

# Linear Algebra

## Chapter 4: Eigenvalues and Eigenvectors

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# Dynamical System

- ▶ For  $A \in \mathbb{R}^{n \times n}$ ,  $\lim_{k \rightarrow \infty} A^k = ?$
- ▶ Try the Octave demos yourselves!

1. Complete graphs (K4.m)
2. Petersen graph (Petersen.m)
3. Cyclic graphs
  - 3.1 Odd number of nodes (C5.m)
  - 3.2 Even number of nodes (C6.m)
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  - ▶ Steady state vector

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# Eigenvalue Problem

- ▶ For a square matrix  $A$ , are there nonzero vectors  $x$  such that  $Ax$  is just a scalar multiplication of  $x$ ? In other words, which nonzero vectors satisfy  $Ax = \lambda x$ ? ( $\lambda \in \mathbb{R}$ ) → “**Eigenvalue problem**”
- ▶ eigen- [áigən]: “own” or “characteristic of”

## Definition

Let  $A$  be an  $n \times n$  matrix. A scalar  $\lambda$  is called an **eigenvalue** of  $A$  if there is a nonzero  $x$  such that  $Ax = \lambda x$ . Such a vector  $x$  is called an **eigenvector** of  $A$  corresponding to  $\lambda$ .

- ▶ Why are they important?
- ▶ Do they exist for *any* matrix?
- ▶ Is there only one eigenvector for an eigenvalue?
- ▶ Is there only one eigenvalue for an eigenvector?
- ▶ Given an eigenvalue, how can we find the corresponding eigenvectors? → Example 4.2
- ▶ How can we find eigenvalues?

# Eigenspace

- ▶ Example 4.2

→ “The set of all eigenvectors corresponding to an eigenvalue  $\lambda$  of an  $n \times n$  matrix  $A$  is just the set of *nonzero* vectors in  $\text{null}(A - \lambda I)$ .”

## Definition

Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be an eigenvalue of  $A$ . The collection of all eigenvectors corresponding to  $\lambda$ , together with the zero vector, is called the **eigenspace** of  $\lambda$  and is denoted by  $E_\lambda$ .

- ▶  $E_\lambda = \text{null}(A - \lambda I)$

# Geometric Interpretation of Eigenvectors

- ▶  $Ax$  and  $\lambda x$  are parallel, i.e.,  $x$  is an eigenvector of  $A$  iff  $T_A$  transforms  $x$  into a parallel vector.
- ▶ Examples: Scaling, reflection (Ex 4.4), rotation
- ▶ Only the direction of an eigenvector matters. (Why?)  
→ Only unit vectors need to be considered. (Fig 4.7)



# Finding Eigenvalues

- ▶ For  $A \in \mathbb{R}^{n \times n}$ , the eigenvectors of  $\lambda$  are the nonzero vectors satisfying  $Ax = \lambda x$ .
  - $Ax - \lambda Ix = \mathbf{0} \rightarrow (A - \lambda I)x = \mathbf{0}$
  - The eigenspace  $E_\lambda$  is the non-trivial null space of  $A - \lambda I$ .
- ▶ For  $2 \times 2$  matrices,
  - a matrix has a non-trivial null space iff it is non-invertible.
  - a matrix is non-invertible iff its determinant is zero.
  - $\lambda$  is an eigenvalue of  $A$  iff  $\det(A - \lambda I) = 0$  (Example 4.5)
- ▶ Can be generalized to any square matrices.
  - What are the determinants for  $n \times n$  matrices?

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# Determinants

- ▶ Notation:  $\det A = |A|$
- ▶  $1 \times 1$  matrices

$$\det A = |a| = a \quad (\text{Not the absolute value})$$

- ▶  $2 \times 2$  matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- ▶  $3 \times 3$  matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

# Determinant of a $3 \times 3$ Matrix

## Definition

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ . Then the **determinant** of  $A$  is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

►  $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

►  $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

►  $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

## Determinant of a $3 \times 3$ Matrix (cont'd)

- ▶ With  $A_{ij}$  defined as the submatrix of  $A$  obtained by deleting row  $i$  and column  $j$ ,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^3 (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

- ▶  $\det A_{ij}$  is called the  $(i, j)$ -**minor** of  $A$ .
- ▶ Computed with respect to the first row.
  - Why row not column? Why the first row?
  - Can be generalized to columns or other rows (The Laplace Expansion Theorem)
- ▶ Another method (See (2) on p.264)

$$\begin{aligned}|A| &= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \\ &\quad - a_{31} a_{22} a_{13} - a_{32} a_{23} a_{11} - a_{33} a_{21} a_{12}\end{aligned}$$

# Determinants of $n \times n$ Matrices

## Definition

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix, where  $n \geq 2$ . Then the **determinant** of  $A$  is the scalar

$$\begin{aligned}\det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

- **Cofactor expansion along the first row:**

With  $(i, j)$ -**cofactor** of  $A$  defined as

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

the definition becomes

$$\det A = \sum_{j=1}^n a_{1j} C_{1j}.$$

# The Laplace Expansion Theorem

## Theorem 4.1: The Laplace Expansion Theorem

The determinant of an  $n \times n$  matrix  $A = [a_{ij}]$ , where  $n \geq 2$ , can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^n a_{ij}C_{ij}$$

(which is the **cofactor expansion along the  $i$ th row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^n a_{ij}C_{ij}$$

(the **cofactor expansion along the  $j$ th column**)

- ▶ Most useful when the matrix contains a row or column with lots of zeros. Why? (Example 4.11)

# Determinants of Triangular Matrices

- ▶ The Laplace expansion theorem is particularly useful when the matrix is (upper or lower) triangular.

## Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if  $A = [a_{ij}]$  is an  $n \times n$  triangular matrix then

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

- ▶ Why? (Example 4.12)



# Computing Determinants

- ▶ Laplace expansion is very inefficient! (p.268)
- ▶ The determinant of a triangular matrix can be easily found.
- ▶ We can compute the determinant of a matrix efficiently from its reduced form.
  - “How does the determinant change after elementary row operations?”

# Properties of Determinants

## Theorem 4.3

Let  $A = [a_{ij}]$  be a square matrix.

- a. If  $A$  has a zero row (column), then  $\det A = 0$ .
- b. If  $B$  is obtained by interchanging two rows (columns) of  $A$ , then  $\det B = -\det A$ .
- c. If  $A$  has two identical rows (columns), then  $\det A = 0$ .
- d. If  $B$  is obtained by multiplying a row (column) of  $A$  by  $k$ , then  $\det B = k\det A$ .
- e. If  $A$ ,  $B$ , and  $C$  are identical except that the  $i$ th row (column) of  $C$  is the sum of the  $i$ th rows (columns) of  $A$  and  $B$ , then  $\det C = \det A + \det B$ .
- f. If  $B$  is obtained by adding a multiple of one row (column) of  $A$  to another row (column), then  $\det B = \det A$ .

# Determinants of Elementary Matrices

- ▶ (b), (d) and (f) of the properties are related to elementary row operations.
- ▶ Example 4.13
- ▶ We can “mix and match” elementary row and column operations.
- ▶ What are the determinants of elementary matrices?

## Theorem 4.4

Let  $E$  be an  $n \times n$  elementary matrix.

- If  $E$  results from interchanging two rows of  $I_n$ , then  $\det E = -1$ .
  - If  $E$  results from multiplying one row of  $I_n$  by  $k$ , then  $\det E = k$ .
  - If  $E$  results from adding a multiple of one row of  $I_n$  to another row, then  $\det E = 1$ .
- ▶ Determinants of all the elementary matrices are nonzero.

# Determinants of Elementary Matrices (cont'd)

- If  $B = EA$ ,  $\det B = ?$

## Lemma 4.5

Let  $B$  be an  $n \times n$  matrix and let  $E$  be an  $n \times n$  elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

- How about  $\det(AB)$  when  $A$  is NOT an elementary matrix?  
→ Theorem 4.8 (p.272)

## Theorem 4.6

A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

# Determinants and Matrix Operations

How can we write the followings in terms of  $\det A$  and  $\det B$ ?

- ▶  $\det(kA) = ?$
- ▶  $\det(A + B) = ?$
- ▶  $\det(AB) = ?$
- ▶  $\det(A^{-1}) = ?$
- ▶  $\det(A^T) = ?$

$\det(kA)$  and  $\det(A + B)$

- $\det(kA)$

### Theorem 4.7

If  $A$  is an  $n \times n$  matrix, then

$$\det(kA) = k^n \det A$$

- See Theorem 4.3(d).
- $\det(A + B)$ 
  - $\det(A + B) = \det A + \det B?$  → No!
  - No general formula. (p.271)

$\det (AB)$ ,  $\det (A^{-1})$  and  $\det (A^T)$

- $\det (AB)$

**Theorem 4.8**

If  $A$  and  $B$  are  $n \times n$  matrices, then

$$\det (AB) = (\det A)(\det B)$$

- $\det (A^{-1})$

**Theorem 4.9**

If  $A$  is invertible, then

$$\det (A^{-1}) = \frac{1}{\det A}$$

- $\det (A^T)$

**Theorem 4.10**

For any square matrix  $A$ ,

$$\det A = \det A^T$$

# Cramer's Rule and the Adjoint

- ▶ What is the relation between determinants and the solution of a linear systems? → Cramer's rule (Theorem 4.11)
- ▶ What is the relation between determinants and the inverse of a matrix? → Adjoint (Theorem 4.12)
- ▶ Notation  
 $A_i(\mathbf{b})$ : the matrix obtained by replacing the  $i$ th column of  $A$  by  $\mathbf{b}$

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix}$$



# Cramer's Rule

## Theorem 4.11: Cramer's Rule

Let  $A$  be an invertible  $n \times n$  matrix and let  $\mathbf{b}$  be a vector in  $\mathbb{R}^n$ . Then the unique solution  $\mathbf{x}$  of the system  $A\mathbf{x} = \mathbf{b}$  is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A} \quad \text{for } i = 1, \dots, n$$

- Requires to compute determinants  
→ Computationally inefficient except small systems.

# Adjoint

1. What is the formula of the inverse of a matrix in terms of determinants?
2. What is the solution of the equation  $AX = I$ ?

$$A\mathbf{x}_1 = \mathbf{e}_1 \quad A\mathbf{x}_2 = \mathbf{e}_2 \quad \cdots \quad A\mathbf{x}_n = \mathbf{e}_n$$

3. By the Cramer's rule,  $x_{ij} = \frac{\det(A_i(\mathbf{e}_j))}{\det A}$
4.  $\det(A_i(\mathbf{e}_j)) = (-1)^{j+i} \det A_{ji} = C_{ji}$  (Why?)
5.  $A^{-1} = X = \frac{1}{\det A} [C_{ji}] = \frac{1}{\det A} [C_{ij}]^T = \frac{1}{\det A} \text{adj } A$

## Adjoint (cont'd)

### Theorem 4.12

Let  $A$  be an invertible  $n \times n$  matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- ▶  $\operatorname{adj} A := [C_{ji}] = [C_{ij}]^T$ : the **adjoint** (or **adjugate**) of  $A$ 
  - ▶  $C_{ij} := (-1)^{i+j} \det A_{ij}$ :  $(i, j)$ -cofactor of  $A$

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# Finding Eigenvalues

- ▶ How to compute eigenvalues of a matrix?
- ▶ How many eigenvalues does a matrix have?
- ▶  $\lambda$  is an eigenvalue of  $A$  iff  $\det(A - \lambda I) = 0$ .

The eigenvalues of a square matrix  $A$  are precisely the solutions  $\lambda$  of the equation

$$\det(A - \lambda I) = 0$$

- ▶ What does  $\det(A - \lambda I)$  look like?
  - A polynomial in  $\lambda$  of degree  $n$   
**(Characteristic polynomial of  $A$ )**
  - At most  $n$  distinct eigenvalues

# Finding Eigenvalues and Eigenvectors

## Procedure

Let  $A$  be an  $n \times n$  matrix.

1. Compute the characteristic polynomial  $\det(A - \lambda I)$  of  $A$ .
2. Find the eigenvalues of  $A$  by solving the characteristic equation  $\det(A - \lambda I) = 0$  for  $\lambda$ .
3. For each eigenvalue  $\lambda$ , find the null space of the matrix  $A - \lambda I$ . This is the eigenspace  $E_\lambda$ , the nonzero vectors of which are the eigenvectors of  $A$  corresponding to  $\lambda$ .
4. Find a basis for each eigenspace.
  - ▶ **Algebraic multiplicity** of an eigenvalue: multiplicity as a root of the characteristic equation.
  - ▶ **Geometric multiplicity** of an eigenvalue  $\lambda$ :  $\dim E_\lambda$
  - ▶ What's the relation between the algebraic & geometric multiplicities? (Example 4.18 & 4.19)  
→ Geometric multiplicity  $\leq$  Algebraic multiplicity  
(Lemma 4.26 on p.303)

# Eigenvalues of Triangular Matrices

- ▶ How does the characteristic equation look like if  $A$  is triangular? (See Theorem 4.2 on p.268)  
 $\rightarrow (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$

## Theorem 4.15

The eigenvalues of a triangular matrix are the entries on its main diagonal.

# What Does Eigenvalues Tell Us?

## Theorem 4.16

A square matrix  $A$  is invertible if and only if 0 is *not* an eigenvalue of  $A$ .

- ▶ Why?
- ▶ ...and there will be more (about the importance of eigenvalues).



# Fundamental Theorem of Invertible Matrices: Ver. 3

## Theorem 3.27

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent:

- a.  $A$  is invertible.
- b.  $Ax = b$  has a unique solution for every  $b$  in  $\mathbb{R}^n$ .
- c.  $Ax = 0$  has only the trivial solution.
- d. The reduced row echelon form of  $A$  is  $I_n$ .
- e.  $A$  is a product of elementary matrices.
- f.  $\text{rank}(A) = n$
- g.  $\text{nullity}(A) = 0$
- h. The column vectors of  $A$  are linearly independent.
  - i. The column vectors of  $A$  span  $\mathbb{R}^n$ .
  - j. The column vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- k. The row vectors of  $A$  are linearly independent.
  - l. The row vectors of  $A$  span  $\mathbb{R}^n$ .
- m. The row vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- n.  $\det A \neq 0$
- o. 0 is not an eigenvalue of  $A$ .

# Eigenvalue of $A^k$ and $A^{-1}$

## Theorem 4.18

Let  $A$  be a square matrix with eigenvalue  $\lambda$  and corresponding eigenvector  $x$ .

- a. For any positive integer  $k$ ,  $\lambda^k$  is an eigenvalue of  $A^k$  with corresponding eigenvector  $x$ .
  - b. If  $A$  is invertible, then  $1/\lambda$  is an eigenvalue of  $A^{-1}$  with corresponding eigenvector  $x$ .
  - c. For any integer  $k$ ,  $\lambda^k$  is an eigenvalue of  $A^k$  with corresponding eigenvector  $x$ .
- 
- ▶ Application: Computing  $A^k x$  where  $x$  is not an eigenvector of  $A$ . (Example 4.21)  
→ Is this possible for any  $x$ ?

## Computing $A^k x$

### Theorem 4.19

Suppose the  $n \times n$  matrix  $A$  has eigenvectors  $v_1, v_2, \dots, v_m$  with corresponding eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . If  $x$  is a vector in  $\mathbb{R}^n$  that can be expressed as a linear combination of these eigenvectors—say,

$$x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$$

then, for any integer  $k$ ,

$$A^k x = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \dots + c_m \lambda_m^k v_m$$

- ▶ When does it work for any  $x$ ?

### Theorem 4.20

Let  $A$  be an  $n \times n$  matrix and let  $\lambda_1, \lambda_2, \dots, \lambda_m$  be distinct eigenvalues of  $A$  with corresponding eigenvectors  $v_1, v_2, \dots, v_m$ . Then  $v_1, v_2, \dots, v_m$  are linearly independent.

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# Why Diagonalize Matrices?

- ▶ Triangular and diagonal matrices are good.
  - How can we relate a square matrix to a triangular or diagonal one keeping the eigenvalues?
- ▶ Gaussian elimination?
  - Eigenvalues are not preserved.
- ▶ Diagonalization

# Similar Matrices

## Definition

Let  $A$  and  $B$  be  $n \times n$  matrices. We say that  $A$  is **similar to**  $B$  if there is an invertible  $n \times n$  matrix  $P$  such that  $P^{-1}AP = B$ . If  $A$  is similar to  $B$ , we write  $A \sim B$ .

- ▶ Equivalent to “ $A = PBP^{-1}$ ” or “ $AP = PB$ .”
- ▶  $P$  depends on  $A$  and  $B$ . Is it unique?

## Theorem 4.21

Let  $A$ ,  $B$  and  $C$  be  $n \times n$  matrices.

- $A \sim A$ . (Reflexivity)
  - If  $A \sim B$ , then  $B \sim A$ . (Symmetry)
  - If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ . (Transitivity)
- ▶ **Equivalent relation**

## Similar Matrices (cont'd)

### Theorem 4.22

Let  $A$  and  $B$  be  $n \times n$  matrices with  $A \sim B$ . Then

- a.  $\det A = \det B$ .
  - b.  $A$  is invertible if and only if  $B$  is invertible.
  - c.  $A$  and  $B$  have the same rank.
  - d.  $A$  and  $B$  have the same characteristic polynomial.
  - e.  $A$  and  $B$  have the same eigenvalues.
- 
- ▶ The converse is not necessarily true. (See Remark on p.300.)
  - ▶ Useful when showing two matrices are not similar. (How?)

# Diagonalization

- ▶ Good if a square matrix is similar to a diagonal matrix.
- ▶ Is it always possible?
- ▶ How can we find  $P$ ?

## Definition

An  $n \times n$  matrix  $A$  is **diagonalizable** if there is a diagonal matrix  $D$  such that  $A$  is similar to  $D$ -that is, if there is an invertible  $n \times n$  matrix  $P$  such that  $P^{-1}AP = D$ .



## Diagonalization (cont'd)

- ▶ How can we find  $D$  and  $P$ ?

### Theorem 4.23

Let  $A$  be an  $n \times n$  matrix. Then  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.

More precisely, there exist an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$  if and only if the columns of  $P$  are  $n$  linearly independent eigenvectors of  $A$  and the diagonal entries of  $D$  are eigenvalues of  $A$  corresponding to the eigenvectors in  $P$  in the same order.

- ▶ Is a non-invertible matrix diagonalizable? (Example 4.26)

## Diagonalization (cont'd)

- ▶ How can we check if the eigenvectors are linearly independent? (See the 2nd remark below Example 4.26)

### Theorem 4.24

Let  $A$  be an  $n \times n$  matrix and let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be distinct eigenvalues of  $A$ . If  $\mathcal{B}_i$  is a basis for the eigenspace  $E_{\lambda_i}$ , then  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_k$  (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

- ▶ We don't have to check the linear independence of eigenvectors associated with different eigenvalues.

### Theorem 4.25

If  $A$  is an  $n \times n$  matrix with  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

- ▶ The converse is not always true.

# The Diagonalization Theorem

## Lemma 4.26

If  $A$  is an  $n \times n$  matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

- ▶ The sum of the algebraic multiplicities is always  $n$ .  
Therefore,  $A$  is diagonalizable when...?

## Theorem 4.27

Let  $A$  be an  $n \times n$  matrix whose distinct eigenvalues are  $\lambda_1, \lambda_2, \dots, \lambda_k$ . The following statements are equivalent:

- $A$  is diagonalizable.
  - The union  $\mathcal{B}$  of the bases of the eigenspaces of  $A$  (as in Theorem 4.24) contains  $n$  vectors.
  - The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.
- ▶ Computing  $A^k$  (Example 4.29)  
 $A^k = PD^kP^{-1}$  for all  $k \geq 1$ .

## Computing $A^k \mathbf{x}$

- ▶ When  $\mathbf{x}$  is an eigenvector of  $A$ . (Theorem 4.18)

$$A^k \mathbf{x} = \lambda^k \mathbf{x}$$

- ▶ When  $\mathbf{x}$  is a linear combination of the eigenvectors of  $A$ . (Theorem 4.19)

$$\mathbf{x} = \sum_{j=1}^m c_j \mathbf{v}_j \rightarrow A^k \mathbf{x} = \sum_{j=1}^m (c_j A^k \mathbf{v}_j) = \sum_{j=1}^m (c_j \lambda_j^k \mathbf{v}_j)$$

- ▶ When  $A$  is diagonalizable. (Example 4.29)

$$A^k \mathbf{x} = (PDP^{-1})^k \mathbf{x} = PD^k P^{-1} \mathbf{x}$$

- ▶ Otherwise... Good luck!

# Outline

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Introduction to Eigenvalues and Eigenvectors

Determinants

Eigenvalues and Eigenvectors of  $n \times n$  Matrices

Similarity and Diagonalization

**Iterative Methods for Computing Eigenvalues**

Applications and the Perron-Frobenius Theorem

# Outline

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# Markov Chains

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \rightarrow \quad \mathbf{x}_k = P^k \mathbf{x}_0$$

- ▶  $P$ : transition matrix
- ▶ All the components in each column of  $P$  add up to 1.  
(Why?)
- ▶  $\mathbf{x}_k$ : **state vector** cf) **probability vector**
- ▶ Here, we will see...
  - ▶ Steady state vector  $\mathbf{x}$ :  $P\mathbf{x} = \mathbf{x}$ .  
→ “Every Markov chain has a unique steady state vector.”  
(p.231)
  - ▶  $\lim_{k \rightarrow \infty} \mathbf{x}_k = \lim_{k \rightarrow \infty} P^k \mathbf{x}_0 = ?$   
→  $\lim_{k \rightarrow \infty} P^k \mathbf{x}_0 = \mathbf{x}$   
:  $\mathbf{x}_k$  converges to  $\mathbf{x}$  regardless of  $\mathbf{x}_0$ .

# Markov Chains (cont'd)

## Theorem 4.30

If  $P$  is the  $n \times n$  transition matrix of a Markov chain, then 1 is an eigenvalue of  $P$ .

- ▶ There always exists a vector  $x$  such that  $Px = x$ .  
→ There always exists a steady state vector. But is it unique?

## Theorem 4.31

Let  $P$  be an  $n \times n$  transition matrix with eigenvalue  $\lambda$ .

- $|\lambda| \leq 1$
- If  $P$  is regular and  $\lambda \neq 1$ , then  $|\lambda| < 1$ .
  - ▶ **Positive matrix:** All the entries are positive.
  - ▶ **Regular matrix:**  $P^k$  is positive for some  $k$ .
  - ▶ If  $P$  is regular,  $-1$  cannot be an eigenvalue.



## Markov Chains (cont'd)

### Lemma 4.32

Let  $P$  is a regular  $n \times n$  transition matrix. If  $P$  is diagonalizable, then the dominant eigenvalue  $\lambda_1 = 1$  has algebraic multiplicity 1.

- ▶ There is only one eigenvector (and its scalar multiplications) such that  $Px = x$ .  
→ the steady state vector is unique.

# Markov Chains (cont'd)

## Theorem 4.33

Let  $P$  be a regular  $n \times n$  transition matrix. Then as  $k \rightarrow \infty$ ,  $P^k$  approaches an  $n \times n$  matrix  $L$  whose columns are identical, each equal to the same vector  $x$ . This vector  $x$  is a steady state probability vector for  $P$ .

- ▶  $\lim_{k \rightarrow \infty} P^k = \begin{bmatrix} x & \cdots & x \end{bmatrix} =: L$   
→  $\lim_{k \rightarrow \infty} P^k$  converges. But what is  $\lim_{k \rightarrow \infty} P^k x_0 = ?$

## Theorem 4.34

Let  $P$  be a regular  $n \times n$  transition matrix, with  $x$  the steady state probability vector for  $P$ , as in Theorem 4.33. Then, for any initial probability vector  $x_0$ , the sequence of iterates  $x_k$  approaches  $x$ .

- ▶  $\lim_{k \rightarrow \infty} P^k x_0 = x$  for any  $x_0$  (initial probability vector)

# Population Growth

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

- ▶ Leslie matrix
- ▶ “The proportion of the population in each class is approaching a steady state.” (p.235)
  - There exists a vector such that  $Lx = \lambda x$  where  $\lambda > 0$ .

## Theorem 4.35

Every Leslie matrix has a unique positive eigenvalue and a corresponding eigenvector with positive components.