

Mathematical Models for Engineering Problems and Differential Equations

Minho Kim

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Chapter 4: Linear Differential Equations of Order Greater Than One

Linear DE of order n

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$$

- ▶ $f_0(x), f_1(x), \dots, f_n(x)$ and $Q(x)$ are continuous functions of x defined on a common interval I and
- ▶ $f_n(x) \not\equiv 0$ in I . (The order is not n otherwise.)
- ▶ *Homogeneous* if $Q(x) \equiv 0$ and *nonhomogeneous* if $Q(x) \not\equiv 0$.

Lesson 18: Complex Numbers and Complex Functions.

Complex numbers

Definition

$$z = x + yj$$

- ▶ j is defined by the relation $j^2 = -1$
- ▶ x : real part of z
- ▶ y : imaginary part of z

Complex numbers (cont'd)

Conjugate of z

$$\bar{z} = x - yj$$

Absolute value of z

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Polar form

$$z = r(\cos \theta + j \sin \theta)$$

- ▶ $r = |z|$
- ▶ $\theta = \text{Arg}z$ (argument of z) is defined to be the *smallest* positive angle satisfying

$$\cos \theta = \frac{x}{|z|}, \quad \sin \theta = \frac{y}{|z|}$$

Algebra of complex numbers

- ▶ $z_1 + z_2 = ?$
- ▶ $z_1 - z_2 = ?$
- ▶ $z_1 z_2 = ?$
- ▶ $z_1 / z_2 = ?$

Exponential, trigonometric, and hyperbolic functions of complex numbers

By the *Taylor* (or *Maclaurin*) series expansions of e^x , $\sin x$ and $\cos x$,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$$

Exponential, trigonometric, and hyperbolic functions of complex numbers (cont'd)

$$e^0 = 1$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$\sin \theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$e^z \neq 0 \quad \text{for any value of } z$$

Exponential, trigonometric, and hyperbolic functions of complex numbers (cont'd)

Hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Lesson 19: Linear Independence of Functions. The Linear DE of Order n

Linear independence of functions

Definition

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$, each defined on a common interval I , is called *linearly dependent* on I , if there exists a set of constants c_1, c_2, \dots, c_n , not *all* zeros, such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$

for every x on I .

→ Any function in the set can be expressed as a linear combination of the rest.

Linear independence and linear DE

- ▶ A homogeneous linear DE has as many linearly independent solutions as the order of its equation. (Theorem 19.3)
- ▶ For a linear DE of order n , we need to
 - ▶ find out n solutions and
 - ▶ show these n solutions are linearly independent.

Existence and uniqueness theorem

Existence and uniqueness theorem

If $f_0(x), f_1(x), \dots, f_n(x)$ and $Q(x)$ are each continuous functions of x on a common interval I , and $f_n(x) \neq 0$ when x is on I , then the linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \dots + f_1(x)y' + f_0(x)y = Q(x)$$

has one and only one solution

$$y = y(x),$$

satisfying the set of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where x_0 is in I , and y_0, y_1, \dots, y_{n-1} are constants.

► Proof in Theorem 65.2

Three important properties

1. The homogeneous linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0 \quad (1)$$

has n linearly independent solutions $y_1(x), y_2(x), \cdots, y_n(x)$.

2. The linear combination of these n solutions

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_1, c_2, \cdots, c_n is a set of n arbitrary constants, is also a solution of (1). (*complementary function* of (1))

3. The function

$$y(x) = y_c(x) + y_p(x),$$

where $y_p(x)$ is a particular solution of the nonhomogeneous linear differential equation corresponding to (1), namely

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x), \quad (2)$$

is an n -parameter family of solutions of (2).

Overview of methods

1. Characteristic equation (Lesson 20)
 - ▶ *Homogeneous* linear DE of order n with constant coefficients
 - ▶ Three cases
2. Method of undetermined coefficients (Lesson 21)
 - ▶ *Nonhomogeneous* linear DE of order n with *constant* coefficients
 - ▶ $Q(x)$ consists of a sum of terms each of which has a *finite* number of linearly independent derivatives, e.g., a , x^k , e^{ax} , $\sin ax$, $\cos ax$, etc. (a constant, k positive integer)
 - ▶ Three cases
3. Variation of parameters (Lesson 22)
 - ▶ $Q(x)$ has an *infinite* number of linearly independent derivatives.
4. Reduction of order method (Lesson 23)
 - ▶ *Nonhomogeneous* linear DE of order n with *nonconstant* coefficients

Lesson 20: Solution of the Homogeneous Linear DE of Order n with Cons

Method #1: Characteristic equation

- ▶ For *homogeneous* linear DE of order n with *constant* coefficients ($a_n \neq 0$):

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

- ▶ If we assume that a possible solution has the form

$$y = e^{mx},$$

we get the *characteristic equation*

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

- ▶ Three cases
 1. All roots are distinct and real. (Lesson 20B)
 2. All roots are real but some repeat. (Lesson 20C)
 3. All roots are imaginary. (Lesson 20D)

Case 1: All roots are real and distinct

The general solution is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Why?

Case 2: All roots are real but some repeat.

- ▶ Find $u(x)$ such that $y_c = u(x)e^{ax}$ is a solution of the DE.
→ Find $u(x)$ for 2nd order case!
- ▶ If the characteristic equation has a root $m = a$ of multiple k , then $e^{ax}, xe^{ax}, \dots, x^{k-1}e^{ax}$ are k linearly independent solutions.

Case 3: All roots are imaginary.

The general solution of a linear DE of order 2, whose characteristic equation has the conjugate roots $\alpha \pm \beta j$, can be written in any of the following forms:

- ▶ $y_c = c_1 e^{(\alpha + \beta j)x} + c_2 e^{(\alpha - \beta j)x}$
- ▶ $y_c = c^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$
- ▶ $y_c = ce^{\alpha x} \sin(\beta x + \delta)$
- ▶ $y_c = ce^{\alpha x} \cos(\beta x - \delta)$

Lesson 21: Solution of the Nonhomogeneous Linear DE of Order n with C

Method #2: Method of undetermined coefficients

- ▶ For *nonhomogeneous* linear DE with constant coefficients.
- ▶ Can be used only if $Q(x)$ consists of a sum of terms each of which has a finite number of linearly independent derivatives, e.g., a , x^k , e^{ax} , $\sin ax$, $\cos ax$, etc. (a constant, k positive integer)
- ▶ Three cases:
 1. When no term in $Q(x)$ is the same as a term in y_c
 2. When $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ of y_c (k nonnegative integer).
 3. When both of the following conditions are fulfilled:
 - ▶ The characteristic equation of the given DE has an r multiple root.
 - ▶ $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ in y_c , where $u(x)$ was obtained from the r multiple root.

Case 1

When no term in $Q(x)$ is the same as a term in y_c
→ A particular solution y_p will be a linear combination of the terms in $Q(x)$ and *all* its linearly independent derivatives.

Case 2

When $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ of y_c (k nonnegative integer).

→ A particular solution y_p will be a linear combination of $x^{k+1}u(x)$ and all its linearly independent derivatives (ignoring constant coefficients). If in addition $Q(x)$ contains terms which belong to Case 1, then the proper terms called for by this case must be included in y_p .

Case 3

When both of the following conditions are fulfilled:

- ▶ The characteristic equation of the given DE has an r multiple root.
- ▶ $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ in y_c , where $u(x)$ was obtained from the r multiple root.

→ A particular solution y_p will be a linear combination of $x^{k+r}u(x)$ and all its linearly independent derivatives. If in addition $Q(x)$ contains terms which belong to Case 1 and 2, then the proper terms called for by these cases must also be added to y_p .

Lesson 22: Solution of the Nonhomogeneous Linear DE by the Method of

Method #3: Method of variation of parameters

- ▶ $Q(x)$ contains terms whose linearly independent derivatives are *infinite* in number.
- ▶ A particular solution

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

of the nonhomogeneous linear DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = Q(x), \quad a_n \neq 0$$

can be obtained from n linearly independent solutions of its related homogeneous equation, y_1, y_2, \dots, y_n , and u'_1, u'_2, \dots, u'_n are the functions obtained by solving simultaneously the following set of equations:

$$u'_1 y_1 + u'_2 y_2 + \cdots + u'_n y_n = 0$$

$$u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n = 0$$

.....

$$u'_1 y_1^{(n-1)} + u'_2 y_2^{(n-1)} + \cdots + u'_n y_n^{(n-1)} = \frac{Q(x)}{a_n}.$$

Lesson 23: Solution of the Linear DE with Nonconstant Coefficients. Red

Method #4: Reduction of order method

- ▶ For the homogeneous linear DE

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0$$

with nonconstant coefficients, we can find one independent solution if the other $n - 1$ independent solutions are known.

- ▶ For second order DE, an independent solution y_2 has the form

$$y_2(x) = y_1(x) \int u(x) dx$$

where $y_1(x)$ is the known solution and

$$u(x) = \frac{e^{-\int \frac{f_1(x)}{f_2(x)} dx}}{y_1^2}.$$

A particular solution can be found by substituting $y_2(x)$ in the nonhomogeneous linear DE.