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Course: AMATH 352

Homework 2

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1 Reading

1. Computer Hardware Dataset : The dataset includes estimated relative performance values of computers from different vendors. It has 9 attributes, including the vendor name, model name, as well as their benchmark statistics such as machine cycle time, minimum main memory, cache memory, and relative performances. We can use a vector of $\underline{x} \in \mathbf{R}^9$ to represent a record in this dataset, which can help us categorize (cluster) models using existing algorithms.

2. Wholesale Customer Dataset : The dataset includes data referring to customers of a wholesale store, consisting of 8 attributes, with 6 continuous ones detailing their annual spending on different types of products and 2 nominal attributes on their channel (either horeca –hotel/restaurant/cafe– or retail) and their religion. We can use a vector of $\underline{x} \in \mathbf{R}^8$ to represent a customer record in the data set. We can use these vectors to help classify and clustering customers into different groups, which would help the store itself improve services and logistics.

2 Exercises

1. Question 1

   We have:

   - $||\underline{x}||_\infty = \max(|x_j|)$
   - $||\underline{x}||_1 = \sum_{j=1}^{d} |x_j|$
   - $d||\underline{x}||_\infty = d\max(|x_j|)$

   For the first pair of inequality, we can easily see that $||\underline{x}||_\infty = \max(|x_j|) <= \sum_{j=1}^{d} |x_j| = ||\underline{x}||_1$.

   The inequality cannot be strict and equality can be achieved, for example with $\underline{x} \in \mathbf{R}^2, \underline{x} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. In this case, we have $||\underline{x}||_\infty = \max(|x_j|) = 1 = \sum_{j=1}^{2} |x_j| = |0| + |1| = 1 = ||\underline{x}||_1$.

   For the second pair of inequality, we can prove that $||\underline{x}||_1 = \sum_{j=1}^{d} |x_j| <= \sum_{k=1}^{d} \max(|x_j|) = d\max(|x_j|)$ because $|x_j| <= \max(|x_j|) \forall x_j$.

   The equal sign can be achieved, for example, with $\underline{x} \in \mathbf{R}^2, \underline{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. In this case, we have $||\underline{x}||_1 = \sum_{j=1}^{d} |x_j| = |1| + |1| = 2 = \sum_{k=1}^{2} \max(|x_j|) = |1| + |1| = d\max(|x_j|)$

2. Question 2

   We have already proved in Homework 1 that

   $$||\underline{x} - \underline{y}||_2^2 + ||\underline{y}||_2^2 = ||\underline{x}||_2^2 + ||\underline{y}||_2^2 - 2\underline{x}^T\underline{y}$$

   Therefore, if $||\underline{x} - \underline{y}||_2^2 + ||\underline{y}||_2^2 + ||\underline{y}||_2^2 = ||\underline{x}||_2^2$ then $2||\underline{y}||_2^2 = 2\underline{x}^T\underline{y}$

   $$\iff \underline{x}^T\underline{y} - ||\underline{y}||_2^2 = 0 \iff \underline{x}^T\underline{y} - \underline{y}^T\underline{y} = 0 \iff (\underline{x} - \underline{y})^T\underline{y} = 0$$

   Two vectors are orthogonal to each other if their inner product is 0. Based on the equation above, we can conclude that $\underline{x} - \underline{y}$ is orthogonal to $\underline{y}$

3. Question 3

   Using the second strategy to show that P^n is a subspace of another linear space:

   We have for any function $f \in P_n(\mathbf{R})$ has a form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, with $a_0, a_1, a_2, ..., a_n \in \mathbf{R}$. Clearly we can see that $P_2(\mathbf{R}) \subset \mathcal{F}(\mathbf{R} : \mathbf{R})$ Proof that the set is closed under addition and scalar multiplication:

   - Addition: Let $f_1(x) = a_0 + a_1x + ... + a_nx^n$ and $f_2(x) = b_0 + b_1x + ... + b_nx^n$ then $f_1(x) + f_2(x) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_n + b_n)x^n \in P_n(\mathbf{R})$
   - Scalar multiplication: $\alpha f_1(x) = \alpha a_0 + \alpha a_1x + ... + \alpha a_nx^n \in P_n(\mathbf{R})$

   The zeroth element of P^n is a constant

4. Question 4

   Assume the following:

   $a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 1.5 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \emptyset,$

   the vectors are only linearly independent if $a = b = c = 0$ is the only option. We have a system of variables:

   $$a - 2b + c = 0 \tag{1}$$

   $$2a + b = 0 \tag{2}$$

   $$3a + 1.5b = 0 \tag{3}$$

   From (2) and (3) we can see that $b = -2a$. Plugging into equation (1): $a + 4a + c = 0 \iff c = -5a$

   Since we can choose non-trivial values $a = -1, b = 2, c = 5$ to plug in the system

   $-1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \\ 1.5 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - 4 + 5 \\ -2 + 2 + 0 \\ -3 + 3 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \emptyset$, the vectors are linearly dependent.

5. Question 5

   The functions $f_1(x) = 1, f_2(x) = \cos \pi t, f_3(x) = \sin \pi t$ are linearly independent one and only when $a(1) + b(\cos \pi t) + c(\sin \pi t) = 0$ with only one solution $a = 0, b = 0, c = 0$

   Let $t = 1$ then $a - b = 0$.

   Let $t = 0$ then $a + b = 0$.

   Let $t = 1/2$ then $a + c = 0$

   From the first pair of equations, the only solution that satisfies $b = a = -b$ is $a = b = 0$. Plugging into the third equation, $c = 0$. Since the only solution is the trivial solution, we can conclude that the three functions are linearly independent.

6. Question 6

   (a) Let there be two functions $f(x)$ and $g(x)$ that belong to this set.

- Checking addition using the sum rule of integrals:

$$\int_0^2 (f+g)(x) = \int_0^2 f(x) + \int_0^2 g(x) = 2+2 = 4 \neq 2$$

  The set is not closed under addition
- Checking scalar multiplication:

$$\int_0^2 \alpha f(x) = \alpha \int_0^2 f(x) = 2\alpha \neq 2 (\alpha \neq 1)$$

  The set is not closed under scalar multiplication unless $\alpha = 1$

Therefore the set is not a linear space

(b) Let there be two functions $f(x)$ and $g(x)$ that belong to this set.

- Checking addition using the sum rule of derivatives:

$$(f' + g')(x) = f'(x) + g'(x) = 2f(x) + 2g(x) = 2(f(x) + g(x)) = 2(f+g)(x)$$

  The set is closed under addition.
- Checking scalar multiplication:

$$\alpha f'(x) = \alpha \cdot 2f(x) = 2(\alpha f(x))$$

  The set is closed under scalar multiplication

Therefore this set is a linear space because it is closed under addition and scalar multiplication, and includes $f(t) = 0$ that satisfies $f'(t) = 0 = 2f(t)$.

7. Question 7

(a) The total number of people in the population: $||\underline{x}||_1$

(b) Let $\alpha = \left[\begin{smallmatrix} 0^{65} \\ 1^{35} \end{smallmatrix}\right]$ with 0^65 as a vector of 65 0's and 1^{35} as a vector of 35 1's.

The total number of people in the population age 65 and over: $\alpha^T \underline{x}$

$$= \begin{bmatrix} 0^{65} \\ 1^{35} \end{bmatrix}^T \underline{x}$$

(c) Let $\alpha = \left[\begin{smallmatrix} 0 \\ 1 \\ 2 \\ \cdots \\ 99 \end{smallmatrix}\right]$ be a vector of length 100. Total 'ages' of the whole population: $\alpha^T \underline{x}$. To calculate the average, divide by the whole population: $\frac{\alpha^T \underline{x}}{||\underline{x}||_1}$

$$= \frac{\begin{bmatrix} 0 \\ 1 \\ 2 \\ \cdots \\ 99 \end{bmatrix}^T \underline{x}}{||\underline{x}||_1}$$