*Proposition 1: Let σ be an element of Sn. Then there exists a positive integer k such that σk = e.*

**Proof**:

There are n! possible permutations of a set of n elements. Therefore, there exists a positive integer k and n such that σk = σn and k < n.

We have σk (σ -1)k = σn (σ -1)k = σn-k = e

Therefore there exists a positive integer such that σ to that integer powered is equal to the identity element.

We are done 😊

*Proposition 2: Let k be the order of 𝜎, then iff k | m.*

**Proof**:

Let k be the order of 𝜎.

We have . Since k is the order of 𝜎, moves all elements around in a circle, which results in the same set. For , must behave the same way.

However, k is different from m that k is the smallest positive integer which satisfy so cycle all elements once. For m, it cycles the elements for some n times, and so m = kn.

Therefore, k | m.

We have k | m, therefore .

We are done 😊

*Proposition 3: Every cycle of length k has order k.*

**Proof**:

Suppose is a n-cycle which has an order of k.

Let

Assume that 0 < k < n

moves to . Since , must be equals to . However, the definition of cycle says that all are different. We see a contradiction here.

Therefore, .

Moreover, moves all in a circle one time. Since , then k = n.

We are done 😊

*Proposition 4: Disjoint cycles commute.*

**Proof**:

Let , where for all and .

Also,

The same goes for . Therefore, disjoint cycles commute.

We are done 😊

*Proposition 5: Let be a permutation of and let ~ be a relation of defined by:*

*for some*

*Prove that ~ is an equivalence relation.*

**Proof**:

Reflexive property:

Let k be the order of . That gives us

Symmetric property:

Let k be the order of . That gives us

Transitive property:

So

Let k be the order of . That gives us

Therefore

And so

Then ~ is an equivalence relation.

*Proposition 6: Every permutation can be written as the product of disjoint cycles.*

**Proof**:

Let be a permutation of S and let be the orbits of . For each , the permutation is a cycle . Since ~ is an equivalence relation, the orbits of are disjointed to each other. Also, . So , as well as any permutation, can be written as the product of disjoint cycles.

*Proposition 7: Every permutation is a product of unique transpositions.*

**Proof**:

For all cycles ,

We are done 😊

*Proposition 8: The Futurama Theorem*

**Proof**:

Let our set to be: {1, 2, 3, 4, 5, 6, … n}and the cycle to be: (1 2 3 4 … n). We can rewrite this as a product of unique cycles with additional points X and Y.

Let’s do (2 Y)(1 X)

(3 Y)(2 X) next.

After that, let’s do (4 Y)(4 X)

From this point onward, we only need to do (k Y)(k X) for k even and (k X)(kY) for k odd until the end.

(5 X)(5 Y)

…

**When n is odd:**

(n X)(n Y)

Now we need n to be in front and swap X and Y for them to be in the correct position.

Do (X Y)(1 Y)

**When n is even:**

(X Y)(1 Y)(n Y)

So, for a cycle (1 2 3 4 … n-1 n), we can rewrite it as:

(X Y)(1 Y)(n X)(n Y)(n – 1 Y)(n – 1 X)… (4 Y)(4 X)(3 Y)(2 X)(2 Y)(1 X) for n odd, and

(X Y)(1 Y)(n Y)(n – 1 X)(n – 1 Y)… (4 Y)(4 X)(3 Y)(2 X)(2 Y)(1 X) for n even.

We can consider the digits in the cycle (1 2 3 4 … n-1 n) as the indexes for any other cycles.

Therefore, any element of Sn can be written as the product of unique elements  
from the set {(1 X),(2 X), … , (n X), (1 Y), (2 Y), …, (n Y), (X Y)}*,* which  
can be thought of as particular transpositions inside of Sn+2.