Computer Vision

About me

- Dr. Dinh Viet Sang
- Lecture at SoICT, HUST
- Email: sangdv@soict.hust.edu.vn
- Room 902 B1 Building

About course

- 12 weeks: lectures
- 3 weeks: Course project presentation

Grading Criteria and Method of Evaluation		
Kind	Percentage	Evaluation Criteria
Examination	50 %	understand basic concepts of image processing and computer vision
Report	%	
Continuous Assessment	50 %	Project: program, report and presentation
Others	%	
Note		

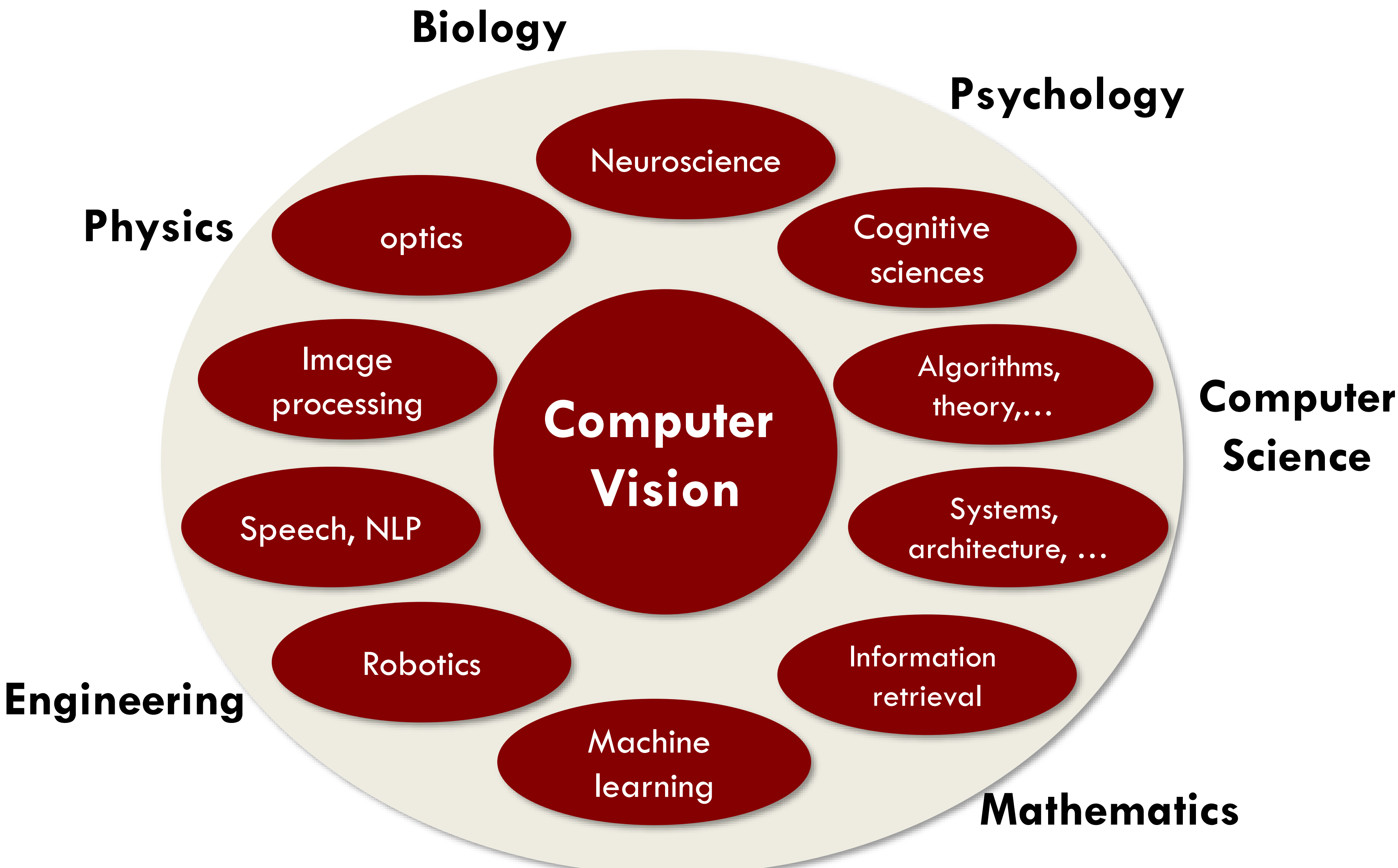
About course

Textbooks				
Title	Author	Publisher	ISBN code	Comment
Computer Vision: Algorithms and Applications	Richard Szeliski	Springer	1-55860-604-1	http://szeliski.org/Book/

Reference books				
Title	Author	Publisher	ISBN code	Comment
Computer Vision: A modern Approach	David A. Forsyth, Jean Ponce		978-0136085928	

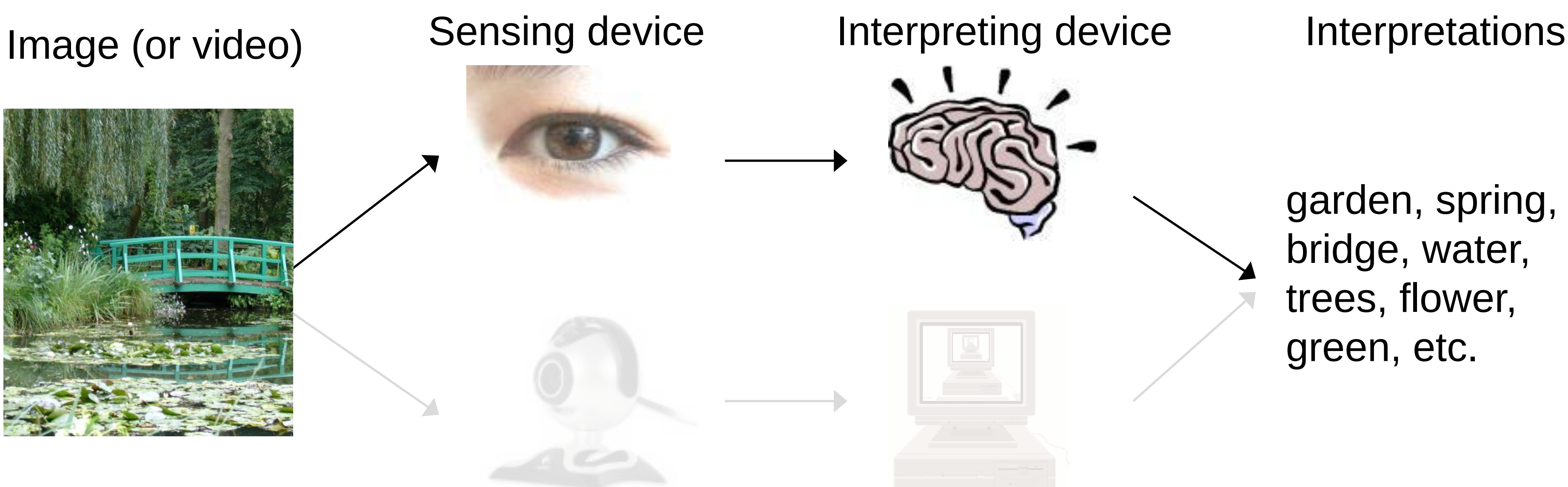
Internet Websites related to the Course
CS131: Computer Vision: Foundations and Applications http://vision.stanford.edu/teaching/cs131_fall1718/syllabus.html

Lecture 1:
Introduction to Computer Vision
Light and Colors



Computer vision brings together a large set of disciplines. Neuroscience can help computer vision by first understanding human vision. Computer vision can be seen as a part of computer science, and algorithm theory or machine learning are essential for developing computer vision algorithms.

What is (computer) vision?



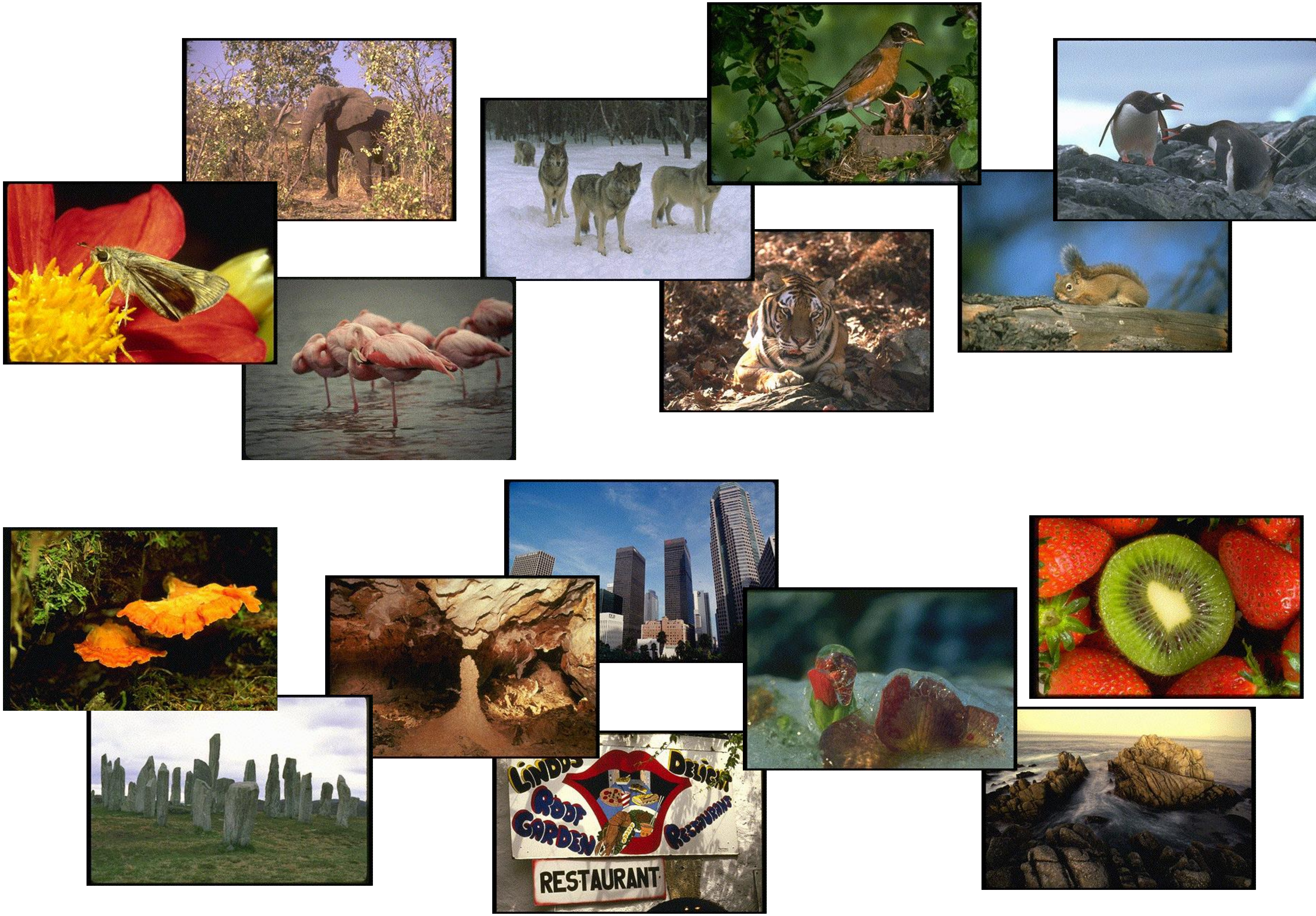
We receive images projected on our retina. Then our brain tries to process the signals and understand images. For instance, in the given image our brain will tries to interpret that here is garden in spring time, there are some objects like bridge, water, trees, flowers, etc..

Human vision is superbly efficient

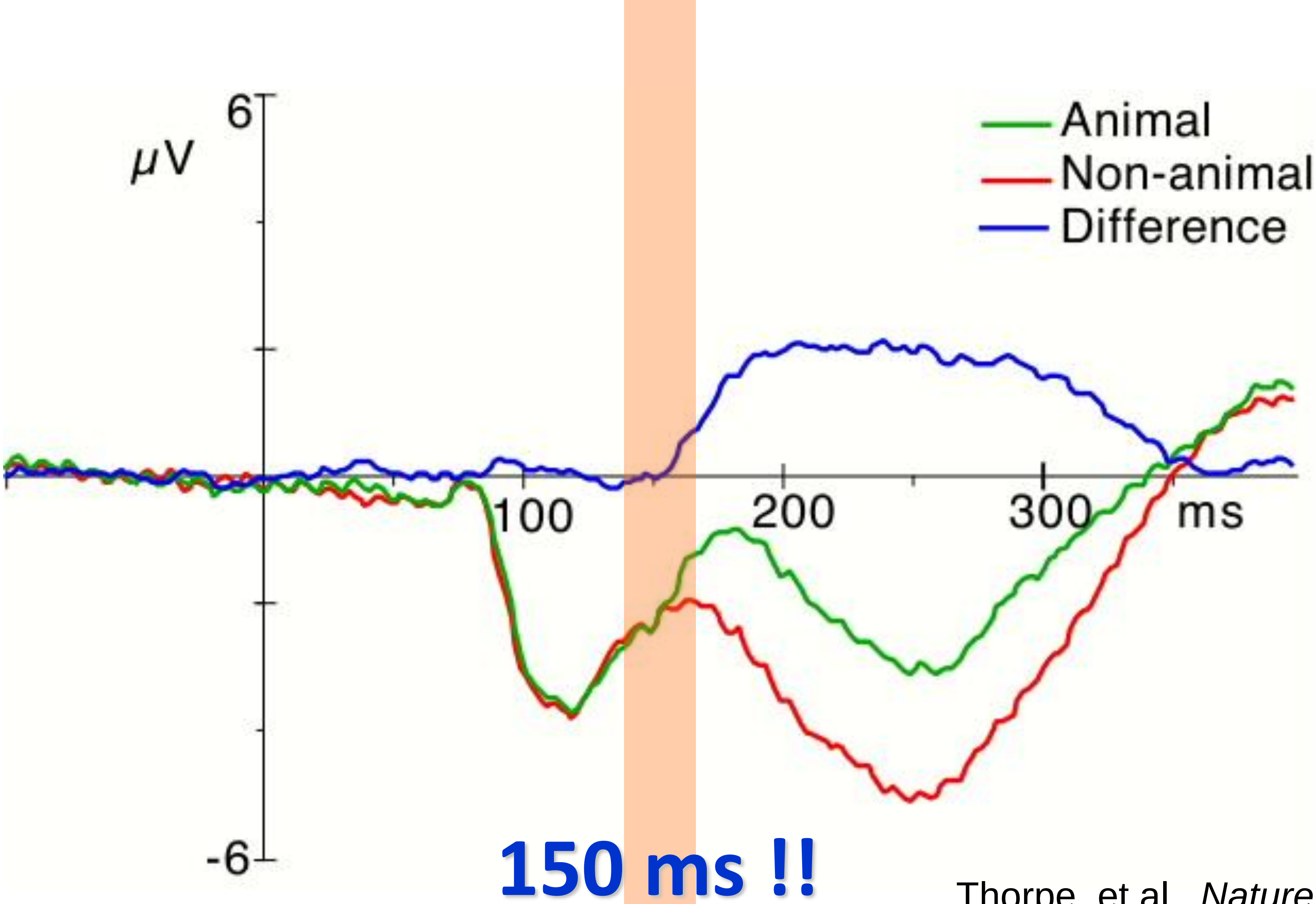
Potter, Biederman, etc. 1970s

100 ms per frame,
Never seen the picture, do not know the person

Can do effortlessly



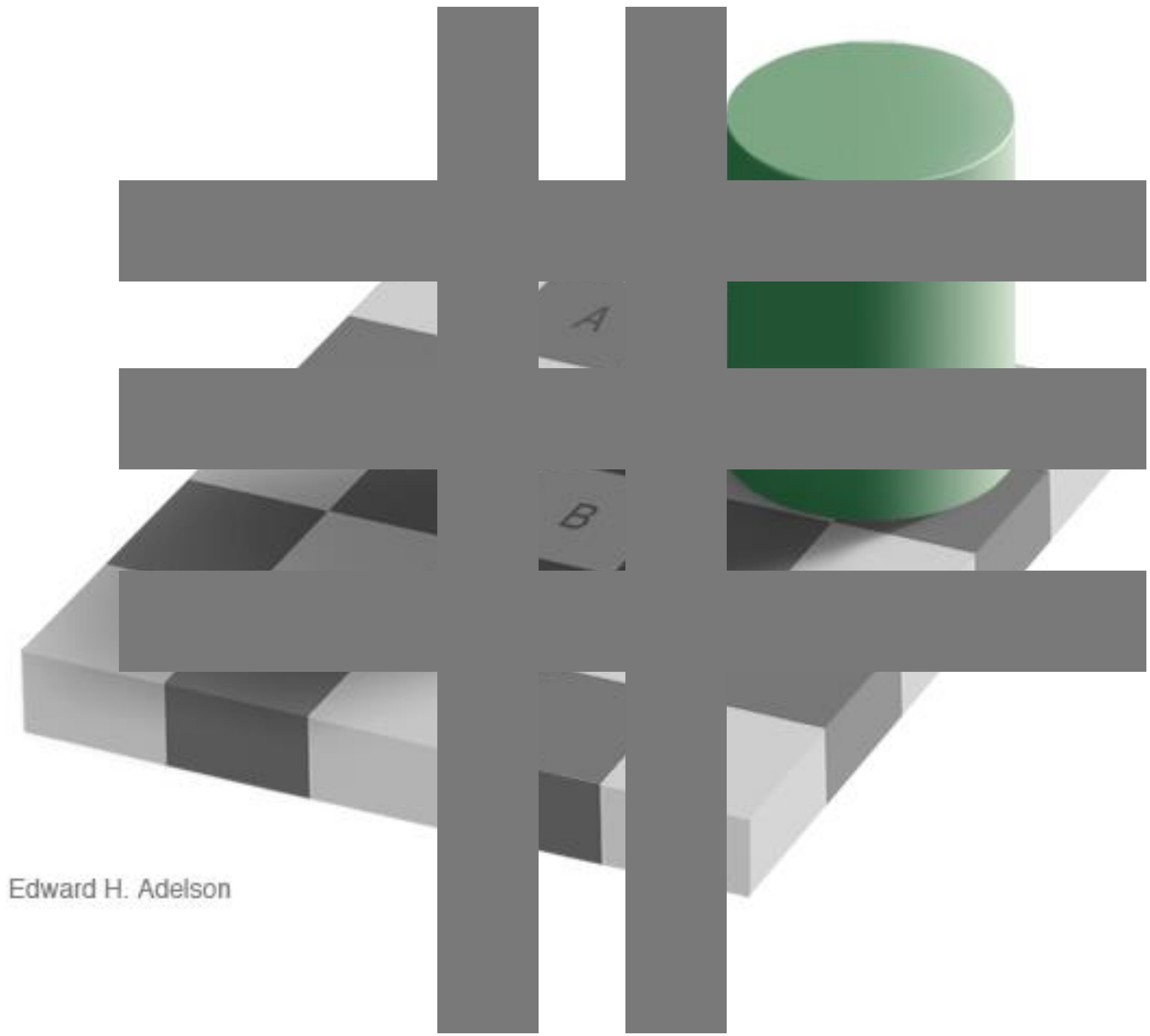
Thorpe, et al. *Nature,* 1996



Thorpe, et al. *Nature,* 1996

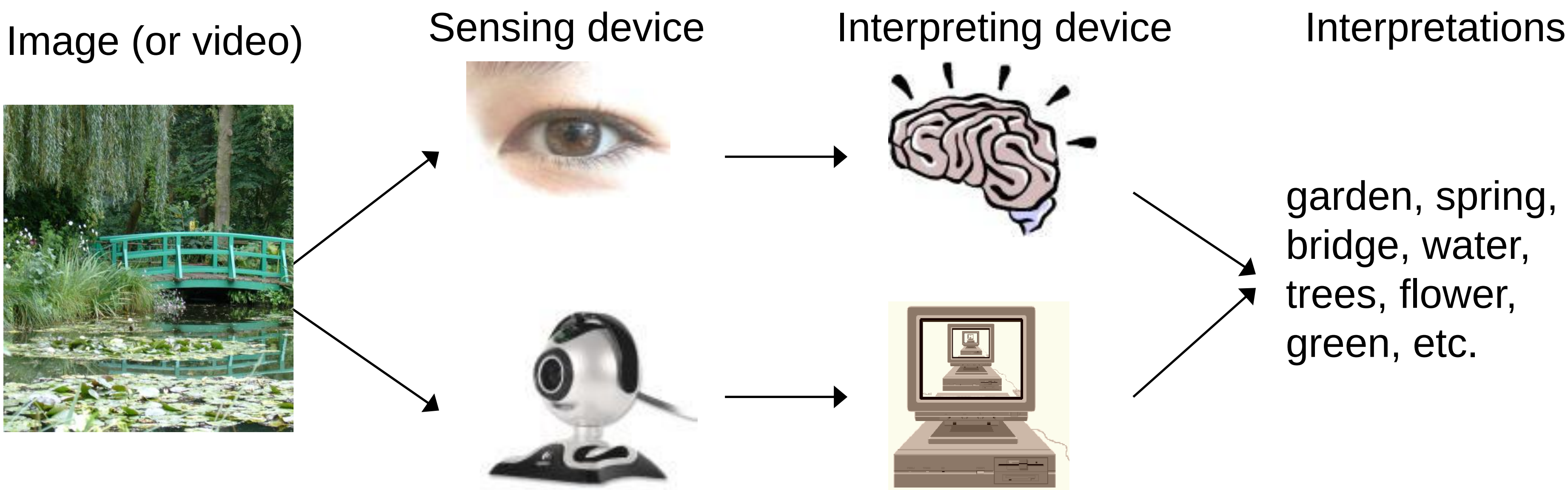
The speed of the human visual system has been measured to around 150ms to recognize an animal from a normal nature scene. Figure shows how the brain responses to images of animals and non-animals diverge after around 150ms.

Perception


A
B
Edward H. Adelson

Imitating birds did not lead humans to planes. Plainly copying nature is not the best way or the most efficient way to learn how to fly. But studying birds made us understand aerodynamics, and understanding concepts like lift allowed us to build planes.
The same might be true with intelligence. Even though it is not possible with today’s technology, simulating a full human brain to create intelligence might still not be the best way to get there. However, neuroscientists hope to get insights at what may be the concepts behind vision, language and other forms of intelligence.

What is (computer) vision?



Computer vision can be defined as a scientific field that extracts information out of digital images. The type of information gained from an image can vary from identification, space measurements for navigation, or augmented reality applications.
Another way to define computer vision is through its applications. Computer vision is building algorithms that can understand the content of images and use it for other applications.

The goal of computer vision

- To bridge the gap between pixels and “meaning”

What we see

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

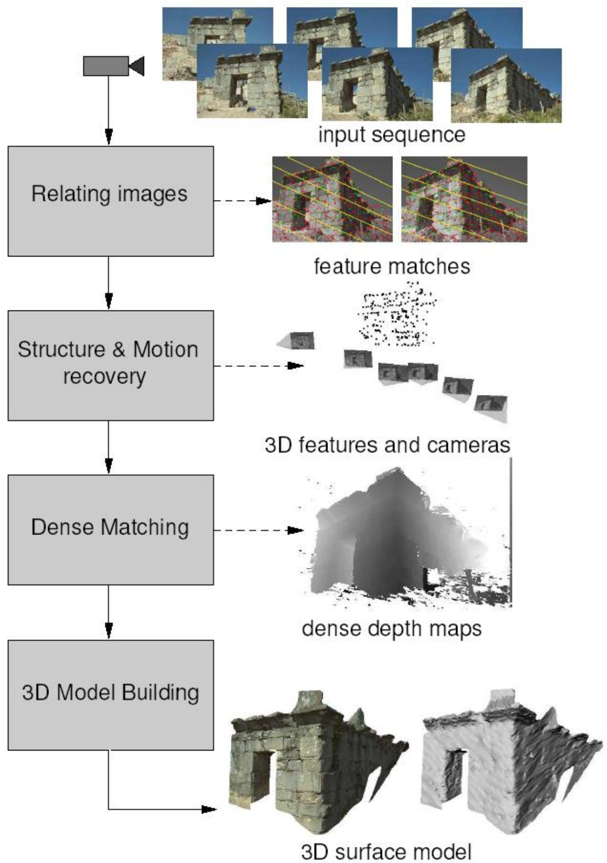
What a computer sees

Source: S. Narasimhan

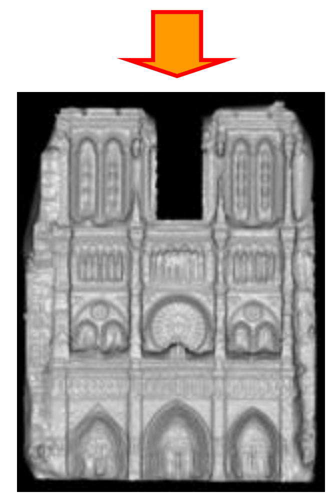
What kind of information can we extract from an image?

- Metric 3D information
- Semantic information

Vision as measurement device



Pollefeys et al.



Goesele et al.

Vision as a source of semantic information

Objects
Activities
Scenes
Locations
Text / writing
Faces
Gestures
Motions
Emotions…

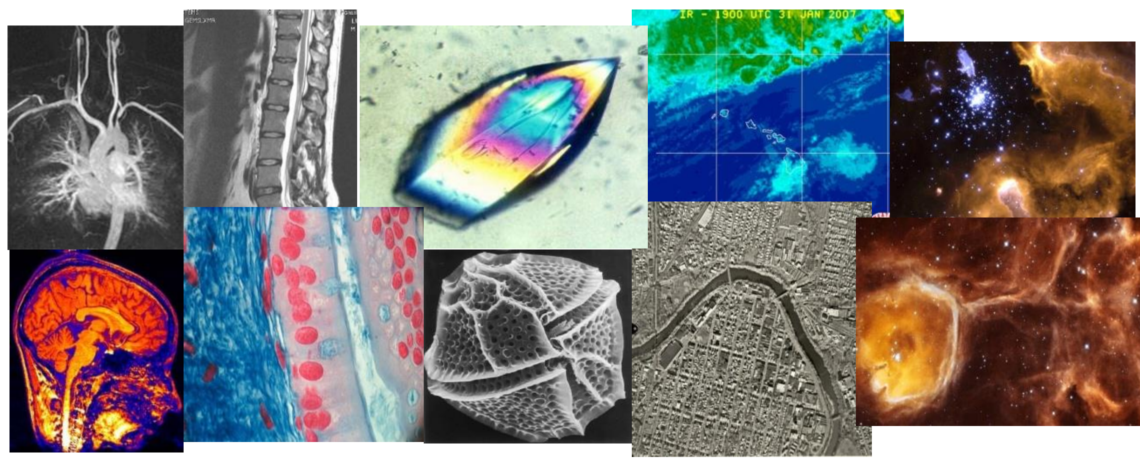
Slide credit: Kristen Grauman

Why study computer vision?

- Vision is useful: Images and video are everywhere!

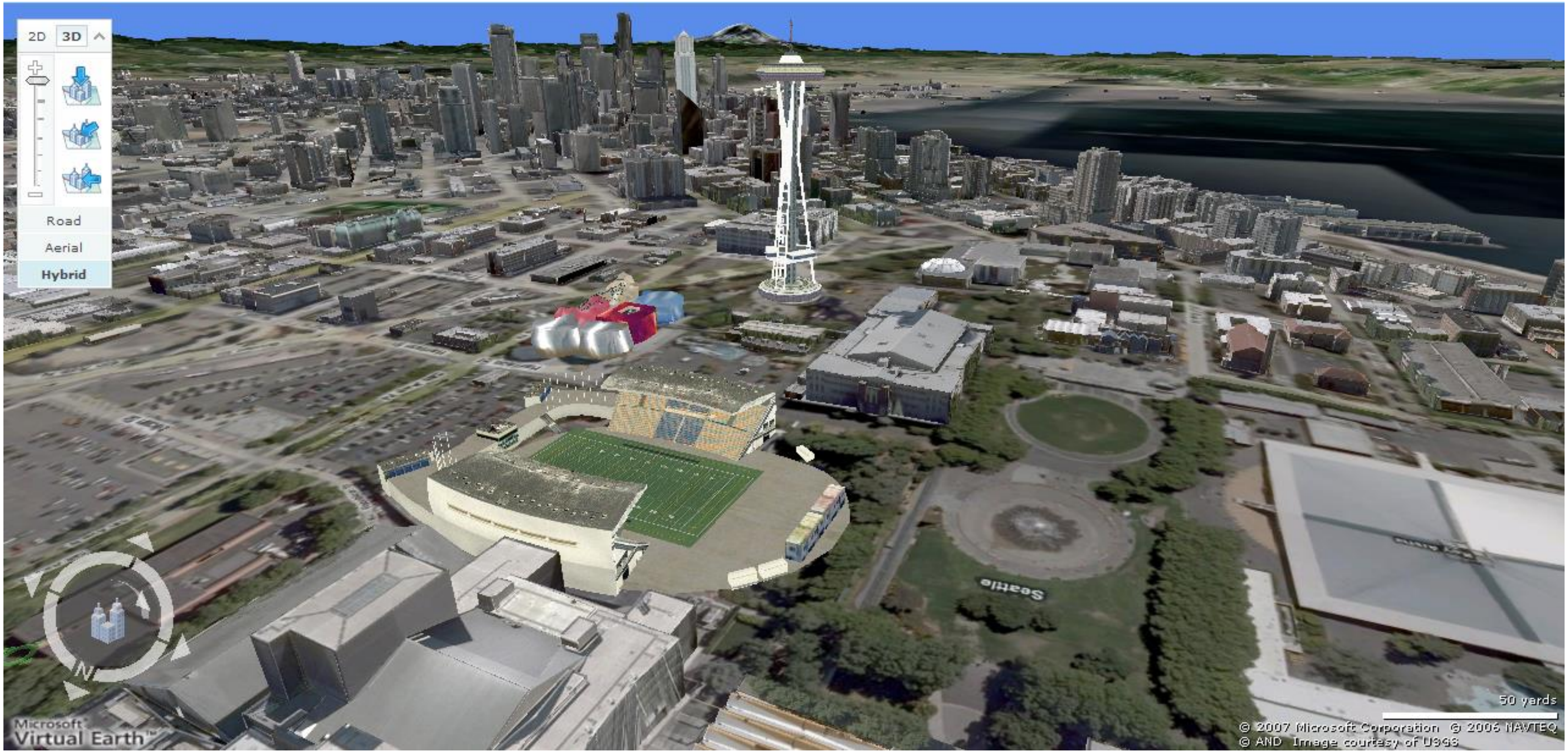
Google Image Search
Google Photos
flickr
webshots
picsearch
YouTube Broadcast Yourself

Surveillance and security



Medical and scientific images

3D urban modeling



Bing maps, Google Streetview

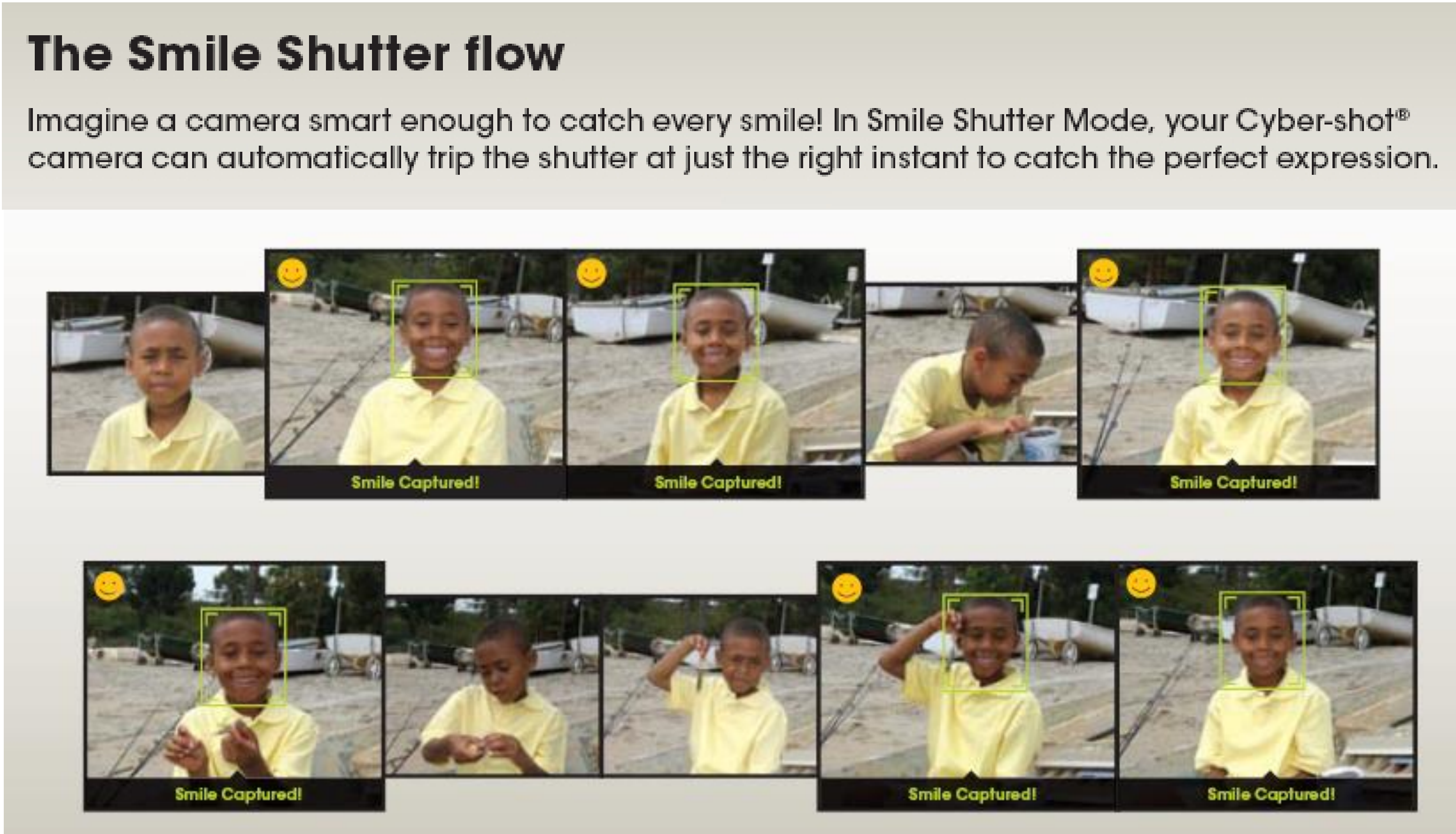
Source: S. Seitz

Face detection

- Many digital cameras now detect faces
  - Canon, Sony, Fuji, …

Source: S. Seitz

Smile detection



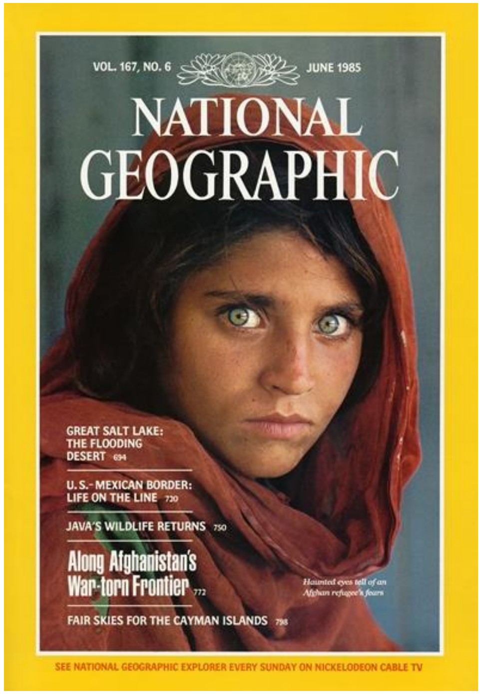
[Sony Cyber-shot® T70 Digital Still Camera](#)

Source: S. Seitz

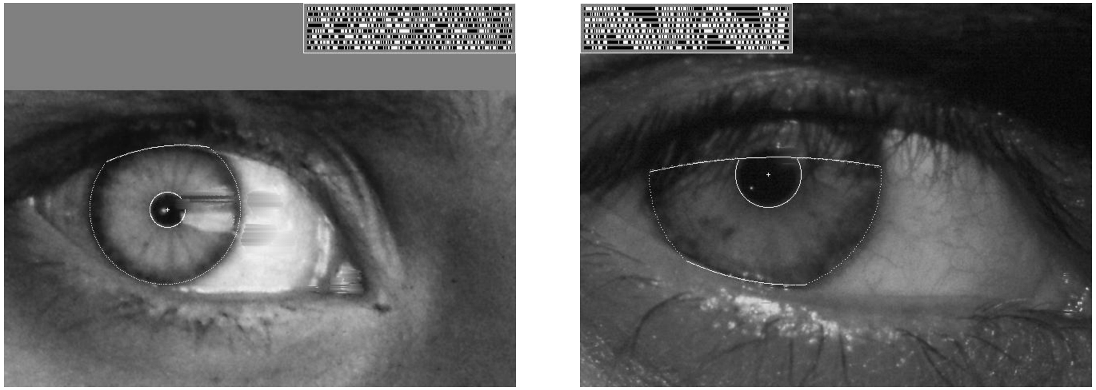
Face recognition: Apple iPhoto software

http://www.apple.com/ilife/iphoto/

Biometrics



How the Afghan Girl was Identified by Her Iris Patterns



Source: S. Seitz

Biometrics

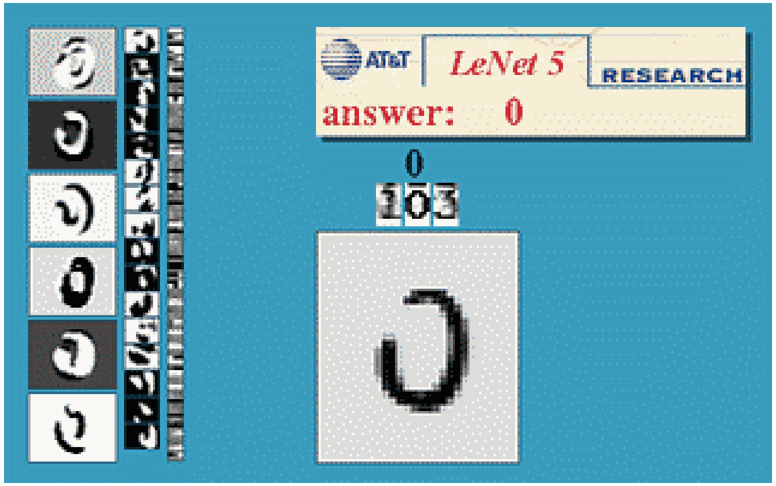
Fingerprint scanners on
many new laptops,
other devices

Face recognition systems now beginning
to appear more widely
iphone X just introduced face recognition

Optical character recognition (OCR)

Technology to convert scanned docs to text

- If you have a scanner, it probably came with OCR software



Digit recognition, AT&T labs

License plate readers
http://en.wikipedia.org/wiki/Automatic_number_plate_recognition

Source: S. Seitz

Vision-based interaction (and games)

Microsoft's Kinect

Sony EyeToy

Assistive technologies

Source: S. Seitz

Augmented Reality

Virtual Reality

Vision for robotics, space exploration

NASA'S Mars Exploration Rover Spirit captured this westward view from atop a low plateau where Spirit spent the closing months of 2007.

Vision systems (JPL) used for several tasks

- Panorama stitching
- 3D terrain modeling
- Obstacle detection, position tracking
- For more, read "Computer Vision on Mars" by Matthies et al.

Source: S. Seitz

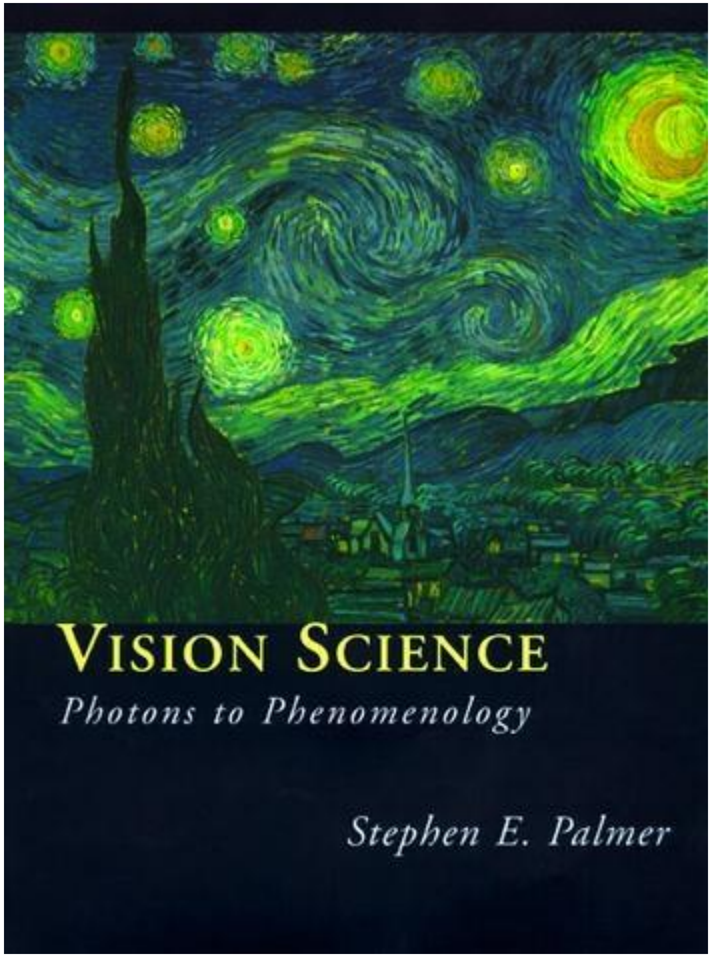
Overview of Color

- Physics of color
- Human encoding of color
- Color spaces
- White balancing

What is color?

- The result of interaction between physical light in the environment and our visual system.

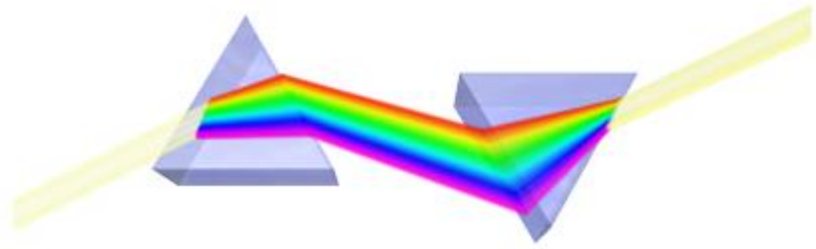
- A *psychological property* of our visual experiences when we look at objects and lights, *not a physical property* of those objects or lights.

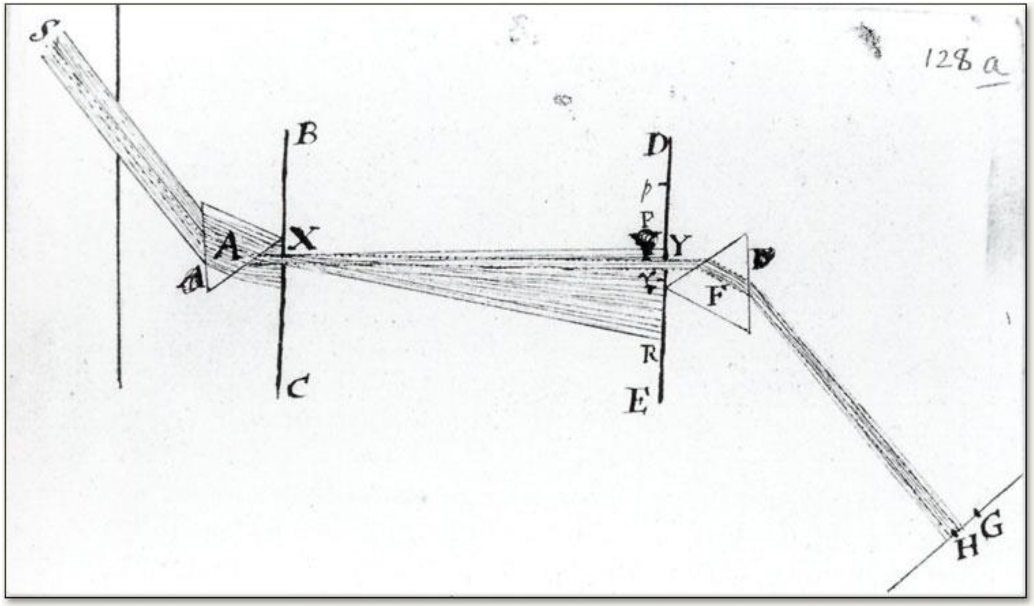


Slide credit: Lana Lazebnik

Color and light

White light: composed of almost equal energy in all wavelengths of the visible spectrum

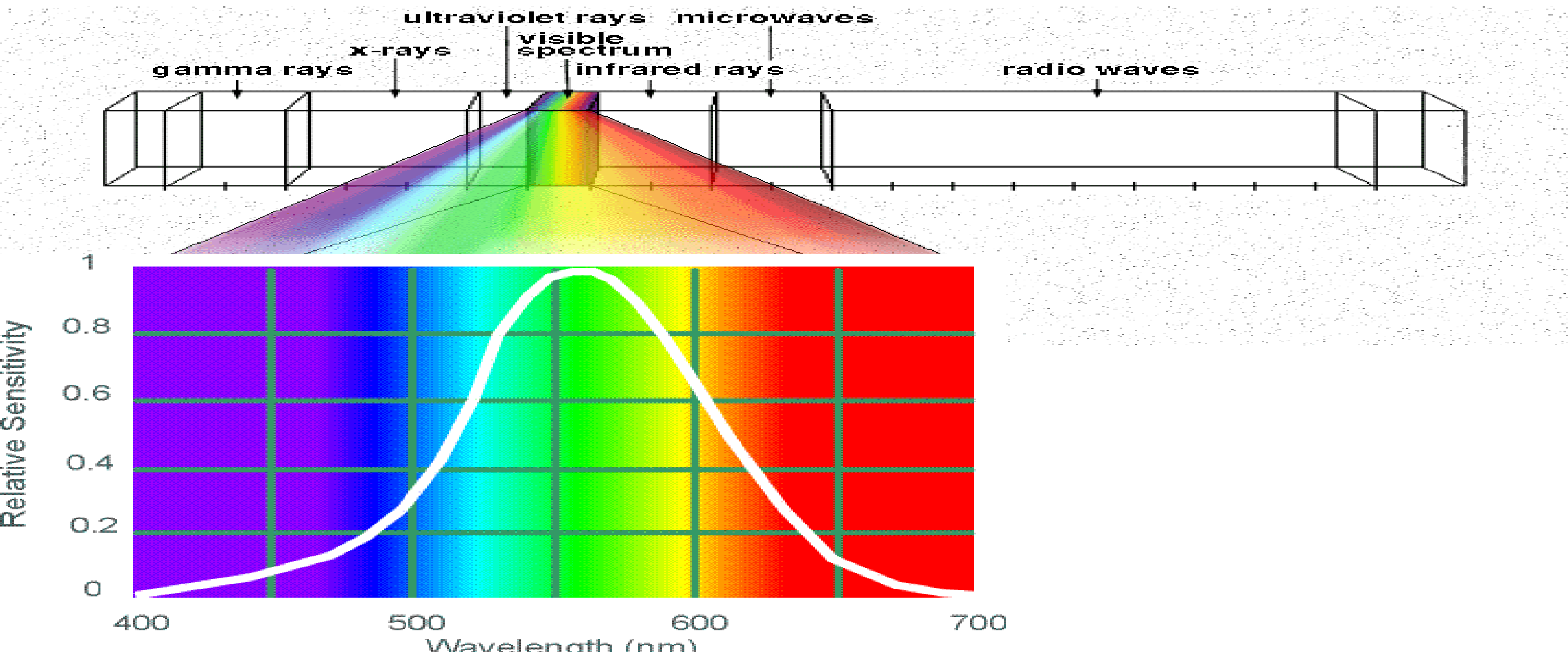




Newton 1665

Image from http://micro.magnet.fsu.edu/

Electromagnetic Spectrum

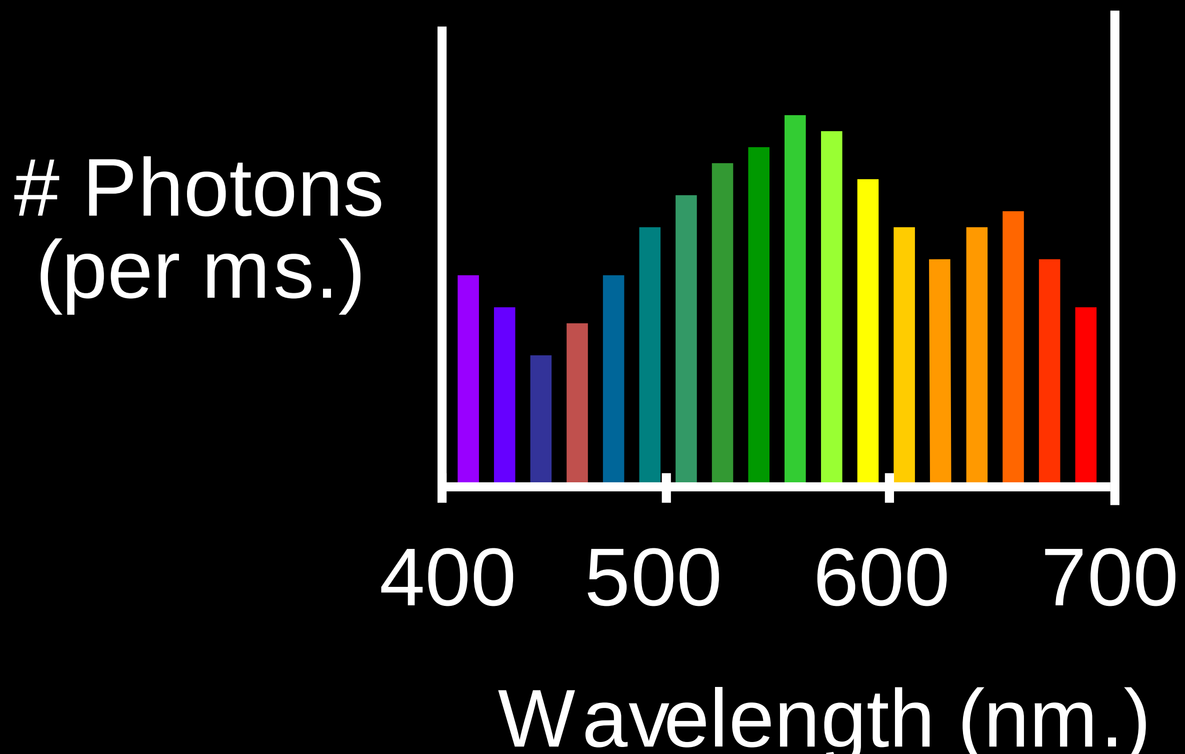


Human Luminance Sensitivity Function

http://www.yorku.ca/eye/photopik.htm

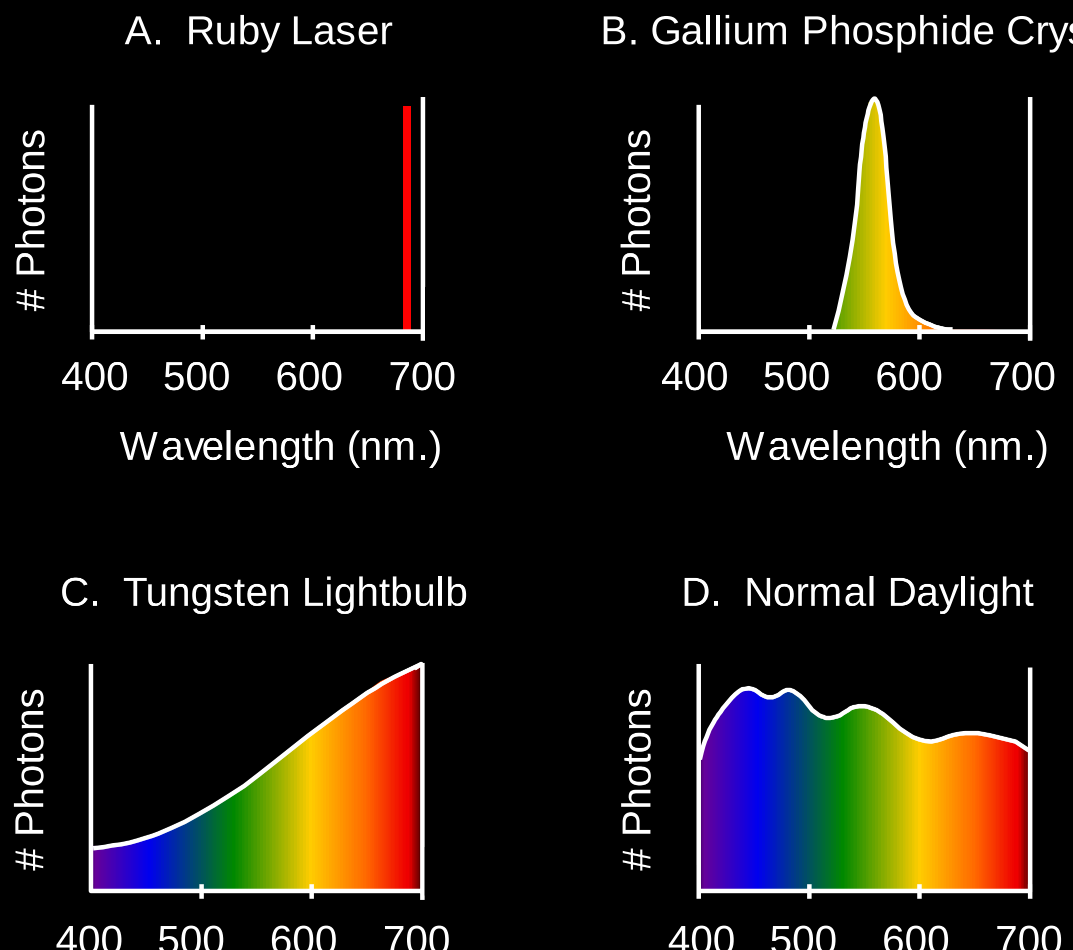
The Physics of Light

Any patch of light can be completely described physically by its spectrum: the number of photons (per time unit) at each wavelength 400 - 700 nm.



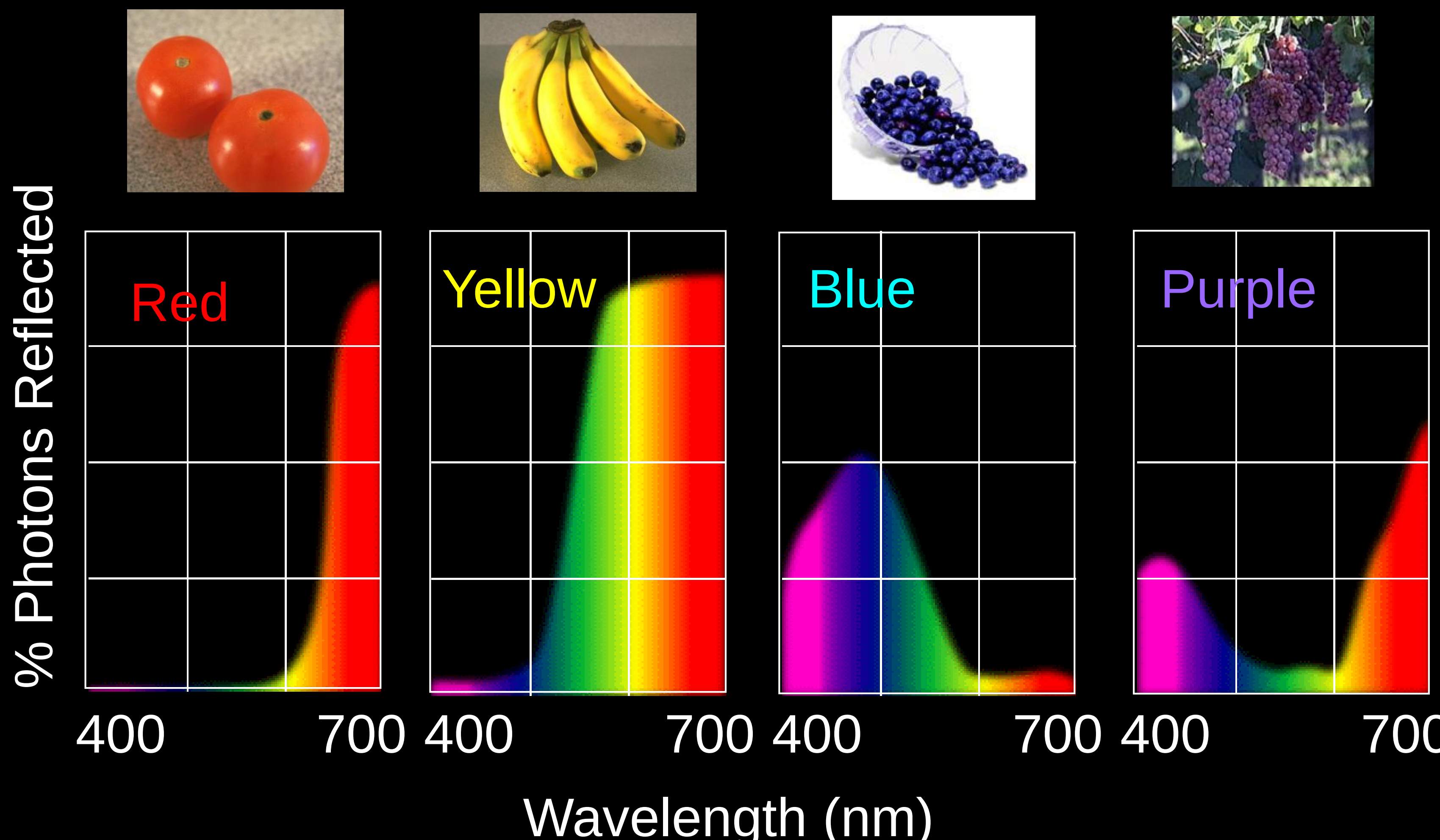
The Physics of Light

Some examples of the spectra of light sources



The Physics of Light

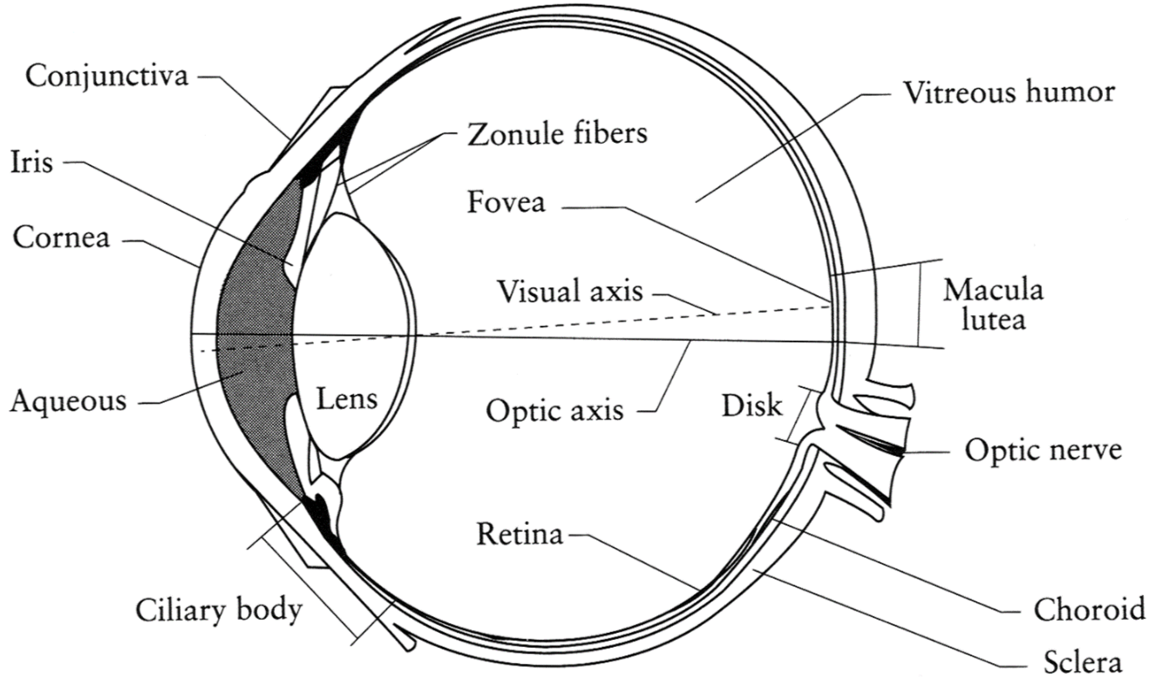
Some examples of the reflectance spectra of surfaces



Overview of Color

- Physics of color
- Human encoding of color
- Color spaces
- White balancing

The Eye

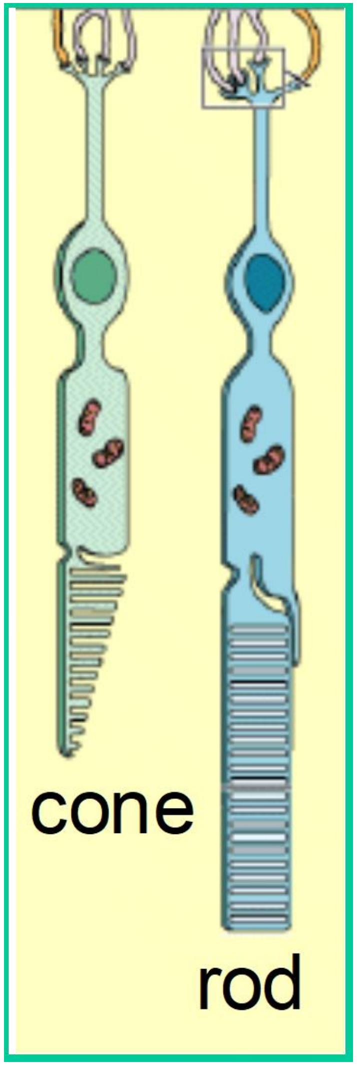


- The human eye is a camera
  - **Iris** - colored annulus with radial muscles
  - **Pupil** - the hole (aperture) whose size is controlled by the iris
  - What's the sensor?
    - photoreceptor cells (rods and cones) in the **retina**

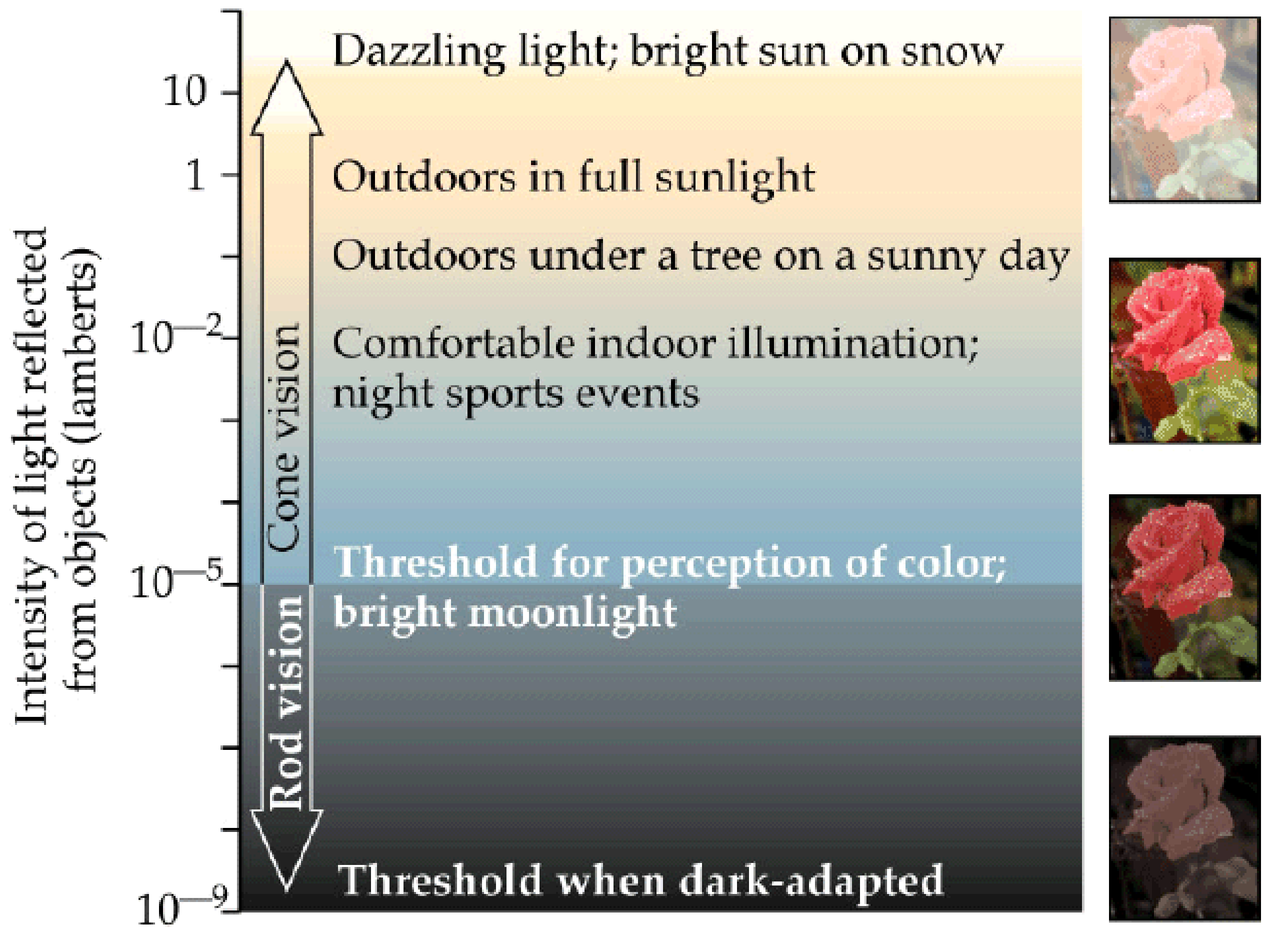
Two types of light-sensitive receptors

Cones
cone-shaped
less sensitive
operate in high light
color vision

Rods
rod-shaped
highly sensitive
operate at night
gray-scale vision

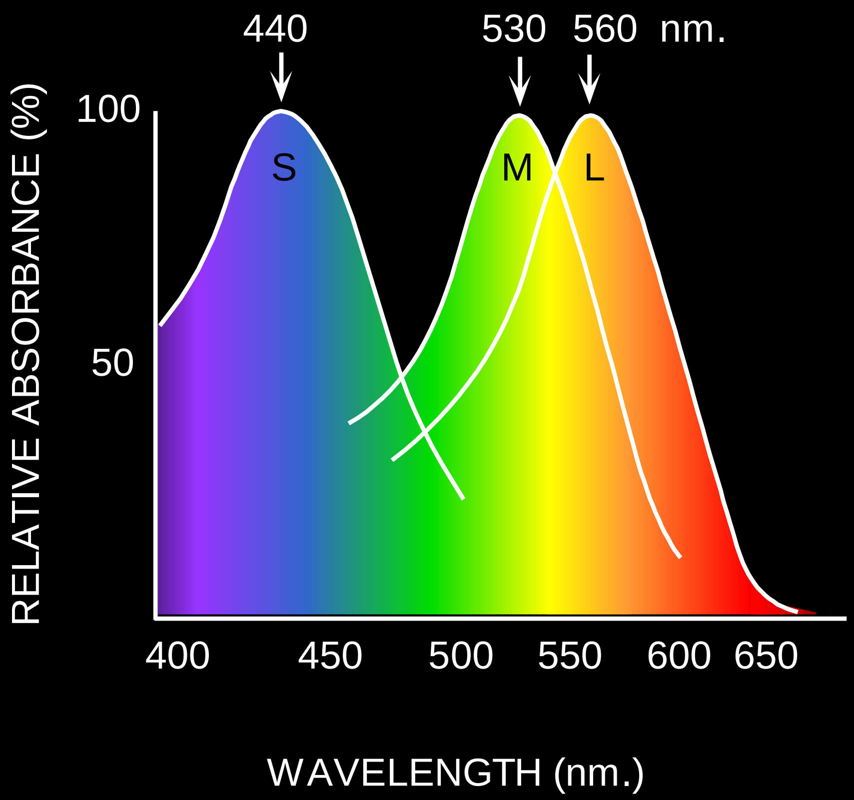


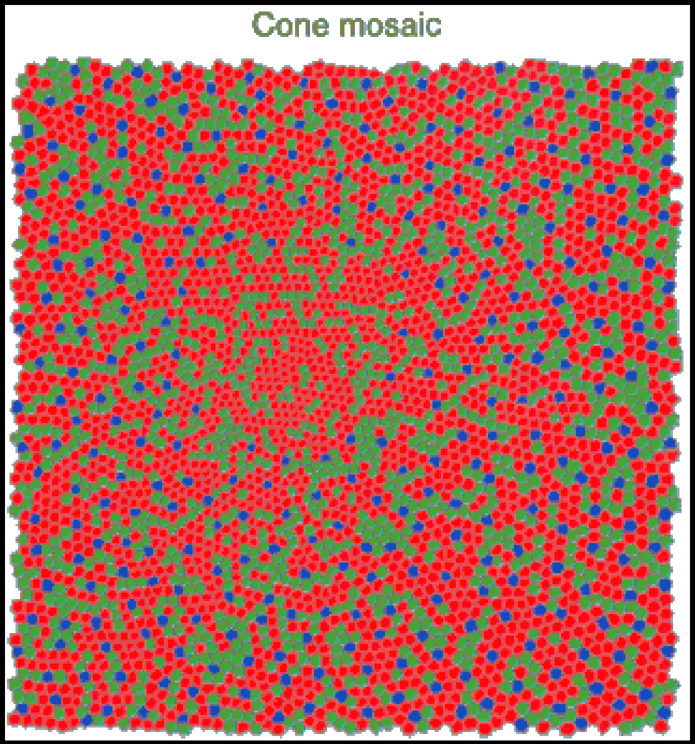
Rod / Cone sensitivity



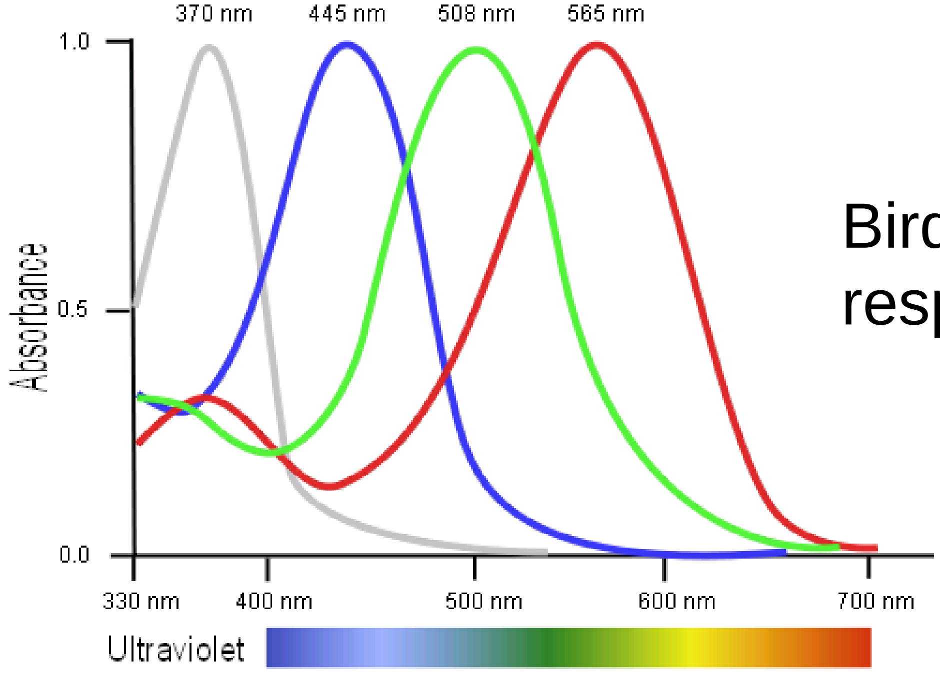
Physiology of Color Vision

Three kinds of cones:





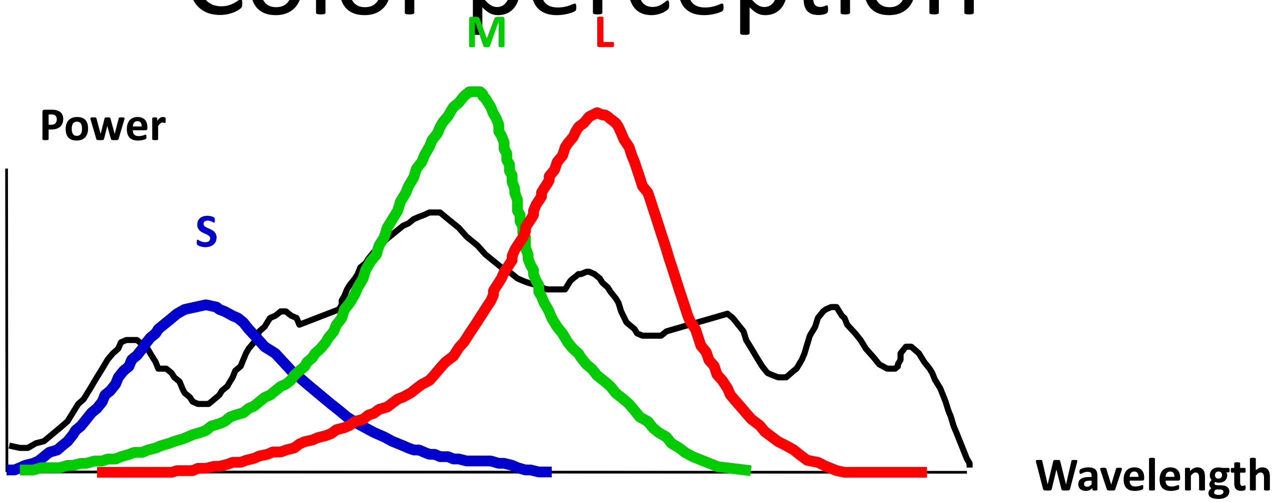
Tetrachromatism



Bird cone responses

- Most birds, and many other animals, have cones for ultraviolet light.
- Some humans seem to have four cones (12% of females).

Color perception



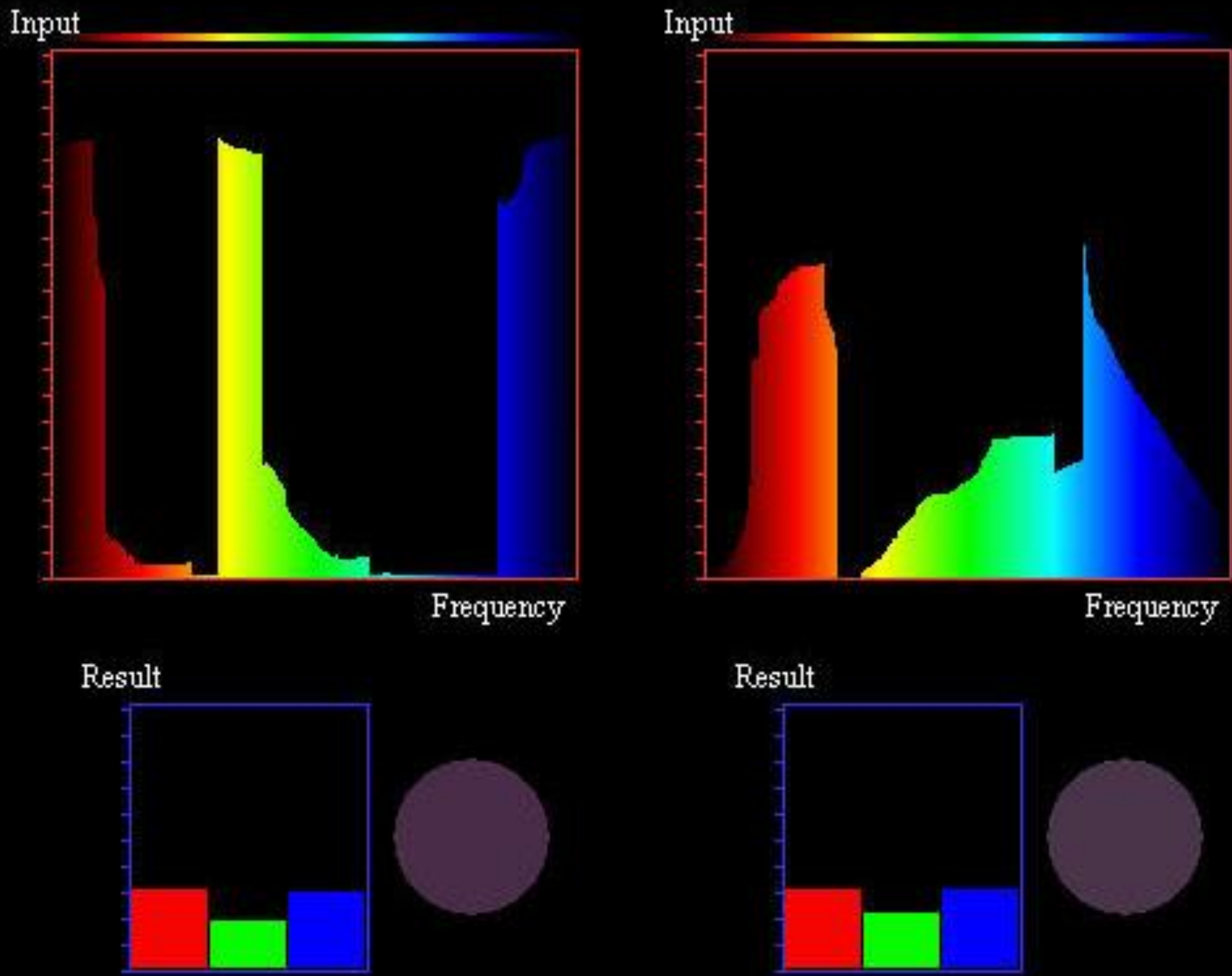
Rods and cones act as filters on the spectrum

– Each cone yields one number

- Q: How can we represent an entire spectrum with 3 numbers?
- A: We can't! Most of the information is lost.
  - As a result, two different spectra may appear indistinguishable
    - » such spectra are known as **metamers**

Slide by Steve Seitz

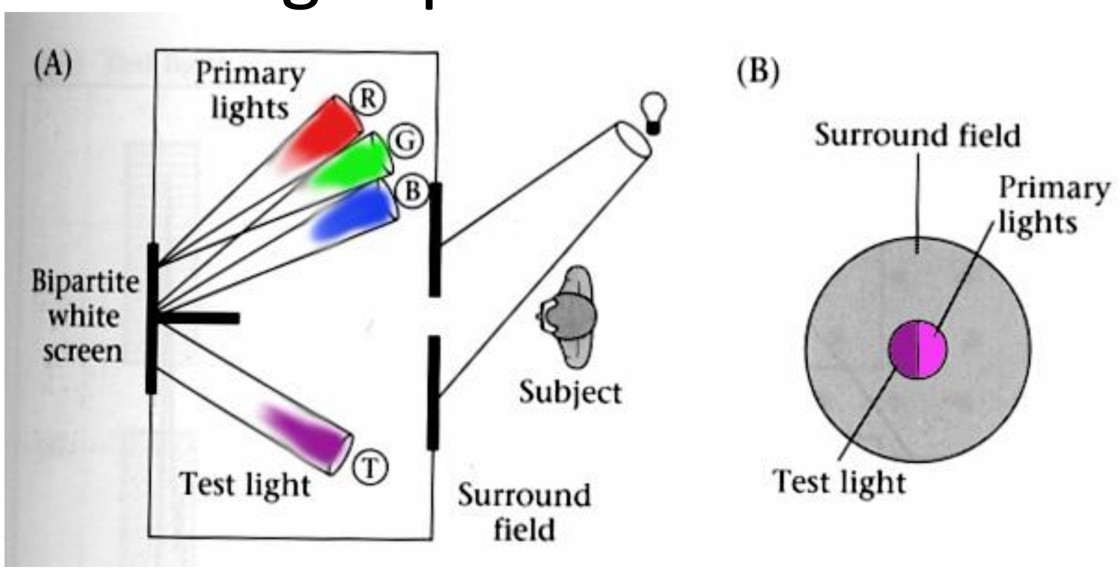
Metamers



by Jeff Beall, Adam Doppelt and John F. Hughes

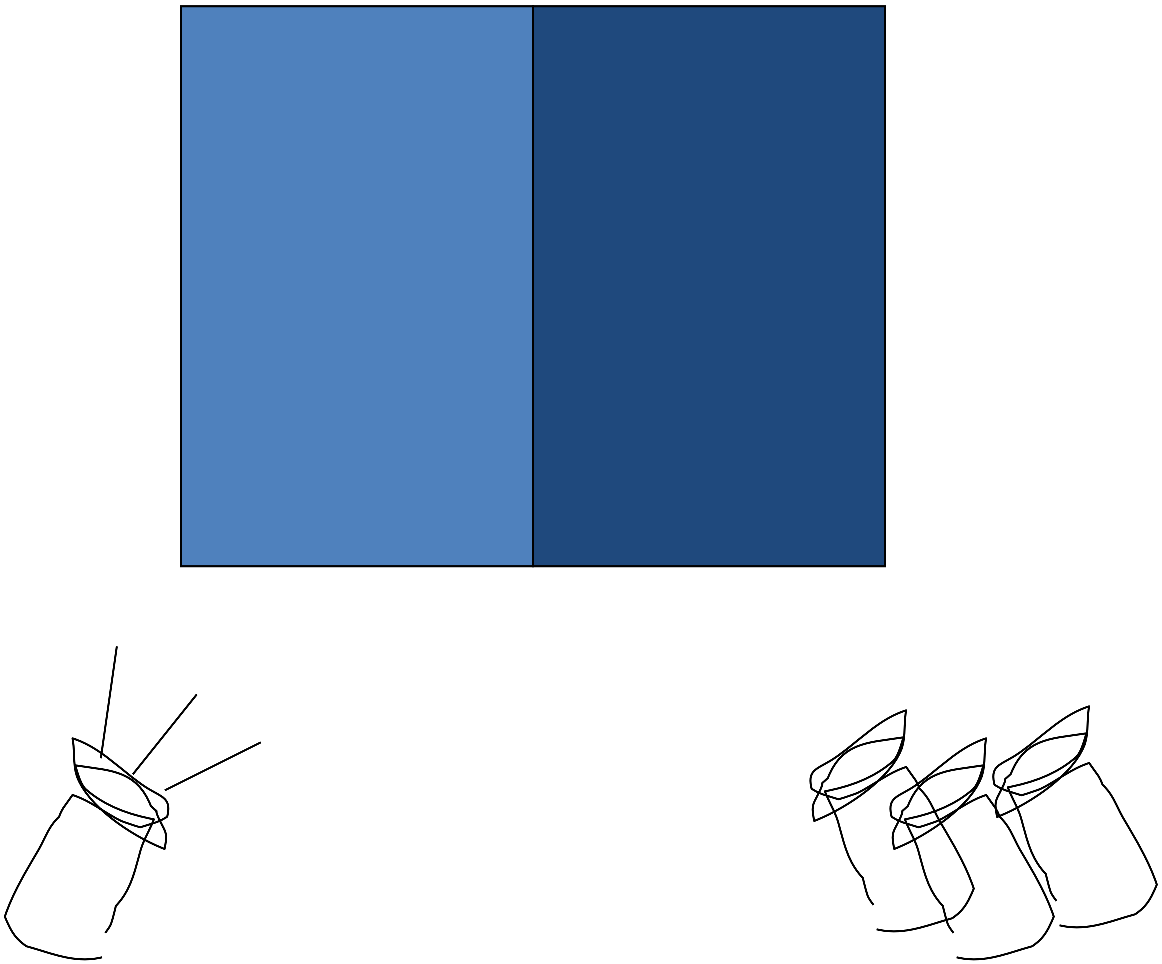
Standardizing color experience

- We would like to understand which spectra produce the same color sensation in people under similar viewing conditions
- Color matching experiments



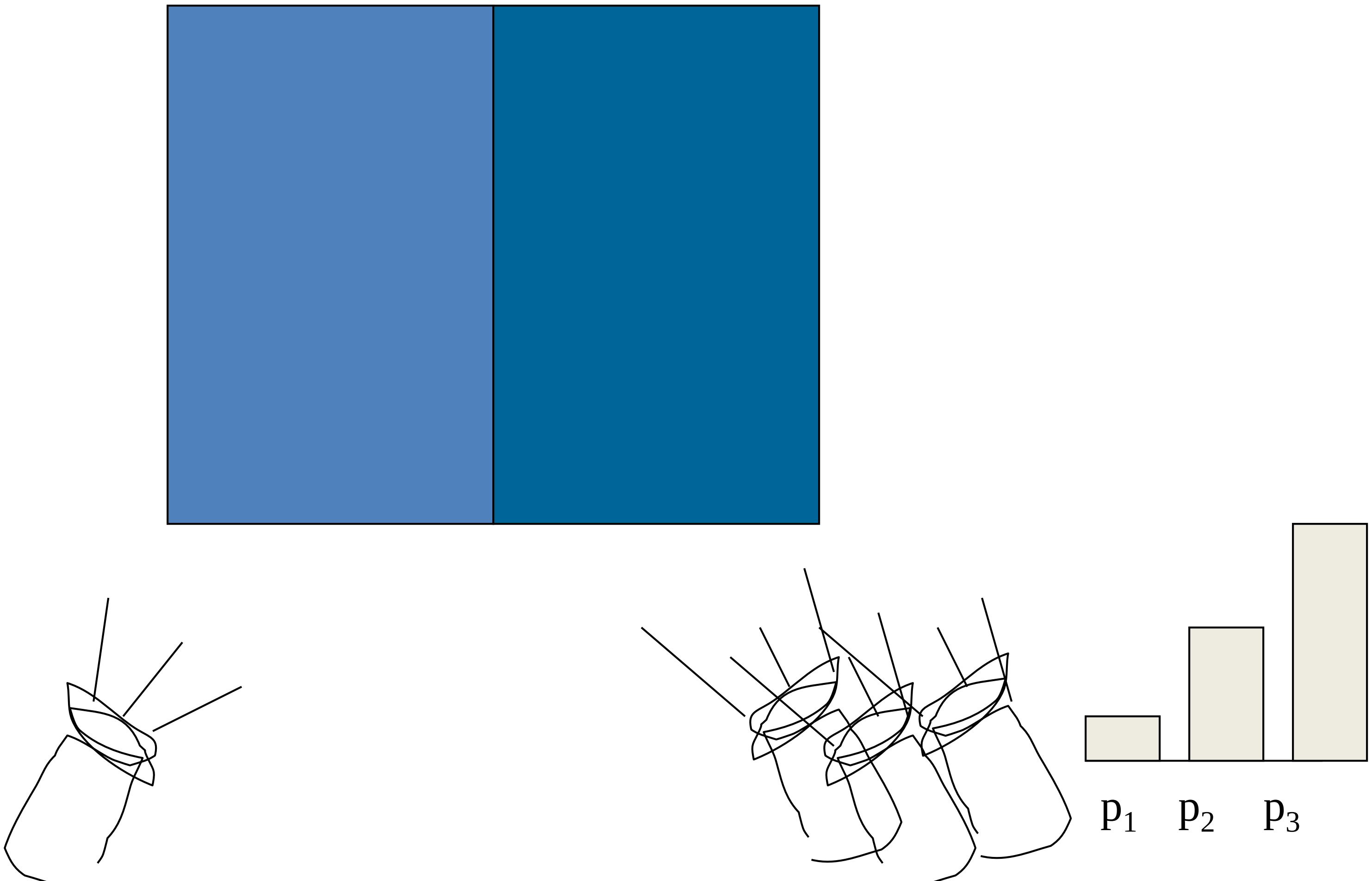
Foundations of Vision, by Brian Wandell, Sinauer Assoc., 1995

Color matching experiment 1



Source: W. Freeman

Color matching experiment 1



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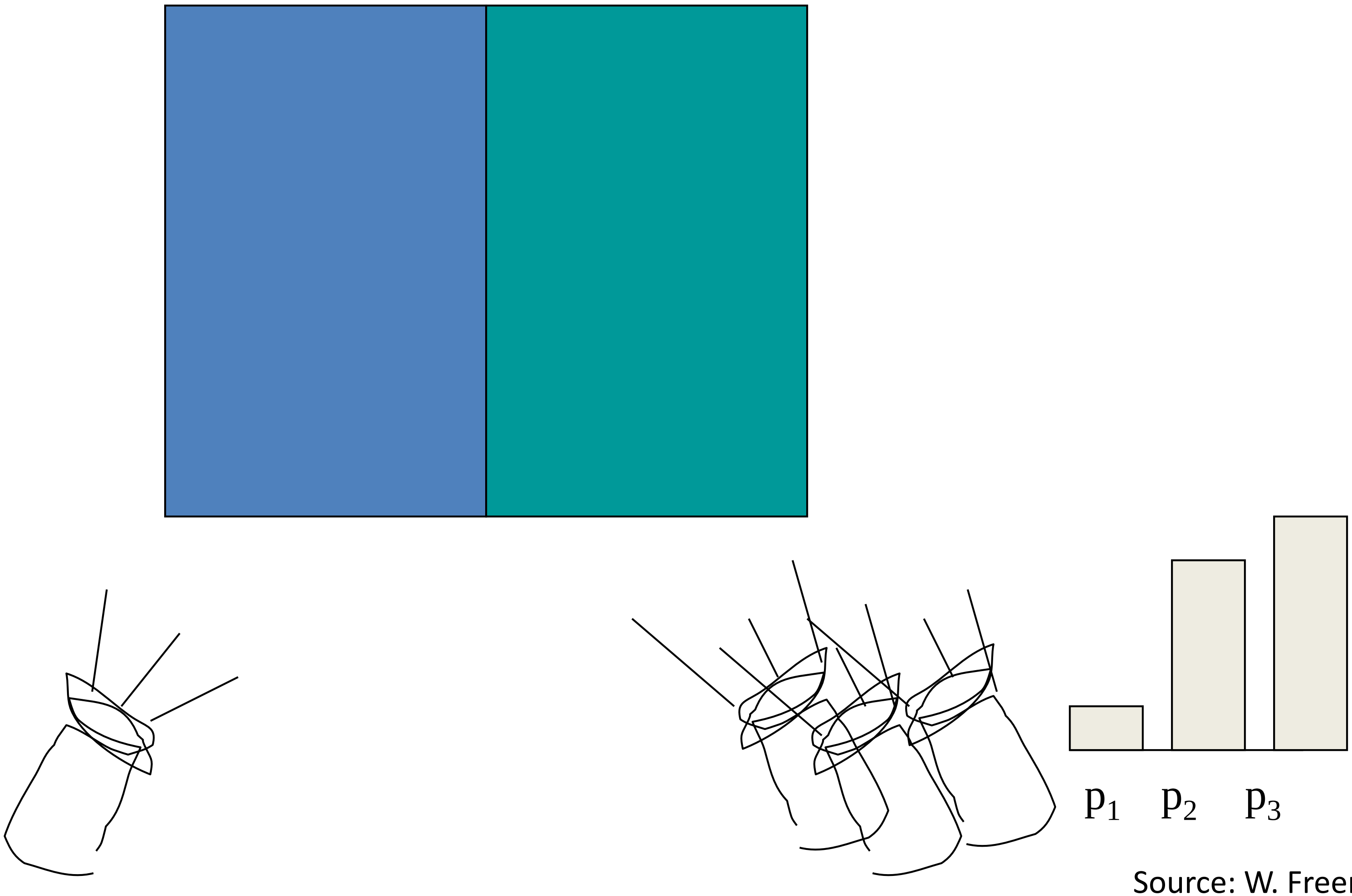
Color matching experiment 1



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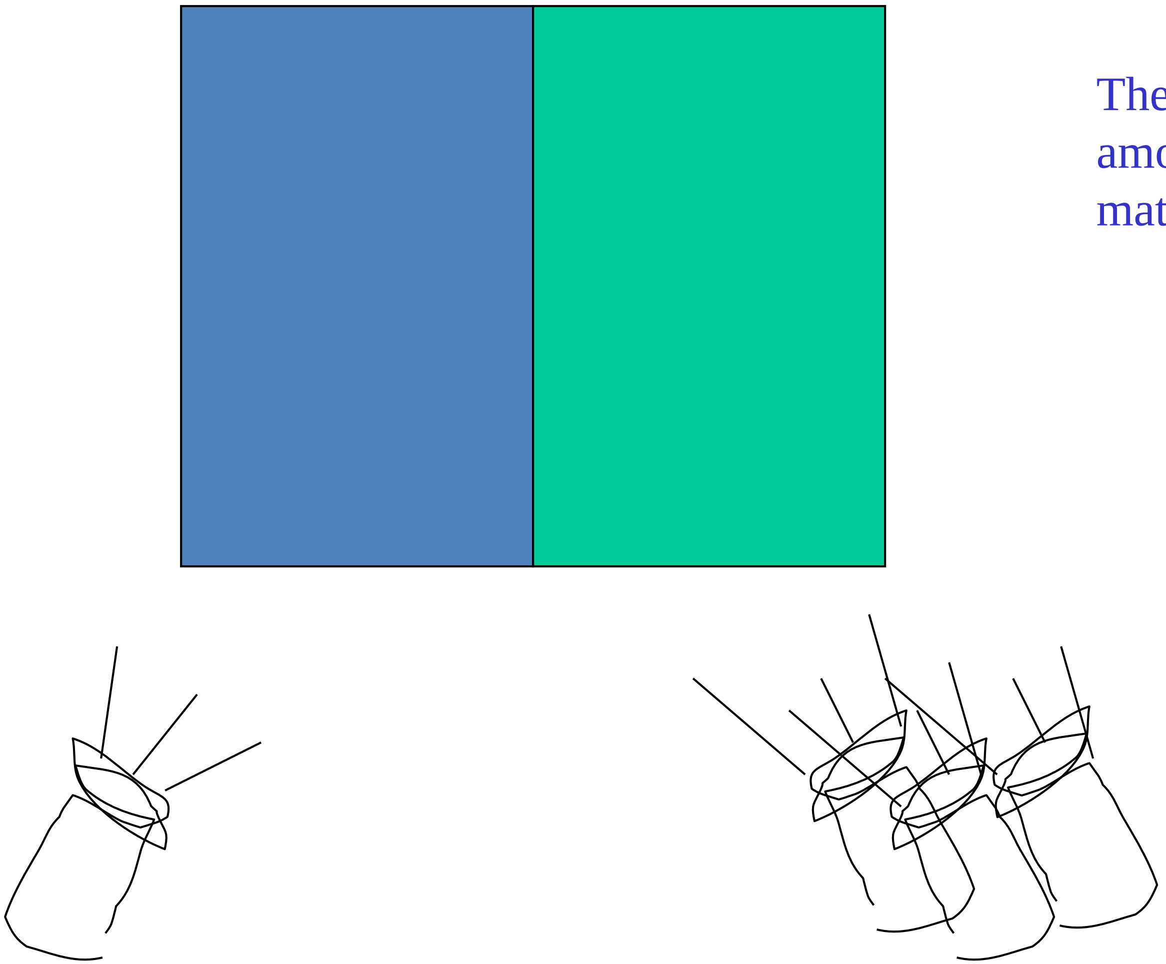
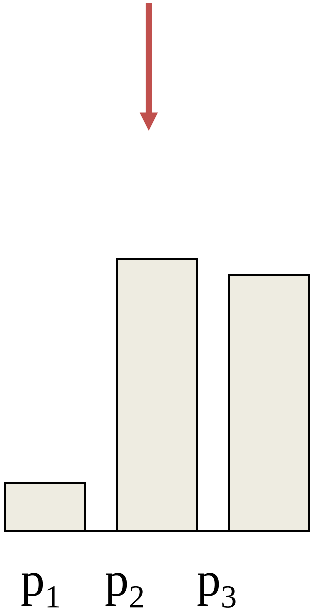
Color matching experiment 1

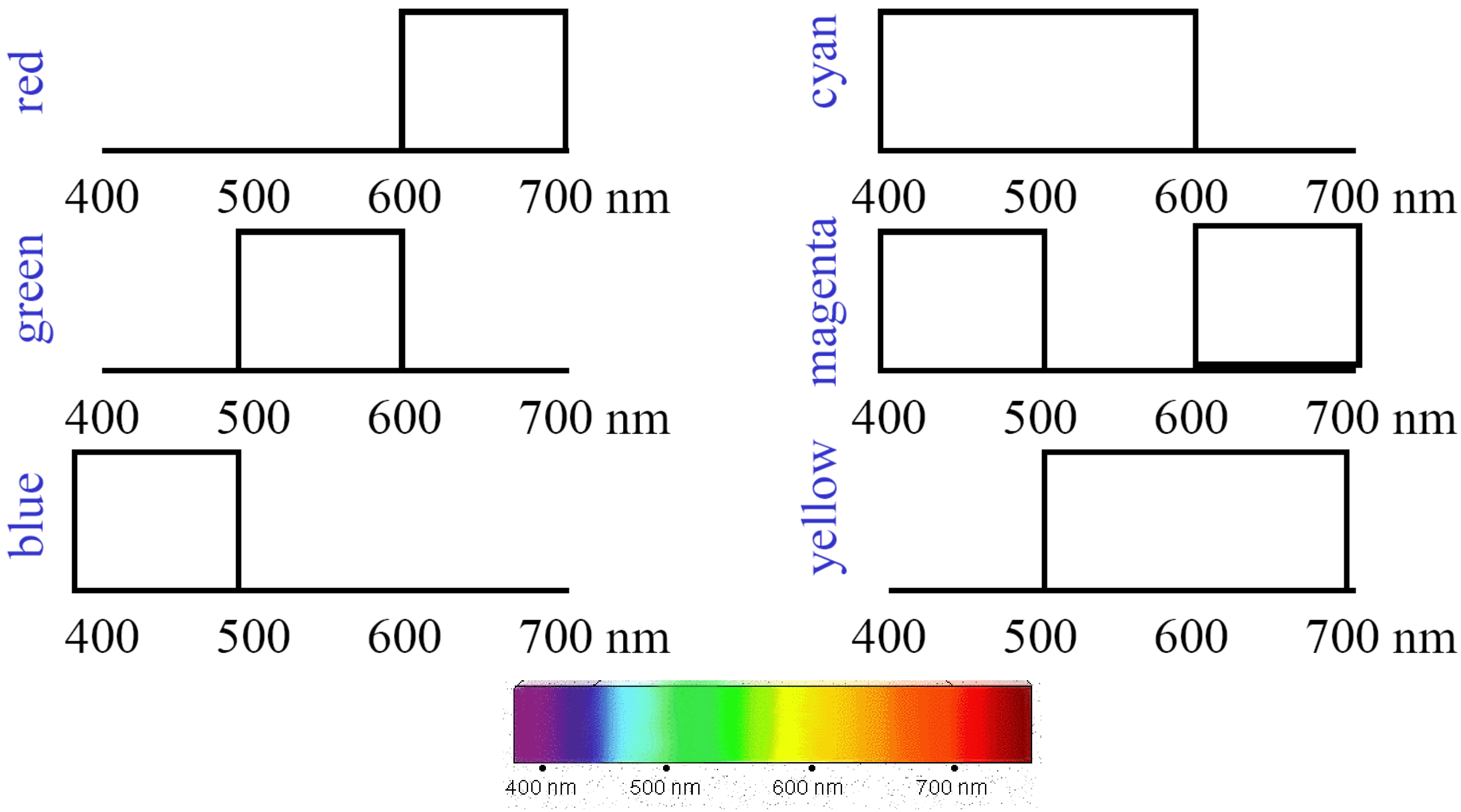


The primary color amounts needed for a match



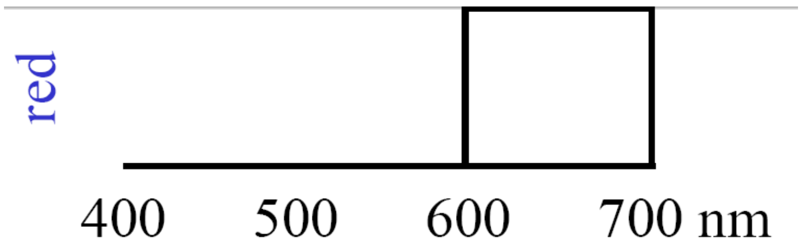
Source: W. Freeman

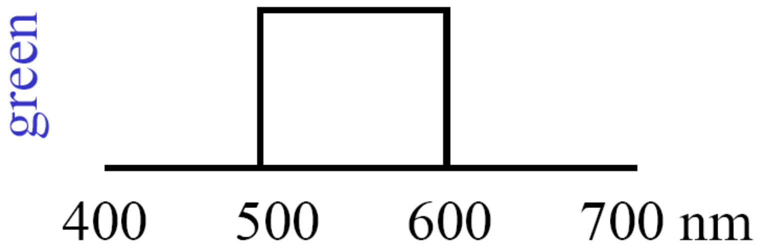
Color mixing

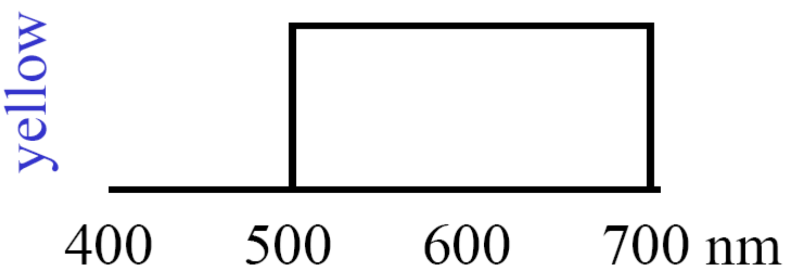


Source: W. Freeman

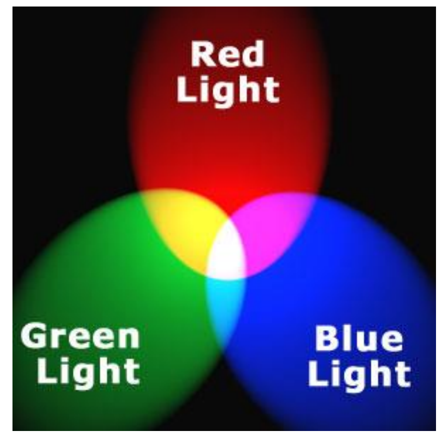
Additive color mixing







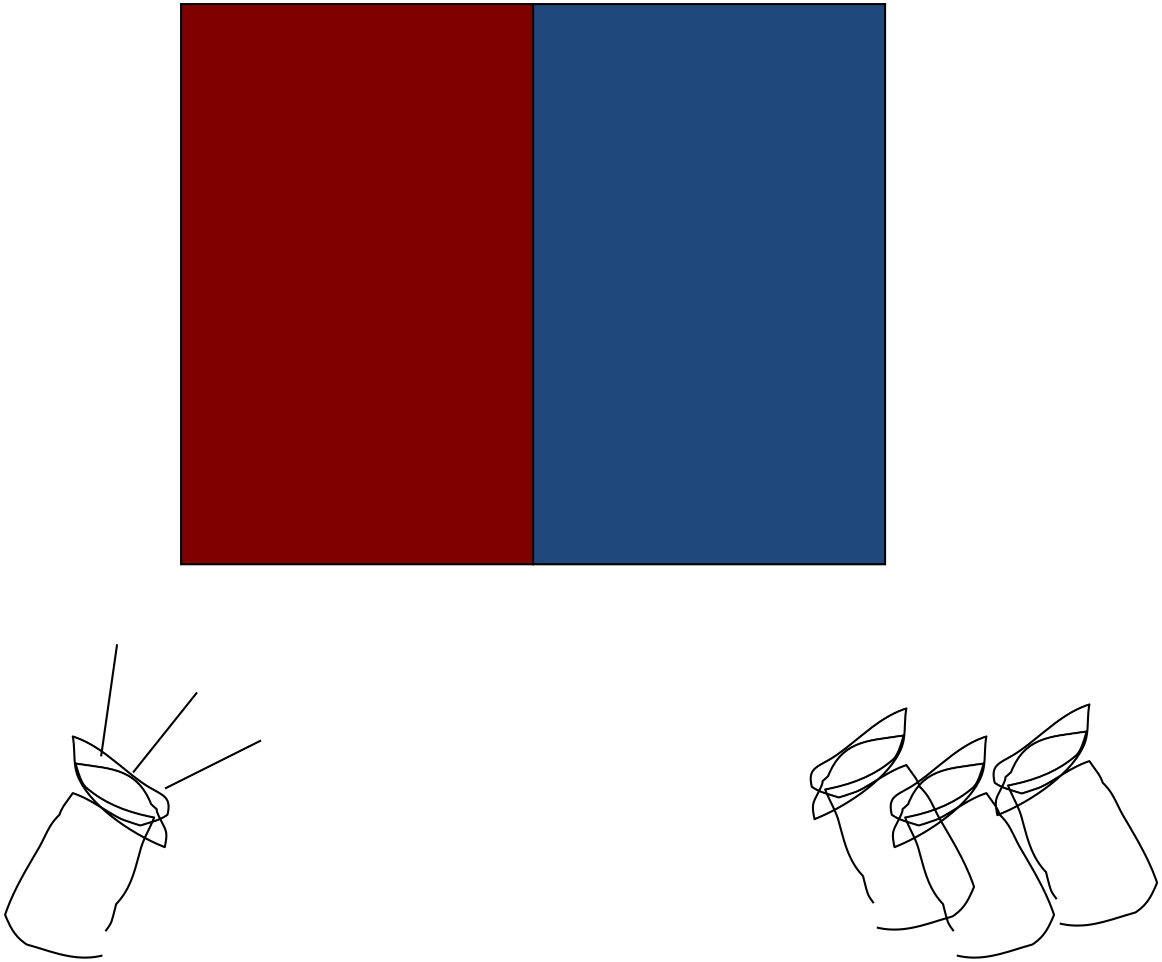
Colors combine by *adding* color spectra



Light *adds* to existing black.

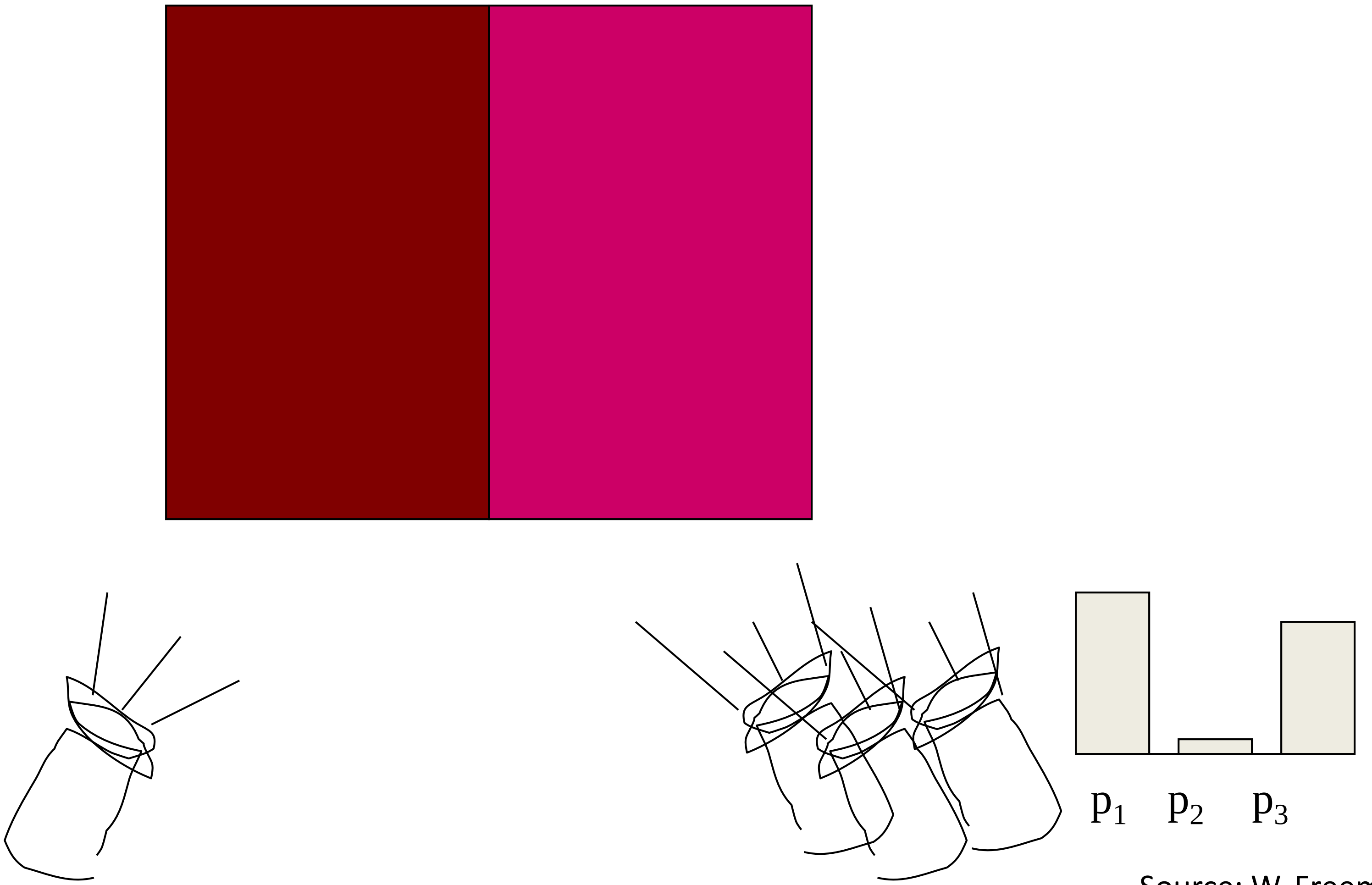
Source: W. Freeman

Color matching experiment 2



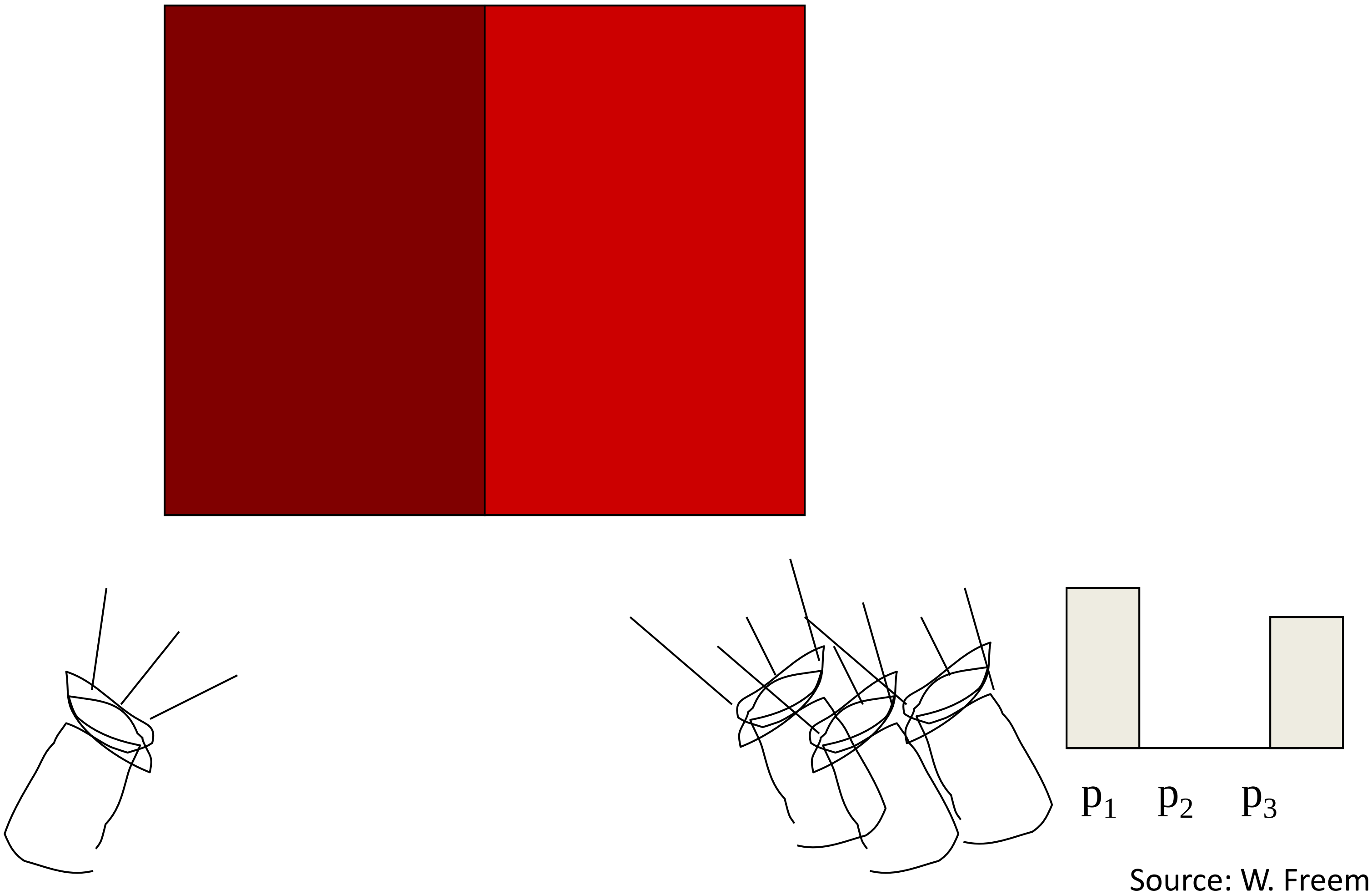
Source: W. Freeman

Color matching experiment 2



Source: W. Freeman

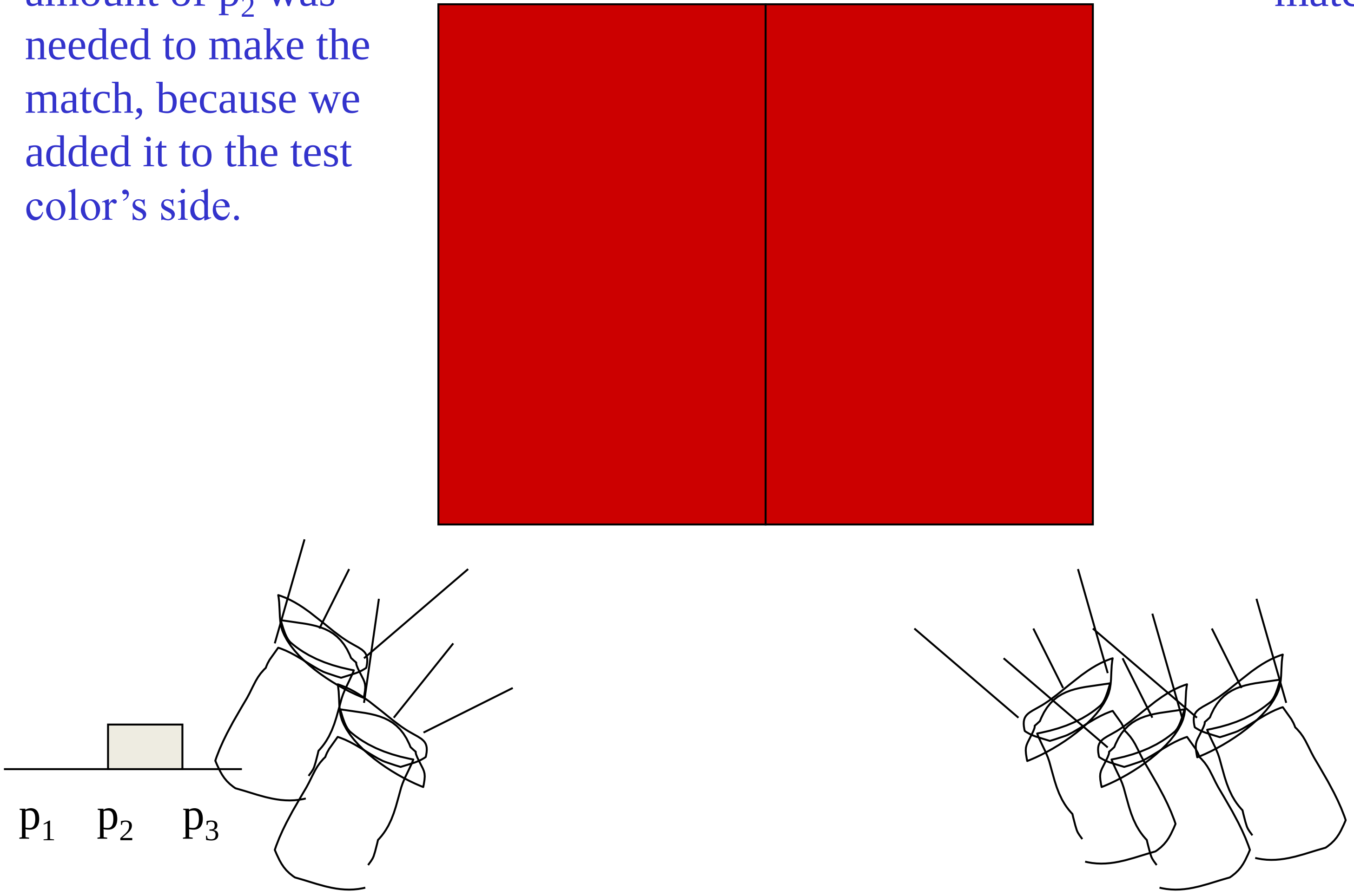
Color matching experiment 2



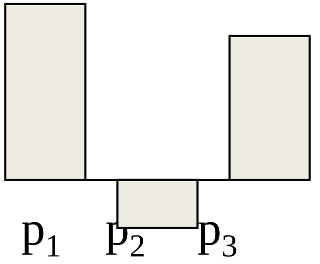
Source: W. Freeman

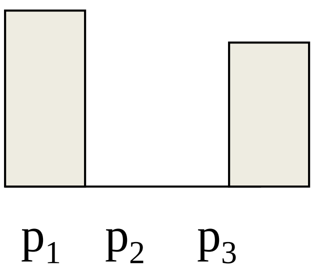
Color matching experiment 2

We say a "negative" amount of p_2 was needed to make the match, because we added it to the test color's side.



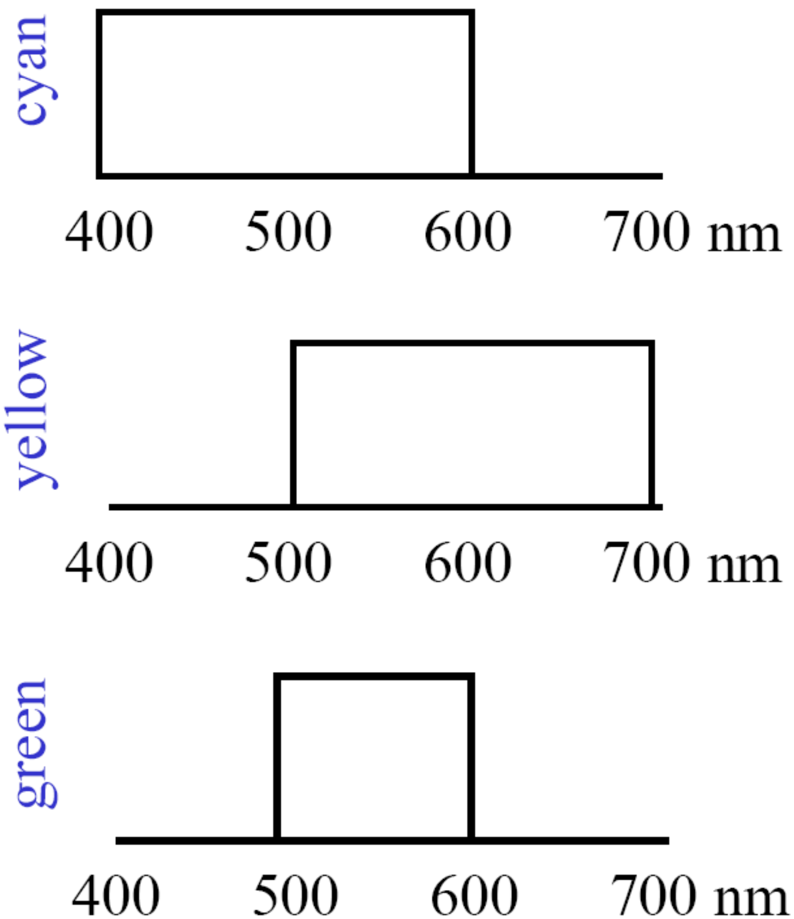
The primary color amounts needed for a match:



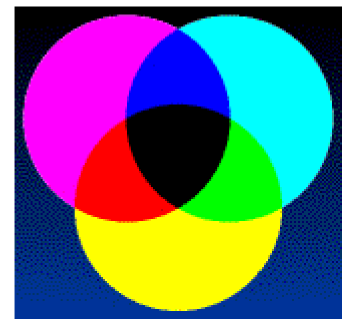


Source: W. Freeman

Subtractive color mixing



Colors combine by *multiplying* color spectra.



Pigments *remove* color from incident light (white).

Source: W. Freeman

Examples of subtractive color systems

- Printing on paper
- Crayons
- Photographic film

Trichromacy

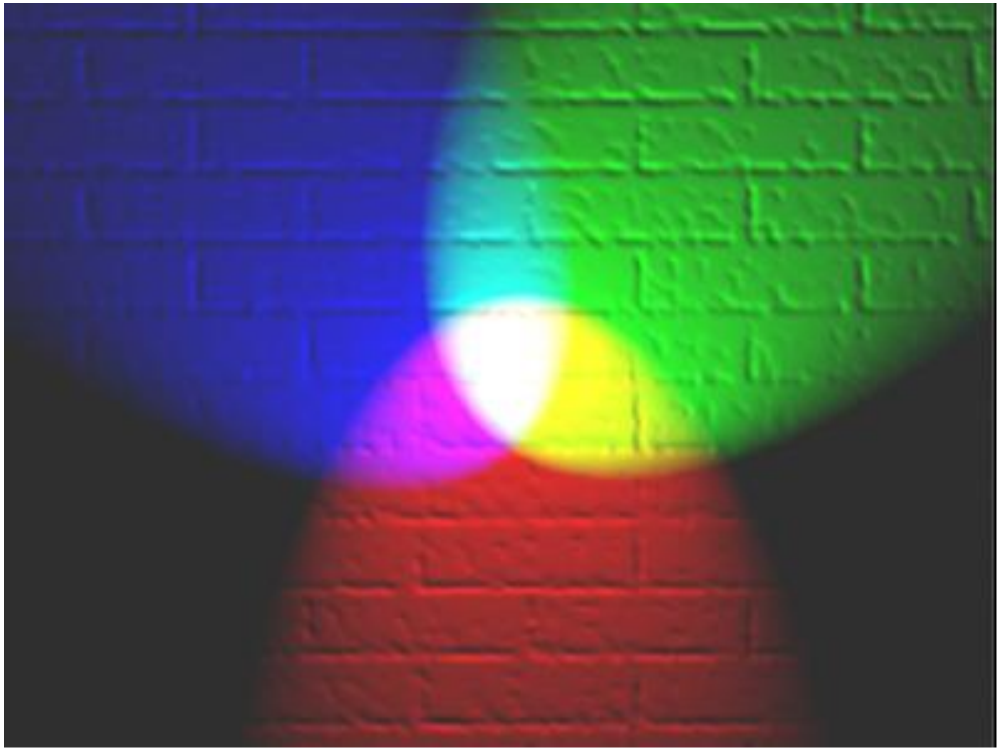
- In color matching experiments, most people can match any given light with three primaries
  - Primaries must be *independent*
- For the same light and same primaries, most people select the same weights
  - Exception: color blindness
- Trichromatic color theory
  - Three numbers seem to be sufficient for encoding color
  - Dates back to 18^{th} century (Thomas Young)

Overview of Color

- Physics of color
- Human encoding of color
- Color spaces
- White balancing

Color spaces

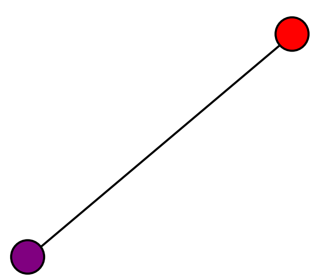
- How can we represent color?



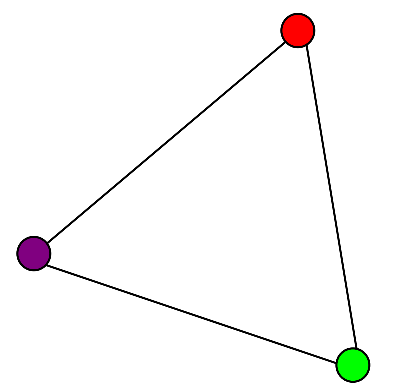
Color space, also known as the color model (or color system), is an abstract mathematical model which describes the range of colors as tuples of numbers, typically as 3 or 4 values or color components (e.g. RGB). A color space may be arbitrary or structured mathematically. Most color models map to an absolute and globally understood system of color interpretation.

Linear color spaces

- Defined by a choice of three *primaries*
- The coordinates of a color are given by the weights of the primaries used to match it

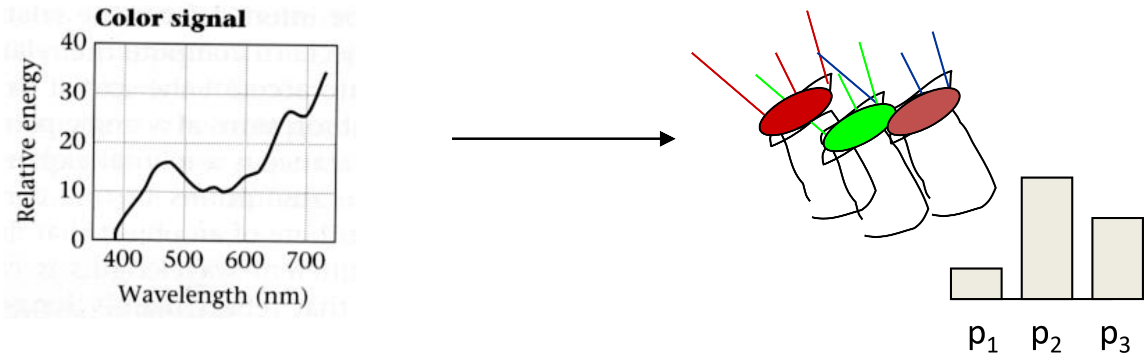


mixing two lights produces colors that lie along a straight line in color space



mixing three lights produces colors that lie within the triangle they define in color space

How to compute the weights of the primaries to match any spectral signal

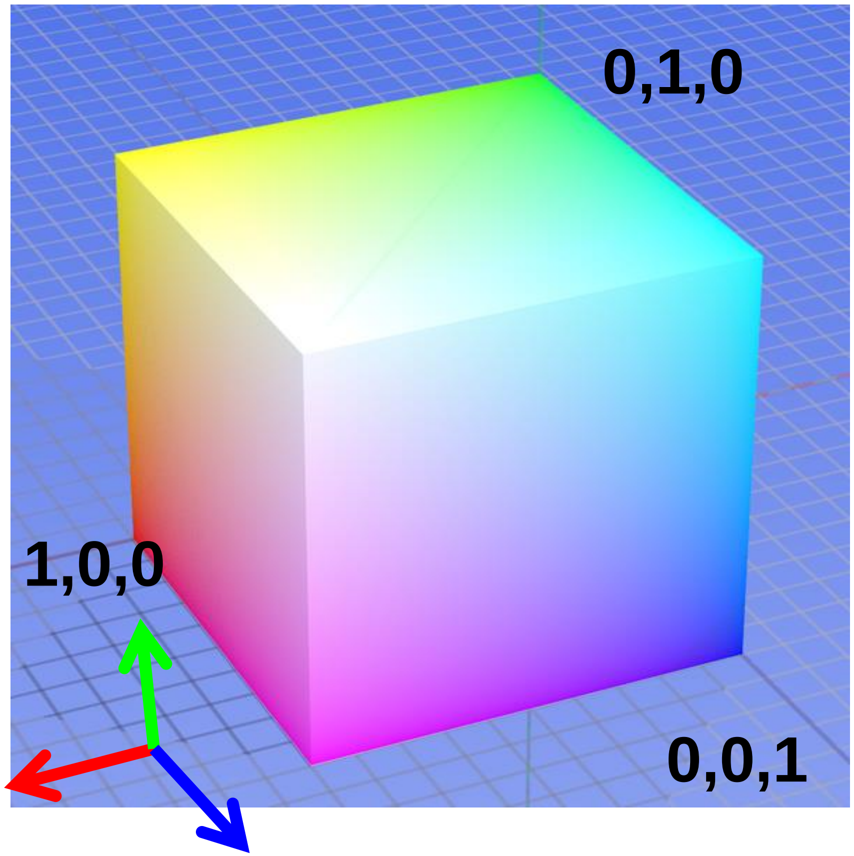


- **Matching functions:** the amount of each primary needed to match a monochromatic light source at each wavelength

Source: W. Freeman

Color spaces: RGB

Default color space



R = 1
(G=0,B=0)

G = 1
(R=0,B=0)

B = 1
(R=0,G=0)

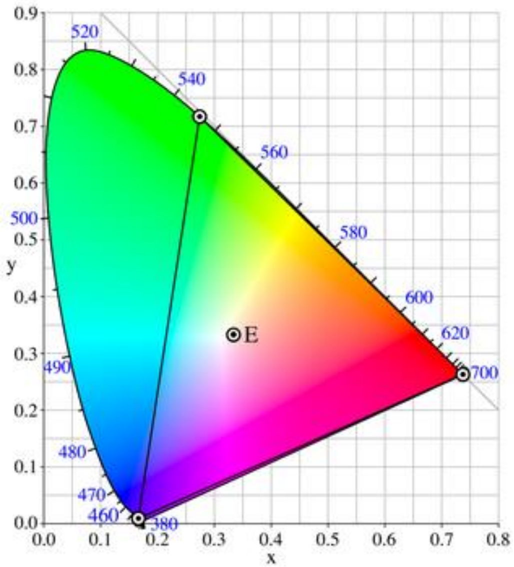
Any color = r*R + g*G + b*B

- Strongly correlated channels
- Non-perceptual

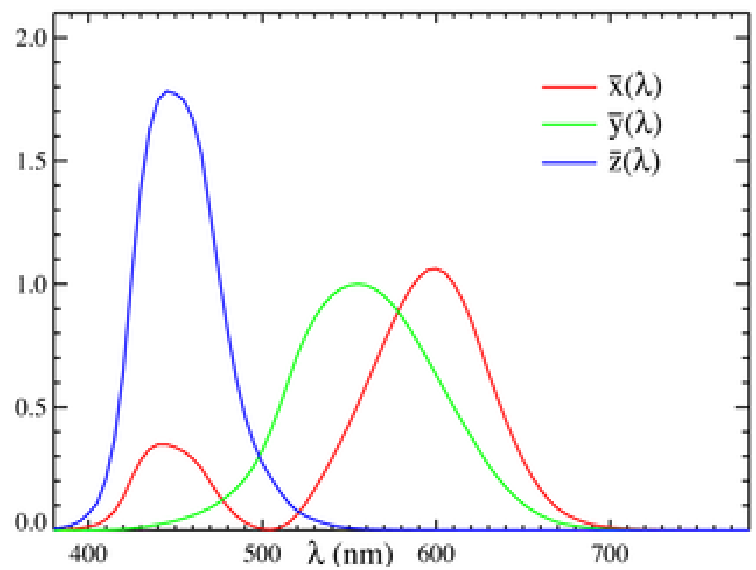
Image from: http://en.wikipedia.org/wiki/File:RGB_color_solid_cube.png

Linear color spaces: CIE XYZ

- Primaries are imaginary, but matching functions are everywhere positive
- The Y parameter corresponds to brightness or *luminance* of a color
- 2D visualization: draw (x,y), where $x = X/(X+Y+Z)$, $y = Y/(X+Y+Z)$

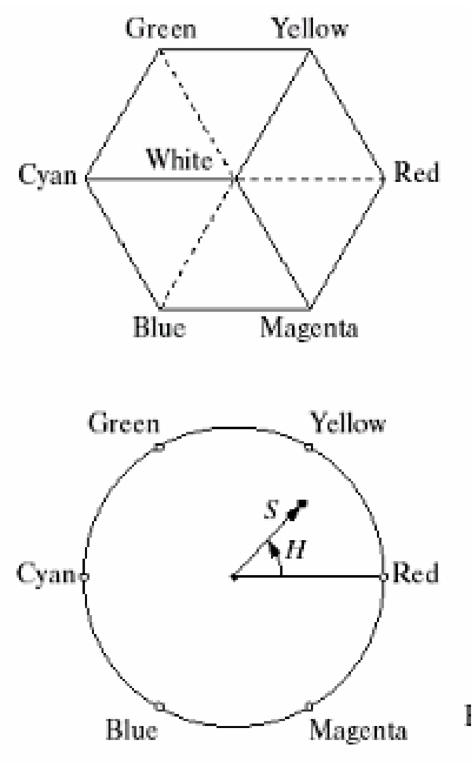


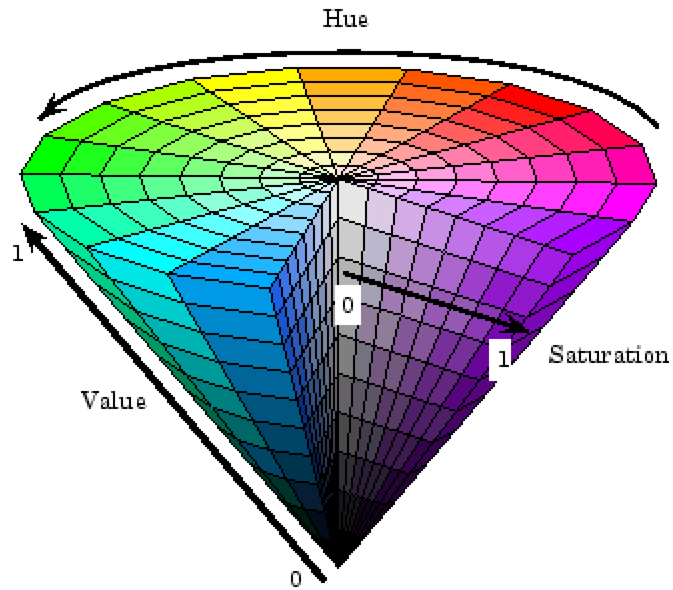
Matching functions

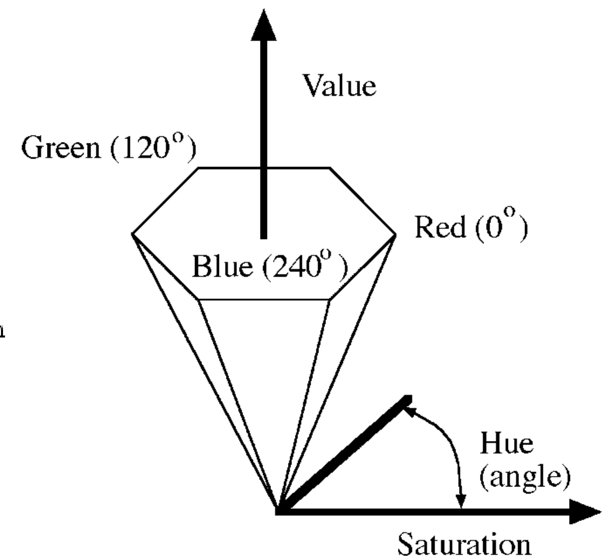


http://en.wikipedia.org/wiki/CIE_1931_color_space

Nonlinear color spaces: HSV







- Perceptually meaningful dimensions: Hue, Saturation, Value (Intensity)
- RGB cube on its vertex

If you had to choose, would you rather go without:

- intensity (‘value’), or
- hue + saturation (‘chroma’)?

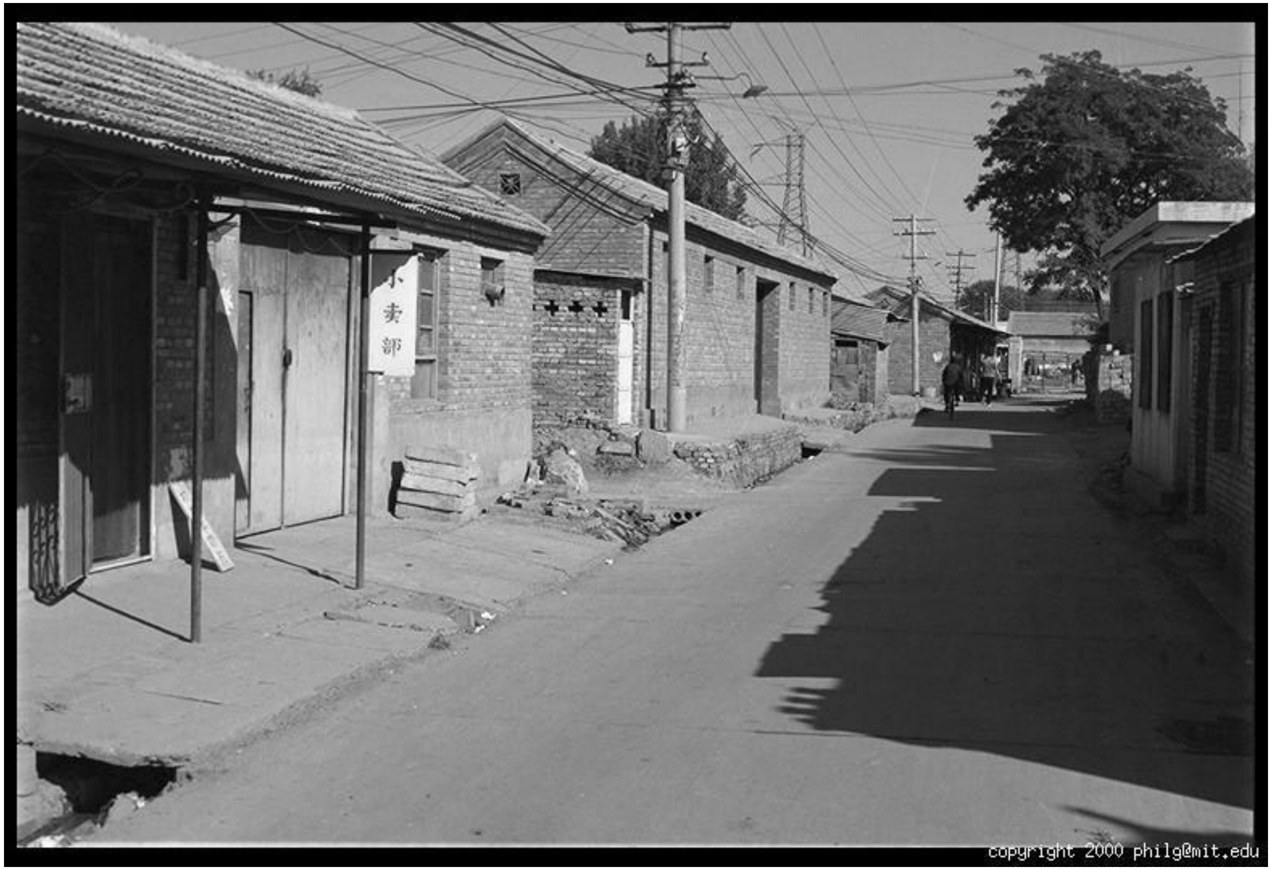
If you had to choose, would you rather go without luminance or chrominance?

Most information in intensity

Only color shown – constant intensity

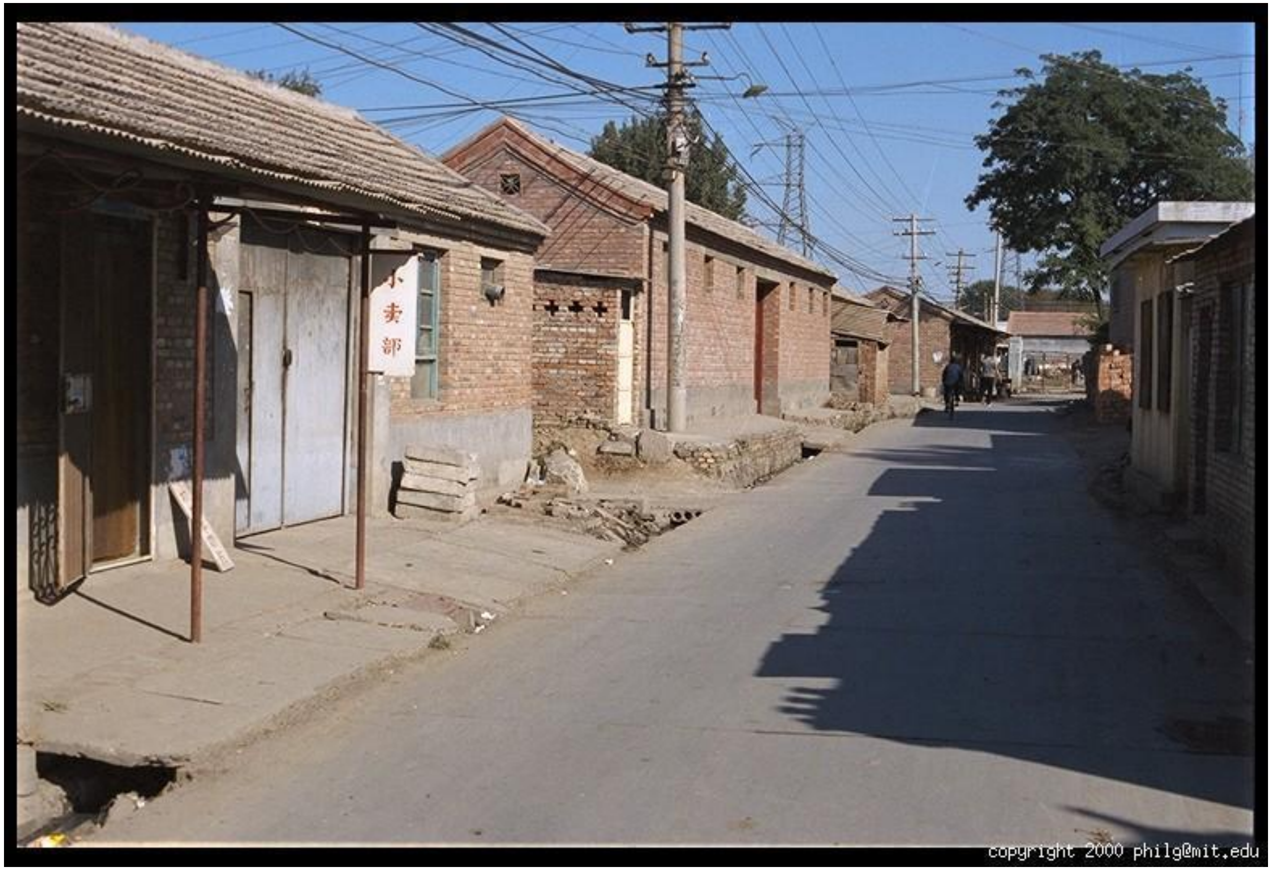
James Hays

Most information in intensity



Only intensity shown – constant color

Most information in intensity



Original image

James Hays

Overview of Color

- Physics of color
- Human encoding of color
- Color spaces
- White balancing

White balance

- When looking at a picture on screen or print, we adapt to the illuminant of the room, not to that of the scene in the picture
- When the white balance is not correct, the picture will have an unnatural color “cast”

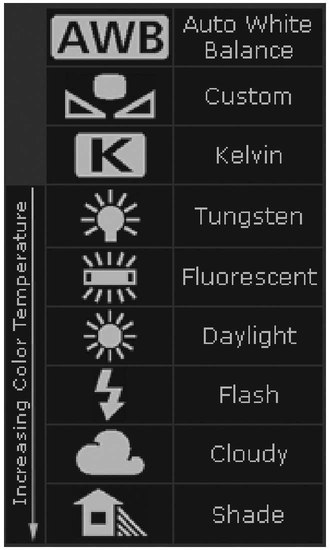
incorrect white balance

correct white balance

http://www.cambridgeincolour.com/tutorials/white-balance.htm

White balance

- Film cameras:
  - Different types of film or different filters for different illumination conditions
- Digital cameras:
  - Automatic white balance
  - White balance settings corresponding to several common illuminants
  - Custom white balance using a reference object



http://www.cambridgeincolour.com/tutorials/white-balance.htm

Slide: F. Durand

White balance

- Von Kries adaptation
  - Multiply each channel by a gain factor
  - A more general transformation would correspond to an arbitrary 3x3 matrix

Slide: F. Durand

White balance

- Von Kries adaptation
  - Multiply each channel by a gain factor
  - A more general transformation would correspond to an arbitrary 3x3 matrix
- Best way: gray card
  - Take a picture of a neutral object (white or gray)
  - Deduce the weight of each channel
    - If the object is recoded as r_w, g_w, b_w use weights $1/r_w$, $1/g_w$, $1/b_w$

Slide: F. Durand

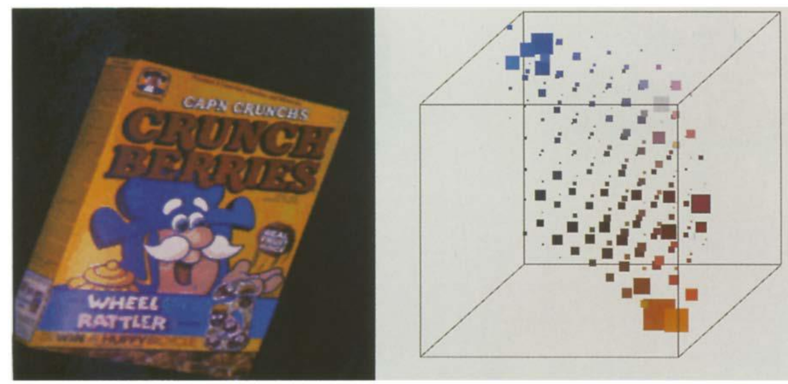
White balance

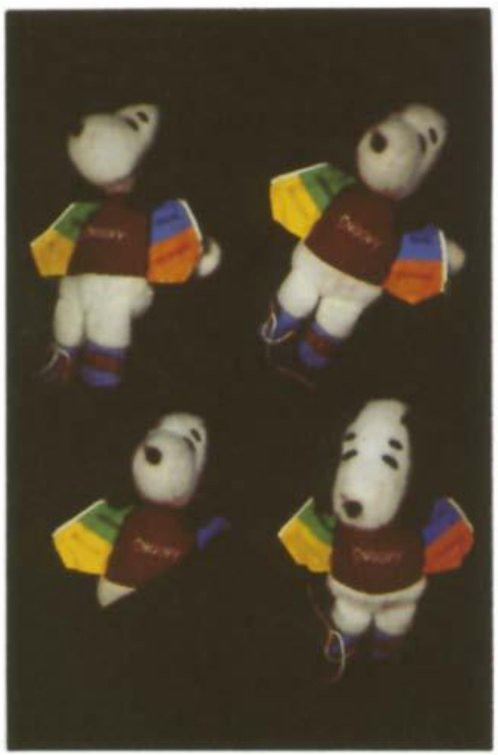
- Without gray cards: we need to "guess" which pixels correspond to white objects
- Gray world assumption
  - The image average r_{ave}, g_{ave}, b_{ave} is gray
  - Use weights $1/r_{ave}$, $1/g_{ave}$, $1/b_{ave}$
- Gamut mapping
  - Gamut: convex hull of all pixel colors in an image
  - Find the transformation that matches the gamut of the image to the gamut of a "typical" image under white light
- Use image statistics, learning techniques

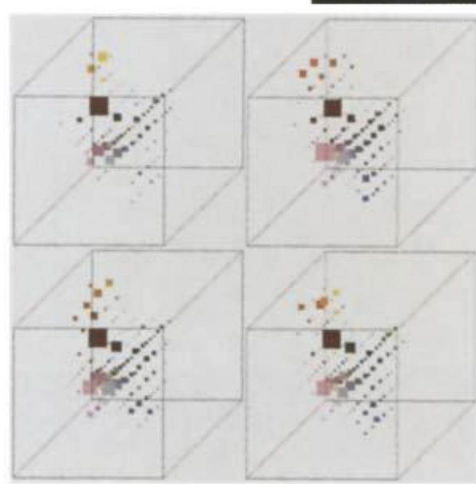
Slide: F. Durand

Uses of color in computer vision

Color histograms for indexing and retrieval



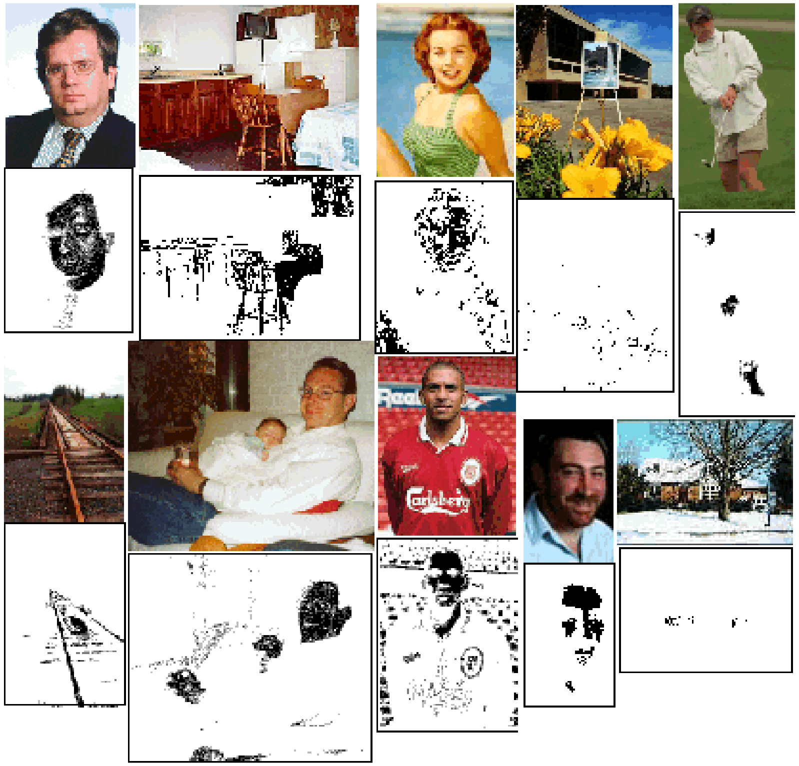




Swain and Ballard, [Color Indexing](), IJCV 1991.

Uses of color in computer vision

Skin detection

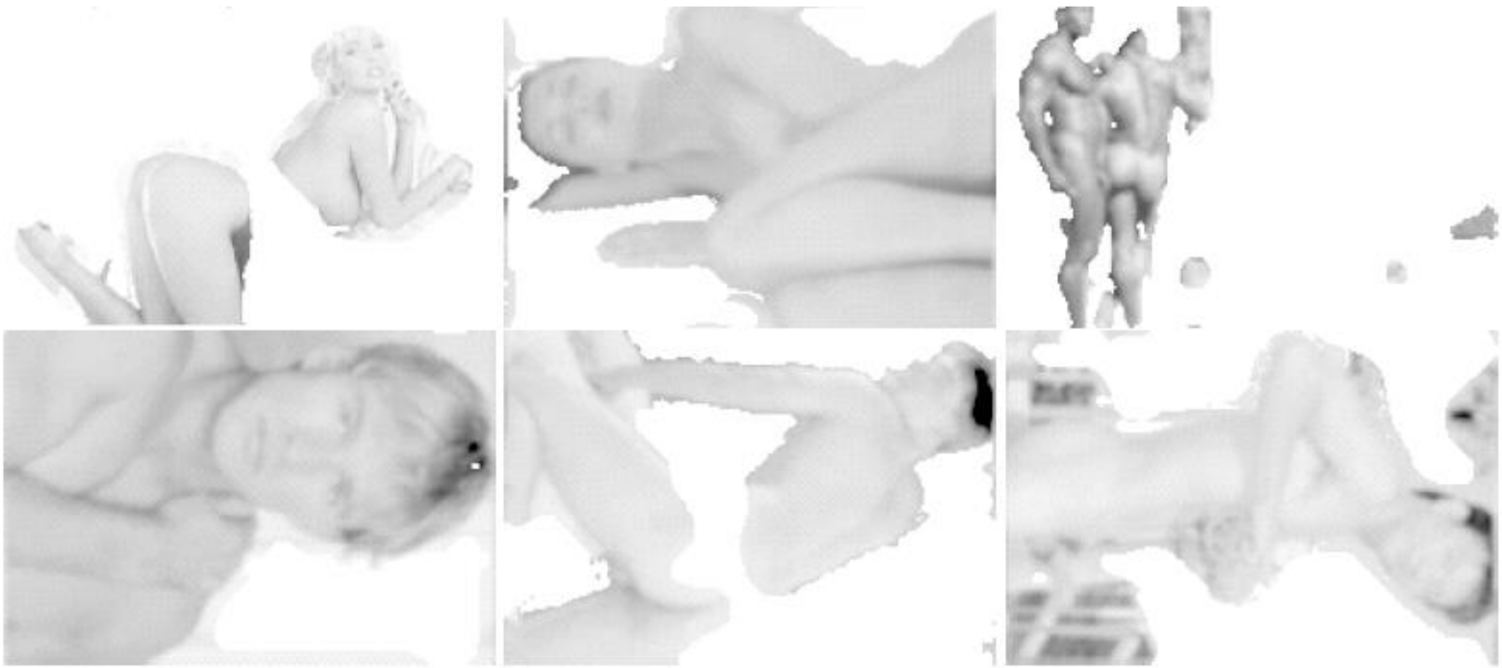


M. Jones and J. Rehg, [Statistical Color Models with Application to Skin Detection](), IJCV 2002.

Source: S. Lazebnik

Uses of color in computer vision

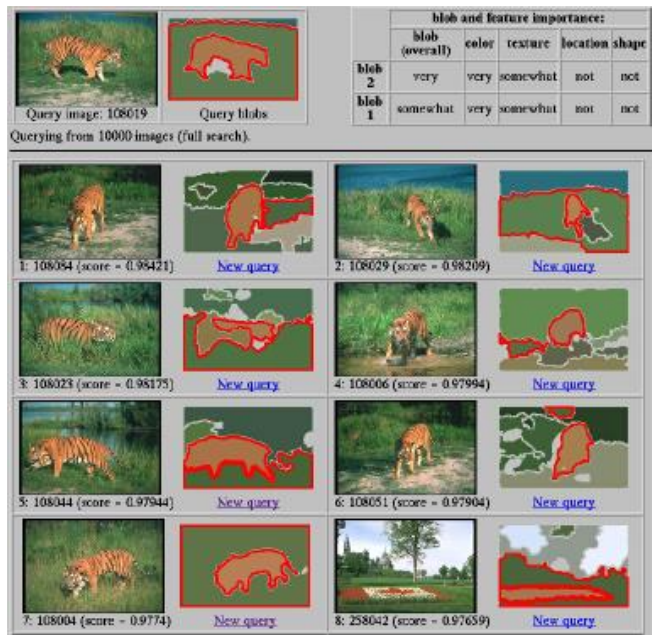
Nude people detection



Forsyth, D.A. and Fleck, M. M., ``Automatic Detection of Human Nudes,'' *International Journal of Computer Vision* , **32** , 1, 63-77, August, 1999

Uses of color in computer vision

Image segmentation and retrieval



C. Carson, S. Belongie, H. Greenspan, and Ji. Malik, Blobworld: Image segmentation using Expectation-Maximization and its application to image querying, ICVIS 1999.

Source: S. Lazebnik

Uses of color in computer vision

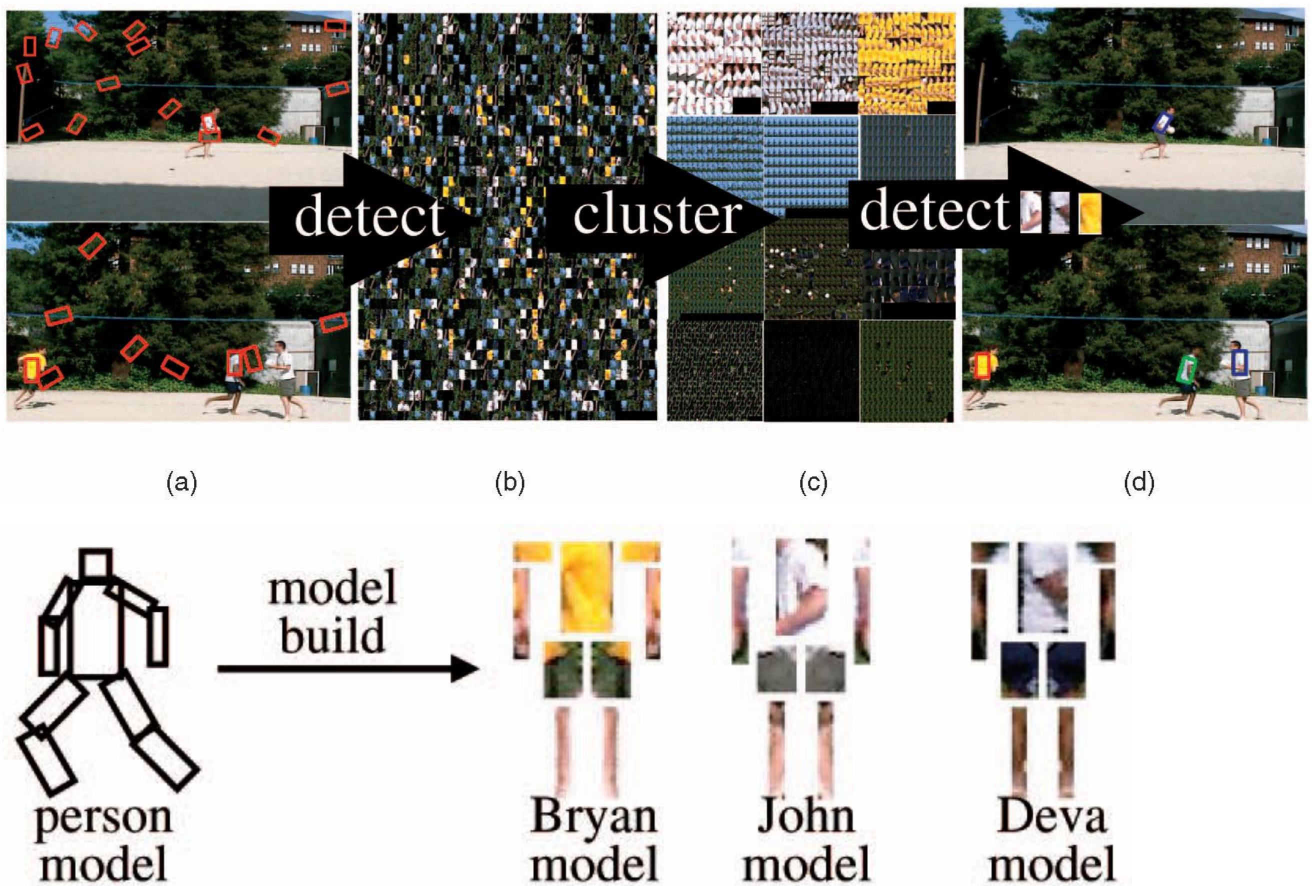
Robot soccer

M. Sridharan and P. Stone, [Towards Eliminating Manual Color Calibration at RoboCup](). RoboCup-2005: Robot Soccer World Cup IX, Springer Verlag, 2006

Source: K. Grauman

Uses of color in computer vision

Building appearance models for tracking



D. Ramanan, D. Forsyth, and A. Zisserman. [Tracking People by Learning their Appearance](). PAMI 2007.

Source: S. Lazebnik

Linear Algebra Primer

Outline

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculus

Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

T denotes the transpose operation

Matrix

- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size   by , i.e. m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

Basic Matrix Operations

- We will discuss:
  - Addition
  - Scaling
  - Dot product
  - Multiplication
  - Transpose
  - Inverse / pseudoinverse
  - Determinant / trace

Vectors

- **Norm** $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$

- More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties:

- **Non-negativity:** For all $x \in \mathbb{R}^n$, $f(x) \geq 0$
- **Definiteness**: f(x) = 0 if and only if x = 0.
- **Homogeneity**: For all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t| f(x)$
- **Triangle inequality**: For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$

Matrix Operations

- **Example Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

- General ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

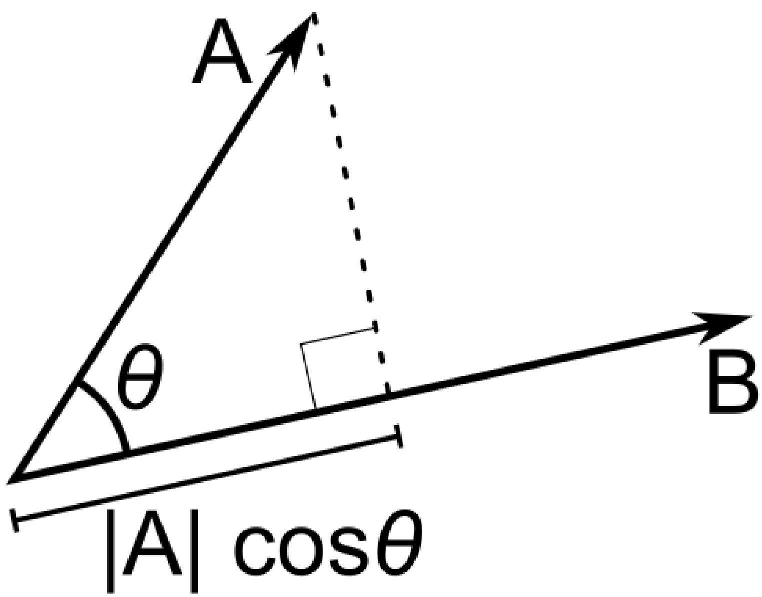
Matrix Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - x·y is also |x||y|Cos(*the angle between x and y*)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

Matrix Operations

- Inner product (dot product) of vectors
  - If B is a unit vector, then A·B gives the length of A which lies in the direction of B



Matrix Operations

- The product of two matrices

$$A \in \mathbb{R}^{m \times n}$$

$$B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

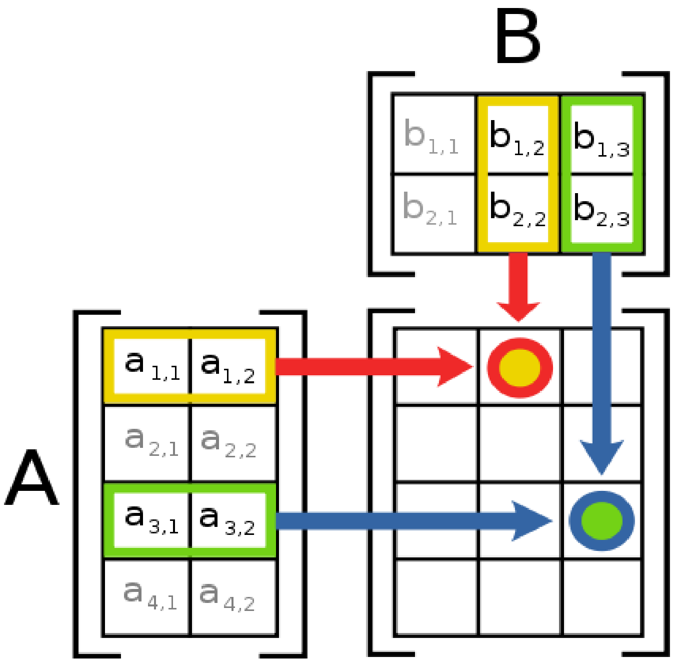
$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}.$$

Matrix Operations

- Multiplication

- The product AB is:



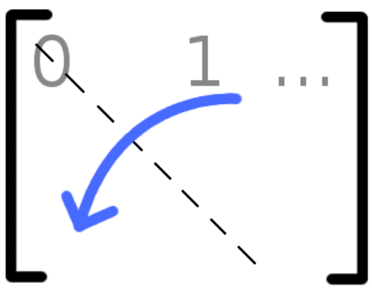
- Each entry in the result is (that row of A) dot product with (that column of B)
- Many uses, which will be covered later

Matrix Operations

- Powers
  - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
  - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

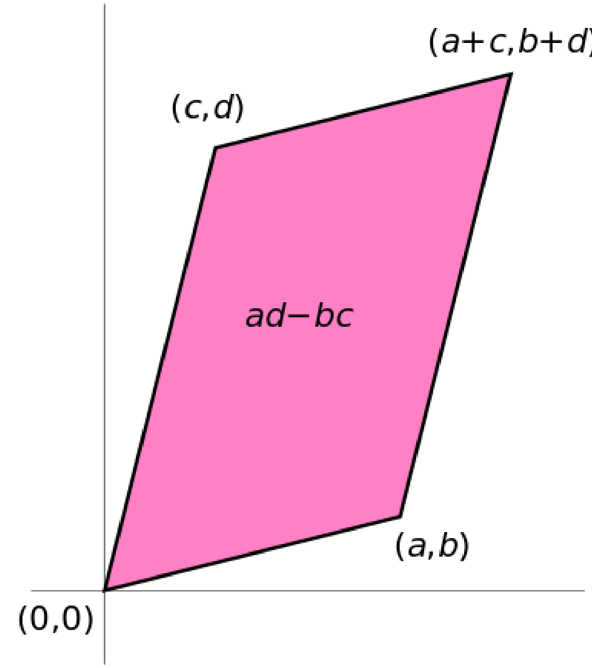
- Determinant
  - $\det(\mathbf{A})$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$
  - Properties:

$$\det(\mathbf{AB}) = \det(\mathbf{BA})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$



Matrix Operations

- Trace

$$\mathrm{tr}(\mathbf{A}) = \text{sum of diagonal elements}$$

$$\mathbf{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  – Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely in this class though.)
  – Properties:

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$
$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

Matrix Operations

- **Vector Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- Matrix norms: Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

Special Matrices

- Identity matrix **I**
  - Square matrix, 1's along diagonal, 0's elsewhere
  - **I** · [another matrix] = [that matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix
  - Square matrix with numbers along diagonal, 0's elsewhere
  - A diagonal · [another matrix] scales the rows of that matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix

$$\mathbf{A}^T = \mathbf{A}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

Inverse

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non-singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

Matrix rank

- Column/row rank

$$\text{col-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent column vectors of } \mathbf{A}$$
$$\text{row-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent row vectors of } \mathbf{A}$$

  – Column rank always equals row rank

- Matrix rank

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$

Matrix rank

- For transformation matrices, the rank tells you the dimensions of the output
- E.g. if rank of **A** is 1, then the transformation

  p'=Ap

  maps points onto a line.
- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+2y \end{bmatrix}$$

All points get mapped to the line y=2x

Matrix rank

- If an m x m matrix is rank m, we say it's "full rank"
  - Maps an m x 1 vector uniquely to another m x 1 vector
  - An inverse matrix can be found
- If rank < m, we say it's "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also doesn't exist for non-square matrices

Outline

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors(SVD)
- Matrix Calculus

Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.

$$Ax = \lambda x, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.
- Applying *A* to the eigenvector only scales the eigenvector by the scalar value *λ*, called an eigenvalue.

$$Ax = \lambda x, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- We want to find all the eigenvalues of A:

$$Ax = \lambda x, \quad x \neq 0.$$

- Which can we written as:

$$Ax = (\lambda I)x \quad x \neq 0.$$

- Therefore:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- We can solve for eigenvalues by solving:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- Since we are looking for non-zero **x**, we can instead solve the above equation as:

$$|(\lambda I - A)| = 0.$$

Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr} A = \sum_{i=1}^{n} \lambda_i.$$

Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i.$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i.$$

Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i.$$

- The determinant of A is equal to the product of its eigenvalues

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- The rank of A is equal to the number of non-zero eigenvalues of A.

Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i.$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i.$$

- The rank of A is equal to the number of non-zero eigenvalues of A.
- The eigenvalues of a diagonal matrix D = diag(d1, . . . dn) are just the diagonal entries d1, . . . dn

Some applications of Eigenvalues

- PageRank
- Schrodinger’s equation
- PCA

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Matrix Calculus – The Gradient

- Let a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ take as input a matrix A of size m × n and returns a real value.
- Then the **gradient** of **f:**

$$\nabla_A f(A) \in \mathbb{R}^{m\times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

Matrix Calculus – The Gradient

- Every entry in the matrix is: $\nabla_A f(A))_{ij} = \frac{\partial f(A)}{\partial A_{ij}}.$
- the size of $\nabla_A f(A)$ is always the same as the size of A. So if A is just a vector x:

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Find: $\dfrac{\partial f(x)}{\partial x_k} = ?$

$$\nabla_x f(x) = ?$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \sum_{i=1}^{n} b_i x_i$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} b_i x_i = b_k.$$

- From this we can conclude that: $\nabla_x b^T x = b.$

Matrix Calculus – The Gradient

- Properties
  - $\nabla_x(f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x)$.
  - For $t \in \mathbb{R}$, $\nabla_x(t\ f(x)) = t\nabla_x f(x)$.

Matrix Calculus – The Hessian

- The Hessian matrix with respect to x, written $\nabla_x^2 f(x)$ or simply as H is the n × n matrix of partial derivatives

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

- Exercise: Why is the Hessian always symmetric?

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}$.
- The Hessian is always symmetric, because

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}.$$

- This is known as Schwarz's theorem: The order of partial derivatives don't matter as long as the second derivative exists and is continuous.

Matrix Calculus – The Hessian

- Note that the hessian is not the gradient of whole gradient of a vector (this is not defined). It is actually the gradient of **every entry** of the gradient of the vector.

$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Eg, the first column is the gradient of $\frac{\partial f(x)}{\partial x_1}$

$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \boxed{\begin{matrix} \frac{\partial^2 f(x)}{\partial x_1^2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} \end{matrix}} & \begin{matrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2^2} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \begin{matrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots \\ \frac{\partial^2 f(x)}{\partial x_n^2} \end{matrix} \end{bmatrix}$$

Exercise

- Example:

consider the quadratic function $f(x) = x^T Ax$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

Exercise

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_i x_j \\ &= \frac{\partial}{\partial x_k}\left[\sum_{i\neq k}\sum_{j\neq k} A_{ij}x_i x_j + \sum_{i\neq k} A_{ik}x_i x_k + \sum_{j\neq k} A_{kj}x_k x_j + A_{kk}x_k^2\right]\end{aligned}$$

Divide the summation into 3 parts depending on whether:

- i == k or
- j == k

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\boxed{\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j} + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \boxed{\sum_{i \neq k} A_{ik} x_i x_k} + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \boxed{\sum_{i \neq k} A_{ik} x_i} + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_i x_j$$

$$= \frac{\partial}{\partial x_k}\left[\sum_{i\neq k}\sum_{j\neq k} A_{ij}x_i x_j + \sum_{i\neq k} A_{ik}x_i x_k + \boxed{\sum_{j\neq k} A_{kj}x_k x_j} + A_{kk}x_k^2\right]$$

$$= \sum_{i\neq k} A_{ik}x_i + \boxed{\sum_{j\neq k} A_{kj}x_j} + 2A_{kk}x_k$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + \boxed{A_{kk} x_k^2} \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + \boxed{2 A_{kk} x_k}
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k \\
&= \sum_{i=1}^{n} A_{ik} x_i + \sum_{j=1}^{n} A_{kj} x_j = 2 \sum_{i=1}^{n} A_{ki} x_i,
\end{aligned}
$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k}\left[\frac{\partial f(x)}{\partial x_\ell}\right] = \frac{\partial}{\partial x_k}\left[\sum_{i=1}^{n} A_{\ell i} x_i\right]$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{n} A_{\ell i} x_i \right]$$

$$= 2A_{\ell k} = 2A_{k\ell}.$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{n} 2 A_{\ell i} x_i \right]$$

$$= 2A_{\ell k} = 2A_{k\ell}.$$

$$\nabla_x^2 f(x) = 2A$$

What we have learned

- Vectors and matrices
  - Basic Matrix Operations
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculate