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Minimum Degree Conditions for the Proper Connection Number of Graphs

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Abstract An edge-coloured graph G is called *properly connected* if any two vertices are connected by a path whose edges are properly coloured. The *proper connection number* of a graph G, denoted by $pc(G)$, is the smallest number of colours that are needed in order to make G properly connected. In this paper, we consider sufficient conditions in terms of the ratio between minimum degree and order of a 2-connected graph G implying that G has proper connection number 2.

Keywords Proper connection number · 2-connected graphs · Minimum degree

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1 Introduction

We use [4] for terminology and notation not defined here and consider simple and undirected graphs G of order $n(G)$ and size $m(G)$, unless specified otherwise.

As an extension of proper colourings and motivated by rainbow connections of graphs, Andrews et al. [1] and, independently, Borozan et al. [2] introduced the concept of proper connections in graphs. An edge-coloured graph G is called *properly connected* if any two vertices $u, v \in V(G)$ are connected by a path whose edges are properly coloured. The proper connection number $pc(G)$ of a graph G is the smallest number of colours needed to colour G properly connected. We say, an edge-colouring c has the *strong property* if for any two vertices $u, v \in V(G)$ there exist two (not necessarily vertex or edge disjoint) properly coloured paths $P_1 : u = w_1w_2 \dots w_k = v$ and $P_2 : u = z_1z_2 \dots z_l = v$ such that $c(w_1w_2) \neq c(z_1z_2)$ and $c(w_{k-1}w_k) \neq c(z_{l-1}z_l)$.

For simplifying notation, let $[k]$ be the set $\{1, 2, \dots, k\}$ for some positive integer k and let $\sqcup$ be the operation symbol for the disjoint union of sets. If G is a graph and $U_1, U_2 \subseteq V(G)$ are two disjoint vertex sets, then $[U_1, U_2]$ denotes the set of edges between vertices of U_1 and vertices of U_2. Furthermore, let G be a graph, $u, v \in V(G)$ be two distinct vertices, and $P : w_1w_2 \dots w_k$ be a path, vertex disjoint from G. We say, we *add the ear* P to G by adding P and the edges uw_1 and vw_k. Hence, for a Θ-*graph* G, there exist a cycle C and a path P such that G is obtained by adding the ear P to C. Moreover, we define a 2-*ear cycle* and a 3-*ear cycle* to be a graph obtained by adding some ear to a Θ-graph or a 2-ear cycle, respectively. For a path P and two of its vertices $x_1, x_2 \in V(P)$, we denote by x_1Px_2 the subpath of P from x_1 to x_2.

Borozan et al. [2] proved some results depending on the connectivity of a graph.

Theorem 1.1 (Borozan et al. [2]) *If G is a 2-connected graph, then there exists an edge-colouring $c : E(G) \to [3]$ having the strong property.*

Furthermore, Borozan et al. [2] introduced a construction to obtain 2-connected graphs having proper connection number 3, for example graph B in Fig. 1. All those graphs contain odd cycles, and we note that 2-connected bipartite graphs have proper connection number at most 2.

Theorem 1.2 (Borozan et al. [2]) *If G is a 2-connected bipartite graph, then there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

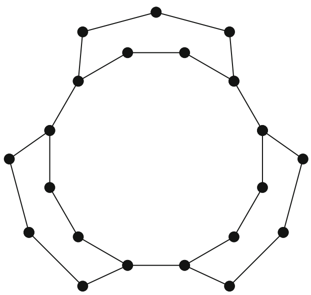

Fig. 1 Graph B with proper connection number 3

By a result of Paulraja in [7], every 3-connected graph G has a 2-connected bipartite spanning graph. Therefore, Borozan et al. deduced the following result.

Theorem 1.3 (Borozan et al. [2]) *If G is a 3-connected graph, then there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

Based on their construction for 2-connected graphs of proper connection number 3, the authors conjectured the following in [2].

Conjecture 1.4 (Borozan et al. [2]) *If G is a graph of connectivity $\kappa(G) = 2$ and minimum degree $\delta(G)$ at least 3, then $pc(G) \leq 2$.*

In this paper, we study sufficient conditions related to the minimum degree and implying a proper connection number at most 2. In particular, we disprove Conjecture 1.4 by constructing a series of 2-connected graphs G_i such that $\delta(G_i) = i$, $n(G_i) = 42i$, and $pc(G_i) \geq 3$.

Theorem 1.5 *For every integer $d \geq 3$, there exists a 2-connected graph G of minimum degree d and order $n = 42d$ such that $pc(G) \geq 3$.*

Furthermore, using our construction technique in a slightly different way, we can prove the following result.

Theorem 1.6 *For all integers $d, k \geq 2$, there exists a connected graph G of minimum degree d and order $n = (d+1)(k+1)$ such that $pc(G) = k$.*

By Theorem 1.5, one cannot bound the minimum degree of a 2-connected graph G from below by a constant such that $pc(G) \leq 2$ follows. Therefore, it is natural to ask for a ratio between minimum degree and order of a 2-connected graph, implying $pc(G) \leq 2$.

Theorem 1.7 *If G is a 2-connected graph of order $n = n(G)$ and minimum degree $\delta(G) > \max\{2, \frac{n+8}{20}\}$, then $pc(G) \leq 2$.*

2 Proofs

2.1 Proof of Theorems 1.5 and 1.6

For the proof of Theorem 1.5, we need the fact that the graph B in Fig. 1 has proper connection number 3 [2]. We will use this graph as a basic tool in our construction. As a further tool for our theorem, we need the following lemma.

Lemma 2.1 *Let $k \geq 3$ be an integer, $K_{k,k}$ be a complete bipartite graph on $2k$ vertices, G be a 2-connected graph of proper connection number at least 3, which is vertex disjoint from $K_{k,k}$, $v \in V(G)$ be one of its vertices of degree at most 3, $\{v_i, i \in [d_G(v)]\}$ be its neighbours, u_1, u_2, u_3 be three distinct vertices of the same partite set of $K_{k,k}$, and G' be the graph obtained from G by removing v and adding the graph $K_{k,k}$ and the edges $u_i v_i$ for $i \in [d_G(v)]$. Now, $pc(G') \geq 3$ and G' is 2-connected.*

Proof Suppose, to the contrary, that $pc(G') \leq 2$. Trivially, G' is non-complete, implying $pc(G') = 2$ by the simple observation that $pc(G') = 1$ if and only if G' is complete [2]. Let w_1 and w_2 be two distinct vertices of G. We now define vertices $x_1, x_2 \in V(G')$ as follows: for $i \in [2]$, if w_i is different from v, then let $x_i = w_i$, otherwise let $x_i = u_1$. Let us consider an edge-colouring c' making G' properly connected. Now $c(e) = c'(e)$ for $e \in E(G) \cap E(G')$ and $c(vv_i) = c'(u_i v_i)$ for $i \in [d_G(v)]$ defines an edge-colouring of G.

By the definition of c', there exists a properly coloured path between x_1 and x_2 in G', say P. Recall, $x_1, x_2 \in (V(G) \backslash \{v\}) \cup \{u_1\}$, implying that no vertex of the added $K_{k,k}$ besides u_1 is an end-vertex of P. If P does not contain an edge of $\{u_i v_i : i \in [d_G(v)]\}$, then one can readily observe that the path P is properly coloured by c in G. If P contains only one edge of $\{u_i v_i : i \in [d_G(v)]\}$, say $u_i v_i$ for $i \in [d_G(v)]$, then $v = w_1$ or $v = w_2$. Hence, either $w_1 = x_1 P v_i v = w_2$ or $w_2 = x_2 P v_i v = w_1$ is a path in G properly coloured by c. If P contains exactly two edges of $\{u_i v_i : i \in [d_G(v)]\}$, renaming vertices if necessary, we may assume $u_1 v_1$ and $u_2 v_2$. Now, since at most one vertex of w_1, w_2 is a vertex of $K_{k,k}$, all internal vertices of $v_1 P v_2$ are vertices of $K_{k,k}$ and the length of $v_1 P v_2$ is even. By our supposition $pc(G') \leq 2$, we conclude $c'(u_1 v_1) \neq c'(u_2 v_2)$. Hence, either $w_1 = x_1 P v_1 v v_2 P x_2 = w_2$ or $w_1 = x_1 P v_2 v v_1 P x_2 = w_2$ is a path in G properly coloured by c. It remains to consider $d_G(v) = 3$ and $u_1 v_1, u_2 v_2, u_3 v_3 \in E(P)$. By the definition of G', $\{u_i v_i : i \in [d_G(v)]\}$ is an edge-cutset. Hence, renaming vertices if necessary, we may assume that $x_1 = u_1$ and that v_3 has the shortest distance on P to x_2. Therefore, $w_1 = v v_3 P x_2 = w_2$ is a path which is properly coloured by c in G.

By the above observations, for every pair of vertices $w_1, w_2 \in V(G)$, we find a properly coloured path, implying that G is properly connected by c. Moreover, the number of colours used by c is at most the number of colours used by c', implying $pc(G) \leq pc(G') = 2 < pc(G)$, a contradiction. Therefore, $pc(G') \geq 3$.

It remains to show the 2-connectivity of G'. Suppose, to the contrary, that G' has a cut-vertex, say x. If $x \in V(G) \cap V(G')$, then $G - x$ is disconnected, a contradiction. Furthermore, if $x \in V(G') \backslash V(G)$, then one can readily observe that $G - v$ is disconnected, a contradiction. Hence, G' is 2-connected. □

We are now able to prove Theorem 1.5. Recall its statement.

Theorem 1.5 *For every integer $d \geq 3$, there exists a 2-connected graph G of minimum degree d and order $n = 42d$ such that $pc(G) \geq 3$.*

Proof Let B be the graph in Fig. 1 of proper connection number 3 and ψ be a bijective function from $[n(B)]$ to $V(B)$, i.e. let ψ be a labelling function for $V(B)$. By choosing $k = d$, an iterative use of the construction described in Lemma 2.1 on $\psi(i)$ (with a new $K_{k,k}$ for each i) for $i \in [n(B)]$ constructs a graph B_d which is 2-connected and has proper connection number at least 3. Furthermore, one can readily observe that $\delta(B_d) = k = d$ and $n(B_d) = 42d$. □

The main idea of the proof of Theorem 1.5 is replacing a vertex by a $K_{k,k}$. If we do not ask for an example which is 2-connected, we may replace vertices by K_k. Therefore, we obtain the next result.

Theorem 1.6 *For all integers $d, k \geq 2$, there exists a connected graph G of minimum degree d and order $n = (d+1)(k+1)$ such that $pc(G) = k$.*

Proof If $k = 2$, then let the graph G consists of 3 pairwise disjoint vertex sets U_1, U_2, U_3 of cardinality $d+1$ such that $G[U_1, U_2]$ is a complete bipartite graph, $G[U_2, U_3]$ a complete bipartite graph minus a perfect matching, and $G[U_1, U_3]$ contains no edges. It can be readily seen that G is connected, non-complete and bipartite, $\delta(G) = d$, $n(G) = 3(d+1)$, and, since $d \geq 2$, G is 2-connected. By Theorem 1.2 and the fact that $pc(G) = 1$ if and only if G is complete [2], $pc(G) = 2$. Hence, we may assume $k \geq 3$.

Let G be the graph obtained from $k+1$ cliques $C_1, C_2, \ldots, C_{k+1}$ of size $d+1$, each containing a labelled vertex $v(C_i)$, by adding edges such that the graph induced by $\{v(C_i) : i \in [k+1]\}$ is a star $K_{1,k}$. Renaming cliques if necessary, we may assume that $v(C_{k+1})$ is the center of the star. Hence, G is connected and $n(G) = (k+1)(d+1)$. Furthermore, it can be easily seen, that there exists exactly one internally vertex disjoint path between $v(C_{i_1})$ and $v(C_{i_2})$ for distinct $i_1, i_2 \in [k+1]$. Hence, any colouring c making G properly connected makes $G[\{v(C_i) : i \in [k+1]\}]$ properly connected. Therefore, $pc(G) \geq pc(K_{1,k}) = k$ and, for our further considerations, we take an k-edge colouring making $G[\{v(C_i) : i \in [k+1]\}]$ properly connected. Moreover, for $i \leq k$, we colour all clique edges of C_i by a colour different from the colour on the edge connecting C_i to $v(C_{k+1})$. Let w be a vertex in C_{k+1} distinct from $v(C_{k+1})$. Now, we colour $v(C_{k+1})w$ by colour 1, for all vertices z in the non-empty set $V(C_{k+1}) \backslash \{v(C_{k+1}), w\}$, we colour $v(C_{k+1})z$ by colour 2, and colour all remaining edges by colour 3. Thus, as one can easily check by a simple case to case analysis, G is properly connected using k colours and, since $pc(G) \geq pc(K_{1,k})$, we deduce $pc(G) = k$. □

2.2 Proof of Theorem 1.7

Before starting to prove Theorem 1.7, we will introduce some helpful results and some further (multi-)graphs. The first result is well-known as Menger's theorem.

Theorem 2.2 (Menger [6]) *If G is a graph, $U_1, U_2 \subseteq V(G)$ are two vertex disjoint sets, then the size of a minimum vertex-cut for U_1 and U_2 equals the maximum number of pairwise vertex disjoint $U_1 - U_2$ paths.*

Please note that the size of a minimum vertex-cut for two disjoint vertex sets U_1 and U_2 is at least the connectivity of the graph. Furthermore, we will not only use the existence of vertex disjoint paths in our proof, but we also need a minimum length of it. A useful result is given by Jackson in [5].

Theorem 2.3 (Jackson [5]) *If S is a 2-connected bipartite graph with bipartition (S_1, S_2) and $u, v \in V(S)$ are two of its vertices, then S contains an $u - v$ path of length at least $2\delta' - 2$, where $\delta' = \min\{d_G(z) : z \in V(S)\backslash\{u, v\}\}$.*

Coming back to the proper connection number of a graph, we may add two further results. The first one follows from the fact that a hamiltonian path can be coloured alternatingly by two colours making the graph properly connected.

Lemma 2.4 (Borozan et al. [2]) *If G is traceable, then $pc(G) \leq 2$.*

Borozan et al. noted that some of their results will be also true if one replaces 2-connectivity by 2-edge-connectivity, for example Theorem 1.2. Although an explicit proof is not given in [2], it is not hard to see that colouring all maximal 2-connected subgraphs G' in a bridgeless (i.e. 2-edge connected) bipartite graph with an edge colouring $c : E(G') \to [2]$ having the strong property gives the next result.

Theorem 2.5 (Borozan et al. [2]) *If G is a bipartite, bridgeless graph, then there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

For simplifying our proof, we introduce graph families. Let G be a multigraph shown in one of the Figs. 2, 3, 4, 5 and 6. We say, a graph G' belongs to the family $\mathcal{S}(G)$ if and only if it can be obtained from G by subdividing edges. We note that thick edges can be seen as the last added ear in an ear decomposition and will play a special role later on.

Right from its definition and by a simple case to case analysis, we obtain the following two facts.

Fact 2.6 *If $\mathcal{G}$ is the set of all 2-ear cycles, then $\mathcal{G} = \bigsqcup_{i \in [4]} \mathcal{S}(S_i)$.*

Fact 2.7 *If $\mathcal{G}$ is the set of all 3-ear cycles, then*

$$\mathcal{G} = \left(\bigcup_{j \in [4]} \mathcal{S}(S_1^j) \right) \cup \left(\bigcup_{j \in [12]} \mathcal{S}(S_2^j) \right) \cup \left(\bigcup_{j \in [13]} \mathcal{S}(S_3^j) \right) \cup \left(\bigcup_{j \in [6]} \mathcal{S}(S_4^j) \right)$$

and the thick edges represent the last added ear.

We are now in a position to prove Theorem 1.7. Recall its statement.

Theorem 1.7 *If G is a 2-connected graph of order $n = n(G)$ and minimum degree $\delta(G) > \max\{2, \frac{n+8}{20}\}$, then $pc(G) \leq 2$.*

Proof At the beginning of our proof let us mention some basic results we will use later on. First of all, we study the proper connection number in traceable graphs such as cycles and Θ-graphs as well as in 2-ear-cycles.

Claim 2.8 *Each cycle, Θ-graph, and 2-ear-cycle has proper connection number at most 2.*

Proof By Lemma 2.4, it remains to consider 2-ear cycles since cycles and Θ-graphs are traceble.

Let G be a 2-ear cycle. Since it is constructable from a Θ-graph, it consists of an even cycle, say $C : u_1u_2 \ldots u_{2l}$ ($l \geq 2$), and two added ears, say R_1 and R_2. Furthermore, renaming ears or vertices if necessary, we may assume that u_i and u_j with $1 \leq i < j \leq 2l$ are the end-vertices of R_1. If an end-vertex of R_2 is on C, say u_k ($k \in [2l]$ and $i = k$ or $j = k$ is possible), then we colour the edges of C alternatingly by two colours, the edges of R_1 and R_2 such that $u_{i-1}u_iR_1u_j$ and $u_{k+1}u_kR_2$ are properly coloured by two colours. By some simple case to case analysis, it can be

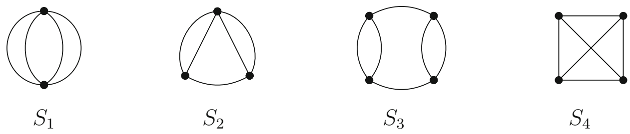

Fig. 2 Graphs S_1, S_2, S_3, S_4

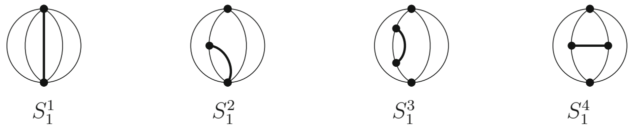

Fig. 3 Graphs $S_1^1, \ldots, S_1^4$

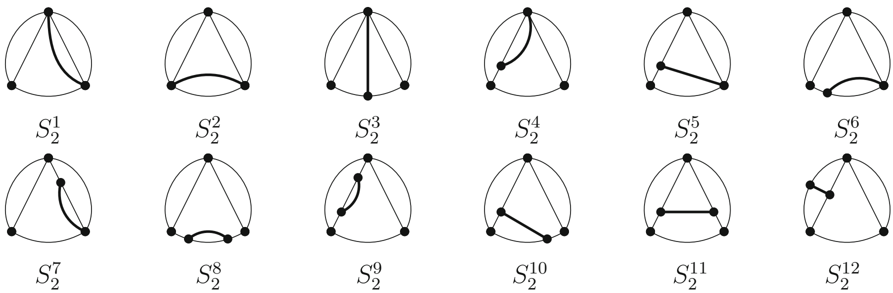

Fig. 4 Graphs $S_2^1, \ldots, S_2^{12}$

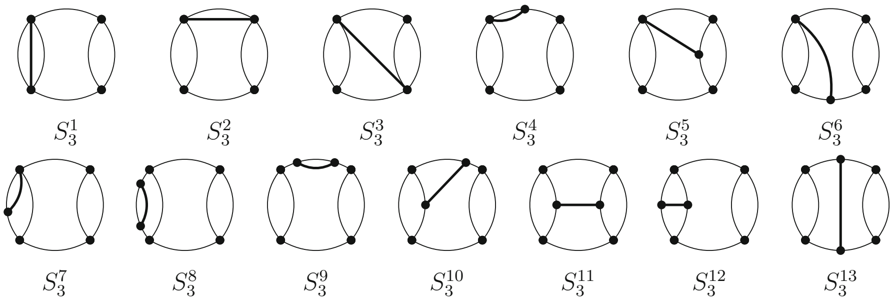

Fig. 5 Graphs $S_3^1, \ldots, S_3^{13}$

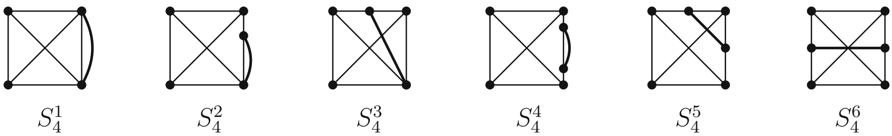

Fig. 6 Graphs $S_4^1, \ldots, S_4^6$

easily seen that this colouring makes G properly connected. If no end-vertex of R_2 is on C, then the end-vertices of R_2 are two distinct vertices of $R_1 - \{u_i, u_j\}$, say u_{k_1} and u_{k_2}. Renaming vertices if necessary, we may assume that u_{k_1} has the smaller distance to u_i on R_1. We colour the edges of C alternatingly by two colours, and the edges of R_1 and R_2 such that $u_{i-1}u_iR_1u_{k_1}R_2u_{k_2}$ and $u_{j+1}u_jR_1u_{k_1}$ are are properly coloured by two colours. Again, by some simple case to case analysis, it can be easily seen that this colouring makes G properly connected. □

Now we are able to prove our theorem. Suppose, to the contrary, that G is a 2-connected graph of order $n = n(G)$, minimum degree $\delta(G) > \max\{2, \frac{n+8}{20}\}$, and proper connection number at least 3. Trivially, any 2-connected graph has a cycle as a subgraph. Furthermore, by ear decomposition, any 2-connected graph which is not a cycle or a Θ-graph has a 2-ear cycle as a subgraph. By our supposition and Claim 2.8, we may assume that G contain 2-ear cycles as subgraphs. Let us take one of largest order, say Q. Note that Q is 2-connected and $pc(Q) \leq 2$ by Claim 2.8. Now we take a subgraph H of G such that

(i) Q is a subgraph of H, H is 2-connected, $pc(H) \leq 2$, and
(ii) subject to (i) , $n(H)$ is maximum.

Clearly, by the existence of Q, we can always find such a graph. In what follows is a series of claims concluding in the implication of the non-existence of G.

Claim 2.9 (Borozan et al. [2]) *Let G' be a graph and H' be a subgraph G of proper connection number at most 2. If $u \in V(G')\backslash V(H')$ has at least two neighbours in $V(H')$, then $pc(G'[V(H') \cup \{u\}]) \leq 2$.*

Claim 2.10 *If u_1, u_2 are two distinct vertices of H and $P : v_1v_2, \ldots v_k$ is a path, vertex disjoint from H, of order $k \geq 1$, then the graph H' obtained by adding edges u_1v_1 and u_2v_k is 2-connected.*

Proof Suppose, to the contrary, that there exists a cut-vertex z in H'. If $z \in V(H)$, then z is a cut-vertex in H, contradicting the assumption on H. Hence, $z \in V(P)$. But trivially, $H' - z$ is connected, a contradiction. Thus, H' is 2-connected. □

More precisely, by Claims 2.9 and 2.10, we may assume that there exists no vertex in $V(G)\backslash V(H)$ having two neighbours in H. Otherwise, we have a contradiction to the maximality of H.

Claim 2.11 *There exists no cycle C in G of even length such that $V(H) \cap V(C) \neq \emptyset$, $(V(G)\backslash V(H)) \cap V(C) \neq \emptyset$, and a colouring of H, using two colours and making H properly connected, restricted to the edges of C makes $C[V(H)]$ properly coloured.*

Proof Suppose, to the contrary, that there exists such a cycle. Note that by the assumption that $C[V(H)]$ is properly coloured, $C[V(H)]$ is connected. Hence, the number of edges between vertices of H and vertices of $V(C)\backslash H$ is exactly two. For simplicity, let us denote $C = w_1w_2 \ldots w_kv_1v_2 \ldots v_l$, where $w_1, w_2 \ldots, w_k \in V(H)$ and $v_1, v_2, \ldots, v_l \in V(G)\backslash V(H)$. Note that possibly $k = 1$ or $l = 1$ but $k + l \geq 4$. By our assumptions, the colouring of H restricted to the edges $w_1w_2, w_2w_3, \ldots, w_{k-1}w_k$

of C gives a proper colouring for the path $w_1w_2\cdots w_k$. By continuing the alternating colouring on C, we obtain a proper colouring of C having the strong property, i.e. two adjacent edges have distinct colours.

Recall, subgraphs C and H are properly connected on their own. Therefore, it remains to show the existence of a properly coloured path between all pairs of vertices $v_i \in V(C)$ and $w \in V(H)\backslash\{w_1, w_2, \ldots, w_k\}$. Since H has proper connection number at most 2, there is a shortest properly coloured path R between w and a vertex $w_j \in V(C)$ in H. We note that $V(R) \cap V(C) = \{w_j\}$. Furthermore, since C is properly coloured with the strong property, we can choose one of the two distinct paths between w_j and v_i in C, say P, in such a way that wRw_jPv_i is properly coloured. Hence, $G[V(H) \cup V(C)]$ has proper connection number at most 2 and is 2-connected by Claim 2.10, contradicting the maximality of H. □

Claim 2.12 *$G - V(H)$ is bipartite.*

Proof Suppose, to the contrary, that C' is an odd cycle in $G - V(H)$. By Menger's Theorem (i.e. Theorem 2.2), there are two vertex disjoint paths P_1 and P_2 between $V(H)$ and $V(C')$. We note that there lengths are possibly one. We denote by x_1 and x_2 the end-vertices of P_1 and P_2 in H, respectively, as well as by z_1 and z_2 the end-vertices of P_1 and P_2 in C', respectively. Since H has proper connection number at most 2, there exists a properly coloured path between x_1 and x_2, say P^{x_1,x_2}. Furthermore, let us denote by R_1 and R_2 the two distinct paths connecting z_1 and z_2 in C'. Hence, by the odd length of C', $C : x_1P_1z_1R_1z_2P_2x_2P^{x_1,x_2}x_1$ or $C : x_1P_1z_1R_2z_2P_2x_2P^{x_1,x_2}x_1$ is an even cycle such that $C[V(H)] = x_1P^{x_1,x_2}x_2$ is properly coloured, contradicting Claim 2.11. □

Claim 2.13 *No vertex of H is adjacent to two vertices of the same component S of $G - V(H)$.*

Proof Suppose, to the contrary, that $u \in V(H)$ has two neighbours v_1 and v_2 in one component S of $G - V(H)$. Now, let R be a shortest path between v_1 and v_2 in $G - V(H)$. By Claim 2.11, $n(uv_1Rv_2u)$ is odd. By Menger's Theorem (2.2), there exists a path, say R', between a vertex of $V(R)$, say v_r, and a vertex of $V(H)$, say u', which does not contain u or any vertex of $V(R)\backslash\{v_r\}$. Let P be a properly coloured path between u and u'. Now, either $C : uv_1Rv_rR'u'Pu$ or $C : uv_2Rv_rR'u'Pu$ is an even cycle such that $C[V(H)] = uP'u'$ is properly coloured, contradicting Claim 2.11. □

Claim 2.14 *There exists no 2-edge-connected subgraph S of $G - V(H)$ such that $|[V(H), V(S)]| \geq 2$.*

Proof Suppose, to the contrary, that there exists such a subgraph. Let us take a 2-edge-connected subgraph S of $G - V(H)$ such that $|[V(H), V(S)]| \geq 2$ and, with respect to this condition, $n(S)$ is minimum. Hence, we may assume that there exist two edges $u_1v_1, u_2v_2 \in [V(H), V(S)]$ such that $u_1, u_2 \in V(H)$ and $v_1, v_2 \in V(S)$. By the maximality of H, Claim 2.9, and Claim 2.10, $v_1 \neq v_2$ and, by Claim 2.13, $u_1 \neq u_2$. Furthermore, by the minimality of S, $G[V(H) \cup V(S)]$ is 2-connected. Otherwise, there exists a cut-vertex, say x, implying, by the 2-connectivity of H, $x \in V(S)$. But now, $G[V(H) \cup V(S)] - x$ consists of $k \geq 2$ components $S_1, \ldots, S_k$, implying that there is an $i \in [k]$ such that $V(S_i) \subseteq V(S)$, $S - V(S_i)$ is 2-edge-connected, and $|[V(H), V(S) \backslash V(S_i)]| \geq 2$, contradicting the minimality of S.

Now by Theorem 2.5, there exists an edge-colouring for S using two colours having the strong property. Let us use such a colouring and an edge-colouring of H making H properly connected by using the same two colours. Clearly, there exists a properly coloured path, say R, between u_1 and u_2 in H. We note that the length of this path is at least 1 since $u_1 \neq u_2$. We extend the colouring of R such that $v_1 u_1 R u_2 v_2$ is properly coloured by 2 colours. Now, H and $G[S]$ are properly connected. If $w \in V(H)$ and $z \in V(S)$, then let R' be a shortest properly coloured path in H from w to a vertex of R, say w'. We note that R' is of possible length 0, i.e. $w = w'$, but $|V(R) \cap V(R')| = 1$. Now, either $wR'w'Ru_1v_1$ or $wR'w'Ru_2v_2$ is a properly coloured path. Renaming vertices and paths if necessary, we may assume the first case. By the strong property of the edge-colouring used for S, we can extend the properly coloured path $wR'w'Ru_1v_1$ to a properly coloured path between w and z. Hence, between any two vertices $w \in V(H)$ and $z \in V(S)$ there is a properly coloured path. Therefore, $G[V(H) \cup V(S)]$ has proper connection number at most 2, contradicting the maximality of H. □

Please note that any 2-connected graph is 2-edge-connected. Therefore, Claim 2.14 remains true if we replace 2-edge-connectivity by 2-connectivity.

Clearly, since G is 2-connected, there are at least two edges between any component S of $G - V(H)$ and $V(H)$. Furthermore, Claim 2.14 implies that S contains a bridge. Let T be the block-cut-vertex-tree of S, i.e. the vertices of T represent all maximal 2-connected graphs in S (also known as blocks) and there is an edge between two vertices of T if and only if the corresponding blocks are connected by a cut-vertex (see [4]). Trivially, T is a graph (and to me more precisely, it is a tree) and contains at least two leaves, say t_1 and t_2. Furthermore, let T_1 and T_2 be the two 2-connected graphs which correspond to t_1 and t_2, respectively.

Claim 2.15 *For $i \in [2]$, $V(T_i) \geq 3$.*

Proof Suppose, to the contrary, there exists an $i \in [2]$ such that $|V(T_i)| \leq 2$. Clearly, since T_i is a block, T_i is a K_2. Furthermore, for $i \in [2]$, $V(T_i)$ contains exactly one cut-vertex, say t_i^S, in S. We denote by t_i^H the second vertex of T_i. By the 2-connectivity of G, t_i^H has at least and, by Claim 2.9, at most one neighbour in $V(H)$. Since t_i^H is no cut-vertex of S, $d_G(t_i^H) = 2$, a contradiction to the fact that $\delta(G) > 2$. □

By Claim 2.15, $|V(T_1)|, |V(T_2)| \geq 3$. Clearly, for $i \in [2]$, $V(T_i)$ contains exactly one cut-vertex, say t_i^S, in S. Therefore, by the 2-connectivity of G, $|[V(T_i), V(H)]| \geq 1$ and we deduce equality by Claim 2.14. Clearly, again by the 2-connectivity of G, t_i^S is distinct from the vertex in T_i incident to the edge in $[V(T_i), V(H)]$, say t_i^H, for $i \in [2]$. Furthermore, since $\delta(G) \geq 3$ and $V(T_i) \geq 3$, $\min\{d_G(v) : v \in V(T_i) \backslash \{t_i^H, t_i^S\}\} \geq \delta(G)$ for $i \in [2]$. By Claim 2.14, we can assume that $t_1^S \neq t_2^S$.

By Theorem 2.3, there exists a path, say P_i, in T_i between t_i^H and t_i^S of length at least $2\delta_i' - 2$, where $\delta_i' = \min\{d_G(t) : t \in V(T_i) \backslash \{t_i^H, t_i^S\}\} \geq \delta(G)$, for $i \in [2]$.

Furthermore, let R be a path connecting t_1^S and t_2^S in S. Let u_1 and u_2 be the neighbours of t_1^H and t_2^H in $V(H)$, respectively. By Claim 2.13, they are distinct. By the 2-connectivity of H and Menger's Theorem (i.e. Theorem 2.2), there are two vertex disjoint paths Q_1 and Q_2 between $\{u_1, u_2\}$ and $V(Q)$. Let u_1, q_1 be the end-vertices of Q_1 and u_2, q_2 be the end-vertices of Q_2. We note that q_1 and q_2 are disjoint and that the lengths of Q_1 and Q_2 are possibly 0, namely $m(Q_i) = 0$ if and only if $u_i \in V(Q)$ for $i \in [2]$. Therefore, $P : q_1 Q_1 u_1 t_1^H P_1 t_1^S R t_2^S P_2 t_2^H u_2 Q_2 q_2$ is a path of length at least $4\delta(G) - 1$ connecting q_1 and q_2.

Let Q' be the graph obtained by adding ear $P - \{q_1, q_2\}$ to Q. Now, we can continue with a fact which can be observed by a small case to case analysis. Recall, the thick edges in Figs. 3, 4, 5 and 6 represent the last added ear.

Fact 2.16 *Any multigraph in* $\{S_1^1, \ldots, S_1^4, S_2^1, \ldots, S_2^{12}, S_3^1, \ldots, S_3^{13}, S_4^1, \ldots, S_4^6\}$ *has* 4 *non-thick edges, say* e_1, e_2, e_3, e_4*, such that* $G - e_k$ *is a multigraph which can be obtained by subdividing edges, if necessary, of a multigraph* S_1, S_2, S_3*, or* S_4 *for* $k \in [4]$.

Using the maximality of Q we deduce our last claim.

Claim 2.17 $n(Q) \geq 16\delta(G) - 6$

Proof By Fact 2.7, there are some i and j such that $Q' \in \mathcal{S}(S_i^j)$ and P correspond to the subdivision of the thick edge. Furthermore, let e_1, e_2, e_3, e_4 be the 4 edges given by Fact 2.16 and $P_{e_1}, P_{e_2}, P_{e_3}, P_{e_4}$ be their corresponding paths in Q'. Since, by Fact 2.16, $S_i^j - e_k$ is a a multigraph which can be obtained by subdividing edges, if necessary, of a multigraph S_1, S_2, S_3, or S_4, we deduce $\mathcal{S}(S_i^j - e_k) \subseteq \mathcal{S}(S_1 - e_k) \cup \mathcal{S}(S_2 - e_k) \cup \mathcal{S}(S_3 - e_k) \cup \mathcal{S}(S_4 - e_k)$ and, by Fact 2.6, $Q' - \{v \in V(P_{e_k}) : d_{P_{e_k}}(v) = 2\}$ is a 2-ear cycle for all $k \in [4]$. Therefore, by the maximality of Q, the lengths of $P_{e_1}, P_{e_2}, P_{e_3}$ and P_{e_4} are at least the length of P. Hence, counting vertices, we obtain the desired result $n(Q) \geq 4[m(P) - 1] + 2 = 16\delta(G) - 6$. □

From the definition of P and Q it follows that $V(P) \cap V(Q) = \{q_1, q_2\}$, implying $n(G) \geq n(P) + n(Q) - 2 = 20\delta(G) - 8 > n(G)$, a contradiction. □

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