Homework #3 Solution

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Excercise 3.1

19 Below is the data given in the problem.

	W.H. #1	W.H. #2
doohichies	200	75
gizmos	150	100
widgets	100	125

(a)

	doohickies	gizmos	widgets
by truck	\$1.50	\$1.00	\$2.00
by train	\$1.75	\$1.50	\$1.00

(b)

Table 1: (a) Number of units of each product shipped to each warehouse and (b) cost of shipping one unit of each product by each transportation.

Let

$$A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$$

then the cost of shipping products to each warehouse can be computed as

$$BA = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$$

	warehouse 1	warehouse 2
by truck	\$650.00	\$462.50
by train	\$675.00	\$406.25

Table 2: Cost of shipping products to each warehouse.

29 The statement "columns of B are linearly dependent" is equivalent to the statement "the linear system $B\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution." Now

let $\boldsymbol{u} \neq \mathbf{0}$ be a non-trivial solution of $B\boldsymbol{x} = \mathbf{0}$. Then, since $(AB)\boldsymbol{u} = A(B\boldsymbol{u}) = \mathbf{0}$, $\boldsymbol{u}$ is a non-trivial solution of the linear system $(AB)\boldsymbol{x} = \mathbf{0}$. Therefore the columns of AB are linearly dependent.

38 Note that, due to the trigonometric equations, (See `http://en.wikipedia.org/wiki/Trigonometric_equation`)

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

(a) $A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$. By the trigonometric equations,

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

therefore

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

(b) First, it is straightforward to show that the equation holds for $n = 1$. Assuming the equation holds for $n = k$,

$$\begin{aligned} A^{k+1} = A^k A &= \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix} \end{aligned}$$

due to the trigonometric equation above and the proof is complete.

Excercise 3.2

27 Let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

(a) $AB = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix}$.

(b) $BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$.

For B to commute to any matrix A, the equation $AB = BA$ must hold for any $x, y, z, w \in \mathbb{R}$. In other words,

$$\begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} cy - bz & bx + (d-a)y - bw \\ -cx + cw + (a-d)z & -cy + bz \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for any $x, y, z, w \in \mathbb{R}$. The solution is therefore $c = 0$, $d = 0$ and $a = d$. In other words, the matrix B must be a scalar multiple of identity matrix.

29 Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are both upper triangular matrices and $C = [c_{ij}] = AB$. Due to the definition of matrix multiplication,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

For $i > j$,

(a) $a_{ik} = 0$ for $1 \leq k < i$ due to the definition of triangular matrices, therefore

$$\sum_{k=1}^{i-1} a_{ik} b_{kj} = 0.$$

(b) $b_{kj} = 0$ for $i \leq k \leq n$ due to the definition of triangular matrices, therefore

$$\sum_{k=i}^{n} a_{ik} b_{kj} = 0.$$

Therefore, for $i > j$,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{n} a_{ik} b_{kj} = 0$$

hence $C = [c_{ij}]$ is an upper triangular matrix.

33 First, the equality holds for $n = 1$ since

$$(A_1)^T = A_1^T.$$

Assume that the equality holds for $n = k$, i.e.,

$$(A_1 \cdots A_k)^T = A_k^T \cdots A_1^T.$$

Then

$$\begin{aligned}(A_1 \cdots A_k A_{k+1})^T &= ((A_1 \cdots A_k) A_{k+1})^T = A_{k+1}^T (A_1 \cdots A_k)^T = A_{k+1}^T (A_k^T \cdots A_1^T) \\ &= A_{k+1}^T A_k^T \cdots A_1^T\end{aligned}$$

therefore the equality holds for all n.

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$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

47 By the Excercise 44(a) and 45,

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$$

but

$$\mathrm{tr}(I_2) = 2$$

therefore the equality cannot hold.

Excercise 3.3

23

$$\begin{aligned}
&ABXA^{-1}B^{-1} = I + A \\
\to &ABXA^{-1} = (I + A)B \\
\to &ABX = (I + A)BA \\
\to &BX = A^{-1}(I + A)BA \\
\to &X = B^{-1}A^{-1}(I + A)BA = (B^{-1}A^{-1} + B^{-1})BA = B^{-1}A^{-1}BA + A
\end{aligned}$$

44(b)

$$A^2 = A \to A^2A^{-1} = AA^{-1} \to A = I.$$

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$$\begin{aligned}
(A^{-1})^T &= (A^T)^{-1} && \text{(Theorem 3.9(d))} \\
&= A^{-1} && \text{(Since } A = A^T\text{)}
\end{aligned}$$

therefore A^{-1} is symmetric.

47 Note that Theorem 3.9(c) requires that both A and B are invertible. Therefore we cannot directly use Theorem 3.9(a).

(a) Since AB is invertible,

$$(AB)(AB)^{-1} = I \to A(B(AB)^{-1}) = I$$

therefore there is a matrix, $C := B(AB)^{-1}$, such that

$$AC = I.$$

By Theorem 3.13, A is invertible.

(b) Again, since AB is invertible,

$$(AB)^{-1}(AB) = I \to ((AB)^{-1}A)B = I$$

therefore there is a matrix, $D := (AB)^{-1}A$, such that

$$DB = I.$$

By Theorem 3.13, B is invertible.

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$$\begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} -(BD^{-1}C)^{-1} & (BD^{-1}C)^{-1}BD^{-1} \\ D^{-1}C(BD^{-1}C)^{-1} & D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} BD^{-1}C(BD^{-1}C)^{-1} & B(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \\ -C(BD^{-1}C)^{-1} + DD^{-1}C(BD^{-1}C)^{-1} & C(BD^{-1}C)^{-1}BD^{-1} + D(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \end{bmatrix}$$

(a)

$$BD^{-1}C(BD^{-1}C)^{-1} = (BD^{-1}C)(BD^{-1}C)^{-1} = I.$$

(b)

$$\begin{aligned} B(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) &= BD^{-1} - BD^{-1}C(BD^{-1}C)^{-1}BD^{-1} \\ &= BD^{-1} - (BD^{-1}C)(BD^{-1}C)^{-1}BD^{-1} \\ &= BD^{-1} - BD^{-1} = O. \end{aligned}$$

(c)

$$-C(BD^{-1}C)^{-1} + DD^{-1}C(BD^{-1}C)^{-1} = -C(BD^{-1}C)^{-1} + C(BD^{-1}C)^{-1} = O.$$

(d)

$$\begin{aligned} &C(BD^{-1}C)^{-1}BD^{-1} + D(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \\ =&C(BD^{-1}C)^{-1}BD^{-1} + I - DD^{-1}C(BD^{-1}C)^{-1}BD^{-1} \\ =&I. \end{aligned}$$

Therefore the result is

$$\begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$$

Excercise 3.5

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$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{\substack{R_1/2 \\ R_2+R_1 \\ R_3-R_1}} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 1 & 3 & 7/2 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) From the reduced row echelon form, a basis for row(A) is

$$\left\{ \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 7/2 \end{bmatrix} \right\}.$$

(b) Since the 1st and 3rd column has leading 1's, a basis for col(A) is the set of the 1st and 3rd column vectors of A:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(c) Since the solution of the linear system $A\boldsymbol{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 &= 2x_2 - x_4 - \frac{1}{2}x_5 \\ x_3 &= -3x_4 - \frac{7}{2}x_5, \end{aligned}$$

with the free parameters $x_2 = s$, $x_4 = t$, and $x_5 = u$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 4t - \frac{1}{2}u \\ s \\ -3t - \frac{7}{2}u \\ t \\ u \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} u.$$

Therefore, a basis for null(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

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$$\begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 + 2R_1 \\ R_3 - aR_1}} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix}$$

(a) If $a \neq -1$,

$$\xrightarrow{\substack{R_2/4(a+1) \\ R_1 - 2R_2 \\ R_3 + 2(a+1)R_2}} \begin{bmatrix} 1 & 0 & a-1 \\ 0 & 1 & 1/2 \\ 0 & 0 & (2-a)(1+a) \end{bmatrix}$$

therefore

$$\operatorname{rank}(A) = \begin{cases} 2 & \text{if } a = 2 \\ 3 & \text{if } a \neq 2 \end{cases}$$

(b) If $a = -1$,

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore $\operatorname{rank}(A) = 1$

Overall,

$$\operatorname{rank}(A) = \begin{cases} 1 & \text{if } a = -1 \\ 2 & \text{if } a = 2 \\ 3 & \text{if } a \notin \{-1, 2\}. \end{cases}$$

50 We need to find the solution of the linear system

$$\begin{bmatrix} 3 & 5 \\ 1 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Since

$$\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{array}\right] \xrightarrow{\substack{R_1/3 \\ R_2 - R_1 \\ R_3 - 4R_1}} \left[\begin{array}{cc|c} 1 & 5/3 & 1/3 \\ 0 & -2/3 & 8/3 \\ 0 & -2/3 & 8/3 \end{array}\right]$$

$$\xrightarrow{\substack{R_2/(-2/3) \\ R_1 - 5/3R_2 \\ R_3 + (2/3)R_2}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

therefore,

$$7 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \boldsymbol{w}$$

and

$$[\boldsymbol{w}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

55 Due to the definition of null space, for any column vector $\boldsymbol{v} \in \operatorname{null}(A)$,

$$A\boldsymbol{v} = \mathbf{0}.$$

A row vector $\boldsymbol{u} \in \operatorname{row}(A)$ can be written as

$$\boldsymbol{u} = \boldsymbol{y}A$$

for some $\boldsymbol{y} \in \mathbb{R}^m$. Since

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}\boldsymbol{v} = (\boldsymbol{y}A)\boldsymbol{u} = \boldsymbol{y}(A\boldsymbol{u}) = \boldsymbol{y}\mathbf{0} = 0,$$

the proof is complete.

57(a) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$ hence $AB \in \mathbb{R}^{m\times k}$.

For any vector $\boldsymbol{u} \in \text{null}(B)$, $B\boldsymbol{u} = \mathbf{0}$, hence $(AB)\boldsymbol{u} = A(B\boldsymbol{u}) = A\mathbf{0} = \mathbf{0}$. Therefore, $\text{null}(B) \subset \text{null}(AB)$ hence $\text{nullity}(B) \leq \text{nullity}(AB)$. Due to The Rank Theorem,

$$\text{rank}(B) + \text{nullity}(B) = k = \text{rank}(AB) + \text{nullity}(AB).$$

Since $\text{nullity}(B) \leq \text{nullity}(AB)$, $\text{rank}(B) \geq \text{rank}(AB)$ and the proof is complete.

59 (a) (i) Due to the Excercise 57(a),

$$\text{rank}(UA) \leq \text{rank}(A).$$

(ii) Due to the Excercise 57(a),

$$\text{rank}(A) = \text{rank}(U^{-1}(UA)) \leq \text{rank}(UA).$$

Therefore $\text{rank}(A) \leq \text{rank}(UA) \leq \text{rank}(A)$ hence $\text{rank}(UA) = \text{rank}(A)$.

(b) (i) Due to the Excercise 58(a),

$$\text{rank}(AV) \leq \text{rank}(A).$$

(ii) Due to the Excercise 58(a),

$$\text{rank}(A) = \text{rank}((AV)V^{-1}) \leq \text{rank}(AV).$$

Therefore, $\text{rank}(A) = \text{rank}(AV)$.

60 Note that we need the condition that $A \neq O$ (zero matrix)

(a) $A = \boldsymbol{u}\boldsymbol{v}^T$ with $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n \Rightarrow \text{rank}(A) = 1$
Due to Excercise 58(a),

$$\text{rank}(A) = \text{rank}(\boldsymbol{u}\boldsymbol{v}^T) \leq \text{rank}(\boldsymbol{u}) = 1.$$

Since $A \neq O$, $\text{rank}(A) \geq 1$. Therefore, $\text{rank}(A) = 1$.

(b) $\text{rank}(A) = 1 \Rightarrow A = \boldsymbol{u}\boldsymbol{v}^T$ with $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$
Since $\text{rank}(A) = 1$, there is only one vector in the basis of $\text{row}(A)$ hence all the rows of A are scalar multiples of a row vector, say, $\boldsymbol{x} \in \mathbb{R}^n$. Therefore,

$$A = \begin{bmatrix} c_1\boldsymbol{x} \\ \vdots \\ c_m\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \boldsymbol{x}$$

hence

$$A = \boldsymbol{u}\boldsymbol{v}^T$$

where

$$\boldsymbol{u} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{x}^T.$$

Excercise 3.6

17 By

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

a vector is transformed as follows:

$$D\boldsymbol{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}.$$

18 Due to Example 3.59 (p.216-217), the projection matrix ($d_1 = d_2 = 1$)

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

26 (b) From the parallelogram in Figure 3.14, we can see that

$$\boldsymbol{x} + F_l(\boldsymbol{x}) = 2P_l(\boldsymbol{x})$$

therefore

$$F_l(\boldsymbol{x}) = 2P_l(\boldsymbol{x}) - \boldsymbol{x}.$$

Due to Example 3.59,

$$[P_l] = \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

therefore

$$\begin{aligned} [F_l] = 2[P_l] - I &= \frac{2}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} 2d_1^2 - (d_1^2 + d_2^2) & 2d_1 d_2 \\ 2d_1 d_2 & 2d_2^2 - (d_1^2 + d_2^2) \end{bmatrix} \\ &= \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 - d_2^2 & 2d_1 d_2 \\ 2d_1 d_2 & -d_1^2 + d_2^2 \end{bmatrix}. \end{aligned}$$

(c) We can decompose the transformation as the following three steps:

(i) Rotate both the line l and the vector $\boldsymbol{x}$ by the angle $-\theta$ such that the line l coincides with the x-axis. Let

$$R_1 := \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

be the rotation matrix.

(ii) Reflect rotated vector $R_{-\theta}\boldsymbol{x}$ with respect to the x-axis. Due to the Example 3.56, the reflection matrix is

$$F := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(iii) Rotate the reflected vector by the angle θ. Let

$$R_2 := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

be the rotation matrix.

Therefore

$$\begin{aligned}
[F_l] = R_2 F R_1 &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}.
\end{aligned}$$

Due to the trigonometic equations,

$$[F_l] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

40 Again, the trigonometric equations says

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[R_\alpha][R_\beta] &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \\
&= [R_{\alpha+\beta}].
\end{aligned}$$

41 Let θ_m be the angle between m and the positive x-axis, and let θ_l be the angle between l and the positive x-axis. Then, due to the Excercise 26(c),

$$[F_m] = \begin{bmatrix} \cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & -\cos 2\theta_m \end{bmatrix} \quad \text{and} \quad [F_l] = \begin{bmatrix} \cos 2\theta_l & \sin 2\theta_l \\ \sin 2\theta_l & -\cos 2\theta_l \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
[F_m \circ F_l] = [F_m][F_l] &= \begin{bmatrix} \cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & -\cos 2\theta_m \end{bmatrix} \begin{bmatrix} \cos 2\theta_l & \sin 2\theta_l \\ \sin 2\theta_l & -\cos 2\theta_l \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta_m \cos 2\theta_l + \sin 2\theta_m \sin 2\theta_l & \cos 2\theta_m \sin 2\theta_l - \sin 2\theta_m \cos 2\theta_l \\ \sin 2\theta_m \cos 2\theta_l - \cos 2\theta_m \sin 2\theta_l & \sin 2\theta_m \sin 2\theta_l + \cos 2\theta_m \cos 2\theta_l \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta_m - \theta_l) & \sin 2(-\theta_m + \theta_l) \\ \sin 2(\theta_m - \theta_l) & \cos 2(\theta_m - \theta_l) \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta_m - \theta_l) & -\sin 2(\theta_m - \theta_l) \\ \sin 2(\theta_m - \theta_l) & \cos 2(\theta_m - \theta_l) \end{bmatrix} = [R_{2(\theta_m - \theta_l)}].
\end{aligned}$$

Since $\theta = \theta_m - \theta_l$, the equality holds.

54 Let R_T be the range of the linear transformation T. We show that $R_T \subset \text{col}([T])$ and $\text{col}([T]) \subset R_T$. Let $\boldsymbol{t}_1, \cdots, \boldsymbol{t}_n$ be the columns of $[T]$ such that

$$[T] = \left[\begin{array}{c:c:c} \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \end{array}\right].$$

(a) $R_T \subset \text{col}([T])$

For any vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$[T]\boldsymbol{x} = \left[\begin{array}{c:c:c} \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{t}_1 + \cdots x_n\boldsymbol{t}_n \in \text{col}([T])$$

(b) $\text{col}([T]) \subset R_T$

Due to the definition of the column space, any vector $\boldsymbol{u} \in \text{col}([T])$ can be written as a linear combination of the columns of $[T]$:

$$\boldsymbol{u} = y_1\boldsymbol{t}_1 + \cdots + y_n\boldsymbol{t}_n$$

therefore

$$\boldsymbol{u} = \left[\begin{array}{c:c:c} \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \end{array}\right] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in R_T.$$