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MAT321: Numerical Methods
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MAT-APC 321: Numerical Methods Homework 5: Numerical Integration

1 Integration

1.1 Suli Mayers 7.1

We consider the equally spaced quadrature points $x_k = a + kh$ for $k = 0, 1, ..., n$ and $n \geq 1$. We consider only the interval $[-1, 1]$. Therefore, we must have $x_k = -1 + k\frac{1-(-1)}{n} = -1 + k\frac{2}{n}$. Therefore, we also observe that:

$$x_{n-k} = -1 + (n-k)\frac{1-(-1)}{n} = -k\frac{2}{n} + 1 = 1 - k\frac{2}{n} = -x_k \tag{1}$$

So, we have for all k, $x_k = -x_{n-k}$. We know that $w_k = \int_{-1}^{1} L_k(x)dx$, with $L_k(x)$ defined as the kth lagrangian polynomial for the given nodes. We observe that:

$$L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \tag{2}$$

$$L_k(-x) = \prod_{i=0, i\neq k}^{n} (\frac{-x - x_i}{x_k - x_i} = \frac{-x + x_{n-i}}{-x_{n-k} + x_{n-i}}) \tag{3}$$

$$\tag{4}$$

Let $j = n - i$. We thus have:

$$L_k(-x) = \prod_{i=0, i\neq k}^{n} \frac{-x + x_{n-i}}{-x_{n-k} + x_{n-i}} = \prod_{j=0, j\neq n-k}^{n} \frac{x - x_j}{x_{n-k} - x_j} = L_{n-k}(x) \tag{5}$$

Thus, we have:

$$w_k = \int_{-1}^{1} L_k(x)dx \tag{6}$$

$$w_{n-k} = \int_{-1}^{1} L_{n-k}(x)dx = \int_{-1}^{1} L_k(-x)dx = \tag{7}$$

$$-\int_{-1}^{1} L_k(-x)d(-x) = -\int_{1}^{-1} L_k(x)dx = \int_{-1}^{1} L_k(x)dx = w_k \tag{8}$$

$$\boxed{w_k = w_{n-k}} \tag{9}$$

as desired.
Expanding this to an arbitrary interval $[a, b]$ instead of $[-1, 1]$, we see that $x_k = a + k\frac{b-a)}{n}$ and $x_{n-k} = a + (n-k)\frac{b-a)}{n} = b - k\frac{b-a)}{n}$, thus giving us:

$$x_{n-k} = -x_k + (a+b) \tag{10}$$

Because this is merely a shift of $(a+b)$ between the quadrature points, we observe that our results from the $[-1, 1]$ interval can be extended to the interval $[a, b]$ because the Lagrange polynomial would not change, giving us $L_k(-x + (a+b)) = L_{n-k}(x)$, so the change of variables in the integral for the quadrature weights w_k and w_{n-k} will remain the same as the results above, thus giving us $w_k = w_{n-k}$, as desired for all $[a, b]$.

1.2 Suli Mayers 7.3

We have the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$ with $0 < \alpha < 1$. We can approximate:

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-\alpha) + w_1 f(\alpha) \tag{11}$$

We know that the formula is exact when f is a polynomial of degree 1. Therefore, the formula must be exact for $f(x) = x$ and $f(x) = 1$, which form a basis for polynomials $f(x)$ of degree 1, so any $f(x)$ can be represented by a linear combination of those two polynomials, so if the formula is exact for $f(x) = x$ and $f(x) = 1$, it will be exact for all polynomials $f(x)$ of degree 1.
1. We have the first case $f(x) = x$:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} xdx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0 \tag{12}$$

$$\int_{-1}^{1} f(x)dx = 0 = w_0 f(-\alpha) + w_1 f(\alpha) = w_0(-\alpha) + w_1(\alpha) \tag{13}$$

$$\rightarrow \alpha(-w_0 + w_1) = 0 \rightarrow w_0 = w_1 \tag{14}$$

2. Further, we have $f(x) = 1$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} dx = x\Big|_{-1}^{1} = 2 = w_0 f(-\alpha) + w_1 f(\alpha) \tag{15}$$

$$\rightarrow w_0 + w_1 = 2 \tag{16}$$

Thus, because $w_0 = w_1$ and $w_0 + w_1 = 2$, we have $\boxed{w_0 = w_1 = 1}$, as desired, regardless of α.
For the formula to be exact for all polynomials of degree 2, we must also have the formula exact for $f(x) = x^2$ as well, in addition to $w_0 = w_1 = 1$.
3. For the case $f(x) = x^2$:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3} \tag{17}$$

$$\frac{2}{3} = w_0 f(-\alpha) + w_1 f(\alpha) = w_0(\alpha^2) + w_1(\alpha^2) = 2\alpha^2 \tag{18}$$

$$\boxed{\alpha = \sqrt{\frac{1}{3}}} \tag{19}$$

To show that the polynomial is also exact for polynomials of degree 3, we have to show equality of the formula for $f(x) = x^3$ as well:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4\Big|_{-1}^{1} = 0 \tag{20}$$

$$0 = w_0 f(-\alpha) + w_1 f(\alpha) = w_0((-\alpha)^3) + w_1(\alpha^3) = -\alpha^3 + \alpha^3 = 0 \tag{21}$$

$$\boxed{\int_{-1}^{1} f(x)dx = w_0 f(-\alpha) + w_1 f(\alpha)} \tag{22}$$

when $f(x)$ is a polynomia of degree 3, as desired.

1.3 Suli Mayers 7.4

The Newton-Cotes formula with $n = 3$ on the interval $[-1, 1]$ is given to be:

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(-1/3) + w_2 f(1/3) + w_3 f(1) \tag{23}$$

We know that the formula is exact for all polynomials of degree 3. Thus, the formula is exact for $f(x) = x^3$, $f(x) = x^2$, $f(x) = x$, and $f(x) = x$, which form a basis for all polynomials of degree 3. Furthermore, from 7.1, we know that Newton-Cotes quadrature for equi-spaced points on $[-1, 1]$, we have $w_k = w_{n-k}$, which gives us $w_0 = w_3$ and $w_1 = w_2$, which gives us the formula now:

$$\int_{-1}^{1} f(x)dx \approx w_0(f(-1) + f(1)) + w_1(f(-1/3) + f(1/3)) \tag{24}$$

1. $f(x) = 1$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} dxx\Big|_{-1}^{1} = 2 \tag{25}$$

$$2 = w_0(f(-1) + f(1)) + w_1(f(-1/3) + f(1/3)) = 2(w_0 + w_1) \tag{26}$$

$$\rightarrow \boxed{2(w_0 + w_1) = 2} \tag{27}$$

2. $f(x) = x$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} xdx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0 \tag{28}$$

$$0 = w_0(f(-1) + f(1)) + w_1(f(-1/3) + f(1/3)) = w_0(0) + w_1(0) = 0 \tag{29}$$

$$\tag{30}$$

3. $f(x) = x^2$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3} \tag{31}$$

$$\frac{2}{3} = w_0(f(-1) + f(1)) + w_1(f(-1/3) + f(1/3)) = w_0(2) + w_1((1/9) + (1/9)) = 2w_0 + \frac{2}{9}w_1 \tag{32}$$

$$w_1 = w_2 \rightarrow \boxed{2w_0 + \frac{2}{9}w_2 = \frac{2}{3}} \tag{33}$$

as desired. 4. We can confirm exactness for $f(x) = x^3$, as well:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4\Big|_{-1}^{1} = 0 \tag{34}$$

$$0 = w_0(f(-1) + f(1)) + w_1(f(-1/3) + f(1/3)) = w_0(0) + w_1(0) = 0 \tag{35}$$

$$\tag{36}$$

Therefore, we have the system of equations ($w_1 = w_2$, $w_0 = w_3$):
$\begin{cases} 2w_0 + \frac{2}{9}w_1 = \frac{2}{3} \\ 2(w_0 + w_1) = 2 \end{cases} \rightarrow \begin{cases} w_0 + \frac{1}{9}w_1 = \frac{1}{3} \\ w_0 + w_1 = 1 \end{cases} \rightarrow \frac{8}{9}w_1 = \frac{2}{3} \rightarrow \boxed{w_1 = w_2 = \frac{3}{4}} \rightarrow \boxed{w_0 = w_3 = \frac{1}{4}}$

1.4 Show

1. Newton-Cotes with $n = 1$:
Consider the interval $[a, b]$. We see that the Lagrange polynomials are:

$$L_0(x) = \frac{x - b}{a - b} \tag{37}$$

$$L_1(x) = \frac{x - a}{b - a} \tag{38}$$

Thus, we have the weights:

$$w_0 = \int_a^b L_0(x)dx = \int_a^b \frac{x - b}{a - b}dx = \frac{b - a}{2} \tag{39}$$

$$w_1 = \int_a^b L_1(x)dx = \int_a^b \frac{x - a}{b - a}dx = \frac{b - a}{2} \tag{40}$$

Thus, we have the formula:

$$\int_a^b f(x)dx \approx w_0 f(x_0) + w_1 f(x_1) = \frac{b - a}{2}(f(a) + f(b)) \tag{41}$$

We see that for $f(x) = 1$, it is exact:

$$\int_a^b f(x)dx = \int_a^b (1)dx = b - a = (\frac{b - a}{2})(1 + 1) = \frac{b - a}{2}(f(a) + f(b)) = w_0 f(x_0) + w_1 f(x_1) \tag{42}$$

We see that for $f(x) = x$, it is exact:

$$\int_a^b f(x)dx = \int_a^b xdx = \frac{1}{2}(b^2 - a^2) = (\frac{b - a}{2})(b + a) = \frac{b - a}{2}(f(a) + f(b)) = w_0 f(x_0) + w_1 f(x_1) \tag{43}$$

Nevertheless, for $f(x) = x^2$, it is not exact:

$$\int_a^b f(x)dx = \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3) \neq (\frac{b - a}{2})(b^2 + a^2) = w_0 f(x_0) + w_1 f(x_1) \tag{44}$$

Therefore, because the formula is not exact for polynomials of degree 2 but exact for polynomials of degree 1, Newton-Cotes with $n = 1$ has degree of precision 1, as desired.

2. We now consider Newton-Cotes with $n = 2$. Therefore, we have the quadrature nodes $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$. Thus, we have the weights (from Lecture Notes):

$$w_0 = w_2 = \int_a^b L_0(x)dx = \frac{b-a}{6} \tag{45}$$

$$w_1 = \int_a^b L_1(x)dx = 4(\frac{b-a}{6}) \tag{46}$$

Thus, we have the formula:

$$\int_a^b f(x)dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) = \frac{b-a}{6}(f(a) + f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) \tag{47}$$

We know that for $n = 2$, we must have degree of precision at least 2 (exact for polynomials of degree 2) because that is how we defined the quadrature rule, but we will confirm anyway.
We see that for $f(x) = 1$, it is exact:

$$\int_a^b f(x)dx = \int_a^b (1)dx = b - a \tag{48}$$

$$\frac{b-a}{6}(f(a) + f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) = \frac{b-a}{6}(2) + 4(\frac{b-a}{6}) = b - a \tag{49}$$

We see that for $f(x) = x$, it is exact:

$$\int_a^b f(x)dx = \int_a^b xdx = \frac{1}{2}(b^2 - a^2) \tag{50}$$

$$\frac{b-a}{6}(f(a) + f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) = \frac{b-a}{6}(a+b) + 4(\frac{b-a}{6})(\frac{a+b}{2}) = \frac{1}{2}(b^2 - a^2) \tag{51}$$

We see that for $f(x) = x^2$, it is exact:

$$\int_a^b f(x)dx = \int_a^b x^2dx = \frac{1}{3}(b^3 - a^3) \tag{52}$$

$$\frac{b-a}{6}(f(a) + f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) = \frac{b-a}{6}(a^2 + b^2) + 4(\frac{b-a}{6})(\frac{(a+b)^2}{4}) = \frac{1}{3}(b^3 - a^3) \tag{53}$$

Furthermore, we see that for $f(x) = x^3$, it is exact:

$$\int_a^b f(x)dx = \int_a^b x^3dx = \frac{1}{4}(b^4 - a^4) \tag{54}$$

$$\frac{b-a}{6}(f(a) + f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) = \frac{b-a}{6}(a^3 + b^3) + 4(\frac{b-a}{6})(\frac{(a+b)^3}{8}) = \frac{1}{4}(b^4 - a^4) \tag{55}$$

Nevertheless, we see that for $f(x) = x^4$, it is not exact:

$$\int_a^b f(x)dx = \int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5) \tag{56}$$

$$\frac{b-a}{6}(f(a)+f(b)) + 4(\frac{b-a}{6})f(\frac{a+b}{2}) = \frac{b-a}{6}(a^4+b^4) + 4(\frac{b-a}{6})(\frac{(a+b)^4}{16}) = \tag{57}$$

$$\frac{1}{24}(5b^5 - 5a^5 + a^4b - b^4a - 2a^3b^2 + 2a^2b^3) \neq \frac{1}{5}(b^5 - a^5) \tag{58}$$

Thereofore, because the formula is not exact for polynomials of degree 4 but exact for polynomials of degree 3, Newton-Cotes with $n = 2$ has degree of precision 3, as desired.

1.5 Implement Golub-Welsch for Gauss-Legendre

We wish to compute and display the nodes and weights of a Gaussian quadrature rule in the case where $w(x) = 1$ over the interval $[-1, 1]$. We first need to find the matrix $\mathcal{J}_{n+1}$ and the corresponding expressions for the three-term recurrence coefficients α_k and β_k for the system of polynomials orthogonal with respect to the weight function. Then, we use the matrix to extract the quadrature nodes and weights. We observe that the roots of ϕ_{n+1} and quadrature nodes are the eigenvalues of the matrix $\mathcal{J}_{n+1}$. Then, we see that the weights can be obtained from $w_j = c_j^2 = \frac{(u_1^{(j)})^2}{\mu^2} = (u_1^{(j)})^2 \int_a^b w(x)dx = 2(u_1^{(j)})^2$ where $u_1^{(j)}$ is the first term of the normalized eigenvector corresponding to the eigenvalue x_j. Furthermore, we know that for the Legendre polynomials, the explicit formulas for α_k and β_k are $\alpha_k = 0$ and $\beta_k = \frac{k^2}{4k^2-1}$. Thus, we implement the algorithm shown below:

```
function[x, w] = gauss_legendre(n)

%determine the three-term recurrence coefficients for the system of
%polynomials
a = zeros(n+1,1);
b = zeros(n+1,1);

%for legendre polynomials of weight w(x) = 1 over interval from -1 to 1, we
%have:
%a_k = 0;
%b_k^2 = k^2/(4k^2-1)

%construct b
for i = 2:(n+1)
    b(i) = sqrt((i-1)^2/(4*(i-1)^2 - 1));
end

%define t be the integral of weight over -1 and 1
t = 2;

%construct the symmetric, tridiagonal matrix J_n+1
J = zeros(n+1,n+1);

J(1,1) = a(1);
J(1,2) = b(2);

```

```
for j = 2:n
    J(j,j-1) = b(j);
    J(j,j) = a(j);
    J(j,j+1) = b(j+1);
end

J(n+1,n) = b(n+1);
J(n+1,n+1) = a(n+1);

%compute the eigenvalues and eigenvectors of J_n+1
[V D] = eig(J);

%find the nodes
x = zeros(1,n+1);

%find the weights
w = zeros(1,n+1);

for k = 1:n+1
    %nodes are eigenvalues of J
    x(k) = D(k,k);

    %weights
    v = V(:,k)./norm(V(:,k));
    w(k) = v(1)^2*t;
end

end
```

We implement the algorithm to compute the nodes and weights for $n \in$ 1,2,3,4,5:

```
%test gauss_legendre

%test for values of n = 1,2,3,4,5
n_vector = [1 2 3 4 5];

for i = 1:length(n_vector)

    [x, w] = gauss_legendre(n_vector(i));

    %print nodes
    fprintf('%.16e\n',x);
    fprintf('\n');
    %print weights
    fprintf('%.16e\n',w);
    fprintf('\n\n');
end
```

We obtain the following output:

```
-5.7735026918962573e-01
5.7735026918962573e-01

1.0000000000000002e+00
```

```
1.0000000000000002e+00


-7.7459666924148329e-01
-4.5102810375396984e-17
7.7459666924148340e-01

5.5555555555555580e-01
8.8888888888888851e-01
5.5555555555555558e-01


-8.6113631159405257e-01
-3.3998104358485637e-01
3.3998104358485637e-01
8.6113631159405246e-01

3.4785484513745418e-01
6.5214515486254576e-01
6.5214515486254632e-01
3.4785484513745413e-01


-9.0617984593866385e-01
-5.3846931010568322e-01
4.1434379230200701e-17
5.3846931010568311e-01
9.0617984593866385e-01

2.3692688505618881e-01
4.7862867049936630e-01
5.6888888888888955e-01
4.7862867049936697e-01
2.3692688505618889e-01


-9.3246951420315172e-01
-6.6120938646626448e-01
-2.3861918608319682e-01
2.3861918608319688e-01
6.6120938646626459e-01
9.3246951420315194e-01

1.7132449237917038e-01
3.6076157304813844e-01
4.6791393457269104e-01
4.6791393457269137e-01
3.6076157304813788e-01
1.7132449237917052e-01
```

We organize the nodes and weights into a more convenient table:

n	nodes	weights
1	-5.77e-01	1.000
	5.77e-01	1.000
2	-7.75e-01	5.56e-01
	-4.51e-17	8.89e-01
	7.75e-01	5.56e-01
3	-8.61e-01	3.48e-01
	-3.40e-01	6.52e-01
	3.40e-01	6.52e-01
	8.61e-01	3.48e-01
4	-9.06e-01	2.37e-01
	-5.38e-01	4.79e-01
	4.14e-17	5.69e-01
	5.38e-01	4.79e-01
	9.06e-01	2.37e-01
5	-9.32e-01	1.71e-01
	-6.61e-01	3.61e-01
	-2.38e-01	4.68e-01
	2.38e-01	4.68e-01
	6.61e-01	3.61e-01
	9.32e-01	1.71e-01

We know that for $n = 5$, we can integrate exactly at least polynomials of degree $n = 5$. Nevertheless, we chose the orthogonal basis and nodes so that with the Legendre polynomials, we can achieve degree of precision $2n+1 = 11$. Therefore, we expect polynomials of degree $\boxed{11}$ to be integrated exactly by the rule $n = 5$. We verify this by integrating monomials up to x^{12}, as shown below:

```
%should have degree of precision of 2n+1
%for the n = 5 case, test integration of monomials

%create function
n_test = 5;
[x_test, w_test] = gauss_legendre(n_test);

%determine max degree
max = 13; %up to degree 12
%determine max integral value
exact = zeros(max,1);
for j = 1:max
    exact(j) = (1/j)*((1)^j - (-1)^j);
end

%compute integral
%create cell of functions of monomials
integral_values = zeros(max,1);
p = cell(max,1);
for i = 1:max
    %create monomonials and store in p
    f = @(x) x^(i-1);
```

```
    p{i} = f;
end

%evaluate approximation of integral for monomials up to maximum degree
for j = 1:max
    approx = zeros(n_test+1,1);

    for k = 1:n_test+1
       approx(k) = p{j}(x_test(k));
    end

    %store approximation value for each monomial in matrix
    integral_values(j) = sum(approx.*w_test');
end

%determine difference between integral and approximation using quadrature
%rule
diff = integral_values - exact;
fprintf('%.16e\n',diff);
```

We obtain the difference between the exact value of the integral and the approximated value for the monomials shown below:

```
x^0: -2.2204460492503131e-16
x^1: -2.7755575615628914e-17
x^2: -2.2204460492503131e-16
x^3: 8.3266726846886741e-17
x^4: -2.2204460492503131e-16
x^5: 2.3592239273284576e-16
x^6: -2.2204460492503131e-16
x^7: 2.6367796834847468e-16
x^8: -2.2204460492503131e-16
x^9: 2.6367796834847468e-16
x^10: -2.7755575615628914e-16
x^11: 2.7755575615628914e-16
x^12: -7.3807866015690449e-04
```

Therefore, we observe that the approximation is no longer exact with the integral for x^{12}, thus giving us only exact integration for polynomials of degree 11, as predicted.

2 Gauss quadrature with Chebyshev polynomials

We have the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ over the interval $[-1, 1]$, which gives us the quadrature rule:

$$\int_{-1}^{1} f(x)w(x)dx \approx \sum_{k=0}^{n} w_k f(x_k) \tag{59}$$

Because the Chebyshev polynomials form a system of orthogonal polynomials with respect to w, the quadrature nodes are the roots of T_n where we define $T_k(x) = \cos(k \arccos(x))$ for $x \in$

[-1,1]. These nodes are obtained by:

$$x_k = \cos(\pi \frac{\frac{1}{2}+k}{n+1}), \text{ for } k = 0, 1, ..., n \tag{60}$$

2.1 Sanity Check

We wish to show that the predicted weights sum up to π: $w_0 + w_1 + ... + w_n = \pi$.
We know that $n \geq 0$, so the quadrature rule must be exact for polynomials of degree 0, so the formula must be exact for $f(x) = 1$ because we know that the quadrature rule, by the way we constructed it, is exact for polynomials of at least degree n. Therefore, for $f(x) = 1$, we have:

$$\int_{-1}^{1} f(x)w(x)dx = \int_{-1}^{1} w(x)dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)\Big|_{-1}^{1} = \pi \tag{61}$$

$$\int_{-1}^{1} f(x)w(x)dx = \sum_{k=0}^{n} w_k f(x_k) = \sum_{k=0}^{n} w_k = \boxed{w_0 + w_1 + ... + w_n = \pi} \tag{62}$$

, as desired.

2.2 Show

We use a change of variables to show that $\int_{-1}^{1} f(x)w(x)dx = \int_0^\pi f(cos(\theta))d\theta$:

$$\int_{-1}^{1} f(x)w(x)dx = -\int_{1}^{-1} f(x)w(x)dx \tag{63}$$

Letting $x = cos(\theta)$, we observe that $dx = -sin(\theta)d\theta$ and $cos(0) = 1$ and $cos(\pi) = -1$, thus giving us: (64)

$$-\int_{1}^{-1} f(x)w(x)dx = \int_{0}^{\pi} f(cos(\theta))w(cos(\theta))sin(\theta)d\theta \tag{65}$$

We know that $w(x) = \frac{1}{\sqrt{1-x^2}}$, so we have $w(cos(\theta)) = \frac{1}{|sin(\theta)|} = \frac{1}{sin(\theta)}$, which gives us:

$$\int_{-1}^{1} f(x)w(x)dx = \int_{0}^{\pi} f(cos(\theta))w(cos(\theta))sin(\theta)d\theta = \int_{0}^{\pi} f(cos(\theta))d\theta \tag{66}$$

, as desired.
We know that the Chebyshev nodes are:

$$x_k = \cos(\pi \frac{\frac{1}{2}+k}{n+1}), \text{ for } k = 0, 1, ..., n \tag{67}$$

Therefore, we have $\theta_k = \pi \frac{\frac{1}{2}+k}{n+1}$ for $k = 0, 1, ..., n$. Therefore, we see that:

$$\theta_0 = \pi \frac{\frac{1}{2}}{n+1} > 0 \tag{68}$$

$$\theta_n = \pi \frac{\frac{1}{2}+n}{n+1} < \pi \tag{69}$$

$$\theta_{k+1} - \theta_k = \frac{\pi}{n+1} \text{ for } k = 0, 1, ..., n \tag{70}$$

Therefore, we see that the $n+1$ values of θ are sampled uniformly in the interval $[0,\pi]$, and the quadrature rule with Chebyshev nodes actually integrates the function $f \circ \cos$ over $[0,\pi]$, sampled uniformly.

2.3 Establish the following identity

$$\sum_{k=0}^{n} \cos(\theta(\frac{1}{2}+k)) = \cos(\frac{(n+1)\theta}{2})\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \tag{71}$$

We see that $\sum_{k=0}^{n} \cos(\theta(\frac{1}{2}+k)) = \cos\frac{\theta}{2} + \cos\frac{3\theta}{2} + \cos\frac{5\theta}{2} + ... + \cos\frac{(2n+1)\theta}{2}$. We can use complex notation to see that $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $z^k = e^{ki\theta} = \cos k\theta + i\sin k\theta$. Therefore, we see that we can represent:

$$\sum_{k=0}^{n} \cos(\theta(\frac{1}{2}+k)) = \mathrm{Re}(z^{\frac{1}{2}} + z^{\frac{3}{2}} + z^{\frac{5}{2}} + ... + z^{\frac{2n+1}{2}}) = \mathrm{Re}(z^{\frac{1}{2}}(z^0 + z^1 + ... + z^n)) = \mathrm{Re}(z^{\frac{1}{2}}(\frac{z^{n+1}-1}{z-1})) \tag{72}$$

We know that $z-1 = z^{1/2}(z^{1/2}-z^{-1/2})$ and $z^{n+1}-1 = z^{\frac{n+1}{2}}(z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}})$. Thus, we have:

$$\mathrm{Re}(z^{\frac{1}{2}}(\frac{z^{n+1}-1}{z-1})) = \mathrm{Re}(z^{\frac{1}{2}}(\frac{z^{\frac{n+1}{2}}(z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}})}{z^{1/2}(z^{1/2}-z^{-1/2})})) = \mathrm{Re}(\frac{z^{\frac{n+1}{2}}(z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}})}{(z^{1/2}-z^{-1/2})}) \tag{73}$$

We see that:

$$z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}} = 2i\sin((\frac{n+1}{2})\theta) \tag{74}$$

$$z^{\frac{1}{2}} - z^{-\frac{1}{2}} = 2i\sin((\frac{1}{2})\theta) \tag{75}$$

$$\frac{z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} = \frac{\sin((\frac{n+1}{2})\theta)}{\sin((\frac{1}{2})\theta)} \tag{76}$$

$$z^{\frac{n+1}{2}} = \cos(\frac{n+1}{2}\theta) + i\sin(\frac{n+1}{2}\theta) \tag{77}$$

So, we can conclude:

$$\sum_{k=0}^{n} \cos(\theta(\frac{1}{2}+k)) = \mathrm{Re}(\frac{z^{\frac{n+1}{2}}(z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}})}{(z^{1/2}-z^{-1/2})}) = \boxed{\cos(\frac{n+1}{2}\theta)\frac{\sin((\frac{n+1}{2})\theta)}{\sin((\frac{1}{2})\theta)} = \frac{\sin((n+1)\theta)}{2\sin(\frac{\theta}{2})}} \tag{78}$$

, as desired.

If the denominator vanishes, we have $\sin(\frac{\theta}{2}) = 0$, so $z^{\frac{1}{2}} = \pm 1$ and consequently $z = 1$. Thus, we see that:

$$\sum_{k=0}^{n} \cos(\theta(\frac{1}{2}+k)) = \mathrm{Re}(z^{\frac{1}{2}}(z^0 + z^1 + ... + z^n)) = \boxed{\pm(n+1)} \tag{79}$$

2.4 Using a siple argument based on orthogonality

We wish to show that:

$$\int_{-1}^{1} T_k(x)w(x)dx = \begin{cases} \pi, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{80}$$

We know that the Chebyshev polynomials are a system of orthogonal polynomials with respect to the weight function $w(x)$. Therefore, we have $\langle T_k(x), T_\ell(x)\rangle = \int_{-1}^{1} T_k(x)T_\ell(x)w(x)dx = 0$ when $k \neq \ell$. Thus, because we know that $T_0(x) = 1$, when $k \neq 0$, we have $\langle T_k(x), T_0(x)\rangle = \int_{-1}^{1} T_k(x)T_0(x)w(x)dx = \int_{-1}^{1} T_k(x)w(x)dx = \boxed{0}$, as desired. For the case $k = 0$, we have: $\langle T_0(x), T_0(x)\rangle = \int_{-1}^{1} w(x)dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)\Big|_{-1}^{1} = \boxed{\pi}$, as desired.

2.5 Argue

We wish to show that the quadrature rule is exact for $f = T_0, T_1, ...T_{2n+1}$. We know that from previous parts, we can represent the quadrature rule as $\int_{-1}^{1} f(x)w(x)dx = \int_0^\pi f(cos(\theta))d\theta$, thus showing that the quadrature rule with the Chebyshev nodes can integrate over the function $g = f \circ \cos$ over the interval $[0, \pi]$ by sampling it uniformly (equi-spaced points). Therefore, we see that we can represent our quadrature rule with the Chebyshev nodes as a Newton-Cotes quadrature rule for n over the interval $[\frac{\pi}{2(n+1)}, \frac{\pi(2n+1)}{2(n+1)}]$. Therefore, because we know that Newton-Cotes rules have degree of precision of at least n, we see that our sampled points θ_k in $[0, \pi]$ or x_i in [-1,1] and weights w_k form a quadrature rule of degree of precision at least n. The Chebyshev polynomials form a system of orthogonal polynomials for the inner product with weight w. Furthermore, we observe that we know that the chosen Chebyshev nodes are defined by:

$$x_k = \cos(\pi\frac{\frac{1}{2} + k}{n+1}), \text{ for } k = 0, 1, ..., n \tag{81}$$

Thus, we see that because $T_k(x) = \cos(k \arccos(x))$, we see that for $k = n + 1$, we have:

$$T_{n+1}(x_k) = \cos\left((n+1)\pi(\frac{\frac{1}{2} + k}{n+1})\right) = \cos\left(\pi(\frac{1}{2} + k)\right), \text{ for } k = 0, 1, ..., n \tag{82}$$

$$\rightarrow T_{n+1}(x_k) = 0, \text{ for } k = 0, 1, ..., n \tag{83}$$

Hence, we see that the Chebyshev nodes x_k are the complete and distinct $n + 1$ roots of T_{n+1}. Therefore, for a function $f(x)$ of degree $2n + 1$ can be represented as $f(x) = p_{2n+1}(x)$, which gives us (for a quotient and remainder function $q(x)$ and $r(x)$ of degree n):

$$p_{2n+1}(x_k) = q(x_k)T_{n+1}(x_k) + r(x_k) = r(x_k) \tag{84}$$

$$\rightarrow \int_{-1}^{1} f(x)w(x)dx = \sum_{k=0}^{n} w_k f(x_k) \tag{85}$$

for $f(x)$ as a polynomial of degree $2n+1$ because we know that thhe quadrature rule has degree of precision at least n. Therefore, because the integration of f over the weight function is exact up to degree $2n + 1$, and the Chebyshev polynomials form a system of orthogonal polynomials with T_k of degree exactly k, f can equal $T_0, T_1, ..., T_{2n+1}$ and the quadrature rule remains exact. Therefore, it follows that the quadrature rule is exact for functions $T_0, T_1, ..., T_n$, as desired.

From the previous part, we know that:

$$\int_{-1}^{1} T_k(x)w(x)dx = \begin{cases} \pi, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{86}$$

We know that for $k = 0, 1, ..., n$, the integration is exact. The quadrature rule states when $f = T_k$ for $0 \leq k \leq n$:

$$\int_{-1}^{1} f(x)w(x)dx = \int_{-1}^{1} T_k(x)w(x)dx = \sum_{i=0}^{n} w_i T_k(x_i) = \begin{cases} \pi, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{87}$$

We see that:

$$T_k(x_i) = \cos(k\pi(\frac{i + \frac{1}{2}}{n+1})) \tag{88}$$

From this, we obtain the $n + 1$ system of linear equations that $w_0, w_1, ..., w_n$ must satisfy:

$$\sum_{i=0}^{n} w_i T_k(x_i) = \sum_{i=0}^{n} w_i \cos(k\pi(\frac{i + \frac{1}{2}}{n+1})) = \begin{cases} \pi, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{89}$$

, which can be represented in Vandermonde matrix form:

$$\boxed{\begin{bmatrix} T_0(x_0) & T_0(x_1) & ... & T_0(x_n) \\ T_1(x_0) & T_1(x_1) & ... & T_1(x_n) \\ T_2(x_0) & T_2(x_1) & ... & T_2(x_n) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ T_n(x_0) & T_n(x_1) & ... & T_n(x_n) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}} \tag{90}$$

2.6 Give a general argument

We observe that the n_1 linear equations produce a transposed Vandermonde matrix as shown above. We know that the Chebyshev nodes x_k are distinct along the interval [-1,1], so the columns of the vandermonde matrix are linearly independent. Let's say that there is a second choice of weights $\overrightarrow{c}$ that satisfies the system of linear equations; then, $\overrightarrow{w - c} = \overrightarrow{b}$ must gives $\overrightarrow{0}$ when multiplied with the Vandermonde matrix. Then, we would get $b_0 T_0(x_k) + b_1 T_1(x_k) + ... b_n T_n(x_k) = 0$. Thus, x_k for all k would be a root of the polynomial $p(x) = b_0 T_0(x) + b_1 T_1(x) + ... b_n T_n(x)$, which is a polynomial of degree n, which can only have at most n distinct roots. Nevertheless, there are $n + 1$ distinct Chebyshev nodes, which is a contradiction. Therefore, $\overrightarrow{b} = \overrightarrow{0}$. Thus, $\overrightarrow{w - c} = \overrightarrow{0} \rightarrow \overrightarrow{w} = \overrightarrow{c}$. Therefore, we see that the Vandermonde matrix is nonsingular, so there can only exist one $\overrightarrow{w}$ weight vector choice that satisfies the $n + 1$ linear system of equations obtained in the previous part, as desired.

2.7 Since the weights are unique

We know that the choice of weights $w_0, w_1, ..., w_n$ are unique. So, let's say that $w_0 = w_1 = ... w_n = \frac{\pi}{n+1}$. We see that for $k = 0$, we have $\sum_{i=0}^{n} w_i T_k(x_i) = \sum_{i=0}^{n} w_i \cos(k\pi(\frac{i+\frac{1}{2}}{n+1})) =$

$\sum_{i=0}^{n} w_i = \sum_{i=0}^{n}(\frac{\pi}{n+1}) = \boxed{\pi}$, as desired.
For the case $k \neq 0$, we have:

$$\sum_{i=0}^{n} w_i T_k(x_i) = \sum_{i=0}^{n} w_i \cos{(k\pi(\frac{i+\frac{1}{2}}{n+1}))} = \frac{\pi}{n+1}\sum_{i=0}^{n}\cos{((k\pi)(\frac{i+\frac{1}{2}}{n+1}))} = \frac{\pi}{n+1}\sum_{i=0}^{n}\cos{(\frac{(k\pi)}{n+1}(i+\frac{1}{2})} = \tag{91}$$

$$(\frac{\pi}{n+1})(\frac{1}{2})\frac{\sin{((n+1)(\frac{k\pi}{n+1}))}}{\sin{(\frac{\frac{k\pi}{n+1}}{2})}} = (\frac{\pi}{n+1})(\frac{1}{2})\frac{\sin{(k\pi)}}{\sin{(\frac{\frac{k\pi}{n+1}}{2})}} = \boxed{0} \tag{92}$$

because we know that $k = 0, 1, ..., n$, so the denominator is never zero. Therefore, because we see that the solution $w_0 = w_1 = ...w_n = \frac{\pi}{n+1}$ satisfies the linear system of equations and we know that the vector is unique, we can conclude that $\boxed{w_0 = w_1 = ...w_n = \frac{\pi}{n+1}}$ is the unique solution to the weights of the quadrature rule with Chebyshev polynomials.

2.8 Experiment

We wish to compute $\int_0^\pi e^{\cos x}dx$. We can do a change of variables:

$$\int_0^\pi e^{\cos x}dx = \int_0^\pi f(\cos\theta)d\theta = \int_{-1}^{1} f(x)w(x)dx \tag{93}$$

$$\text{Therefore, we can define the function } f \text{ as: } f(x) = e^x \tag{94}$$

$$\tag{95}$$

Thus, we have:

$$\int_0^\pi e^{\cos x}dx = \int_{-1}^{1}(\frac{e^x}{\sqrt{1-x^2}})dx \approx \sum_{k=0}^{n} w_k f(x_k) = \frac{\pi}{n+1}\sum_{k=0}^{n} f(x_k) = \frac{\pi}{n+1}\sum_{k=0}^{n} e^{x_k} \tag{96}$$

where the Chebyshev nodes are defined by:

$$x_k = \cos(\pi\frac{\frac{1}{2}+k}{n+1}), \text{ for } k = 0, 1, ..., n \tag{97}$$

We implement the computation below:

```
%experiment Gauss quadrature with Chebyshev polynomials
%evalue integral from 0 to pi of e^(cos(x))

%input the number of integration points (n+1 total points) to output error
function[x] = gauss_chebyshev(n)

%establish weight:
w = pi/(n+1);
%initiate sum
sum = 0;

for i = 1:n+1
    %find the Chebyshev nodes at x(i-1)
```

```
    x = cos(pi/(n+1)*(1/2+(i-1)));

    %update sum
    sum = sum + exp(x);
end
%experimental value
integral = w*sum;

%theoretical value:
theoretical = pi*besseli(0,1);

%relative error:
rel_err = (theoretical - integral)/(theoretical);
x = rel_err;

%print
fprintf('%.16e\n',x);

end
```

We experiment with different values of n (different number of integration points) to obtain an error close to machine precision:

```
%test experiment
err = 1; %set random initial error value
i  = 0; % number of integration points - 1

%iterate until error gets close to machine precision
while err > 10^(-15)

    i = i+1;
    err = gauss_chebyshev(i);

    %print error and number of integration points
    fprintf('%d\n',i+1);
    fprintf('%.16e\n',err);
    fprintf('\n');
end
```

This gives us the output:

```
2
4.3236622413135983e-03
--------------------
3
3.5525261919883754e-05
--------------------
4
1.5734764187791083e-07
--------------------
5
4.3488218574644227e-10
```

```
--------------------
6
8.2052589538503435e-13
--------------------
7
8.9321093524020618e-16
--------------------
```

Therefore, we observe that we need $\boxed{7}$ integration points to get a relative error close to machine precision. This is surprisingly a small number to get a near exact approximation of the integral.