Problem 2

a) Using Taylor expansion, I got:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3} f'''(x) + \dots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3} f'''(x) + \dots$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{3} h^3 f'''(x) + \dots$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{8}{3} h^3 f'''(x) + \dots$$

. $f(x-2h) - f(x+2h) = -4hf'(x) - \frac{16}{3} h^3 f'''(x) + O(h^4)$

$\Leftrightarrow \frac{1}{12h}\left[f(x-2h) - f(x+2h)\right] = -\frac{1}{3} f'(x) - \frac{4}{9} h^3 f'''(x)$

. $f(x+h) - f(x-h) = 2hf'(x) + \frac{2}{3} h^3 f'''(x) + O(h^4)$

$\Leftrightarrow \frac{2}{3h}\left[f(x+h) - f(x-h)\right] = \frac{4}{3} f'(x) + \frac{4}{9} h^3 f'''(x)$

I got: $f'(x) = -\frac{1}{3} f'(x) - \frac{4}{9} h^3 f'''(x) + \frac{4}{3} f'(x) + \frac{4}{9} h^3 f'''(x)$

$$= \frac{1}{12h}\left[f(x-2h) - f(x+2h)\right] + \frac{2}{3h}\left[f(x+h) - f(x-h)\right]$$

$$= \frac{1}{12h}\left[f(x-2h) - f(x+2h) + 8f(x+h) - 8f(x-h)\right]$$

$$= \frac{1}{12h}\left[f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)\right]$$

In conclusion, $f'(x) = \frac{1}{12h}\left[f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)\right]$.