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MAT321: Numerical Methods
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Due: 3 October 2018

Homework 1: $f(x) = 0$ and IEEE floating points

1 Solving $f(x) = 0$ with f: $\mathbb{R} \to \mathbb{R}$

1.1 Convergence rate of g_3

We are given the function $f(x) = e^x - 2x - 1$. We find the roots of $f(x)$, we consider the simple iteration on $g(x) = e^x - x - 1$. We observe that our function $g(x)$ is a contraction by the Mean Value Theorem. We define the sequence $x_{k+1}= g(x_k)$. We define $\xi = \lim_{k\to\infty} x_k$. To prove that the sequence x_{k+1} converges quadratically to $\xi = 0$, by Definition 1.7, we have to show that with $\varepsilon_k = \mid x_k - \xi \mid$ for $k = 0, 1, 2, ...$, there exists a $\mu > 0$ such that $\mu = \lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q}$ for $q = 2$.

We see that:

$$\varepsilon_k =\mid x_k - \xi \mid=\mid x_k \mid \tag{1}$$

So,

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \lim_{k\to\infty} \frac{\mid x_{k+1} \mid}{\mid x_k \mid^q} = \lim_{k\to\infty} \frac{\mid g(x_k) \mid}{\mid x_k \mid^q} = \lim_{k\to\infty} \frac{\mid e^{x_k} - x_k - 1 \mid}{\mid x_k \mid^q} = \lim_{x_k\to\xi} \frac{\mid e^{x_k} - x_k - 1 \mid}{\mid x_k \mid^q} \tag{2}$$

Because $\xi = 0$, and the limit goes to 0 in both the numerator and denominator, we can use L'Hopital's Rule to find the left and right limits to solve for μ:

$$\lim_{x_k\to 0} \frac{\mid e^{x_k} - x_k - 1 \mid}{\mid x_k \mid^q} = \lim_{x_k\to 0^-} \frac{e^{x_k} - x_k - 1}{(-1)^q x_k^q} = \lim_{x_k\to 0^+} \frac{e^{x_k} - x_k - 1}{x_k^q} \tag{3}$$

$$\mu = \lim_{x_k\to 0^-} \frac{e^{x_k} - x_k - 1}{(-1)^q x_k^q} = \lim_{x_k\to 0^-} \frac{e^{x_k} - 1}{(-1)^q q x_k^{q-1}} = \lim_{x_k\to 0^-} \frac{e^{x_k}}{(-1)^q q(q-1) x_k^{q-2}} = \frac{1}{2} \tag{4}$$

$$\mu = \lim_{x_k\to 0^+} \frac{e^{x_k} - x_k - 1}{x_k^q} = \lim_{x_k\to 0^+} \frac{e^{x_k} - 1}{q x_k^{q-1}} = \lim_{x_k\to 0^+} \frac{e^{x_k}}{q(q-1) x_k^{q-2}} = \frac{1}{2} \tag{5}$$

We see that by equations (4) and (5), the only possible value for q such that μ takes on a real value > 0 ($\mu = 1/2$) is when $q = 2$. Therefore, the sequence x_{k+1} converges to $\mu = 0$ quadratically (defined by Definition 1.7).

1.2 Suli and Mayers Exercise 1.10

We see that:

$$\psi(x_{k-1}) = \lim_{x_k\to\xi} \varphi(x_k, x_{k-1}) = \lim_{x_k\to\xi} \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)} = \lim_{x_k\to\xi} \frac{\frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)} - \xi}{(x_k - \xi)(x_{k-1} - \xi)} \tag{6}$$

$$\lim_{x_k\to\xi} \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k) - \xi f(x_{k-1}) + \xi f(x_k)}{(f(x_{k-1}) - f(x_k))((x_k - \xi)(x_{k-1} - \xi))} \tag{7}$$

Since $f(\xi) = 0$, the limit in both numerator and denominator goes to 0, we can use l'Hopital's Rule to find the limit:

$$\lim_{x_k\to\xi} \frac{f(x_{k-1}) - x_{k-1} f'(x_k) + \xi f'(x_k)}{-f'(x_k)((x_k - \xi)(x_{k-1} - \xi)) + (f(x_{k-1}) - f(x_k))(x_{k-1} - \xi)} \tag{8}$$

$$\psi(x_{k-1}) = \frac{f(x_{k-1}) - x_{k-1}f'(\xi) + \xi f'(\xi)}{(f(x_{k-1}) - f(\xi))(x_{k-1} - \xi)} = \boxed{\frac{f(x_{k-1}) - x_{k-1}f'(\xi) + \xi f'(\xi)}{(f(x_{k-1}))(x_{k-1} - \xi)}} \tag{9}$$

So, we see that to find $\lim_{x_{k-1}\to\xi} \psi(x_{k-1})$, we have to use L'Hopital's Rule (twice) since both numerator and denominator approaches 0 with x_{k-1} approaches ξ. Thus,

$$\lim_{x_{k-1}\to\xi} \psi(x_{k-1}) = \lim_{x_{k-1}\to\xi} \frac{f'(x_{k-1}) - f'(\xi)}{x_{k-1}f'(x_{k-1}) + f(x_{k-1}) - \xi f'(x_{k-1})} \tag{10}$$

$$= \lim_{x_{k-1}\to\xi} \frac{f''(x_{k-1})}{x_{k-1}f''(x_{k-1}) + 2f'(x_{k-1}) - \xi f''(x_{k-1})} = \boxed{\frac{f''(\xi)}{2f'(\xi)}} \tag{11}$$

Now, we assume that:

$$\lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid^q} = A \tag{12}$$

So,

$$\lim_{k\to\infty} \frac{\mid x_k - \xi \mid}{\mid x_{k-1} - \xi \mid^q} = A \Rightarrow \lim_{k\to\infty} \frac{\mid x_k - \xi \mid^{1/q}}{\mid x_{k-1} - \xi \mid} = A^{1/q} \tag{13}$$

So,

$$\lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid^{q-1/q} \mid x_{k-1} - \xi \mid} = A^{\frac{1+q}{q}} \tag{14}$$

$$\lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid^{q-1/q-1} \mid x_k - \xi \mid\mid x_{k-1} - \xi \mid} = A^{\frac{1+q}{q}} \tag{15}$$

$$= \lim_{k\to\infty} \frac{1}{\mid x_k - \xi \mid^{q-1/q-1}} \lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid\mid x_{k-1} - \xi \mid} \tag{16}$$

We know that

$$\lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid\mid x_{k-1} - \xi \mid} =\mid \frac{f''(\xi)}{2f'(\xi)} \mid \tag{17}$$

So, we have:

$$\lim_{k\to\infty} \mid x_k - \xi \mid^{q-1/q-1} = A^{\frac{q}{1+q}} \mid \frac{f''(\xi)}{2f'(\xi)} \mid \tag{18}$$

We see that because we know $A > 0$, we must have $\boxed{q - 1/q - 1 = 0}$ because if $q-1/q-1 \neq 0$, then the left limit of equation (18) would not exist because it would approach either 0 or ∞

So, we have:

$$q - 1 - 1/q = 0 \Rightarrow \boxed{q = \frac{1}{2}(1 + \sqrt{5})} \tag{19}$$

and

$$A^{\frac{1+q}{q}} = \frac{f''(\xi)}{2f'(\xi)} \tag{20}$$

So, we can now solve for A:

$$A = (A^{\frac{1+q}{q}})^{\frac{q}{q+1}} = (\frac{f''(\xi)}{2f'(\xi)})^{\frac{q}{q+1}} \tag{21}$$

Therefore,

$$\lim_{k\to\infty} \frac{\mid x_{k+1} - \xi \mid}{\mid x_k - \xi \mid^q} = A = \boxed{(\frac{f''(\xi)}{2f'(\xi)})^{\frac{q}{q+1}}} \tag{22}$$

Now, we wish to implement Newton's Method (obtained from lecture notes) for the function $f(x) = \frac{e^{cos(x)}}{2+x} - 1$ with a stopping criteria to check for the number of iterations, the residual size, and the increment size in comparison to machine accuracy to compute a root for the function.

```
%Implement Newton's Method for given function with an increment stopping
%criteria
f = @(x) exp(cos(x))/(2+x) - 1; %equation definition
df = @(x) -exp(cos(x))*sin(x)/(2+x) - exp(cos(x))/(2+x)^2; %first order
%derivative of f
x0 = 0; %initial guess
% choose a few integers, and can converge to root

N = 50; %Maximum number of iterations %stopping criteria #1
tolerance = 1E-16; %Convergence tolerance %stopping criteria #2

x = x0; %set initial integer
fprintf('x = %+.16e, \t f(x) = %+.16e\n',x,f(x));

x_before = x0;

root = 0; %define root

%Iterate until convergence
%Newton's Method algorithm
for k = 1 : N %stopping criteria 1

    x = x - f(x) / df(x); %simple iteration
    fprintf('x = %+.16e, \t f(x) = %+.16e\n',x,f(x));

    error1 = f(x); %stopping criteria 2: if the error is less than tolerance
    if (abs(error1) <= tolerance)
        break;
    end

    error2 = abs(x - x_before)/abs(x); %stopping criteria 3: if the relative
    %error is less than the tolerance
    if(error2 <= tolerance)
        break;
    end
    x_before = x; %update x_before

end

root = x; %final root obtained
fprintf('root = %+.16e',root);
```

Initializing with integers around the root, such as 0, 1, and 2, we obtain the root of the function $f(x)$: $\xi = \boxed{4.5515854822042595 \cdot 10^{-01}}$

We now implement the Secant Method (obtained from lecture notes) for the function $f(x) = \frac{e^{cos(x)}}{2+x} - 1$ with a stopping criteria to check for the number of iterations, the residual size, and the increment size in comparison to machine accuracy to compute a root for the function.

```
%Implement Secant Method to find the roots of the function
f = @(x) exp(cos(x))/(2+x) - 1; %No need for the derivative of f

%Initialize initial two points
x0 = 0.0;
x1 = 1.0;

%root
root;

%Find f at the initial points
f0 = f(x0);
f1 = f(x1);

fprintf('x = %+.16e, \t f(x) = %+.16e\n', x1, f1);

N = 50; %Maximum number of iterations %stopping criteria #1
tolerance = 1E-16; %Convergence tolerance %stopping criteria #2

%Iterate until convergence
for k = 1 : N

    error1 = abs(x1 - x0)/abs(x0); %stopping criteria: if the error is less than tolerance
    if (error1 <= tolerance)
        break;
    end

    error2 = f(x1);
    if (abs(error2) <= tolerance)
        break;
    end

    %Compute the next iterate
    x2 = x1 - f1 * (x1 - x0) / (f1 - f0);

    %Evaluate the function there with a call to f
    f2 = f(x2);

    fprintf('x = %+.16e, \t f(x) = %+.16e\n', x2, f2);

    x0 = x1; %update the x and f
    f0 = f1;
    x1 = x2;
    f1 = f2;

end

root = x; %final root obtained
fprintf('root = %+.16e',root);
```

Initializing with integers around the root, such as $x_0 = 0$ and $x_1 = 1$, we obtain the root of the function $f(x)$: $\xi = \boxed{4.5515854822042590 \cdot 10^{-01}}$

2 IEEE floating points

2.1 Difference of big numbers

We first consider the case of $c_1 = a^2 - b^2$:

$$(a \odot a) \ominus (b \odot b) = (a^2(1+\varepsilon_1)) \ominus (b^2(1+\varepsilon_2)) = ((a^2 + \varepsilon_1 a^2) - (b^2 + \varepsilon_2 b^2))(1+\varepsilon_3) \tag{23}$$

$$= a^2 + \varepsilon_1 a^2 - b^2 - \varepsilon_2 b^2 + \varepsilon_3 a^2 - \varepsilon_3 b^2 + \varepsilon_3 \varepsilon_1 a^2 - \varepsilon_3 \varepsilon_2 b^2 = (a^2 - b^2) + \varepsilon_1 a^2 - \varepsilon_2 b^2 + \varepsilon_3 (a^2 - b^2) + O(\varepsilon_{mach}^2) \tag{24}$$

Hence, we can bound the error (by using the Triangle Inequality):

$$\mid (a \odot a) \ominus (b \odot b) - (a^2 - b^2) \mid \leq \varepsilon_{mach}(\mid a^2 \mid + \mid b^2 \mid + \mid a^2 - b^2 \mid) + O(\varepsilon_{mach}^2) \tag{25}$$

$$\mid \frac{(a \odot a) \ominus (b \odot b) - (a^2 - b^2)}{a^2 - b^2} \mid \leq \varepsilon_{mach}(\frac{\mid a^2 \mid + \mid b^2 \mid}{\mid a^2 - b^2 \mid} + 1) + O(\varepsilon_{mach}^2) \tag{26}$$

We now consider the case of $c_1 = (a+b)(a-b)$:

$$(a \oplus b) \otimes (a \ominus b) = (a+b)(1+\varepsilon_1) \otimes (a-b)(1+\varepsilon_2) = (a+b)(1+\varepsilon_1)(a-b)(1+\varepsilon_2)(1+\varepsilon_3) \tag{27}$$

$$= (a^2 - b^2)(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_2\varepsilon_3) \tag{28}$$

Hence, we can bound the error:

$$\mid (a \oplus b) \otimes (a \ominus b) - (a^2 - b^2) \mid \leq \varepsilon_{mach}(3 \mid a^2 - b^2 \mid) + O(\varepsilon_{mach}^2) \tag{29}$$

In relative terms, we have:

$$\mid \frac{(a \oplus b) \otimes (a \ominus b) - (a^2 - b^2)}{a^2 - b^2} \mid \leq 3 \mid \varepsilon_{mach} \mid + O(\varepsilon_{mach}^2) \tag{30}$$

Therefore, we see that the second formula (c_2) is significantly more trustworthy than c_1 because the former is bounded by a $3\varepsilon_{mach}$ consistently regardless of our choice for a and b. We see that for c_1, when we choose a and b to be numbers that are close to each other, the relative error will become large because we divide by $\mid a^2 - b^2 \mid$, which is multiplied by the ε_{mach}, thus potentially giving a very large error.

2.2 Distance between points on a circle

We wish to first demonstrate that $dist_1(x,y) = dist_2(x,y)$ mathematically. We let $dist_1 = \theta_1$, and $dist_2 = \theta_2$. So, we see that:

$$dist_1(x,y) = \theta_1 = arccos(x^T y) \Rightarrow cos(\theta_1) = x^T y = x \cdot y \tag{31}$$

We also see that:

$$dist_2(x,y) = \theta_2 = 2arcsin(1/2 \mid\mid x - y \mid\mid) \Rightarrow sin(\frac{\theta_2}{2}) = \frac{1}{2} \mid\mid x - y \mid\mid \tag{32}$$

Because we know that $x \cdot x = y \cdot y = 1$ because x and y lie on the unit circle, so their magnitudes from the origin must be 1, we see that:

$$sin(\frac{\theta_2}{2}) = \frac{1}{2}\frac{|| x - y ||}{|| x ||} \Rightarrow cos(\theta_2) = 1 - 2sin^2(\frac{\theta_2}{2}) = 1 - 2\frac{\frac{1}{4} || x - y ||^2}{|| x ||^2} \tag{33}$$

$$cos(\theta_2) = 1 - 2sin^2(\frac{\theta_2}{2}) = 1 - \frac{1}{2}\frac{(x - y) \cdot (x - y)}{x \cdot x} = 1 - \frac{1}{2}\frac{x \cdot x - 2x \cdot y + y \cdot y}{x \cdot x} = 1 - \frac{1}{2}(2 - 2x \cdot y) \tag{34}$$

Therefore, we conclude that

$$cos(\theta_2) = x \cdot y = cos(\theta_1) \Rightarrow \theta_1 = \theta_2 \Rightarrow \boxed{dist_1(x, y) = dist_2(x, y)} \tag{35}$$

Now, we wish to confirm that $y(x, v, t)$ lies on the unit circle and t is the distance between x and y. So, we wish to show that $y \cdot y = 1$:

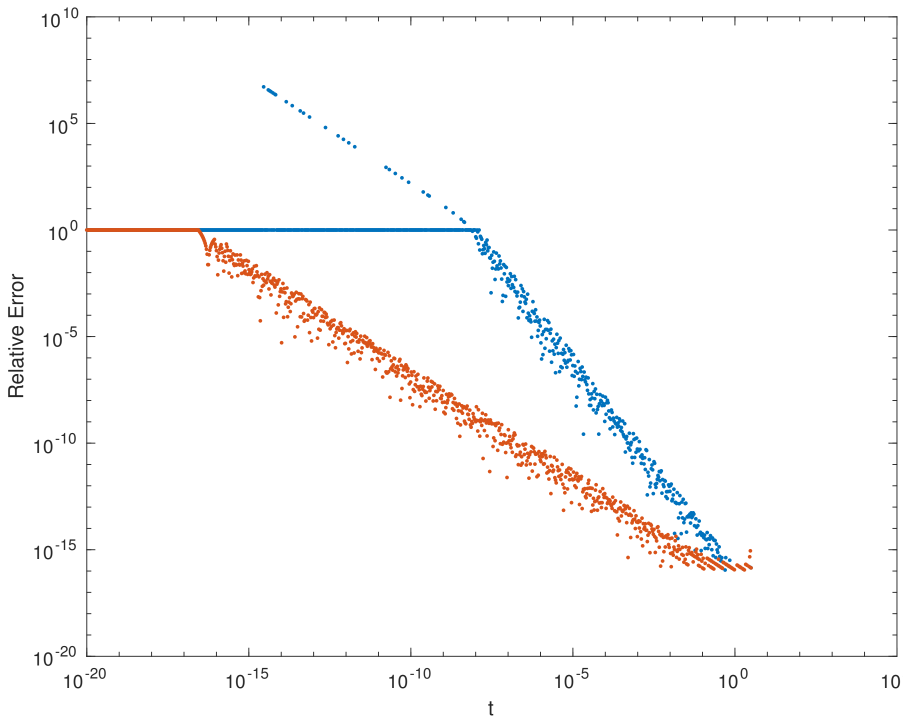
$$(cos(t)x + sin(t)v) \cdot (cos(t)x + sin(t)v) = cos^2(t)x \cdot x + 2sin(t)cos(t)x \cdot v + sin^2(t)y \cdot y = cos^2(t) + sin^2(t) = \boxed{1} \tag{36}$$

Furthermore,

$$dist_1(x, y) = arccos(x^T y) = arccos(x \cdot (cos(t)x + sin(t)v)) = arccos(cos(t)) = \boxed{t} \tag{37}$$

Our MATLAB code to produce the plot is shown below:

```
n = 2;
x = randn(n, 1);
x = x / norm(x) %x vector

v = randn(n, 1);
v = v - (x'*v)*x;
v = v / norm(v)%v vector

y = @(x,v,t) cos(t)*x + sin(t)*v; % define y vector

dist1 = @(x, y) acos(dot(x,y)); %define dist1
dist2 = @(x,y) 2 * asin(0.5 * sqrt(dot((x-y),(x-y)))); %define dist2

t = logspace(-20, pi, 1000); %create t vector

%create g1 and g2 vectors
g1 = [];
g2 = [];

for k = 1:1000
g1(k) = abs(t(k) - dist1(x, y(x,v,t(k))))/t(k);
g2(k) = abs(t(k) - dist2(x, y(x,v,t(k))))/t(k);
end

figure
loglog(t,g1,'.')
hold on
loglog(t, g2, '.')
legend('g1','g2')
xlabel('t')
ylabel('Relative Error')
hold off
```



We plot the relative errors in a log-log plot for $g1(t)$ and $g2(t)$. We observe that the function g2 is preferred because the relative error of the function is consistently lower than g1.

Now, we wish to show that for $dist_1$,

1. $\mid \frac{fl(x^Ty)-x^Ty}{x^Ty} \mid\leq (1+\frac{|x|^T|y|}{|x^Ty|})\varepsilon_{mach}$: We see that the floating point arithmetic of the dot product can be depicted as: $fl(x^Ty) = (x_1 \otimes y_1) \oplus (x_2 \otimes y_2)$. This becomes:

$$(x_1y_1)(1+\varepsilon_1) \oplus (x_2y_2)(1+\varepsilon_2) = (x_1y_1 + \varepsilon_1x_1y_1 + x_2y_2 + \varepsilon_2x_2y_2)(1+\varepsilon_3) \tag{38}$$

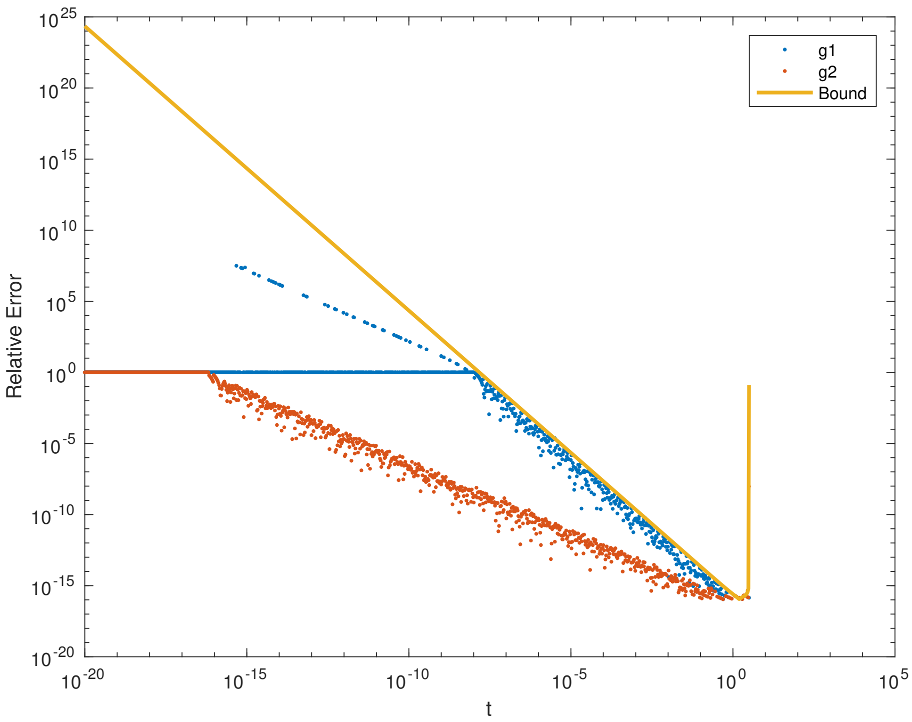
$$(x_1y_1 + x_2y_2) + \varepsilon_1x_1y_1 + \varepsilon_2x_2y_2 + \varepsilon_3(x_1y_1 + x_2y_2) + O(\varepsilon_{mach}^2) \tag{39}$$

$$\mid fl(x^Ty) - (x_1y_1 + x_2y_2) \mid\leq \varepsilon_{mach}(\mid x_1y_1 \mid + \mid x_2y_2 \mid + \mid (x_1y_1 + x_2y_2) \mid) + O(\varepsilon_{mach}^2) \tag{40}$$

$$\mid \frac{fl(x^Ty) - (x_1y_1 + x_2y_2)}{x_1y_1 + x_2y_2} \mid\leq \varepsilon_{mach}(\frac{\mid x_1y_1 \mid + \mid x_2y_2 \mid}{\mid (x_1y_1 + x_2y_2) \mid} + 1) + O(\varepsilon_{mach}^2) \tag{41}$$

$$\mid \frac{fl(x^Ty) - (x^Ty)}{x^Ty} \mid\leq \varepsilon_{mach}(\frac{\mid x \mid^T\mid y \mid}{\mid x^Ty \mid} + 1) \tag{42}$$

2. Hence, we see that when x and y are sufficiently close, we observe that $\mid x^Ty \mid \to \mid x^Tx \mid = \mid x \mid^2$ and $\mid x \mid^T\mid y \mid \to \mid x \mid^T\mid x \mid = \mid x_1 \mid^2 + \mid x_2 \mid^2 = \mid x \mid^2$. Thus, we observe that when x is sufficiently close to y, $\mid \frac{fl(x^Ty)-(x^Ty)}{x^Ty} \mid\leq \varepsilon_{mach}(\frac{|x|^2}{|x|^2}+1) = 2\varepsilon_{mach}$, so $fl(x^Ty) = x^Ty(1+2\varepsilon)$



with $| \varepsilon | \leq \varepsilon_{mach}$, as desired.

3. We see that $fl(dist_1(x,y)) = fl(arccos(x^Ty)) = \arccos(fl(x^Ty))(1+\varepsilon_1))$ by the given identity of the relative accuracy of arccos. From part 2, we see that $fl(x^Ty) = x^Ty(1+2\varepsilon)$, so we see that:

$$fl(dist_1(x,y)) = arccos(x^Ty(1+2\varepsilon_2))(1+\varepsilon_1) = arccos(x^Ty + 2\varepsilon_2 x^Ty)(1+\varepsilon_1) \tag{43}$$

Using the Taylor expansion identity or arccos, we obtain:

$$arccos(x^Ty + 2\varepsilon_2 x^Ty) = (arccos(x^Ty) - \frac{1}{\sqrt{((1-(x^Ty)^2)}}(2\varepsilon_2 x^Ty) + O(\varepsilon_{mach}^2))(1+\varepsilon_1) \tag{44}$$

$$= arccos(x^Ty) - \varepsilon_2(\frac{2x^Ty}{\sqrt{1-(x^Ty)^2}}) + \varepsilon_1 arccos(x^Ty) + O(\varepsilon_{mach}^2) \tag{45}$$

Therefore,

$$| \frac{fl(dist_1(x,y)) - arccos(x^Ty)}{arccos(x^Ty)} | \leq \varepsilon_{mach}(1 + \frac{2x^Ty}{\sqrt{1-(x^Ty)^2} \mid arccos(x^Ty) \mid}) + O(\varepsilon_{mach}^2) \tag{46}$$

We are given that $t = dist_1(x,y) = arccos(x^Ty)$, so $cos(t) = x^Ty$. Thus,

$$| \frac{fl(dist_1(x,y)) - arccos(x^Ty)}{arccos(x^Ty)} | \leq \varepsilon_{mach}(1 + \frac{2(cos(t))}{\sqrt{1-cos^2(t)} \mid t \mid}) + O(\varepsilon_{mach}^2) \tag{47}$$

$$| \frac{fl(dist_1(x,y)) - arccos(x^Ty)}{arccos(x^Ty)} | \leq \varepsilon_{mach}(1 + \frac{2(cos(t))}{\mid sin(t) \mid\mid t \mid}) + O(\varepsilon_{mach}^2) \tag{48}$$

$$| \frac{fl(dist_1(x,y)) - dist_1(x,y)}{dist_1(x,y)} | \leq \boxed{\varepsilon_{mach}(1 + \frac{2}{|\,(t)tan(t)\,|}) + O(\varepsilon_{mach}^2)} \tag{49}$$

as desired.

2.3 Extra Credit

For when x is very close to y so $dist_1(x, y) \to 0$, we observe that $x^T y \to || x ||^2 = 1$ because x lies on the unit circle. Thus, when x is close to y, t becomes very small, and the distance between x and y can be approximated by $dist_1(x, y) = arccos(x^T y) = arccos(x^T x) = arccos(1)$ when x and y are the same point. For when x and y are close, the next approximation we can possibly do is $arccos(1 - \varepsilon_{mach})$ because each increment can only be ε_{mach}. We can Taylor expand:

$$t' = arccos(1 - \varepsilon_{mach}) \tag{50}$$

$$cos(t') = 1 - \frac{t^2}{2!} = 1 - \varepsilon_{mach} \Rightarrow t' = \sqrt{2\varepsilon_{mach}} \tag{51}$$

Thus, around $t = 0$, $t' - t = \sqrt{2\varepsilon_{mach}}$, so because $\varepsilon_{mach} \approx 1.11 \cdot 10^{-16}$, $\sqrt{2\varepsilon_{mach}} \propto 10^{-8}$. Thus, the minimum value of t that we can represent accurately for $dist_1$ is when $t \geq 10^{-8}$. When t is smaller than this, the relative error becomes very large/inconsistent because the minimum separation between x and y is ε_{mach}. Thus, this explains the change in slope of the relative error when $t \leq 10^{-8}$ on our graph with the smaller values of t even having a relative error of 1. Therefore, we observe that we cannot calculate the distance between x and y accurately using $dist_1$ if they are separated by $t \leq 10^{-8}$. When $x^T y$ is between 1 and $1 - \varepsilon_{mach}$, then $arccos(x^T y)$ will be rounded off to $arccos(1) = 0$, so that explains the constant relative error of 1 because the absolute error is $((t - 0)/ \mid t \mid) = 1$ when $t \leq 10^{-8}$.