Mathematical Models for Engineering Problems and Differential Equations

Minho Kim

November 11, 2009

Table of contents

Chapter 5: Operators and Laplace Transform

Lesson 24: Differential and Polynomial Operators.

Operator

Definition

A mathematical device which converts one function into another.

Examples:

- differentiating operator: $f(x) \to f'(x)$
- integrating operator: $f(x) \to F(x) = \int_{x_0}^{x} f(t)dt$

Differential Operator

$$D^n y = y^{(n)}$$

Polynomial Operator

- A linear combination of differential operators of orders 0 to n

$$P(D) = a_0 + a_1 D + a_2 D + \cdots + a_n D^n, \quad a_n \neq 0$$

- $P(D)y := a_n y^{(n)} + \cdots + a_1 y' + a_0 y, \quad y_n \neq 0$ A linear differential equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = Q(x), \quad a_n \neq 0$$

  can be expressed as

$$P(D)y = Q(x).$$

- Linear: $P(D)(c_1 y_1 + c_2 y_2) = c_1 P(D) y_1 + c_2 P(D) y_2$
  1. If $P(D)y_1 = 0, \cdots, P(D)y_n = 0$ then $P(D)(c_1 y_1 + \cdots + c_n y_n) = 0$.
  2. If $P(D)y_c = 0$ and $P(D)y_p = Q(x)$ then $P(D)(y_c + y_p) = Q(x)$.

Polynomial Operator (cont'd)

- Principle of superposition:
  If $P(D)y_{1p} = Q_1(x)$, $P(D)y_{2p} = Q_2(x)$, $\cdots$, $P(D)y_{np} = Q_n(x)$, then
  $P(D)(y_{1p} + y_{2p} + \cdots + y_{np}) = Q_1(x) + Q_2(x) + \cdots + Q_n(x)$.

Algebraic Properties of Polynomial Operators

- $(P_1 + P_2)(D)y := P_1(D)y + P_2(D)y$
- $[h(x)P(D)]y := h(x)[P(D)y]$
- $[P_1(D)P_2(D)]y := P_1(D)[P_2(D)y]$
- Commutitative:

$$[P_1(D)P_2(D)]y = [P_2(D)P_2(D)]y$$

- Associative:

$$P_1(D)[P_2(D)P_3(D)] = [P_1(D)P_2(D)]P_3(D) = P_1(D)P_2(D)P_3(D)$$

- Distributive:

$$P_1(D)[P_2(D) + P_3(D)] = P_1(D)P_2(D) + P_1(D)P_3(D)$$

Factoring of Polynomial Operators

If

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \quad a_n \neq 0,$$

where $a_0, a_1, \cdots, a_n$ are constants, then

$$P(D) = a_n (D - r_1)(D - r_2) \cdots (D - r_n),$$

where $r_1, r_2, \cdots, r_n$ are the real or imaginary roots of the characteristic equation of $P(D)y = 0$.

Exponential Shift Theorem for Polynomial Operators

Theorem

If

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \quad a_n \neq 0$$

then

$$P(D)(u(x)e^{ax}) = e^{ax} P(D+a) u(x).$$

Corollary

$$(D-a)^n (u(x)e^{ax}) = e^{ax} D^n u(x)$$

Corollary

$$P(D)(ce^{ax}) = ce^{ax} P(a)$$

Solving a Linear D.E. by Means of Polynomial Operators

1. Factor $P(D)$ then the D.E. is expressed as

$$P(D)y = (D - r_1)(D - r_2)\cdots(D - r_n)y = Q(x)$$

2. Let $u_1 = (D - r_2)\cdots(D - r_n)y$ then the D.E. becomes

$$(D - r_1)u_1 = Q(x).$$

3. Find u_1 then we get a D.E.

$$(D - r_2)(D - r_3)\cdots(D - r_n)y = u_1(x).$$

4. Let $u_2 = (D - r_3)\cdots(D - r_n)y$ then the D.E. becomes

$$(D - r_2)u_2 = u_1(x).$$

5. Repeat.

Lesson 25: Inverse Operators.

Inverse Operators

Definition

Let $P(D)y = Q(x)$ where $Q(x)$ is the special function consisting only of such terms as b, x^k, $\sin ax$, $\cos ax$, and a finite number of combinations of such terms. (a, b constants and k a positive integer) Then the *inverse operator* of $P(D)$, written as $P^{-1}(D)$ or $1/P(D)$, is defined as an operator which, when operating on $Q(x)$, will give the particular solution y_p of the D.E. that contains no constant multiples of a term in the complementary function y_c, i.e.,

$$P^{-1}(D)Q(x) = y_p \quad \text{or} \quad \frac{1}{P(D)}Q(x) = y_p.$$

- What does "$D^{-n}Q(x)$" mean?
- $P^{-1}(D)(0) = ?$
- $P(D)[P^{-1}(D)Q] = ?$

Solving "$P(D)y = Q(x)$" by Means of Inverse Operators

1. $Q(x) = bx^k$ and $P(D) = D - a_0$
2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$
3. $Q(x) = be^{ax}$ and $P(a) \neq 0$
4. $Q(x) = b \sin ax$ or $b \cos ax$
5. $Q(x) = u(x)e^{ax}$, $u(x)$ is a polynomial in x
6. $Q(x) = be^{ax}$ and $P(a) = 0$
7. $Q(x) = Q_1(x) + \cdots + Q_n(x)$

1. $Q(x) = bx^k$ and $P(D) = D - a_0$

1. The D.E.:
$$y' - a_0 y = bx^k, \quad a_0 \neq 0.$$
2. This is the Case 1. of Lesson 21A; "No term of $Q(x)$ is the same as as term of y_c. In thie case, y_p is a linear combination of the terms in $Q(x)$ and *all* its linearly independent derivatives."
3. Trial function: $y_p = A_k x^k + A_{k-1}x^{k-1} + \cdots + A_1 x + A_0$.
4. We get
$$y_p = -\frac{b}{a_0}\left(x^k + \frac{k}{a_0}x^{k-1} + \frac{k(k-1)}{a_0^2}x^{k-2} + \cdots + \frac{k!}{a_0^k}\right), \quad a_0 \neq 0.$$
5. The same result can be obtained as follows:
$$y_p = \frac{1}{D - a_0}(bx^k) = \frac{1}{-a_0\left(1 - \frac{D}{a_0}\right)}(bx^k)$$
$$= -\frac{1}{a_0}\left(1 + \frac{D}{a_0} + \frac{D^2}{a_0} + \cdots + \frac{D^k}{a_0}\right)(bx^k).$$

2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$

1. In general, if

$$P(D) = D^r(a_n D^{n-r} + \cdots + a_r).$$

2. By the inverse operator, we can solve as follows:
   - differentiating followed by integrating:

$$y_p = \frac{1}{D^r}\left[\frac{1}{a_n D^{n-r} + \cdots + a_{r+1}D + a_r}(bx^k)\right], \quad a_r \neq 0.$$

   - integrating followed by differentiating:

$$y_p = \frac{1}{a_n D^{n-r} + \cdots + a_{r+1}D + a_r}\left[\frac{1}{D^r}(bx^k)\right], \quad a_r \neq 0.$$

   $\rightarrow$ May introduce terms that are constant multiples of terms in y_c.

2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$: Example

Example 25.36

$$y'' - 2y' = 5, \quad (D^2 - 2D)y = 5.$$

1. First, note that

$$\frac{1}{D-2} = -\frac{1}{2}\left(1 + \frac{D/2}{1-D/2}\right) = -\frac{1}{2}\left(1 + \frac{D}{2} + \frac{D^2/4}{1-D/2}\right)$$
$$= -\frac{1}{2}\left(1 + \frac{D}{2} + \frac{D^2}{4} + \frac{D^3}{8} + \cdots + \frac{D^n}{2^n} + \cdots\right)$$

2. $y_c = c_1 e^2 x + c_2$.

2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$: Example (cont'd)

Example 25.36

$$y'' - 2y' = 5, \quad (D^2 - 2D)y = 5.$$

3. Method #1

$$y_p = \frac{1}{D}\left[\frac{1}{D-2}(5)\right] = \frac{1}{D}\left[-\frac{1}{2}\left(1 + \frac{D}{2} + \cdots\right)(5)\right] = \frac{1}{D}\left(-\frac{5}{2}\right)$$
$$= -\frac{5}{2}x.$$

4. Method #2

$$y_p = \frac{1}{D-2}\left[\frac{1}{D}(5)\right] = -\frac{1}{2}\left(1 + \frac{D}{2} + \frac{D^2}{4} + \cdots\right)(5x)$$
$$= -\frac{5}{2}x - \frac{5}{4}.$$

3. $Q(x) = be^{ax}$ and $P(a) \neq 0$

The particular solution is

$$y_p = \frac{1}{P(D)} be^{ax} = \frac{be^{ax}}{P(a)}, \quad P(a) \neq 0.$$

Can be proved by setting the trial function $y_p = Ae^{ax}$ and finding A. (Try it.)

4. $Q(x) = b \sin ax$ or $b \cos ax$

1. Set $Q(x) = be^{jax}$.
2. Solve it by matching real/imaginary part.

5. $Q(x) = u(x)e^{ax}$, $u(x)$ is a polynomial in x

The particular solution is

$$y_p = \frac{1}{P(D)}u(x)e^{ax} = e^{ax}\frac{1}{P(D+a)}u(x).$$

Can be proved by “exponential shift theorem”:

$$P(D)(u(x)e^{ax}) = e^{ax}P(D+a)(u(x))$$

6. $Q(x) = be^{ax}$ and $P(a) = 0$

1. $P(D) = (D - a)^r F(D), \quad F(a) \neq 0.$
2. The particular solution is

$$y_p = \frac{1}{(D-a)^r F(D)}(be^{ax}) = \frac{1}{(D-a)^r}\left[\frac{1}{F(D)}(be^{ax})\right].$$

3. Applying "type 3" method, we get

$$y_p = \frac{1}{(D-a)^r}\left[\frac{b}{F(a)}e^{ax}\right].$$

4. By the "exponential shift theorem",

$$y_p = e^{ax}\frac{1}{D^r}\left[\frac{b}{F(a)}\right] = \frac{e^{ax}bx^r}{r!F(a)}, \quad F(a) \neq 0.$$

7. $Q(x) = Q_1(x) + \cdots + Q_n(x)$

By "principle of superposition",

$$y_p = \frac{1}{P(D)}Q(x) \equiv \frac{1}{P(D)}Q_1(x) + \cdots + \frac{a}{P(D)}Q_n(x).$$

Lesson 26: Solution of a Linear Differential Equation by Means of the Par

Partial Fraction Expansion Theorem

Example:
For polynomials $P(x)$ and

$$Q(x) = (x+a)(x^3+b)(x^2+c)^2(x+d)^3,$$

where the degree of $P(x)$ is less than $Q(x)$,

$$\begin{aligned}\frac{P(x)}{Q(x)} =& \frac{A}{x+a} + \frac{Bx^2+Cx+D}{x^3+b} \\ &+ \frac{Ex+F}{x^2+c} + \frac{Gx+H}{(x^2+c)^2} + \frac{I}{x+d} + \frac{J}{(x+d)^2} + \frac{K}{(x+d)^3}.\end{aligned}$$

- Degree of each numerator?
- Types of denominators?

Partial Fraction Expansion Theorem (cont'd)

For

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_k) \cdots (x - r_n)$$

where $r_1, r_2, \cdots, r_n$ are *distinct*,

$$\frac{1}{f(x)} = \frac{1}{f'(r_1)(x - r_1)} + \cdots + \frac{1}{f'(r_n)(x - r_n)}.$$

Solving a L.D.E. by Means of the Partial Fraction Expansion of Inverse Operators

1. The D.E.:

$$P(D)y = Q(x).$$

2. Partial fraction expand:

$$\frac{1}{P(D)} = \frac{1}{P_1(D)} + \cdots + \frac{1}{P_k(D)}.$$

3. The particular solution is

$$y_p = \frac{Q(x)}{P_1(D)} + \cdots + \frac{Q(x)}{P_k(D)}.$$

Lesson 27: The Laplace Transform. Gamma Function.

Improper Integral

- $\int_0^\infty f(x)dx$.
- Existence and convergence:

$$\int_0^\infty f(x)dx = \lim_{\substack{h\to\infty \\ \epsilon\to 0}} \int_\epsilon^h f(x)dx = L.$$

Theorem

If the improper integral

$$\int_0^\infty e^{-sx} f(x)dx, \quad 0 \le x < \infty,$$

converges for a value of $s = s_0$, then it converges for every $s > s_0$.

Laplace Transform

$$\mathcal{L}\left[f(x)\right] = F(s) = \int_0^\infty e^{-sx} f(x)dx.$$

Assuming that $f(x)$ is a function for which the improper integral on the right converges.

Properties:

- Linear: $\mathcal{L}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}\left[f_1\right] + c_2 \mathcal{L}\left[f_2\right]$.
- Invertible: $\mathcal{L}\left[f_1\right] = \mathcal{L}\left[f_2\right]$ if and only if $f_1 = f_2$.
  $\rightarrow$ Inverse Laplace transform: If $\mathcal{L}\left[f(x)\right] = F(s)$, then $\mathcal{L}^{-1}\left\{F(s)\right\} = f(x)$.

Laplace Transform Method

1. For a L.D.E.

$$a_n y^{(n)}(x) + \cdots + a_1 y'(x) + a_0 = f(x),$$

   we get

$$\mathcal{L}\left[a_n y^{(n)} + \cdots + a_1 y'(x) + a_0 y\right]$$
$$= a_n \mathcal{L}\left[y^{(n)}\right] + \cdots a_1 \mathcal{L}\left[y'\right] + a_0 \mathcal{L}\left[y\right] = \mathcal{L}\left[f(x)\right], \quad s > s_0.$$

2. Assuming (27.28)

$$\lim_{x\to\infty} e^{-sx} y^{(k)}(x) = 0, \quad k = 0, 1, 2, \cdots, n-1, \quad s > s_0,$$

   we get

$$\mathcal{L}\left[y'\right] = -y(0) + s\mathcal{L}\left[y\right].$$

   In general, (27.33)

$$\mathcal{L}\left[y^{(n)}\right] = s^n \mathcal{L}\left[y\right]$$
$$-\left(y^{(n-1)}(0) + s y^{(n-2)}(0) + \cdots + s^{n-2} y'(0) + s^{n-1} y(0)\right).$$

   $\rightarrow s^n \mathcal{L}\left[y\right]$ + Polynomial of s!

Laplace Transform Method (cont'd)

3. We can rearrange as (27.4 and 27.41)

$$\mathcal{L}[y] = F(s), \quad s > s_0$$

then

$$y = \mathcal{L}^{-1}\{F(s)\}.$$

Note

- $F(s)$ can be evaluated if initial conditions $y(0), y'(0), \cdots, y^{(n-1)}(0)$ are given.
- The Laplace transform method has changed the original differential equation involving derivatives, to an algebraic equation involving a function of s.

Laplace Transform of Simple Functions

$$\mathcal{L}[k] = \frac{k}{s} \quad s > 0.$$

$$\mathcal{L}[x^n] = \frac{n!}{s^{n+1}} \quad s > 0, n = 1, 2, \cdots .$$

$$\mathcal{L}[e^{ax}] = \frac{1}{s-a}, \quad s > a.$$

$$\mathcal{L}[\sin ax] = \frac{a}{s^2 + a^2}.$$

...and more. (See the table on p.306)

Theorem

If $F(s) = \mathcal{L}[f(x)], \quad s > s_0,$

$$F^{(n)}(s) = (-1)^n \mathcal{L}[x^n f(x)] = (-1)^n \int_0^\infty e^{-sx} x^n f(x) dx, \quad s > s_0.$$

$\rightarrow \mathcal{L}[x^n f(x)]$ can be obtained from $\mathcal{L}[f(x)]$.

Faltung (Folding) Theorem

Theorem

If

$$F(s) = \mathcal{L}[f(x)] \quad \text{and} \quad G(s) = \mathcal{L}[g(x)],$$

then

$$\mathcal{L}\left[\int_0^x f(x-t)g(t)dt\right] = \mathcal{L}\left[\int_0^x f(t)g(x-t)dt\right]$$
$$= \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)] = F(s) \cdot G(s).$$

Laplace Transform of $x^{1/2}$ and $x^{n-1/2}$?

We need Gamma function!

The Gamma Function

- Extension of factorial function to real and complex numbers.
- $\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx, \quad k > 0.$
- $\Gamma(1) = 1 \quad$ and $\quad \Gamma(k+1) = k\Gamma(k)$
  $\rightarrow \Gamma(n+1) = n! \quad n$ integer.
- $\Gamma(k) = \dfrac{\Gamma(k+n)}{k(k+1)\cdots(k+n-1)}, \quad k \neq 0, -1, \cdots, -(n-1).$
  $\rightarrow$ Extended to negative values of k, provided that
  $k \neq 0, -1, \cdots, -(n-1)$.
  Example: $\Gamma(-1/2) = -2\Gamma(1/2)$.
- Euler's reflection formula

$$\Gamma(1-k)\,\Gamma(k) = \frac{\pi}{\sin(\pi k)}$$

- The Laplace transform $\mathcal{L}[x^n] = \dfrac{n!}{s^{n+1}} \quad s > 0$, can be extendedn to any n, except negative integers.

The Gamma Function (cont'd)