Solution for homework #3-2

June 18, 2012

- Excercises 3-2

  31 $(A^r)^T = (AA\cdots AA)^T = A^T A^T \cdots A^T A^T = (A^T)^r$.

  34 (i) $(AA^T)^T = (A^T)^T A^T = AA^T$.

  (ii) $(A^T A)^T = A^T (A^T)^T = A^T A$.

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  17(b) Since $A^{-1}B^{-1} = (BA)^{-1}$, $(AB)^{-1} = (BA)^{-1}$ therefore $AB = BA$.
  In other words, "$(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$."

  42 (a) Multiplying A^{-1} on the left of both sides,

  $$A^{-1}AB = B = A^{-1}O = O.$$

  (b)

  $$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

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  $$\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} + BO & A(-A^{-1}BD^{-1}) + BD^{-1} \\ OA^{-1} + DO & O(-A^{-1}BD^{-1}) + DD^{-1} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

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  $$\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3-(3/2)R_1]{R_2-2R_1} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 1 & 11/2 \end{bmatrix} \qquad \left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & * & 1 \end{bmatrix} \right)$$

  $$\xrightarrow{R_3-(-1/4)R_2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} \qquad \left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & -1/4 & 1 \end{bmatrix} \right)$$

  Therefore

  $$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & -1/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

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  $$\begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

  Pre-multiplying the permutation

  $$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

first, the Gaussian elimination becomes

$$PA = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4-(-1)R_2} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & * & 1 \end{bmatrix})$$

$$\xrightarrow{R_4-0R_2} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad (L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix})$$

Threrefore

$$A = P^T LU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow[R_3-R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow[R_2-R_3]{(-1/2)R_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i)

$$\text{row}(A) = \text{span}\left(\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\right)$$

(ii)

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \right)$$

(iii) The solution of the homogeneous system is

$$\begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

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$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3+(1/3)R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 4/3 & 1 \end{bmatrix}$$

Therefore,

$$\text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = \text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

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$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & -3 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}\right) \neq \mathbb{R}^3,$$

the vectors do not span $\mathbb{R}^3$.

58 For any $\boldsymbol{x} \in \text{col}(AB)$, there exists a vector $\boldsymbol{y}$ such that $\boldsymbol{x} = (AB)\boldsymbol{y}$. Since $(AB)\boldsymbol{y} = A(B\boldsymbol{y})$, $\boldsymbol{x} \in \text{col}(A)$. Threrefore $\text{col}(AB) \subseteq \text{col}(A)$ and hence

$$\dim \text{col}(AB) = \text{rank}(AB) \leqslant \dim \text{col}(A) = \text{rank}(A).$$

63 For any vector $\boldsymbol{x} \in \operatorname{col}(A)$, there exists a vector $\boldsymbol{y}$ such that $\boldsymbol{x} = A\boldsymbol{y}$. Therefore, $A\boldsymbol{x} = A^2\boldsymbol{y} = O\boldsymbol{y} = \mathbf{0}$ and hence $\boldsymbol{x} \in \operatorname{null}(A)$. In other words, $\operatorname{col}(A) \subseteq \operatorname{null}(A)$ and hence

$$\dim \operatorname{col}(A) = \operatorname{rank}(A) \leqslant \dim \operatorname{null}(A) = \operatorname{nullity}(A).$$

On the other hand, by the rank theorem,

$$\operatorname{nullity}(A) = n - \operatorname{rank}(A) \geqslant \operatorname{rank}(A)$$

therefore

$$\operatorname{rank}(A) \leqslant n/2.$$

- Excercises 3-6

42 (a) From Example 3.59,

$$[P] = \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

therefore

$$[P \circ P] = [P]^2 = \frac{1}{(d_1^2 + d_2^2)^2} \begin{bmatrix} d_1^4 + (d_1 d_2)^2 & d_1 d_2 (d_1^2 + d_2^2) \\ d_1 d_2 (d_1^2 + d_2^2) & (d_1 d_2)^2 + d_2^4 \end{bmatrix} = \frac{d_1^2 + d_2^2}{(d_1^2 + d_2^2)^2} \begin{bmatrix} d_1^2 + d_2^2 & d_1 d_2 \\ d_1 d_2 & d_1^2 + d_2^2 \end{bmatrix} = P$$

(b) Since

$$[P] \begin{bmatrix} d_2 \\ -d_1 \end{bmatrix} = \mathbf{0},$$

$\operatorname{nullity}([P]) \neq 0$ and hence $[P]$ is not invertible.

44 Let ℓ be a line with its vector form

$$\boldsymbol{p} + t\boldsymbol{d}.$$

Then

$$[T](\boldsymbol{p} + t\boldsymbol{d}) = [T]\boldsymbol{p} + t[T]\boldsymbol{d}$$

therefore the line ℓ is transformed to another line of which vector form is

$$[T]\boldsymbol{p} + t[T]\boldsymbol{d}.$$