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On two conjectures about the proper connection number of graphs[☆]

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ABSTRACT

A path in an edge-colored graph is called proper if no two consecutive edges of the path receive the same color. For a connected graph G, the proper connection number $pc(G)$ of G is defined as the minimum number of colors needed to color its edges so that every pair of distinct vertices of G is connected by at least one proper path in G. In this paper, we consider two conjectures on the proper connection number of graphs. The first conjecture states that if G is a noncomplete graph with connectivity $\kappa(G) = 2$ and minimum degree $\delta(G) \geq 3$, then $pc(G) = 2$, posed by Borozan et al. (2012). We give a family of counterexamples to disprove this conjecture. However, from a result of Thomassen it follows that 3-edge-connected noncomplete graphs have proper connection number 2. Using this result, we can prove that if G is a 2-connected noncomplete graph with $diam(G) = 3$, then $pc(G) = 2$, which solves the second conjecture we want to mention, posed by Li and Magnant (2015).

1. Introduction

All graphs in this paper are simple, finite and undirected. We follow [2] for graph theoretical notation and terminology not defined here. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of v. For a subset $U \subseteq V(G)$, let $N(U) = \left(\bigcup_{v\in U} N(v)\right) \setminus U$. For any two disjoint subsets X and Y of $V(G)$, we use $E(X, Y)$ to denote the set of edges of G that have one end in X and the other in Y. Denote by $|E(X, Y)|$ the number of edges in $E(X, Y)$. An (X, Y)-path is a path which starts at a vertex of X, ends at a vertex of Y, and whose internal vertices belong to neither X nor Y.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of G are assigned different colors by c, then c is called a *proper (edge-)coloring*. For a graph G, the minimum number of colors needed in a proper coloring of G is referred to as the *edge-chromatic number* of G and denoted by $\chi'(G)$. A path of an edge-colored graph G is said to be a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if for any two vertices there is a rainbow path of G connecting them. An edge-coloring of a connected graph is a *rainbow connecting coloring* if it makes the graph rainbow connected. For a connected graph G, the *rainbow connection number* $rc(G)$ of G is defined to be the smallest number of colors that are needed in order

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to make G rainbow connected. The concept of rainbow connection of graphs was introduced by Chartrand et al. [5] in 2008. Readers who are interested in this topic can see [11,12] for a survey.

Motivated by the rainbow coloring and proper coloring in graphs, Andrews et al. [1] and Borozan et al. [3] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored with the same color. An edge-coloring of a connected graph G is a *proper-path coloring* if every pair of distinct vertices of G is connected by a proper path in G. If k colors are used, then c is referred to as a *proper-path k-coloring*. An edge-colored graph G is called *proper connected* if every pair of distinct vertices of G is connected by a proper path. For a connected graph G, the *proper connection number* of G, denoted by $pc(G)$, is defined as the smallest number of colors that are needed in order to make G proper connected.

The proper connection of graphs has the following application background. When building a communication network between wireless signal towers, one fundamental requirement is that the network be connected. If there cannot be a direct connection between two towers A and B, say for example if there is a mountain in between, there must be a route through other towers to get from A to B. As a wireless transmission passes through a signal tower, to avoid interference, it would help if the incoming signal and the outgoing signal do not share the same frequency. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then, the number of frequencies needed to assign frequencies to the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Let G be a nontrivial connected graph of order n (number of vertices) and size m (number of edges). Then the proper connection number of G has the following clear bounds:

$$1 \le pc(G) \le \min\{rc(G), \chi'(G)\} \le m.$$

Furthermore, $pc(G) = 1$ if and only if $G = K_n$, $pc(G) = m$ if and only if $G = K_{1,m}$ is a star of size m.

Given an edge-colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1 and v_s, we denote by $start(P)$ the color of the first edge in the path, i.e., $c(v_1v_2)$, and by $end(P)$ the color of the last edge in the path, i.e., $c(v_{s-1}v_s)$. If P is just the edge v_1v_s, then $start(P) = end(P) = c(v_1v_s)$.

Definition 1.1 ([3]). Let c be an edge-coloring of G that makes G proper connected. We say that G has the *strong property under c* if for any pair of vertices $u, v \in V(G)$, there exist two proper paths P_1, P_2 connecting them (not necessarily disjoint) such that $start(P_1) \ne start(P_2)$ and $end(P_1) \ne end(P_2)$.

Next we list the following four lemmas, which will be used in this work.

Lemma 1.1 ([1]). *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $pc(G) \le pc(H)$. In particular, $pc(G) \le pc(T)$ for every spanning tree T of G.*

In fact, Lemma 1.1 also states that the proper connection number is monotonic under adding edges.

Lemma 1.2 ([3]). *If G is a 2-connected graph, then $pc(G) \le 3$. Furthermore, there exists a 3-edge-coloring c of G such that G has the strong property under c.*

Lemma 1.3 ([3,9]). *If G is a connected bridgeless bipartite graph, then $pc(G) \le 2$. Furthermore, there exists a 2-edge-coloring c of G such that G has the strong property under c.*

Lemma 1.4 ([1]). *Let G be a connected graph and v a vertex not in G. If $pc(G) = 2$, then $pc(G \cup v) = 2$ as long as $d(v) \ge 2$, that is, we connect v to G by using at least two edges.*

For more details we refer to [1,3,6–8,13] and a dynamic survey [10].

The first conjecture we will consider in this paper is as follows, which was posed by Borozan et al. in [3].

Conjecture 1.1 ([3]). *If $\kappa(G) = 2$ and $\delta(G) \ge 3$, then $pc(G) = 2$.*

As observed in the example of [[3], Proposition 3], in which the graph G satisfies that $\kappa(G) = 2$ and $\delta(G) = 2$ but $pc(G) = 3$, the bound on $\delta(G)$ in Conjecture 1.1 would be sharp if the conjecture is true.

The second conjecture we will consider is as follows, which was posed by Li and Magnant in [10].

Conjecture 1.2 ([10]). *If G is a 2-connected noncomplete graph with $diam(G) = 3$, then $pc(G) = 2$.*

In Section 2 we disprove Conjecture 1.1. In Section 3 we prove Conjecture 1.2.

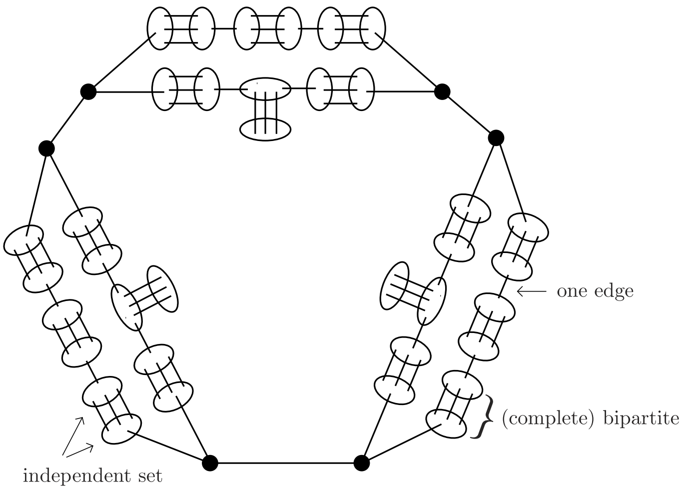

Fig. 1. Counterexamples for Conjecture 1.1.

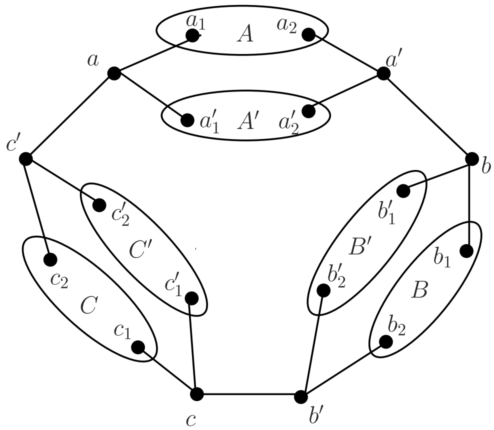

Fig. 2. Labeling of the graph G.

2. Disprove Conjecture 1.1

Theorem 2.1. *Let G be a connected graph as shown in Fig. 1 such that every (complete) bipartite graph contains at least one vertex. Then, $pc(G) = 3$. When every complete bipartite graph contains a copy of $K_{3,3}$, we have disproved Conjecture 1.1.*

Proof. First, we label the graph G as in Figs. 2 and 3 for simplicity where subgraphs correspond to the subgraphs in Fig. 1. From Lemma 1.2, we have that $pc(G) \le 3$. Thus, it is sufficient to show that $pc(G) \ne 2$. Assume, to the contrary, that we have a 2-edge-coloring c which makes G proper connected. Then, for any two vertices of G, there is a proper path connecting them.

Claim 2.1. *In each pair (A, A'), (B, B') and (C, C'), say (A, A'), there exists a vertex v in $A \cup A'$ such that either all the proper paths from v to $G \setminus (A \cup A')$ go through the edge ac' rather than $a'b$ or all the proper paths from v to $G \setminus (A \cup A')$ go through the edge $a'b$ rather than ac'.*

Proof. Suppose, to the contrary, that every vertex in $A \cup A'$ has proper paths to $G \setminus (A \cup A')$ through ac' and also has proper paths to $G \setminus (A \cup A')$ through $a'b$. Let $f = v_1v_2$ and $f' = v_3v_4$ be the two cut-edges in $G[A]$ with f the closer edge to a and f' the closer edge to a'. Let $e = u_1u_2$ and $e' = u_3u_4$ be the two cut-edges in $G[A']$ with e the closer edge to a and e' the closer edge to a'. Also assume the vertices with lower index on each of these edges are closer to a.

If aa_1 and f have different colors, then all the proper paths from v_2 to $G \setminus A$ must go through a_2a'. Thus, $a'a'_2$ and $a'b$ have the same color. So a'_2 has no proper path to $G \setminus (A \cup A')$ through $a'b$, a contradiction. Hence, aa_1 and f have the same color. Similarly, f and f' as well as f' and a_2a' have the same color. Thus, aa_1, f, f' and a_2a' all have the same color. If $a'a_2$ and $a'b$

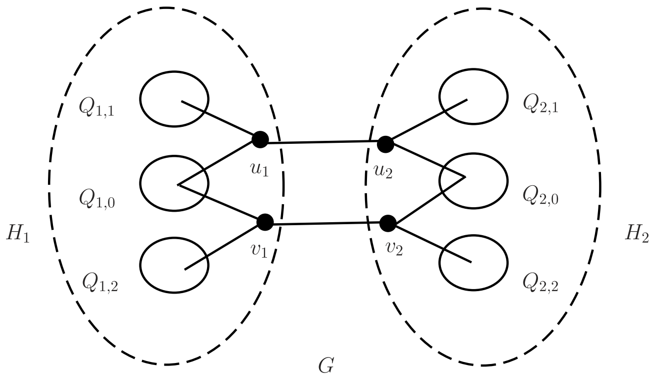

Fig. 3. The graph G in Case 1.

have the same color, then there is no proper path from a_2 to $G \setminus (A \cup A')$ through $a'b$ since the parity of any path from a to a' passing through A is different from the parity of any path from a to a' passing through A'. Thus, $a'a_2$ and $a'b$ have different colors and symmetrically, aa_1 and ac' have different colors.

If e has the opposite color from aa'_1, then u_2 cannot possibly have a proper path leaving A' through the edge a'_1a. Then, since a_2a' and $a'b$ have different colors, the edge a'_2a' must share one of those colors. This means that u_2 has a proper path to $G \setminus (A \cup A')$ through only one of ac' or $a'b$, a contradiction. On the other hand, this means that e must have the same color as aa'_1. Similarly, we have that e' must have the same color as $a'a'_2$. If e and e' have the same color, then u_1 cannot possibly have a proper path leaving A' through the edge a'_2a. Thus, u_1 has no proper path to $G \setminus (A \cup A')$ through one of ac' or $a'b$, a contradiction. Now we have that e and e' have different colors. By the symmetry, we suppose that aa'_1 and aa_1 have the same color and $a'a'_2$ and $a'b$ have the same color. Thus, any vertex of A' has no proper path to $G \setminus (A \cup A')$ through $a'b$, which is a contradiction. So, there exists a vertex $v \in A \cup A'$ such that either all the proper paths from v to $G \setminus (A \cup A')$ go through the edge ac' rather than $a'b$ or all the proper paths from v to $G \setminus (A \cup A')$ go through the edge $a'b$ rather than ac'. Of course, the same argument holds in $B \cup B'$ and $C \cup C'$ to complete the proof of Claim 2.1. $\square$

Let a^* be the vertex in $A \cup A'$ resulting from Claim 2.1 and similarly define b^* and c^*. By the pigeonhole principle, there exists a pair of these paths that leave their respective sets in the same direction. More specifically, we may assume without loss of generality that all the proper paths from a^* to $G \setminus (A \cup A')$ go through only ac' (and not $a'b$) and all the proper paths from b^* to $G \setminus (B \cup B')$ go through only ba' (and not $b'c$). Then there can be no proper path from a^* to b^*. $\square$

Remark. A few months after we submitted our this paper, we knew from a manuscript [4] that Brause et al. showed that for every integer $d \geq 3$, there exists a 2-connected graph G of minimum degree d and order $n = 42d$ such that $pc(G) \geq 3$. By taking $d = 3$, their smallest example would have an order 126. But, our smallest example has an order 114 by taking every complete bipartite graph in Fig. 1 as $K_{3,3}$. So, our examples are different from theirs. Their paper [4] is still in refereeing.

3. Proof of Conjecture 1.2

Thomassen [14] observed that given a graph G which is at least $(2k-1)$-edge-connected, then G contains a bipartite spanning subgraph H for which H is k-edge-connected. Combining with Lemma 1.3, we have the following Theorem.

Theorem 3.1. *If G is a 3-edge-connected noncomplete graph, then $pc(G) = 2$ and there exists a 2-edge-coloring c of G such that G has the strong property under c.*

In the following, we will use Theorem 3.1 to give a confirmative proof for Conjecture 1.2.

Theorem 3.2. *If G is a 2-connected graph with $diam(G) = 3$, then $pc(G) = 2$.*

Proof. If G is 3-edge-connected, Theorem 3.1 implies that $pc(G) = 2$. So, we may assume $\kappa'(G) = 2$, where $\kappa'(G)$ denotes the edge-connectivity of G. We distinguish the following two cases to proceed the proof.

Case 1: There is a 2-edge-cut S of G such that each component of $G - S$ has at least three vertices.

Then, let $S = \{u_1u_2, v_1v_2\}$ be the 2-edge-cut of G and H_1, H_2 be the components of $G \setminus S$ such that $|H_i| \geq 3$ for $i = 1, 2$, where $u_1, v_1 \in H_1$ and $u_2, v_2 \in H_2$. Since G is 2-connected, we have that $u_1 \neq v_1$ and $u_2 \neq v_2$. Let $Q_i = H_i \setminus \{u_i, v_i\}$ for $i = 1, 2$. Since $diam(G) = 3$, we know that any vertex $w \in Q_i$ must be adjacent to at least one vertex of $\{u_i, v_i\}$ for $i = 1, 2$. For each Q_i $(i = 1, 2)$, define the subsets $Q_{i,0} = N(u_i) \cap N(v_i) \cap Q_i$, $Q_{i,1} = (N(u_i) \cap Q_i) \setminus Q_{i,0}$ and $Q_{i,2} = (N(v_i) \cap Q_i) \setminus Q_{i,0}$.

Since $diam(G) = 3$, we have that at least one of $Q_{1,1}$ and $Q_{2,2}$ is empty. Without loss of generality, we may assume that $Q_{2,2}$ is empty. Similarly, at least one of $Q_{1,2}$ and $Q_{2,1}$ is empty. So, there are two subcases to deal with.

Subcase 1.1: $Q_{1,2}$ is empty.

If $Q_{1,0}$ is empty, then u_1 is a cut-vertex of G, a contradiction. So, $Q_{1,0}$ is nonempty. Similarly, $Q_{2,0}$ is nonempty. Let $G_0 = G[\{u_1, v_1, u_2, v_2\} \cup Q_{1,0} \cup Q_{2,0}]$. The graph G_0 contains a 2-connected bipartite spanning subgraph. So, $pc(G_0) = 2$ from Lemmas 1.1 and 1.3.

Let G_1 be a subgraph of G obtained by adding a vertex to G_0 which has at least 2 edges into G_0. Furthermore, let G_i be a subgraph of G obtained by adding a vertex to G_{i-1} which has at least 2 edges connecting to G_{i-1}. By Lemma 1.4, $pc(G_i) = 2$ for all i. We claim that such a sequence of subgraphs of G exists, and we can find a spanning subgraph of G by repeating this procedure. In order to prove this, suppose that G_i is the largest such subgraph of G and suppose that there exists a vertex $v \in G \setminus G_i$. Assume, without loss of generality, that $v \in Q_{1,1}$. Since G is 2-connected, we have that there is a 2-fan from v to G_i. So we can find a path from v to G_i in H_1 which is not vu_1. Let w be the last vertex on this path which is not in G_i. We know that w must be adjacent to u_1. This means that $d_{G_i}(w) \geq 2$, and so we may set $G_{i+1} = G_i \cup w$ to get a contradiction. Therefore, we find a spanning subgraph G' of G and $pc(G') = 2$. Then we can get that $pc(G) = 2$ by Lemma 1.1. This completes the proof.

Subcase 1.2: $Q_{2,1}$ is empty.

If $E(Q_{1,1}, Q_{1,2}) = \emptyset$, then we can get $pc(G) = 2$ by a similar argument as in Subcase 1. Now we have $|E(Q_{1,1}, Q_{1,2})| \geq 1$. Let M be a maximum matching of $E(Q_{1,1}, Q_{1,2})$, and let $G'_0 = G[\{u_1, u_2, v_1, v_2\} \cup M \cup Q_{2,0}]$. Note that $G[\{u_2, v_2\} \cup Q_{2,0}]$ contains a 2-connected bipartite spanning subgraph if $|Q_{2,0}| \geq 2$. From Lemma 1.3 we know that $G[\{u_2, v_2\} \cup Q_{2,0}]$ has a 2-edge-coloring with the strong property if $|Q_{2,0}| \geq 2$. If $|M| \geq 2$, then $G[M \cup \{u_1, v_1\}]$ contains a 2-connected bipartite spanning subgraph, and we can also get that $G[M \cup \{u_1, v_1\}]$ has a 2-edge-coloring with the strong property by Lemma 1.3. If $|M| = 1$ and $|Q_{2,0}| = 1$, then we have that G'_0 is a cycle possibly with some chords. Thus, $pc(G'_0) = 2$. If $|M| = 1$ and $|Q_{2,0}| \geq 2$, then by Lemma 1.4 we know $pc(G'_0) = 2$. If $|M| \geq 2$, then we can easily get that $pc(G[M \cup \{u_1, v_1, u_2, v_2\}]) = 2$ since $G[M \cup \{u_1, v_1\}]$ has a 2-edge-coloring with the strong property. Thus, from Lemma 1.4 we have $pc(G'_0) = 2$. Therefore, $pc(G'_0) = 2$.

Let G'_1 be a subgraph of G obtained by adding a vertex to G'_0 which has at least 2 edges connecting to G'_0. Furthermore, let G'_i be a subgraph of G obtained by adding a vertex to G'_{i-1} which has at least 2 edges connecting to G'_{i-1}. By Lemma 1.4, $pc(G'_i) = 2$ for all i. We claim that such a sequence of subgraphs of G exists, and we can find a spanning subgraph of G by repeating this procedure. In order to prove this, suppose that G'_i is the largest such subgraph of G and suppose that there exists a vertex $v \in G \setminus G'_i$. Let $Q'_{1,1} = Q_{1,1} \setminus V(G'_i)$, $Q'_{1,2} = Q_{1,2} \setminus V(G'_i)$. According to the construction of G'_0, we have $E(Q'_{1,1}, Q'_{1,2}) = \emptyset$. Certainly, every vertex adjacent to both u_1 and v_1 is in G'_i. This means $v \in Q'_{1,1} \cup Q'_{1,2}$. Without loss of generality, we assume $v \in Q'_{1,1}$. Since G is 2-connected, we have that there is a 2-fan from v to G'_i. So, we can find a path from v to G'_i in H_1 which is not vu_1. Let w be the last vertex on this path which is not in G'_i. We know that w must be adjacent to u_1. This means that $d_{G'_i}(w) \geq 2$, and so we may set $G'_{i+1} = G'_i \cup w$ to get a contradiction. This completes the proof.

Case 2: For every 2-edge-cut S of G, $G - S$ has a component with at most two vertices.

If G does not have an even cycle, then $G = C_7$ since G is 2-connected and $diam(G) = 3$. It follows that $pc(G) = 2$. Thus, we suppose G contains an even cycle. Let $H = H(X, Y)$ be a maximal 2-edge-connected bipartite subgraph of G. We claim that H contains all the vertices with degree at least 3 of G. Assume, to the contrary, that there is a vertex $v \in V(G) \setminus V(H)$ with $d_G(v) \geq 3$. We know that there exist three edge-disjoint (v, H)-paths P_1, P_2, P_3. If this is not the case, then we can find a 2-edge-cut S' of G such that each component of $G - S'$ has at least three vertices, which contradicts the assumption. For $1 \leq i \leq 3$, let y_i be the terminal vertex of P_i and let x_i be the last intersection vertex of P_i outside H. Note that y_1, y_2, y_3 may not be distinct and at least two of x_1, x_2, x_3 coincide. Without loss of generality, we assume $x_1 = x_2$. If $x_1 = x_2 = x_3$, then let $P'_i = x_1 P_i y_i (1 \leq i \leq 3)$. We can see that P'_1, P'_2, P'_3 are three internally disjoint (x_1, H)-paths in G. Now we consider $x_1 \neq x_3$. Let y be the last intersection vertex of P_1 and P_3 outside H, and let $P'_1 = x_1 P_1 y_1$, $P'_2 = x_1 P_2 y_2$, and $P'_3 = x_1 P_1 y P_3 y_3$. It is easy to see that P'_1, P'_2, P'_3 are three internally disjoint (x_1, H)-paths in G. By the pigeonhole principle, we have that at least two of y_1, y_2 and y_3 belong to the same partite of H. Without loss of generality, suppose $y_1, y_2 \in X$. If the lengths of P'_1 and P'_2 have the same parity, then $H \cup P'_1 \cup P'_2$ is a 2-edge-connected bipartite graph with more vertices than H, a contradiction. Now we have that the lengths of P'_1 and P'_2 have different parities. Without loss of generality, we assume that the lengths of P'_1 and P'_3 have the same parity. If $y_3 \in X$, let $H' = H \cup P'_1 \cup P'_3$; if $y_3 \in Y$, let $H' = H \cup P'_2 \cup P'_3$. We have that H' is a 2-edge-connected bipartite graph with more vertices than H, which contradicts to the maximality of H.

If H is a spanning subgraph of G, then $pc(G) = 2$ by Lemmas 1.1 and 1.3. Otherwise, by the assumption of Case 2, the components of $G - H$ have the following two types: (1) an isolated vertex; (2) an edge. Let $A_1, \ldots, A_p, B_1, \ldots, B_q$ be the components of $G - H$ such that $|A_i| = 1$ $(0 \leq i \leq p)$ and $|B_j| = 2$ $(0 \leq j \leq q)$, where p, q are nonnegative integers, and $p = 0$ or $q = 0$ means that there is no A_i-type component or B_j-type component. Let $N(B_j) = \{a_j, b_j\}$. Then $a_j \neq b_j$ since G is 2-connected. If a_j and b_j are in different partite sets of H, then $B_j \cup H$ is also a 2-edge-connected bipartite graph, which contradicts to the maximality of H. So, for each B_j we have that a_j and b_j are in the same partite set of H. Let $C(a, b) = \{B_i | N(B_i) = \{a, b\}, 1 \leq i \leq q\}$. Since H is a 2-edge-connected bipartite graph, it follows from Lemma 1.3 that H has a 2-edge-coloring c which makes H have the strong property under c. If $|C(a, b)| \geq 2$, then $G[V(C(a, b)) \cup \{a, b\}] - ab$ is a 2-edge-connected bipartite graph. Thus, there is a 2-edge-coloring c such that $G[V(C(a, b)) \cup \{a, b\}] - ab$ has the strong property under c by Lemma 1.3. Now we color the edges of $G \setminus \{A_1, \ldots, A_p\}$ with two colors $\{1, 2\}$. Firstly, we color the edges of H such that H has the strong property under this coloring. Then, we color the edges of $G[X]$ and $G[Y]$ with color 2. If $|C(a, b)| \geq 2$, then we color the edges of $G[V(C(a, b)) \cup \{a, b\}] - ab$ such that $G[V(C(a, b)) \cup \{a, b\}] - ab$ has the strong property under this coloring. If $|C(a,b)| = 1$, then $G[V(C(a,b)) \cup \{a,b\}] - ab$ is a path P with length 3. Thus, we color the two pendant edges of P with color 1 and the central edge of P with color 2.

Next, we will show that this 2-edge-coloring c makes $G \setminus \{A_1, \ldots, A_p\}$ proper connected. Let u, v be any two vertices of $G \setminus \{A_1, \ldots, A_p\}$. If both u and v are in H, then there is already a proper path connecting them in H. If one of u, v is in H, without loss of generality, let $u \in H$, then v has a neighbor v' in H. Since H has the strong property under c, it follows that there is a proper path P connecting u and v' in H such that $end(P) \neq c(vv')$, and $P \cup \{vv'\}$ is a proper path connecting u, v. If $\{u, v\} \in V(C(a,b))$, then there is already a proper path connecting them in $G[V(C(a,b)) \cup \{a,b\}] - ab$. Suppose that $u \in V(C(a,b))$ and $v \in V(C(a',b'))$. Since $diam(G) = 3$, we have $E(\{a,b\}, \{a',b'\}) \neq \emptyset$. If $\{a,b\}$ and $\{a',b'\}$ are in the same partite set of H, then we consider the following two cases based on the relationship between b and b'. Without loss of generality, let $aa' \in E(\{a,b\}, \{a',b'\})$. If either $b \neq b'$, or $b = b'$ and $|C(a,b)| = 1$, or $b = b'$ and $|C(a',b')| = 1$, then it is easy to see that there is a proper path P_1 connecting u and a with $end(P_1) = 1$ in $G[V(C(a,b)) \cup \{a,b\}] - ab$, and there is also a proper path P_2 connecting v and a' with $end(P_2) = 1$ in $G[V(C(a',b')) \cup \{a',b'\}] - a'b'$. Thus, $P_1 \cup \{aa'\} \cup P_2$ is a proper path connecting u and v. If $b = b'$ and $|C(a,b)| \geq 2$, $|C(a',b')| \geq 2$, then it is easy to check that there is a proper path P_1 connecting u and b with $end(P_1) = 1$ in $G[V(C(a,b)) \cup \{a,b\}] - ab$, and there is also a proper path P_2 connecting v and b' with $end(P_2) = 2$ in $G[V(C(a',b')) \cup \{a',b'\}] - a'b'$. Thus, $P_1 \cup P_2$ is a proper path connecting u and v. Now suppose that $\{a,b\}$ and $\{a',b'\}$ are in different partite set of H. It is easy to see that there is already a proper $(\{a,b\}, \{a',b'\})$-path P in H with $start(P) = end(P) = 2$. Without loss of generality, let a, a' be the two endvertices of P. Therefore, P contains no b and b'. It is easy to check that there is a proper path P_1 connecting u and a with $end(P_1) = 1$ in $G[V(C(a,b)) \cup \{a,b\}] - ab$, and there is also a proper path P_2 connecting v and a' with $end(P_2) = 1$ in $G[V(C(a',b')) \cup \{a',b'\}] - a'b'$. Thus, $P_1 \cup P \cup P_2$ is a proper path connecting u and v. Hence, $G \setminus \{A_1, \ldots, A_p\}$ is proper connected. Since $d_G(v) = 2$ for each vertex v of $A_i (1 \leq i \leq p)$, it follows that $pc(G) = 2$ by Lemma 1.4, completing the proof. □

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References

[1] E. Andrews, E. Laforge, C. Lumduanhom, P. Zhang, On proper-path colorings in graphs, J. Combin. Math. Combin. Comput. 97 (2016) 189–207.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, in: GTM, vol. 244, Springer, 2008.
[3] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, Zs. Tuza, Proper connection of graphs, Discrete Math. 312 (2012) 2550–2560.
[4] C. Brause, T.D. Doan, I. Schiermeyer, Minimum degree conditions for the proper connection number of graphs, Manuscript.
[5] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem 133 (2008) 85–98.
[6] R. Gu, X. Li, Z. Qin, Proper connection number of random graphs, Theoret. Comput. Sci. 609 (2016) 336–343.
[7] F. Huang, X. Li, Z. Qin, C. Magnant, Minimum degree condition for proper connection number 2, Theoret. Comput. Sci. (2017) in press. http://dx.doi.org/10.1016/j.tcs.2016.04.042.
[8] F. Huang, X. Li, S. Wang, Proper connection numbers of complementary graphs, Bull. Malays. Math. Sci. Soc. (2017) in press. http://dx.doi.org/10.1007/s40840-016-0381-8.
[9] F. Huang, X. Li, S. Wang, Proper connection number and 2-proper connection number of a graph, arXiv:1507.01426 [math.CO].
[10] X. Li, C. Magnant, Properly colored notions of connectivity - a dynamic survey, Theory Appl. Graphs 0 (2015). Art.2. Updated in December 2016.
[11] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, Graphs Combin. 29 (2013) 1–38.
[12] X. Li, Y. Sun, Rainbow Connections of Graphs, Springer Briefs in Math., Springer, New York, 2012.
[13] X. Li, M. Wei, J. Yue, Proper connection number and connected dominating sets, Theoret. Comput. Sci. 607 (2015) 480–487.
[14] C. Thomassen, Paths, circuits and subdivisions, in: Selected Topics in Graph Theory, vol. 3, Academic Press, San Diego, CA, 1988, pp. 97–131.