TMC: Nonlinear Equation

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Motivation

- Why?

$$ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}$$

- But

$$ax^5+bx^4+cx^3+dx^2+ex+f=0 \quad \Rightarrow x=?$$

$$\sin x+x=0 \quad \Rightarrow x=?$$

- **Nonlinear Equation Solvers**
  - **Bracketing**
    - Bisection
    - False Position
  - **Graphical**
  - **Open Methods**
    - Newton Raphson
    - Secant

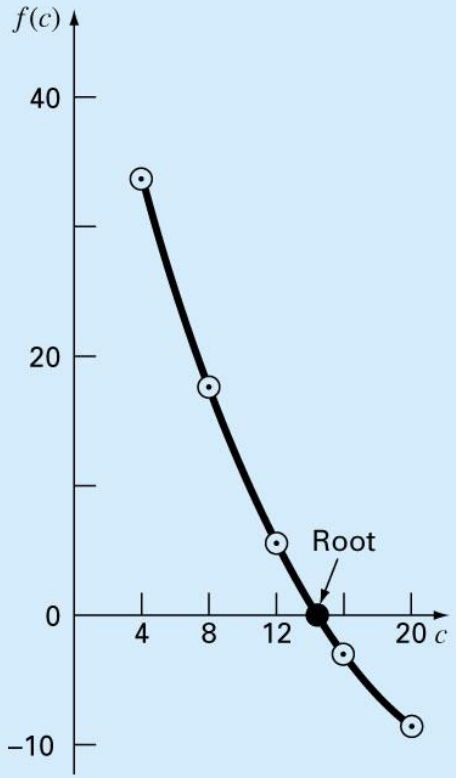
All Iterative

PART A

BRACKETING METHODS

Bracketing Methods

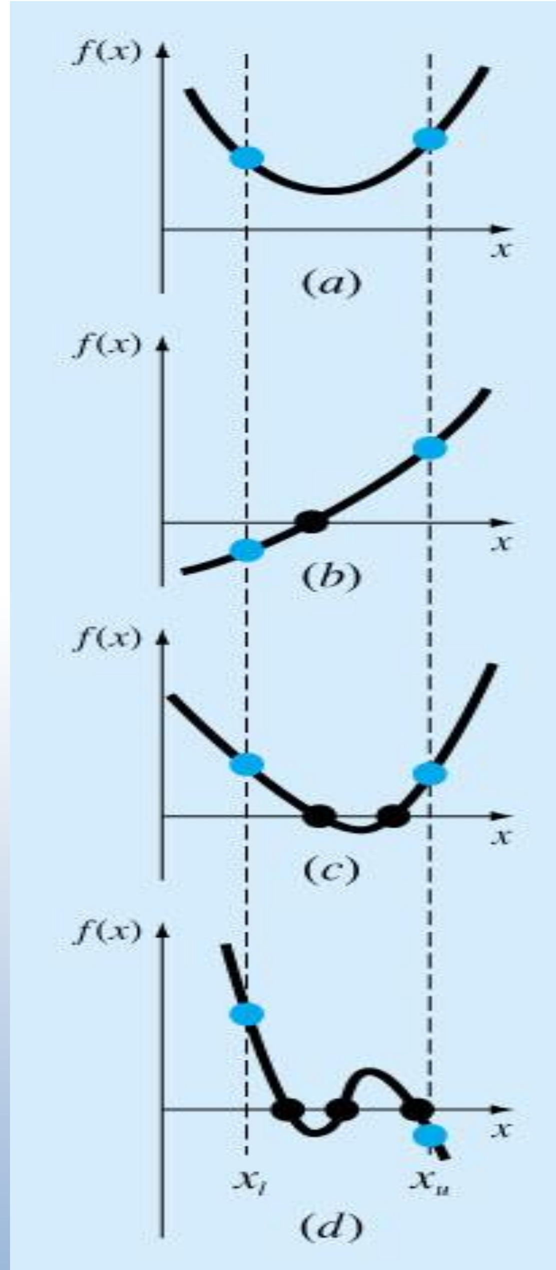
(Or, two point methods for finding roots)



Two initial guesses for the root are required. These guesses must “bracket” or be on either side of the root.

If one root of a real and continuous function, $f(x)=0$, is bounded by values $x=x_l$, $x=x_u$ then

$f(x_l) \cdot f(x_u) < 0$. (The function changes sign on opposite sides of the root)



Parts (*a*) and (*c*) indicate that if both *f* (x_l) and *f* (x_u) have the same sign, either there will be no roots or there will be an even number of roots within the interval.

Parts (*b*) and (*d*) indicate that if the function has different signs at the end points, there will be an odd number of roots in the interval

1. The Bisection Method

For the arbitrary equation of one variable, $f(x)=0$

1. Pick x_l and x_u such that they bound the root of interest, check if $f(x_l).f(x_u) <0$.

2. Estimate the root by evaluating $f[(x_l+x_u)/2]$.

3. Find the pair
   - If $f(x_l). f[(x_l+x_u)/2]<0$, root lies in the lower interval, then $x_u=(x_l+x_u)/2$ and go to step 2.
   - If $f(x_l). f[(x_l+x_u)/2]>0$, root lies in the upper interval, then $x_l= [(x_l+x_u)/2$, go to step 2.
   - If $f(x_l). f[(x_l+x_u)/2]=0$, then root is $(x_l+x_u)/2$ and terminate.

4. Compare ε_s with ε_a

5. If $\varepsilon_a < \varepsilon_s$, stop. Otherwise repeat the process.

Termination Criteria and Error Estimates

An approximate percent relative error ε_a can be calculated

$$\varepsilon_a = \left|\frac{x_r^{new} - xro^{ld}}{x_r^{new}}\right| 100\%$$

where

x_r^{new} is the root for the present iteration and
x_r^{old} is the root from the previous iteration.

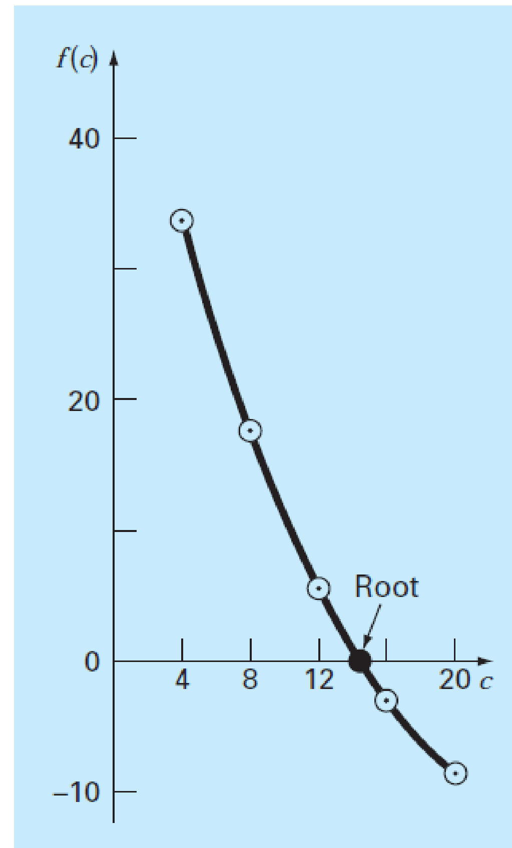
When ε_a becomes less than a prespecified stopping criterion ε_s, the computation is terminated.

Example #1

Determine the root of the following equation

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$

c	f(c)
4	34.115
8	17.653
12	6.067
16	−2.269
20	−8.401



Example #1 (cont.)

The first step in bisection is to guess two values of the unknown that give values for f (c) with different signs. From the above figure, we can see that the function changes sign between values of 12 and 16. Therefore, the initial estimate of the root x_r lies at the midpoint of the interval

$$x_r = \frac{12 + 16}{2} = 14$$

Next we compute the product of the function value at the lower bound and at the midpoint:

$$f(12)\ f(14) = 6.067(1.569) = 9.517$$

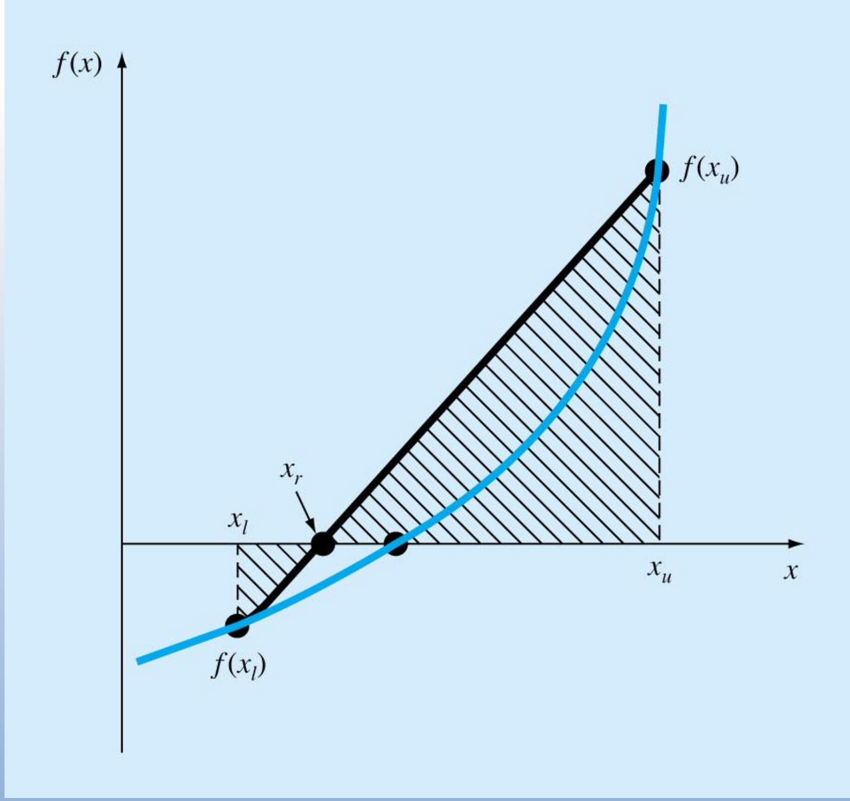
Consequently, the root must be located between 14 and 16. The method can be repeated until the result is accurate enough to satisfy your needs.

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	12	16	14		5.279
2	14	16	15	6.667	1.487
3	14	15	14.5	3.448	1.896
4	14.5	15	14.75	1.695	0.204
5	14.75	15	14.875	0.840	0.641
6	14.75	14.875	14.8125	0.422	0.219

Thus, after six iterations ε_a finally falls below ε_s = 0.5%, and the computation can be terminated.

The False-Position Method (Regula-Falsi)

If a real root is bounded by x_l and x_u of $f(x)=0$, then we can approximate the solution by doing a linear interpolation between the points $[x_l, f(x_l)]$ and $[x_u, f(x_u)]$ to find the x_r value such that $l(x_r)=0$, $l(x)$ is the linear approximation of $f(x)$.



Procedure

1. Find a pair of values of x, x_l and x_u such that $f_l=f(x_l) <0$ and $f_u=f(x_u) >0$.
2. Estimate the value of the root from the following formula

$$x_r = \frac{x_l f_u - x_u f_l}{f_u - f_l}$$

   and evaluate $f(x_r)$.
3. Use the new point to replace one of the original points, keeping the two points on opposite sides of the x axis.

   If $f(x_r)<0$ then $x_l=x_r$ == > $f_l=f(x_r)$
   If $f(x_r)>0$ then $x_u=x_r$ == > $f_u=f(x_r)$
   If $f(x_r)=0$ then you have found the root and need go no further!

4. See if the new x_l and x_u are close enough for convergence to be declared. If they are not, go back to step 2.

Example #2

Use the false-position method to determine the root of the same equation investigated in Example #1

As in Example #1, initiate the computation with guesses of x_l =12 and x_u =16.

First iteration:

$$x_l = 12 \qquad f(x_l) = 6.0699$$
$$x_u = 16 \qquad f(x_u) = -2.2688$$
$$x_r = 16 - \frac{-2.2688(12 - 16)}{6.0669 - (-2.2688)} = 14.9113$$

which has a true relative error of 0.89 percent.

Second iteration:

$$f(x_l)\, f(x_r) = -1.5426$$

Therefore, the root lies in the first subinterval, and x_r becomes the upper limit for the next iteration, $x_u = 14.9113$:

$$x_l = 12 \qquad f(x_l) = 6.0699$$
$$x_u = 14.9113 \qquad f(x_u) = -0.2543$$
$$x_r = 14.9113 - \frac{-0.2543(12 - 14.9113)}{6.0669 - (-0.2543)} = 14.7942$$

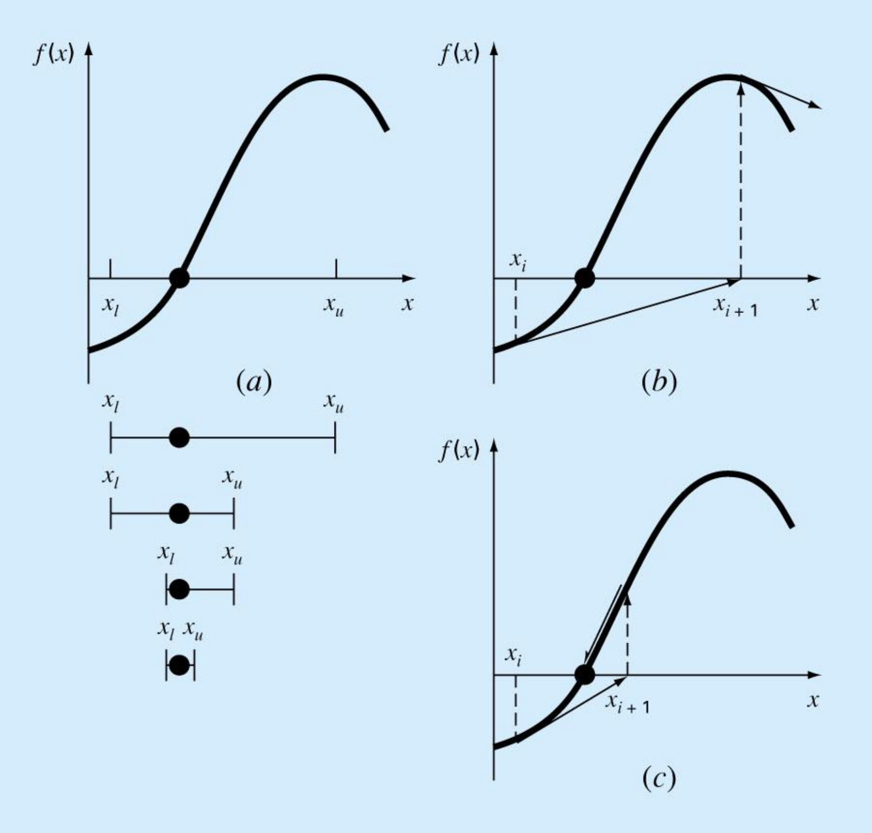
which has true and approximate relative errors of 0.09 and 0.79 percent. Additional iterations can be performed to refine the estimate of the roots.

PART B

OPEN METHODS

Open Methods

Open methods are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root.



Simple Fixed-point Iteration

Rearrange the function so that x is on the left side of the equation:

$$f(x) = 0 \quad \Rightarrow \quad g(x) = x$$

$$x_k = g(x_{k-1}) \qquad x_o \text{ given}, \mathrm{k} = 1, 2, \ldots$$

•Bracketing methods are “convergent”.

•Fixed-point methods may sometime “diverge”, depending on the stating point (initial guess) and how the function behaves.

Example :

$$f(x) = x^2 - x - 2 \qquad\qquad x \succ 0$$

$$g(x) = x^2 - 2$$

$$or$$

$$g(x) = \sqrt{x+2}$$

$$or$$

$$g(x) = 1 + \frac{2}{x}$$

$$\vdots$$

Example #2

Use simple fixed-point iteration to locate the root of

$$f(x) = e^{-x} - x.$$

The function can be separated as

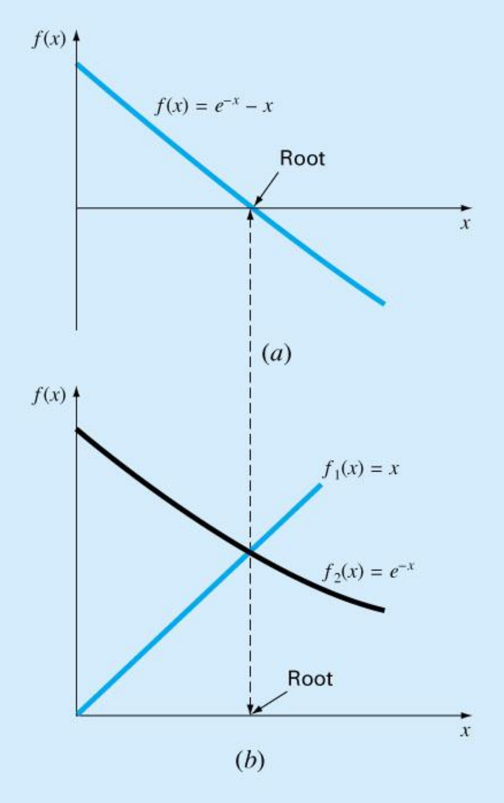
$$x_{i+1} = e^{-x_i}$$

Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute

i	x_i	ε_a (%)	ε_t (%)
0	0		100.0
1	1.000000	100.0	76.3
2	0.367879	171.8	35.1
3	0.692201	46.9	22.1
4	0.500473	38.3	11.8
5	0.606244	17.4	6.89
6	0.545396	11.2	3.83
7	0.579612	5.90	2.20
8	0.560115	3.48	1.24
9	0.571143	1.93	0.705
10	0.564879	1.11	0.399

Thus, each iteration brings the estimate closer to the true value of the root: 0.56714329.

Convergence



x=g(x) can be expressed as a pair of equations:

y1=x

y2=g(x) (component equations)

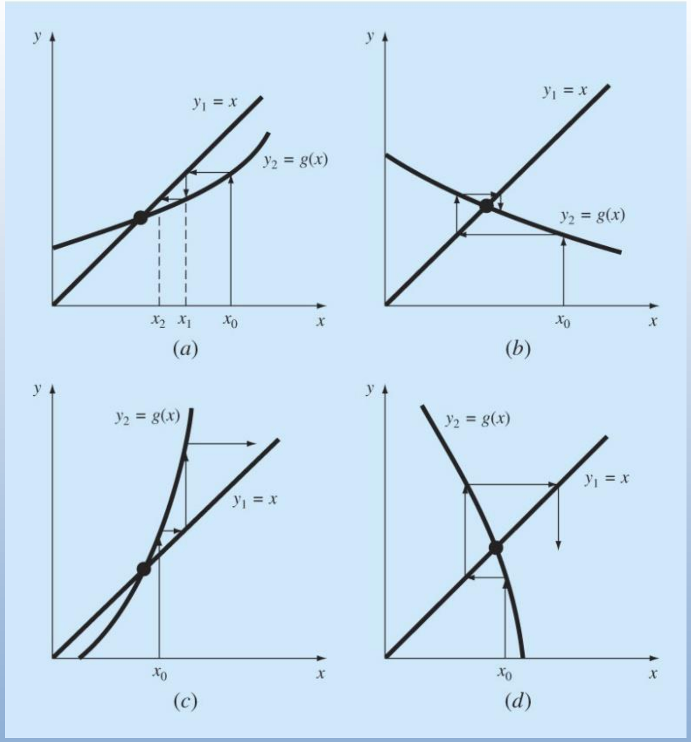
Plot them separately.

Conclusion

Fixed-point iteration converges if

$$\left|g'(x)\right| \prec 1 \quad \text{(slope of the line f(x) = x)}$$

When the method converges, the error is roughly proportional to or less than the error of the previous step, therefore it is called "linearly convergent."



1. Newton-Raphson Method

Most widely used method.

Based on Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

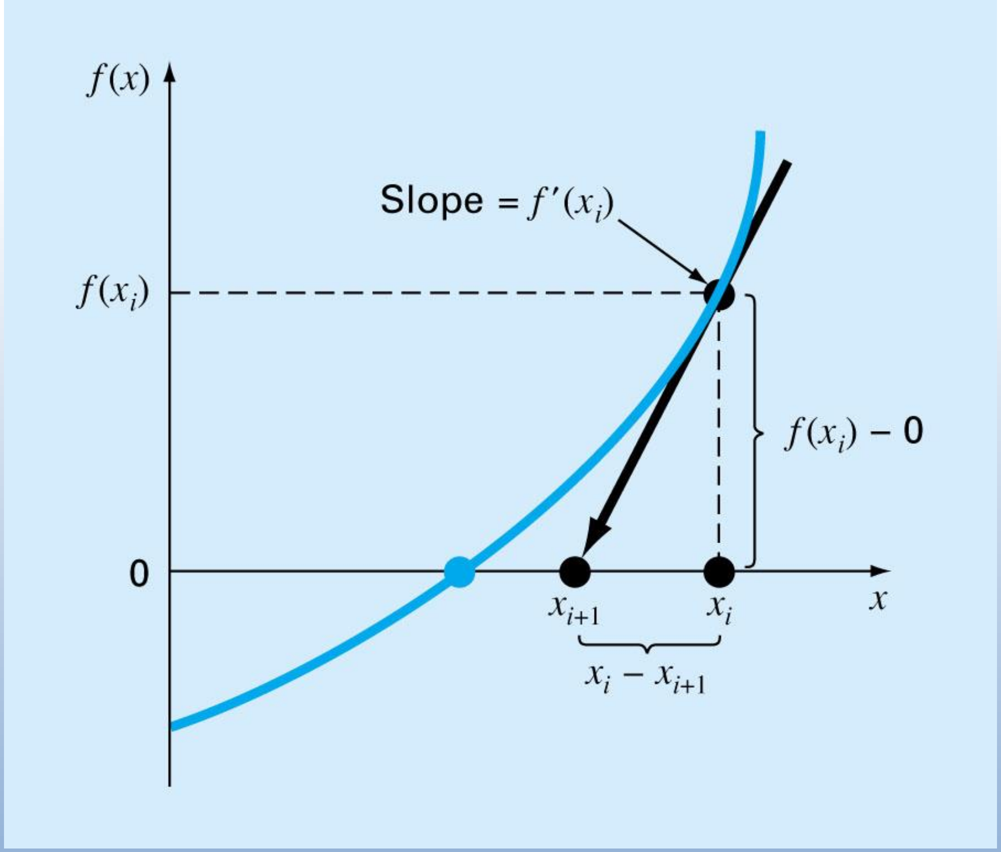
The root is the value of $\mathrm{x}_{\mathrm{i+1}}$ when $\mathrm{f}(\mathrm{x}_{\mathrm{i+1}}) = 0$

Rearranging,

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

A convenient method for functions whose derivatives can be evaluated analytically. It may not be convenient for functions whose derivatives cannot be evaluated analytically.



Example #3

Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$.

Solution. The first derivative of the function can be evaluated as

$$f'(x) = -e^{-x} - 1$$

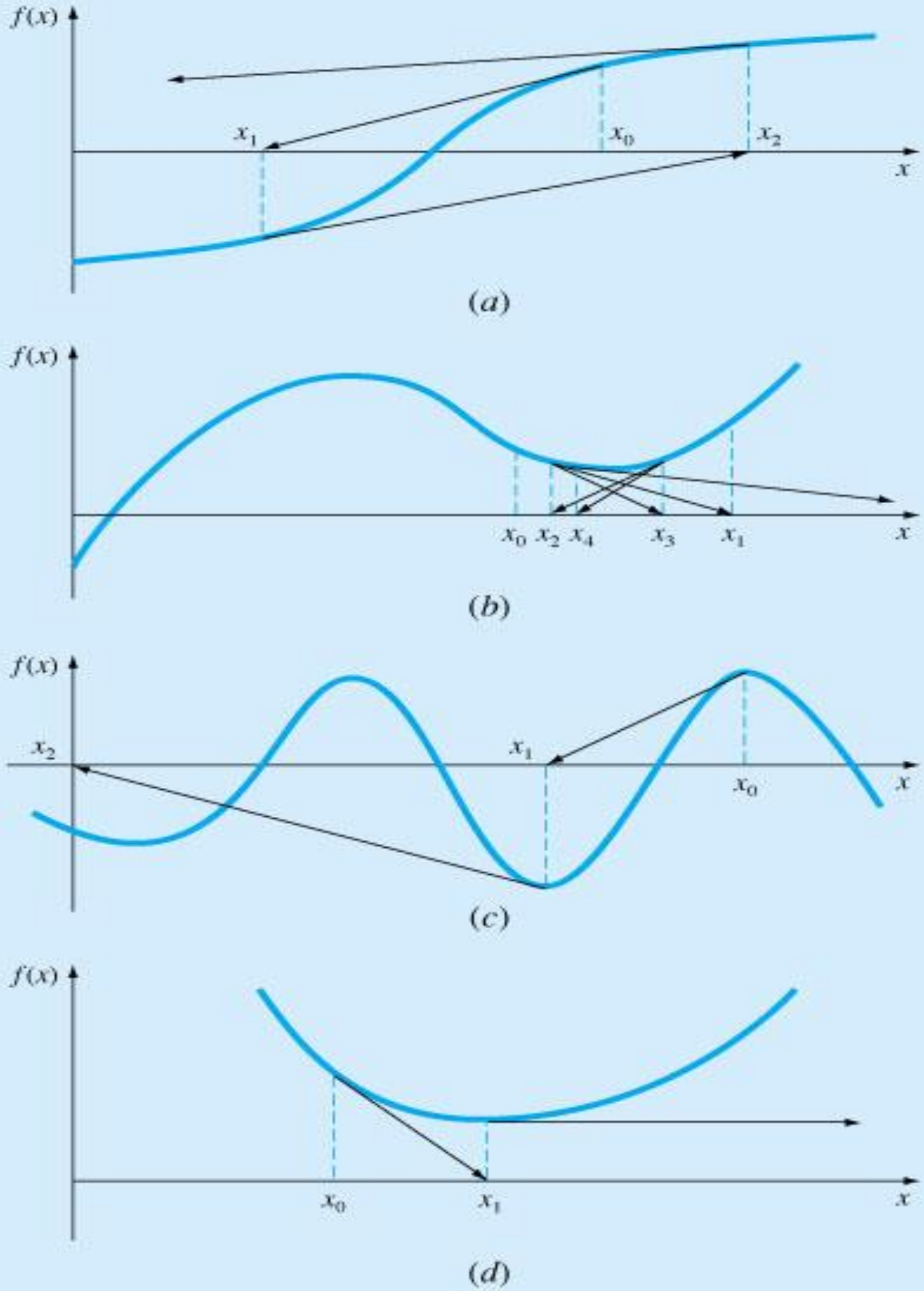
which can be substituted along with the original function into Eq. (6.6) to give

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute

i	x_i	ε_t (%)
0	0	100
1	0.500000000	11.8
2	0.566311003	0.147
3	0.567143165	0.0000220
4	0.567143290	$< 10^{-8}$

Four cases where the Newton-Raphson method exhibits poor convergence



2. Secant Method

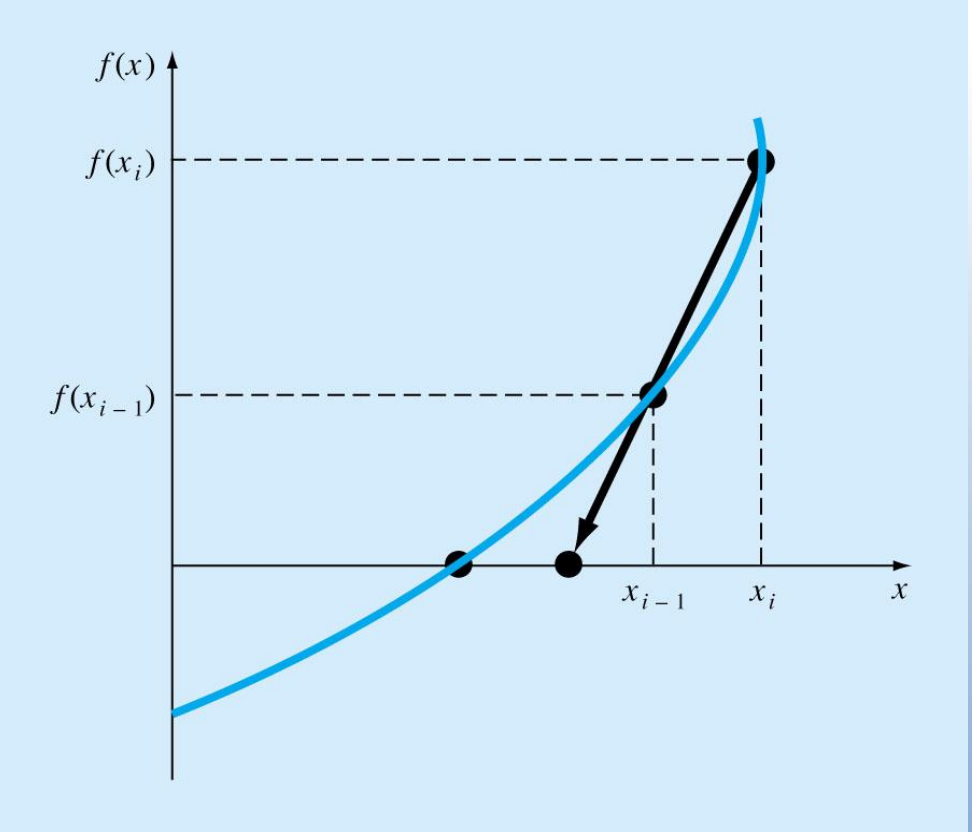
A slight variation of Newton's method for functions whose derivatives are difficult to evaluate. For these cases the derivative can be approximated by a backward finite divided difference.

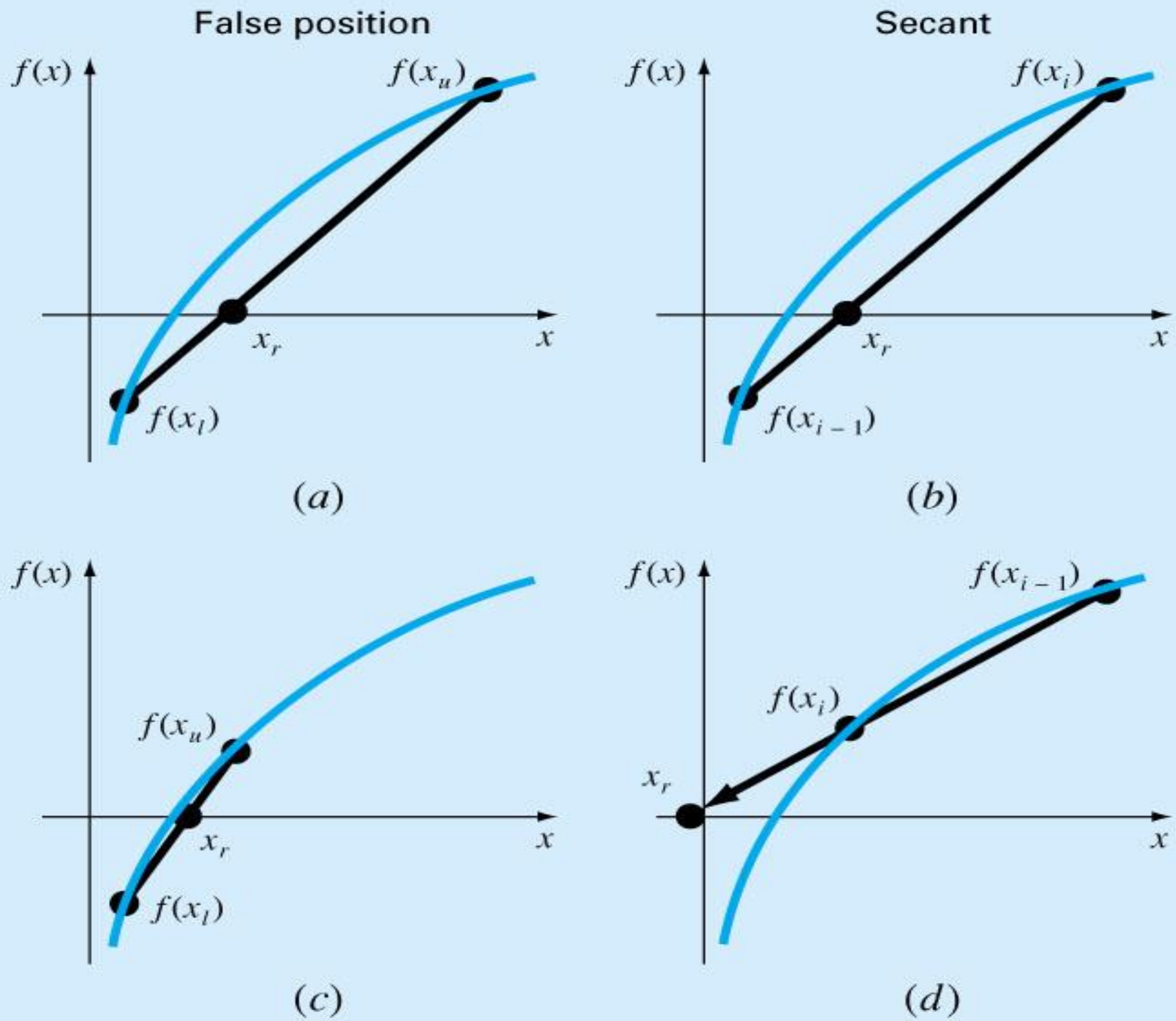
$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \qquad i = 1,2,3,\ldots$$

• Requires two initial estimates of x , e.g, x_o, x_1. However, because f(x) is not required to change signs between estimates, it is not classified as a “bracketing” method.

• The scant method has the same properties as Newton’s method. Convergence is not guaranteed for all x_o, f(x).




False position
Secant
$f(x)$
$f(x_u)$
x_r
x
$f(x_l)$
$f(x_i)$
$f(x_{i-1})$
(a)
(b)
(c)
(d)

Example #4

Use the secant method to estimate the root of $f(x)=e^{-x}-x$. Start with initial estimates of $x_{-1}=0$ and $x_0=1.0$.

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1}=0 \qquad f(x_{-1})=1.00000$$

$$x_0=1 \qquad f(x_0)=-0.63212$$

$$x_1=1-\frac{-0.63212(0-1)}{1-(-0.63212)}=0.61270 \qquad \varepsilon_t=8.0\%$$

Second iteration:

$$x_0=1 \qquad f(x_0)=-0.63212$$

$$x_1=0.61270 \qquad f(x_1)=-0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2=0.61270-\frac{-0.07081(1-0.61270)}{-0.63212-(-0.07081)}=0.56384 \qquad \varepsilon_t=0.58\%$$

Third iteration:

$$x_1=0.61270 \qquad f(x_1)=-0.07081$$

$$x_2=0.56384 \qquad f(x_2)=0.00518$$

$$x_3=0.56384-\frac{0.00518(0.61270-0.56384)}{-0.07081-(-0.00518)}=0.56717 \qquad \varepsilon_t=0.0048\%$$

Multiple Roots

None of the methods deal with multiple roots efficiently, however, one way to deal with problems is as follows:

Set $$u(x_i) = \frac{f(x_i)}{f'(x_i)}$$

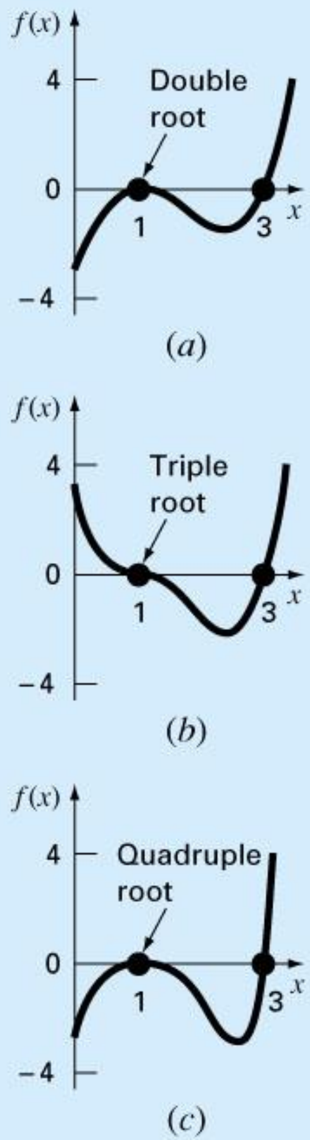
This function has roots at all the same locations as the original function

Then find $$\mathrm{x}_{\mathrm{i+1}} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$u'(x) = \frac{f'(x)f'(x) - f(x)f''(x)}{\left[f'(x)\right]^2}$$

$$\Rightarrow \mathrm{x}_{\mathrm{i+1}} = x_i - \frac{f(x_i)f'(x_i)}{\left[f'(x)\right]^2 - f(x)f''(x)}$$

Multiple Roots



“Multiple root” corresponds to a point where a function is tangent to the x axis.

Difficulties

Function does not change sign at the multiple root, therefore, cannot use bracketing methods.

Both f(x) and f′(x)=0, division by zero with Newton’s and Secant methods.

Example #4

Use both the standard and modified Newton-Raphson methods to evaluate the multiple root of f(x) = (x − 3)(x − 1)(x − 1), with an initial guess of $x_0 = 0$.

The first derivative of f(x) is f'(x) = 3x2 − 10x + 7, and therefore, the standard Newton-Raphson method for this problem is

$$x_{i+1} = x_i - \frac{x_i^3 - 5x_i^2 + 7x_i - 3}{3x_i^2 - 10x_i + 7}$$

which can be solved iteratively for

i	x_i	ε_t (%)
0	0	100
1	0.4285714	57
2	0.6857143	31
3	0.8328654	17
4	0.9133290	8.7
5	0.9557833	4.4
6	0.9776551	2.2

As anticipated, the method is linearly convergent toward the true value of 1.0.

For the modified method, the second derivative is $f''(x) = 6x - 10$, and the iterative relationship is [Eq. (6.16)]

$$x_{i+1} = x_i - \frac{(x_i^3 - 5x_i^2 + 7x_i - 3)(3x_i^2 - 10x_i + 7)}{(3x_i^2 - 10x_i + 7)^2 - (x_i^3 - 5x_i^2 + 7x_i - 3)(6x_i - 10)}$$

which can be solved for

i	x_i	ε_t (%)
0	0	100
1	1.105263	11
2	1.003082	0.31
3	1.000002	0.00024

Systems of Linear Equations

To this point, we have focused on the determination of the roots of a single equation. A related problem is to locate the roots of a set of simultaneous equations,

$$
\begin{aligned}
&f_1(x_1, x_2, x_3, \ldots, x_n) = 0 \\
&f_2(x_1, x_2, x_3, \ldots, x_n) = 0 \\
&\vdots \\
&f_n(x_1, x_2, x_3, \ldots, x_n) = 0
\end{aligned}
$$

Fixed-Point Iteration

Use fixed-point iteration to determine the roots of

u(x, y) = x2 + xy − 10 = 0

v(x, y) = y + 3xy2 − 57 = 0

Note that a correct pair of roots is x = 2 and y = 3. Initiate the computation with guesses of x = 1.5 and y = 3.5.

We have

$$x = \sqrt{10 - xy}$$

$$y = \sqrt{\frac{57-y}{3x}}$$

Now the results are more satisfactory:

$$x = \sqrt{10 - 1.5(3.5)} = 2.17945$$

$$y = \sqrt{\frac{57-3.5}{3(2.17945)}} = 2.86051$$

$$x = \sqrt{10 - 2.17945(2.86051)} = 1.94053$$

$$y = \sqrt{\frac{57-2.86051}{3(1.94053)}} = 3.04955$$

Thus, the approach is converging on the true values of x = 2 and y = 3.

Newton-Raphson

Taylor series expansion of a function of more than one variable

$$u_{i+1} = u_i + \frac{\partial u_i}{\partial x}(x_{i+1} - x_i) + \frac{\partial u_i}{\partial y}(y_{i+1} - y_i)$$

$$v_{i+1} = v_i + \frac{\partial v_i}{\partial x}(x_{i+1} - x_i) + \frac{\partial v_i}{\partial y}(y_{i+1} - y_i)$$

The root of the equation occurs at the value of x and y where u_{i+1} and v_{i+1} equal to zero.

$$\frac{\partial u_i}{\partial x} x_{i+1} + \frac{\partial u_i}{\partial y} y_{i+1} = -u_i + x_i \frac{\partial u_i}{\partial x} + y_i \frac{\partial u_i}{\partial y}$$

$$\frac{\partial v_i}{\partial x} x_{i+1} + \frac{\partial v_i}{\partial y} y_{i+1} = -v_i + x_i \frac{\partial v_i}{\partial x} + y_i \frac{\partial v_i}{\partial y}$$

A set of two linear equations with two unknowns that can be solved for.

$$x_{i+1} = x_i - \frac{u_i \dfrac{\partial v_i}{\partial y} - v_i \dfrac{\partial u_i}{\partial y}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}$$

$$y_{i+1} = y_i - \frac{u_i \dfrac{\partial v_i}{\partial x} - v_i \dfrac{\partial u_i}{\partial x}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}$$

Determinant of the *Jacobian* of the system.

Example #5

Use the multiple-equation Newton-Raphson method to determine roots of

u(x, y) = x2 + xy − 10 = 0

v(x, y) = y + 3xy2 − 57 = 0

Note that a correct pair of roots is x = 2 and y = 3. Initiate the computation with guesses of x = 1.5 and y = 3.5.

Solution. First compute the partial derivatives and evaluate them at the initial guesses of x and y:

$$\frac{\partial u_0}{\partial x} = 2x + y = 2(1.5) + 3.5 = 6.5 \qquad \frac{\partial u_0}{\partial y} = x = 1.5$$

$$\frac{\partial v_0}{\partial x} = 3y^2 = 3(3.5)^2 = 36.75 \qquad \frac{\partial v_0}{\partial y} = 1 + 6xy = 1 + 6(1.5)(3.5) = 32.5$$

Thus, the determinant of the Jacobian for the first iteration is

$$6.5(32.5) - 1.5(36.75) = 156.125$$

The values of the functions can be evaluated at the initial guesses as

$$u_0 = (1.5)^2 + 1.5(3.5) - 10 = -2.5$$

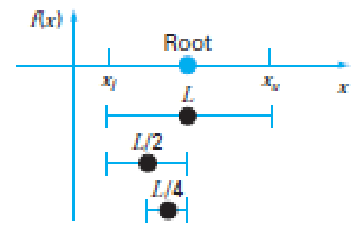
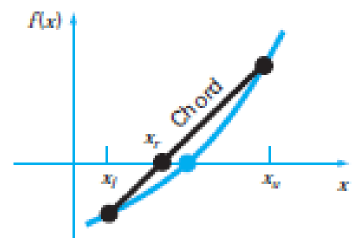
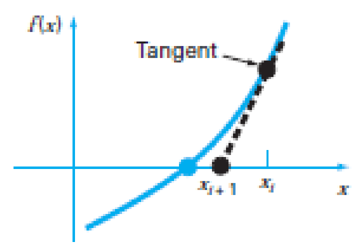
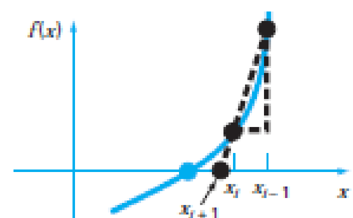
$$v_0 = 3.5 + 3(1.5)(3.5)^2 - 57 = 1.625$$

These values can be substituted

$$x = 1.5 - \frac{-2.5(32.5) - 1.625(1.5)}{156.125} = 2.03603$$

$$y = 3.5 - \frac{1.625(6.5) - (-2.5)(36.75)}{156.125} = 2.84388$$

Thus, the results are converging to the true values of $x = 2$ and $y = 3$. The computation can be repeated until an acceptable accuracy is obtained.

Method	Formulation	Graphical Interpretation	Errors and Stopping Criteria
		Bracketing methods:	
Bisection	$x_r = \dfrac{x_l + x_u}{2}$ If $f(x_l)f(x_r) < 0,\ x_u = x_r$ $f(x_l)f(x_r) > 0,\ x_l = x_r$		Stopping criterion: $\left\lvert \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}} \right\rvert 100\% \le \varepsilon_s$
False position	$x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$ If $f(x_l)f(x_r) < 0,\ x_u = x_r$ $f(x_l)f(x_r) > 0,\ x_l = x_r$		Stopping criterion: $\left\lvert \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}} \right\rvert 100\% \le \varepsilon_s$
Newton-Raphson	$x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$		Stopping criterion: $\left\lvert \dfrac{x_{i+1} - x_i}{x_{i+1}} \right\rvert 100\% \le \varepsilon_s$ Error: $E_{i+1} = O(E_i^2)$
Secant	$x_{i+1} = x_i - \dfrac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$		Stopping criterion: $\left\lvert \dfrac{x_{i+1} - x_i}{x_{i+1}} \right\rvert 100\% \le \varepsilon_s$

PART C

ROOTS OF POLYNOMIALS

Roots of Polynomials

The roots of polynomials such as

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Follow these rules:

1.For an n^{th} order equation, there are n real or complex roots.

2.If *n* is odd, there is at least one real root.

3.If complex root exist in conjugate pairs (that is, *l+mi* and *l-mi*), where *i=sqrt(-1).*

Conventional Methods

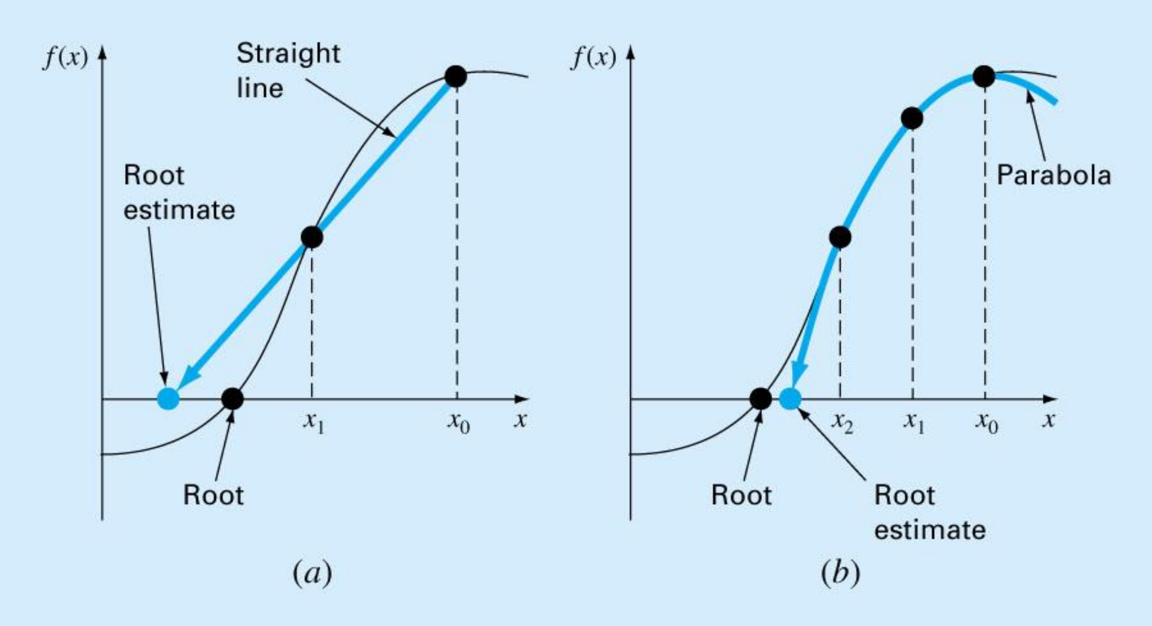
The efficacy of bracketing and open methods depends on whether the problem being solved involves complex roots. If only real roots exist, these methods could be used.

However, finding good initial guesses complicates both the open and bracketing methods, also the open methods could be susceptible to divergence.

Special methods have been developed to find the real and complex roots of polynomials – Müller and Bairstow methods.

Müller Method

Müller's method obtains a root estimate by projecting a parabola to the x axis through three function values.

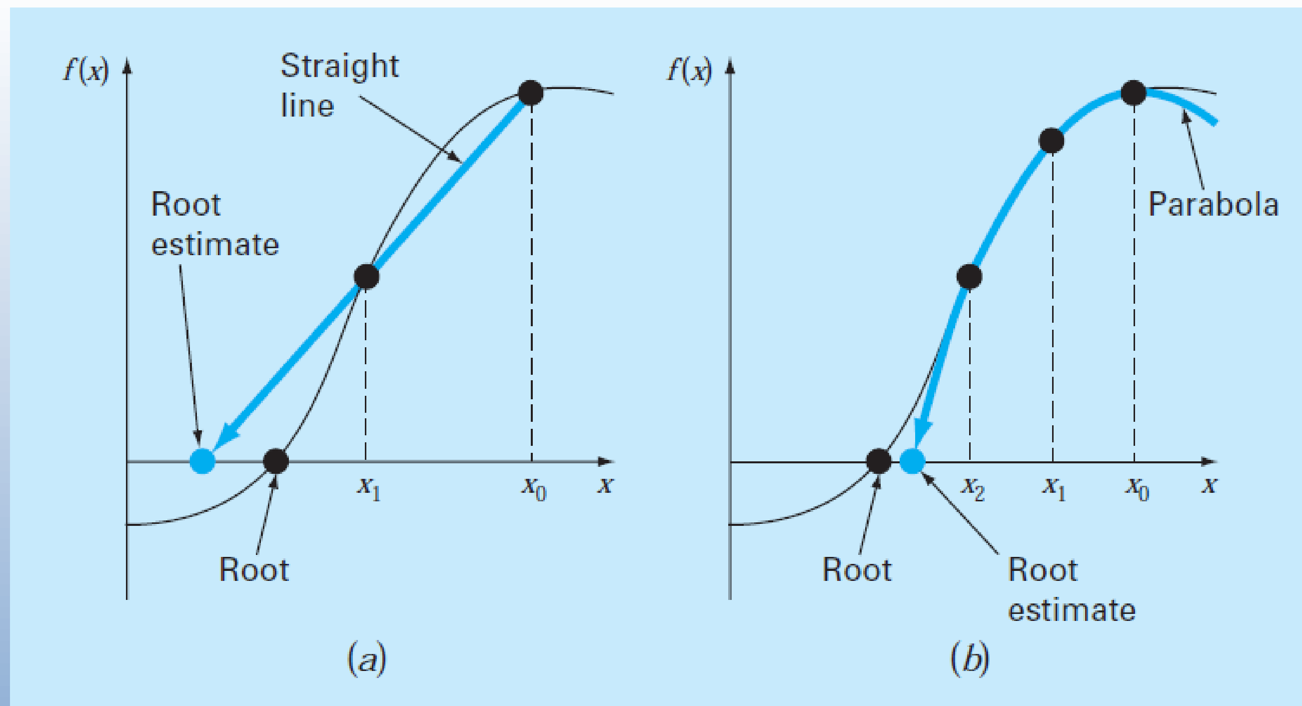


Müller Method

The method consists of deriving the coefficients of parabola that goes through the three points:

1. Write the equation in a convenient form:

$$f_2(x) = a(x - x_2)^2 + b(x - x_2) + c$$



A comparison of two related approaches for locating roots: (a) the secant method and (b) Müller's method.

2. The parabola should intersect the three points *[x_o, $f(x_o)$], [x_1, $f(x_1)$], [x_2, $f(x_2)$].* The coefficients of the polynomial can be estimated by substituting three points to give

$$f(x_o) = a(x_o - x_2)^2 + b(x_o - x_2) + c$$

$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$

$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

3. Three equations can be solved for three unknowns, *a, b, c*. Since two of the terms in the 3rd equation are zero, it can be immediately solved for $c=f(x_2)$.

$$f(x_o) - f(x_2) = a(x_o - x_2)^2 + b(x_o - x_2)$$

$$f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2)$$

If

$h_o = x_1 - x_o \qquad h_1 = x_2 - x_1$

$$\delta_o = \frac{f(x_1) - f(x_o)}{x_1 - x_o} \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We have

$$x_2 - x_0 = h_1 + h_0$$

$$\delta_0 h_0 = f(x_1) - f(x_0)$$

$$\delta 1 h 1 = f(x_2) - f(x_1)$$

$$\Rightarrow f(x_2) - f(x_0) = \delta 0 h 0 + \delta 1 h 1$$

So,

$$(h_o + h_1)b - (h_o + h_1)^2 a = h_o \delta_o + h_1 \delta_1$$

$$h_1 b - h_1^2 a = h_1 \delta_1$$

The result can be summarized as

$$a = \frac{\delta_1 - \delta_o}{h_1 + h_o} \qquad b = a h_1 + \delta_1 \qquad c = f(x_2)$$

Roots can be found by applying an alternative form of quadratic formula:

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

The error can be calculated as

$$\varepsilon_a = \left| \frac{x_3 - x_2}{x_3} \right| 100\%$$

±term yields two roots, the sign is chosen to agree with b. This will result in a largest denominator, and will give root estimate that is closest to x_2.

Once x_3 is determined, the process is repeated using the following guidelines:

1. If only real roots are being located, choose the two original points that are nearest the new root estimate, x_3.
2. If both real and complex roots are estimated, employ a sequential approach just like in secant method, x_1, x_2, and x_3 to replace x_0, x_1, and x_2.

Example #6

Use Müller's method with guesses of x0, x1, and x2 = 4.5, 5.5, and 5, respectively, to determine a root of the equation f(x) = x3 − 13x − 12

Note that the roots of this equation are −3, −1, and 4.

Solution. First, we evaluate the function at the guesses

$$f(4.5) = 20.625 \qquad f(5.5) = 82.875 \qquad f(5) = 48$$

which can be used to calculate

$$h_0 = 5.5 - 4.5 = 1 \qquad h_1 = 5 - 5.5 = -0.5$$

$$\delta_0 = \frac{82.875 - 20.625}{5.5 - 4.5} = 62.25 \qquad \delta_1 = \frac{48 - 82.875}{5 - 5.5} = 69.75$$

These values in turn can be substituted

$$a = \frac{69.75 - 62.25}{-0.5 + 1} = 15 \qquad b = 15(-0.5) + 69.75 = 62.25 \qquad c = 48$$

The square root of the discriminant can be evaluated as

$$\sqrt{62.25^2 - 4(15)48} = 31.54461$$

Then, because $|62.25 + 31.54451| > |62.25 - 31.54451|$,

$$x_3 = 5 + \frac{-2(48)}{62.25 + 31.54451} = 3.976487$$

and develop the error estimate

$$\varepsilon_a = \left|\frac{-1.023513}{3.976487}\right| 100\% = 25.74\%$$

i	x_r	ε_a (%)
0	5	
1	3.976487	25.74
2	4.00105	0.6139
3	4	0.0262
4	4	0.0000119

Because the error is large, new guesses are assigned; x_0 is replaced by x_1, x_1 is replaced by x_2, and x_2 is replaced by x_3. Therefore, for the new iteration,

$$x_0 = 5.5 \qquad x_1 = 5 \qquad x_2 = 3.976487$$

2. Bairstow's Method

Bairstow's method is an iterative approach loosely related to both Müller and Newton Raphson methods.

It is based on dividing a polynomial by a factor x-t:

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$f_{n-1}(x) = b_1 + b_2 x + b_3 x^2 + \ldots + b_n x^{n-1}$$

with a reminder $\mathrm{R} = \mathrm{b}_{\mathrm{o}}$, the coefficients are calculated by recurrence relationship

$$b_n = a_n$$

$$b_i = a_i + b_{i+1} t \quad i = n-1 \text{ to } 2$$

To permit the evaluation of complex roots, Bairstow's method divides the polynomial by a quadratic factor x^2-rx-s:

$$f_{n-2}(x) = b_2 + b_3 x + \ldots + b_{n-1} x^{n-3} + b_n x^{n-2}$$

$$R = b_1(x-r) + b_o$$

Using a simple recurrence relationship

$$b_n = a_n$$

$$b_{n\text{-}1} = a_{n\text{-}1} + rb_n$$

$$b_i = a_i + rb_{i+1} + sb_{i+2} \qquad i = n\text{-}2 \text{ } to \text{ } 0$$

For the remainder to be zero, b_o and b_1 must be zero. However, it is unlikely that our initial guesses at the values of r and s will lead to this result, a systematic approach can be used to modify our guesses so that b_o and b_1 approach to zero.

Using a similar approach to Newton Raphson method, both b_o and b_1 can be expanded as function of both r and s in Taylor series.

Example

$$
\begin{array}{r}
4x^2 - 9x + 5 \\
2x + 8 \overline{\smash{\big)}\, 8x^3 + 14x^2 - 62x + 40} \\
\underline{-(8x^3 + 32x^2)} \qquad\qquad\quad \\
-18x^2 - 62x \qquad \\
- \qquad\qquad\qquad \\
\underline{(-18x^2 - 72x)} \qquad \\
10x + 40 \\
\underline{-(10x + 40)} \\
0
\end{array}
$$

$$
\begin{array}{r}
3x^2 - x - 8 \\
2x - 9 \overline{\smash{\big)}\, 6x^3 - 29x^2 - 7x + 72} \\
\underline{-(6x^3 - 27x^2)} \qquad\qquad\quad \\
-2x^2 - 7x \qquad \\
- \qquad\qquad\qquad \\
\underline{(-2x^2 + 9x)} \qquad \\
-16x + 72 \\
- \qquad\quad \\
\underline{(-16x + 72)} \\
0
\end{array}
$$

$$b_1(r+\Delta r, s+\Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s$$

$$b_o(r+\Delta r, s+\Delta s) = b_o + \frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s$$

assuming that the initial guesses are adequately close to the values of r and s at roots. The changes in Δs and Δr needed to improve our guesses will be estimated as

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1$$

$$\frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s = -b_o$$

If partial derivatives of the b's can be determined, then the two equations can be solved simultaneously for the two unknowns Δr and Δ s.

Partial derivatives can be obtained by a synthetic division of the b's in a similar fashion the b's themselves are derived:

$$c_n = b_n$$

$$c_{n-1} = b_{n-1} + rc_n$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \qquad i = n-2 \; to \; 2$$

$$where$$

$$\frac{\partial b_o}{\partial r} = c_1 \qquad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \qquad \frac{\partial b_1}{\partial s} = c_3$$

Then

$$\left.\begin{aligned} c_2\Delta r + c_3\Delta s &= -b_1 \\ c_1\Delta r + c_2\Delta s &= -b_o \end{aligned}\right\}$$

Solved for **Δ** r and **Δ** s, in turn are employed to improve the initial guesses.

At each step the error can be estimated as

$$\left|\varepsilon_{a,r}\right| = \left|\frac{\Delta r}{r}\right| 100\%$$

$$\left|\varepsilon_{a,s}\right| = \left|\frac{\Delta r}{r}\right| 100\%$$

- The values of the roots are determined by

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2}$$

- At this point three possibilities exist:

  1. *The quotient is a third-order polynomial or greater*. The previous values of r and s serve as initial guesses and Bairstow's method is applied to the quotient to evaluate new r and s values.

  2. *The quotient is quadratic*. The remaining two roots are evaluated directly, using the above eqn.

  3. *The quotient is a 1^{st} order polynomial*. The remaining single root can be evaluated simply as $x=-s/r$.

Example #7

Employ Bairstow’s method to determine the roots of the polynomial

f5(x) = x5 − 3.5x4 + 2.75x3 + 2.125x2 − 3.875x + 1.25

Use initial guesses of r = s=−1 and iterate to a level of εs = 1%.

Solution.

$b_5 = 1 \quad b_4 = -4.5 \quad b_3 = 6.25 \quad b_2 = 0.375 \quad b_1 = -10.5$
$b_0 = 11.375$

$c_5 = 1 \quad c_4 = -5.5 \quad c_3 = 10.75 \quad c_2 = -4.875 \quad c_1 = -16.375$

Thus, the simultaneous equations to solve for Δr and Δs are

$$-4.875\Delta r + 10.75\Delta s = 10.5$$
$$-16.375\Delta r - 4.875\Delta s = -11.375$$

which can be solved for $\Delta r = 0.3558$ and $\Delta s = 1.1381$. Therefore, our original guesses can be corrected to

$$r = -1 + 0.3558 = -0.6442$$
$$s = -1 + 1.1381 = 0.1381$$

and the approximate errors can be evaluated

$$|\varepsilon_{a,r}| = \left|\frac{0.3558}{-0.6442}\right| 100\% = 55.23\% \qquad |\varepsilon_{a,s}| = \left|\frac{1.1381}{0.1381}\right| 100\% = 824.1\%$$

Next, the computation is repeated using the revised values for r and s.

$b_5 = 1 \quad b_4 = -4.1442 \quad b_3 = 5.5578 \quad b_2 = -2.0276 \quad b_1 = -1.8013$
$b_0 = 2.1304$

$c_5 = 1 \quad c_4 = -4.7884 \quad c_3 = 8.7806 \quad c_2 = -8.3454 \quad c_1 = 4.7874$

Therefore, we must solve

$$-8.3454\Delta r + 8.7806\Delta s = 1.8013$$
$$4.7874\Delta r - 8.3454\Delta s = -2.1304$$

Example #7

for $\Delta r = 0.1331$ and $\Delta s = 0.3316$, which can be used to correct the root estimates as

$$r = -0.6442 + 0.1331 = -0.5111 \qquad |\varepsilon_{a,r}| = 26.0\%$$
$$s = 0.1381 + 0.3316 = 0.4697 \qquad |\varepsilon_{a,s}| = 70.6\%$$

The computation can be continued, with the result that after four iterations the method converges on values of $r = -0.5$ ($|\varepsilon_{a,r}| = 0.063\%$)and $s = 0.5$ ($|\varepsilon_{a,s}| = 0.040\%$).

$$x = \frac{-0.5 \pm \sqrt{(-0.5)^2 + 4(0.5)}}{2} = 0.5, -1.0$$

At this point, the quotient is the cubic equation

$$f(x) = x^3 - 4x^2 + 5.25x - 2.5$$

Bairstow's method can be applied to this polynomial using the results of the previous step, $r = -0.5$ and $s = 0.5$, as starting guesses. Five iterations yield estimates of $r = 2$ and $s = -1.249$, which can be used to compute

$$x = \frac{2 \pm \sqrt{2^2 + 4(-1.249)}}{2} = 1 \pm 0.499i$$

At this point, the quotient is a first-order polynomial that can be directly evaluated

Any Questions?

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