^2+(u_1v_2)^2+(u_2v_1)^2+(u_2v_2)^2]-[(u_1v_1)^2+(u_2v_2)^2+2u_1v_1u_2v_2]\\
&=(u_1v_2)^2+(u_2v_1)^2-2u_1v_1u_2v_2\\
&=(u_1v_2-u_2v_1)^2\geq 0
\end{aligned}$$

therefore

$$(u_1v_1+u_2v_2)^2\leq(u_1^2+u_2^2)(v_1^2+v_2^2).$$

(b) The analogue in $\mathbb{R}^3$ is

$$(u_1v_1+u_2v_2+u_3v_3)^2\leq(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2).$$

We can prove it using the same method:

$$\begin{aligned}
&(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2\\
=&[(u_1v_1)^2+(u_1v_2)^2+(u_1v_3)^2(u_2v_1)^2+(u_2v_2)^2+(u_2v_3)^2(u_3v_1)^2+(u_3v_2)^2+(u_3v_3)^2]\\
&-[(u_1v_1)^2+(u_2v_2)^2+(u_3v_3)^2+2(u_1v_1u_2v_2+u_2v_2u_3v_3+u_3v_3u_1v_1)]\\
=&[(u_1v_2)^2+(u_2v_1)^2-2(u_1v_1u_2v_2)]+[(u_2v_3)^2+(u_3v_2)^2-2(u_2v_2u_3v_3)]\\
&+[(u_3v_1)^2+(u_1v_3)^2-2(u_3v_3u_1v_1)]\\
=&(u_1v_2-u_2v_1)^2+(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2\geq 0.
\end{aligned}$$

66

$$\begin{aligned}
\|\operatorname{proj}_{\boldsymbol{u}}(\boldsymbol{v})\| &= \left\|\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}\right\|\\
&=\left|\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\right|\|\boldsymbol{u}\|\\
&=\frac{|\boldsymbol{u}\cdot\boldsymbol{v}|}{\|\boldsymbol{u}\|^2}\|\boldsymbol{u}\|\\
&=\frac{|\boldsymbol{u}\cdot\boldsymbol{v}|}{\|\boldsymbol{u}\|}\\
&\leq\|\boldsymbol{v}\|
\end{aligned}$$

therefore

$$|\boldsymbol{u}\cdot\boldsymbol{v}|\leq\|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

67 Since $\boldsymbol{u}$ and $\boldsymbol{v}-c\boldsymbol{u}$ are orthogonal,

$$\boldsymbol{u}\cdot(\boldsymbol{v}-c\boldsymbol{u})=0.$$

Therefore,

$$\boldsymbol{u}\cdot\boldsymbol{v}-c\boldsymbol{u}\cdot\boldsymbol{u}=0$$

hence

$$c=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}.$$

68 First, note that that, for any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\|+\|\boldsymbol{v}\|.$$

Proof:

$$\begin{aligned}(\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2-\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2-(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) \\ &= \|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|-[\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2+2\boldsymbol{u}\cdot\boldsymbol{v}] \\ &= 2(\|\boldsymbol{u}\|\|\boldsymbol{v}\|-\boldsymbol{u}\cdot\boldsymbol{v}) \geq 0\end{aligned}$$

since, due to the Cauchy-Schwarz Inequality,

$$\boldsymbol{u}\cdot\boldsymbol{v} \leq |\boldsymbol{u}\cdot\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

For $n=1$, the inequality becomes

$$\|\boldsymbol{v}_1\| \leq \|\boldsymbol{v}_1\|$$

which is true.

Assuming the inequality is satisfied for $n=k$, i.e.,

$$\|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k\| \leq \|\boldsymbol{v}_1\|+\cdots+\|\boldsymbol{v}_k\|.$$

According to the inequality proved above,

$$\|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k+\boldsymbol{v}_{k+1}\| \leq \|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k\|+\|\boldsymbol{v}_{k+1}\|.$$

Since

$$\|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k\| \leq \|\boldsymbol{v}_1\|+\cdots+\|\boldsymbol{v}_k\|$$

by the assumption, we get

$$\|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k+\boldsymbol{v}_{k+1}\| \leq \|\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k\|+\|\boldsymbol{v}_{k+1}\| \leq \|\boldsymbol{v}_1\|+\cdots+\|\boldsymbol{v}_k\|+\|\boldsymbol{v}_{k+1}\|$$

therefore the inequality is satisfied for $n=k+1$ the proof is complete.

Exercise 1.3

26 Let

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is the point which is equidistant from two points

$$P=\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad Q=\begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

Then we have the equation

$$(x - x_p)^2 + (y - y_p)^2 + (z - z_p)^2 = (x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2$$

which can be converted to

$$\begin{aligned}(x_p - x_q)x + (y_p - y_q)y + (z_p - z_q)y &= \frac{1}{2}\left[x_p^2 + y_p^2 + z_p^2 - (x_q^2 + y_q^2 + z_q^2)\right] \\ \boldsymbol{x} \cdot (\boldsymbol{p} - \boldsymbol{q}) &= \frac{1}{2}(\|\boldsymbol{p}\|^2 - \|\boldsymbol{q}\|^2)\end{aligned}$$

where $\boldsymbol{x}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are the point vectors corresponding to X, P and Q, respectively. Substituting the values for P and Q, we get

$$\boldsymbol{x} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 10.$$

42 (This proof is different from the one I gave in the class.)

First, note that distance equation on p.41 can be converted to

$$\frac{|\boldsymbol{n} \cdot \boldsymbol{p} - d|}{\|\boldsymbol{p}\|}$$

where

$$\boldsymbol{n} := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \boldsymbol{p} := \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

Let $\boldsymbol{p}$ be the point vector on the plane $\boldsymbol{n} \cdot \boldsymbol{x} = d_1$. Then the distance from $\boldsymbol{p}$ to the plane $\boldsymbol{n} \cdot \boldsymbol{x} = d_2$ is given as

$$\frac{|\boldsymbol{n} \cdot \boldsymbol{p} - d_2|}{\|\boldsymbol{n}\|}.$$

Since $\boldsymbol{p}$ is on the plane $\boldsymbol{n} \cdot \boldsymbol{x} = d_1$, we have $\boldsymbol{n} \cdot \boldsymbol{p} = d_1$ therefore the distance is

$$\frac{|d_1 - d_2|}{\|\boldsymbol{n}\|}.$$

47 Since $\boldsymbol{n}$ is orthogonal to every vector $\boldsymbol{p} = \boldsymbol{v} - c\boldsymbol{n} \in \mathcal{P}$,

$$\boldsymbol{n} \cdot (\boldsymbol{v} - c\boldsymbol{n}) = \boldsymbol{n} \cdot \boldsymbol{v} - c\boldsymbol{n} \cdot \boldsymbol{n} = 0 \quad \rightarrow \quad c = \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{\boldsymbol{n} \cdot \boldsymbol{n}}$$

therefore

$$\boldsymbol{p} = \boldsymbol{v} - c\boldsymbol{n} = \boldsymbol{v} - \left(\frac{\boldsymbol{n} \cdot \boldsymbol{v}}{\boldsymbol{n} \cdot \boldsymbol{n}}\right)\boldsymbol{n}.$$