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Homework 4

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1 Exercises

1. Question 1

   $C^1(\mathbf{R})$ is a set of functions that guarantee at least one derivative.

   We know that D is a function that maps f to its derivative f'. Since it already stated as a function, we have to prove its linearity. In linear algebra, a linear function has to be closed under scalar multiplication and addition.

   Let $f(x), g(x) \in C^1(\mathbf{R})$. By the properties of differentiation, we have:

$$\begin{aligned} D(f(x)+g(x)) &= \frac{d}{dx}(f(x)+g(x)) \\ &= \frac{d}{dx}(f(x)) + \frac{d}{dx}((g(x)) \\ &= D(f(x)) + D(g(x)) = f'(x) + g'(x) \end{aligned} \tag{1}$$

   For $f(x) \in C^1(\mathbf{R})$ and scalar $r \in \mathbf{R}$, we also have:

$$\begin{aligned} D(rf(x)) &= \frac{d}{dx}(rf(x)) \\ &= r\frac{d}{dx}(f(x)) \\ &= rD(f(x)) = rf'(x) \end{aligned} \tag{2}$$

   From (1) and (2), we can see that function D is a linear function from $C^1(\mathbf{R})$ to $C^0(\mathbf{R})$

2. Question 2

   (a) $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2/3 & 2 \end{bmatrix}, n = 2, m = 3$

   To find the rank of this matrix, find dim(R(A))

$$\alpha\underline{a}_1 + \beta\underline{a}_2 = \emptyset \to \begin{cases} 2\alpha + 6\beta = 0 \\ \alpha + 3\beta = 0 \\ 2/3\alpha + 2\beta = 0 \end{cases} \to \alpha = -3\beta$$

   Since the matrix is dependent, we only need one vector to make up the basis of the matrix. Therefore, dim(R(A)) = Rank(A)= 1, Nullity(A)=n-Rank(A)=2-1=1 since n=2=Rank(A)+Nullity(A)

   (b) $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, n = 2, m = 2$

   To find the rank of this matrix, find dim(R(B))

$$\alpha\underline{a}_1 + \beta\underline{a}_2 = \emptyset \to \begin{cases} \alpha + 2\beta = 0 \\ 3\alpha + 4\beta = 0 \end{cases} \to \begin{cases} 2\beta = 0 \\ \alpha + 2\beta = 0 \end{cases} \to \alpha = \beta = 0$$

Since the set of equations returns only the trivial solution then we can conclude $\underline{a}_1, \underline{a}_2$ are linearly independent meaning that they together form a basis for $\mathbf{R}^3$, R(B)=$\mathbf{R}^3$, dim(R(B))=Rank(B)=2 and Nullity(A)=n-Rank(A)=2-2=0

(c) $C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}, n = 3, m = 2$

To find the rank of the matrix, check the linear independence of the column vectors of the matrix

$$\alpha\underline{a}_1 + \beta\underline{a}_2 + \gamma\underline{a}_3 = \emptyset \rightarrow \begin{cases} \alpha + 2\gamma = 0 \\ -1\alpha + 3\beta + \gamma = 0 \end{cases} \rightarrow \begin{cases} \alpha + 2\gamma = 0 \\ 3\beta + 3\gamma = 0 \end{cases} \rightarrow \begin{cases} \alpha = -2\gamma \\ \beta = -\gamma \end{cases}$$

Therefore, the solution is not trivial. Check the independence of the subset of vectors. We can see that the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ are linearly independent from each other. Check:

$$\alpha\underline{a}_1 + \beta\underline{a}_2 = \emptyset \rightarrow \begin{cases} \alpha = 0 \\ -1\alpha + 3\beta = 0 \end{cases} \rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

Therefore, dim(R(A))= Rank(A)=2 and Nullity(A)=n-Rank(A)=3-2=1

3. Question 3

Since $\underline{u} \in \mathbf{R}^n$ and $\underline{v} \in \mathbf{R}^m$ then $A = \underline{uv}^T \in \mathbf{R}^{n\times m}$ and this is the outer product of the two vectors $\underline{u}, \underline{y}$

Therefore, the matrix has the shape $A = \begin{bmatrix} u_1v_1 & u_1v_2 & ... & u_1v_m \\ u_2v_1 & u_2v_2 & ... & u_2v_m \\ ... & ... & ... & ... \\ u_nv_1 & u_nv_2 & ... & u_nv_m \end{bmatrix}$

We can see that the columns of the matrix are all proportional to the first column and the difference are just the scalars $(v_1, v_2, ..., v_m)$, thus they are all linearly dependent on a single column, hence the matrix is rank one (rank(A)=1). The range of this matrix is $R(A) = span\{\underline{u}\} = span\left\{\begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix}\right\}$

4. Question 4

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix}$$

This matrix has been reduced to echelon form, with the three pivots showing in column 1, 2 and 4. Therefore, the basis for the range of A is $\left\{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}\right\}$

We define the null space of A as the set $N(A) = \{\underline{x} \in \mathbf{R}^5 | A\underline{x} = 0\}$. Define $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

To obtain all solutions $A\underline{x} = 0$, note that x_3, x_5 are free variables (non-pivoted) in the matrix. Set $x_3 = s$ and $x_5 = t$. Then we have:

$$A\underline{x} = \begin{bmatrix} 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s \\ x_4 \\ t \end{bmatrix} = 0$$

$$\to \begin{cases} 3x_1 + x_4 + 2t = 0 \\ 2x_2 - 2x_3 + x_5 = 0 \\ 4x_4 + t = 0 \end{cases} \quad \to \begin{cases} x_1 = -2/3t + 1/12t = -7/12t \\ x_2 = -t/2 + s \\ x_3 = s \\ x_4 = -1/4t \\ x_5 = t \end{cases}$$

The non-trivial solutions are given by:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7/12t \\ -t/2 + s \\ s \\ -1/4t \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7/12 \\ -1/2 \\ 0 \\ -1/4 \\ 1 \end{bmatrix}$$

A basis for the null space of A is given by $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7/12 \\ -1/2 \\ 0 \\ -1/4 \\ 1 \end{bmatrix} \right\}$

5. Question 5

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the inverse of A, $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

From the class lecture we have $A \times A^{-1} = A^{-1} = A^{-1}A = I_{2\times 2}$ hence we can define A^{-1} as a 2×2 matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

We have:

$$A^{-1}A = I \to \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{cases} ax_1 + cx_2 = 1 \\ bx_1 + dx_2 = 0 \\ ax_3 + cx_4 = 0 \\ bx_3 + dx_4 = 1 \end{cases}$$

$$\begin{cases} b(ax_1 + cx_2) = b \\ a(bx_1 + dx_2) = 0 \end{cases} \quad \to (abx_1 + adx_2) - (abx_1 + bcx_2) = -b \to x_2(ad - bc) = -b \to x_2 = \frac{-b}{ad - bc} \quad (3)$$

$$\begin{cases} b(ax_3 + cx_4) = 0 \\ a(bx_3 + dx_4) = a \end{cases} \quad \to (abx_3 + adx_4) - (abx_3 - bcx_4) = a \to x_4(ad - bc) = a \to x_4 = \frac{a}{ad - bc} \quad (4)$$

$$\begin{cases} d(ax_1 + cx_2) = d \\ c(bx_1 + dx_2) = 0 \end{cases} \quad \to (adx_1 + cdx_2) - (bcx_1 - cdx_2) = d \to x_1(ad - bc) = d \to x_1 = \frac{d}{ad - bc} \quad (5)$$

$$\begin{cases} d(ax_3 + cx_4) = 0 \\ c(bx_3 + dx_4) = c \end{cases} \rightarrow (adx_3 + cdx_4) - (bcx_3 - cdx_4) = -c \rightarrow x_3(ad - bc) = -c \rightarrow x_3 = \frac{-c}{ad - bc} \quad (6)$$

From equations 3, 4, 5, and 6, the inverse of A, A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Factor out the denominator, we can conclude the formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$