Thesis for Master's Degree

Formation Control and Guidance using Bearing-only Measurements

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Formation Control and Guidance using Bearing-only Measurements

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Dedicated to

my family.

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Abstract

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Chapter 2

Preliminaries

2.1 Bearing measurement

Consider a team of N autonomous agents on the plane. We use $p_i \in \mathbb{R}^2, i \in \{1, ..., N\}$ to represent agent's i position and denote $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{2n}$ as depicted in Fig. 2.1. We can define the bearing angle and bearing vector as follows:

Definition 2.1 *If the agent i and j are not collocated, the relative bearing from i to j is the counter clockwise angle β ($0 \leq \beta < 2\pi$) between the x-axis and the line segment connecting i and j. Moreover, we can construct the relative bearing vector*

$$\hat{u}_{ij} := \hat{u}(p_j - p_i) = \frac{p_j - p_i}{\|p_j - p_i\|} = \mathbf{1}\angle\beta, \tag{2.1}$$

where $\mathbf{1}$ is the unit vector of the x-axis.

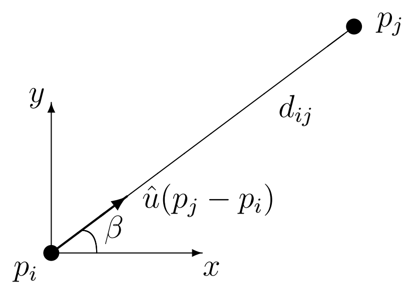

Figure 2.1. The bearing vector

Remark 2.2 *The bearing vector defined in Eq.* (2.1) *requires no distance information.*

2.2 Algebraic graph theory

In this section, some related results on algebraic graph theory is briefly summarized. Graph theory will be used in chapter **??** and **??** to describe the sensing topology of the multi-agent system.

A directed graph is defined as a pair $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of ordered pairs of the nodes, called edges. In formation control problem, each agent is represented by a node in the graph. A directed edge, denoted by $(i, j) \in \mathcal{E}$, exists if agent i measures the bearing to agent j. We call agent j a neighbor of agent i and denote the neighbor set of i by $\mathcal{N}_i$.

A directed path is defined as a sequence of edges $(i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k)$ in $\mathcal{E}$, where $i_1, i_2, ..., i_k \in \mathcal{V}$. i_1 and i_k are referred as the start vertex and the end vertex, respectively. A directed cycle is a directed path which has the same start vertex and end vertex. An acyclic directed graph is a directed graph which has no directed cycle. More results on graph theory can be found from [**?**].

2.3 Input-to-state stability and cascade systems

In this section, we present basics of Input-to-state stability theory. Input-to-state stability is a powerful tool to study stability of cascade systems. Consider the following system

$$\dot{x} = f(x, u) \tag{2.2}$$

where $f : D_x \times D_u \mapsto \mathbb{R}^n$ is locally Lipschitz in x and u, and $D_x \subseteq \mathbb{R}^n$ and $D_u \subseteq \mathbb{R}^m$ are domains containing $x = 0$ and $u = 0$, respectively. Input-to-state stability is defined as follows

Definition 2.3 *[?] The system (2.2) is locally input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function γ, and positive constants k_x and k_u such that for any initial state $x(0)$ with $\|x(0)\| < k_x$ and any input $u(t)$ with $\sup\limits_{0\le\tau\le t} \|u(\tau)\| < k_u$, the solution $x(t)$ exists and satisfies*

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\sup_{0\le\tau\le t} \|u(\tau)\|, t) \ge 0. \tag{2.3}$$

The following theorem presents a sufficient condition for locally ISS

Theorem 2.4 *[?] If there exists a neighborhood U of $(x = 0, u = 0)$ such that the function $f(x, u)$ in (2.2) is continuously differentiable and the unforced system $\dot{x} = f(x, 0)$ is asymptotically stable in U, then the system (2.2) is locally input-to-state stable.*

Based on ISS, a sufficient condition for local asymptotic stability of cascade systems is provided in Theorem 2.5

Theorem 2.5 *[?] For the cascade system*

$$\dot{x}_1 = f_1(x_1, x_2), \tag{2.4a}$$

$$\dot{x}_2 = f_2(x_2), \tag{2.4b}$$

where $f_1 : D_{x_1} \times D_{x_2} \mapsto \mathbb{R}^{n_1}$ and $f_2 : D_{x_2} \mapsto \mathbb{R}^{n_2}$ are locally Lipschitz in x_1 and x_2, if the system (2.4a), with x_2 as input, is locally ISS and the origin of the system (2.4b) is locally asymptotically stable, the origin of the cascade system (2.4) is locally asymptotically stable.

Chapter 3

Quadrotor systems

3.1 Hardware platform

3.2 Operation of the systems

Chapter 4

Angle-based control of directed acyclic formations with three leaders

In this chapter, we study angle-based control for directed acyclic formations. Consider a group of single-integrator modeled agents on the plane. The agent group consists of three leader agents and the remaining follower agents. The leader agents are able to measure their absolute positions while the remaining agents measure the bearing of their three neighbors with respect to the x-axis. The bearing angle sensing topology is directed and acyclic. By adopting a direct position control law for the leader and an angle-based control law for follower agents, we drive the agents to their desired locations. Based on input-to-state stability notion, we show that the desired locations are locally asymptotically stable.

4.1 Problem formulation

Consider the following N single-integrator modeled agents on the plane

$$\dot{p}_i = u_i, \quad i = 1, \ldots, N, \tag{4.1}$$

where $p_i \in \mathbb{R}^2$ and $u_i \in \mathbb{R}^2$ denote the position and control input of agent i, respectively. Without loss of generality, we assume that agents 1, 2, and 3 are leaders while the remaining agents are followers. We further assume that the leaders are able to measure their own

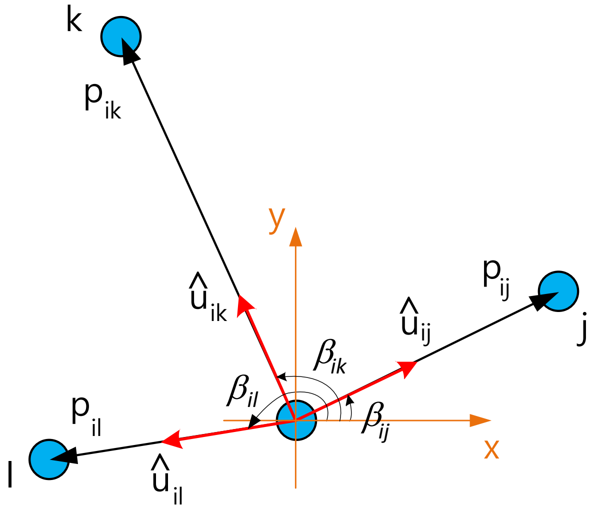

Figure 4.1. Bearing measurements topology of follower i.

positions while the followers only measure the bearing angle of their exactly three neighbors. Let the bearing angle sensing topology be modeled by a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and all followers share a common coordinate frame.

Let follower agent i have agents j, k, and l as its neighbors. We denote by β_{ij} the bearing angle of agent j from the x-axis. From bearing measurements, agent i can construct three corresponding bearing vectors $\hat{u}_{ij}, \hat{u}_{ik}$ and $\hat{u}_{ij}$ as depicted in Fig. **??**.

Let $p_i^* \in \mathbb{R}^2$ be desired position of agent i, where $i = 1, \ldots, N$. Before stating the formation control problem, we summarize standing assumptions:

Assumption 4.1 *For the single-integrator modeled agents* (**??**)*, we assume*

- *Leader* $i \in \{1, 2, 3\}$ *measures* p_i*;*
- *Follower* $i \in \{4, 5, ..., N\}$ *has exactly three neighbors and it measures the bearing angles of its neighbors from the* x*-axis;*

- *The bearing angle measuring topology is modeled by a directed acyclic graph $\mathcal{G}$;*
- *Let agents j, k, and l be the neighbors of agent i. Then any three of p_i^*, p_j^*, p_k^*, and p_l^* are not collinear.*

We are now ready to state the angle–based formation control problem for the single–integrator modeled agents. (**??**).

Problem 4.1 *Consider the single-integrator modeled agents* (**??**)*. Let desired global position p_i^* be given to leader i and desired bearing vectors $\hat{u}_{ij}^*$, $j \in \mathcal{N}_i$ be given to follower i. Under Assumption* **??***, design control laws for the leaders and followers such that $p^* = [p_1^{*T} \cdots p_N^{*T}]^T$ is asymptotically stable with respect to* (**??**).

4.2 Control strategy and stability analysis

4.2.1 Proposed Control Strategy

We propose a formation control strategy that allows the leaders to directly control their positions while the followers actively control the bearing angles to their neighbors.

For $1 \leq i \leq j \leq N$, we introduce the notations:

$$\mathcal{V}_{[i:j]} := \{i, ..., j\}, p_{[i:j]} = [p_i^T \dots p_j^T]^T.$$

Consider the single-integrator modeled agents (**??**). Since the leaders measure their positions and know their destinations, it is natural to use the following control law:

$$u_i = k_L(p_i^* - p_i), \tag{4.2}$$

where $k_L > 0$ is a constant.

For the follower $i \in \mathcal{V}_{[4:N]}$, we adopt the angle-based control law that has been proposed in [**?**]. To introduce the control law, we define the orthogonal vector of $\hat{u}_{ik}$ as

$$\hat{u}_{ik}^{\perp} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{u}_{ik}.$$

The control law for the followers can be designed as

$$\dot{p}_i = u_i = -\sum_{j \in \mathcal{N}_i} (\hat{u}_{ij}^{*T} \hat{u}_{ij}^{\perp}) \hat{u}_{ij}^{\perp}. \tag{4.3}$$

Let $\tilde{p}_i := p_i - p_i^*$. Then the error dynamics of the agents can be described by

$$\dot{\tilde{p}}_i = -k_L \tilde{p}_i, i \in \mathcal{V}_{[1:3]}, \tag{4.4a}$$

$$\dot{\tilde{p}}_i = f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}), i \in \mathcal{V}_{[4:N]}, \tag{4.4b}$$

where

$$f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}) := -\sum_{j \in \mathcal{N}_i} (\hat{u}_{ij}^{*T} \hat{u}_{ij}^{\perp}) \hat{u}_{ij} \tag{4.5}$$

4.2.2 Stability analysis

In below, we show that the origin $\tilde{p} = [\tilde{p}_1^T, \ldots, \tilde{p}_N^T]^T = 0$ is locally asymptotically stable under Assumption **??** [**?**]. Follower i can be described as

$$\dot{\tilde{p}}_i = f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}), \tag{4.6a}$$

$$\dot{\tilde{p}}_{[1:i-1]} = f_{[1:i-1]}(\tilde{p}_{[1:i-1]}), \tag{4.6b}$$

where

$$
f_{[1:i-1]}(\tilde{p}_{[1:i-1]}) := \begin{bmatrix} -k_L \tilde{p}_{[1:3]} \\ f_4(\tilde{p}_4, \tilde{p}_{[1:3]}) \\ \vdots \\ f_{i-1}(\tilde{p}_{i-1}, \tilde{p}_{[1:i-2]}) \end{bmatrix}. \tag{4.7}
$$

Remark 4.1 *The system* (**??**) *has the form of a cascade system* (2.4)*, where* $\tilde{p}_{[1:i-1]}$ *is the input to (**??**a).*

Our stability analysis consists of three steps as follows:

- *First Step*: We show that the origin $\tilde{p}_{[1:3]} = 0$ is exponentially stable with respect to $\dot{\tilde{p}}_{[1:3]} = -k_L \tilde{p}_{[1:3]}$.
- *Second Step*: We show that (**??**) is locally ISS with $\tilde{p}_{[1:i-1]}$ as input. Based on Theorem 2.5, $\tilde{p}_{[1:i]}$ is locally asymptotically stable with respect to (**??**) if $\tilde{p}_{[1:i-1]} = 0$ is locally asymptotically stable with respect to (**??**b).
- *Third Step*: Finally we show that the origin is locally asymptotically stable with respect to $\dot{\tilde{p}}_{[1:N]} = f_{[1:N]}(\tilde{p}_i, \tilde{p}_{[1:N]})$ based on mathematical induction.

The first step is obvious [**?**]. For the second step, we show local stability of the following unforced dynamics of (**??**a)

$$
\dot{\tilde{p}}_i = f_i(\tilde{p}_i, 0), \ i \in \mathcal{V}_{[4:N]}. \tag{4.8}
$$

We have the following lemmas:

Lemma 4.2 *[?, Theorem 6] Given three desired bearing vectors $\hat{u}_{ij}, \hat{u}_{ik}, \hat{u}_{il}$. If three neighbors j, k, l are stationary, the system (**??**) is globally asymptotically stable.*

Lemma 4.3 *Let Assumption **??** hold. For $i \in \mathcal{V}_{[4:N]}$, the origin $\tilde{p}_i = 0$ is locally asymptotically stable with respect to (**??**).*

Proof: The unforced error dynamics (**??**) implies that all agents $1, \ldots, i-1$ are at their desired positions, which are non-collinear as a result of Assumption **??**. It then follows from Lemma **??** that $\tilde{p}_i = 0$ is asymptotically stable with respect to (**??**). ■

The following lemma reveals locally ISS of (**??**a):

Lemma 4.4 *Let Assumption **??**. For $i \in \mathcal{V}_{[4:N]}$, (**??**a) is locally input-to-state stable with $\tilde{p}_{[1:i-1]}$ as input.*

Proof: From Lemma **??**, $\tilde{p}_i = 0$ is locally asymptotically stable with respect to (**??**). Further f_i defined in (**??**) is continuously differentiable in $\tilde{p}_i$ and $\tilde{p}_{[1:i-1]}$. Thus, it follows from Theorem 2.4 that (**??**a) is locally ISS with $\tilde{p}_{[1:i-1]}$ as input. ■

Finally, we have the following theorem:

Theorem 4.5 *Let Assumption **??** hold. The origin $\tilde{p} = 0$ is locally asymptotically stable with respect to (**??**).*

Proof: Consider the following cascade system

$$\dot{\tilde{p}}_4 = f_4(\tilde{p}_4, \tilde{p}_{[1:3]}), \tag{4.9a}$$

$$\dot{\tilde{p}}_{[1:3]} = -k_L \tilde{p}_{[1:3]}. \tag{4.9b}$$

It is obvious that the origin $\tilde{p}_{[1:3]} = 0$ is exponentially stable with respect to (**??**b). Based on *Lemma* **??**, (**??**a) is locally stable with $\tilde{p}_{[1:3]}$ as input. It then follow from Theorem 2.5 that $\tilde{p}_{[1:4]} = 0$ is locally asymptotically stable with respect to (**??**).

Next, suppose that, for any $i \in \mathcal{V}_{[L+1,N]}$, (**??**b) is locally asymptotically stable. *Lemma* **??** guarantees the locally ISS stability of (**??**a) with $\tilde{p}_{[1:i-1]} = 0$ as input. Thus, from Theorem 2.5, we have $\tilde{p}_{[1:i]} = 0$ is also locally asymptotically stable with respect to (**??**b).

As a result of mathematical induction, for any $i \in \mathcal{V}_{[L+1,N]}$, $\tilde{p}_{[1:i]} = 0$ is locally asymptotically stable with respect to (**??**). Thus, we can conclude that $\tilde{p} = 0$ is locally asymptotically stable with respect to (**??**). ■

4.3 Simulations

We consider a group of ten agents whose sensing topology is depicted in Fig. **??**.

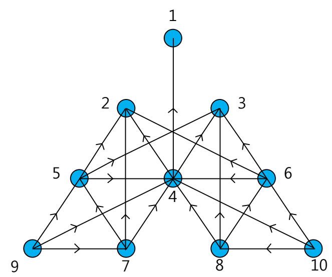

Figure 4.2. Bearing measurement topology of a ten-agents system.

The desired positions are given as $p_1^* = (0, \frac{3\sqrt{3}}{2}), p_2^* = (-\frac{1}{2}, \sqrt{3}), p_3^* = (\frac{1}{2}, \sqrt{3}), p_4^* = (0, \frac{\sqrt{3}}{2}), p_5^* = (-1, \frac{\sqrt{3}}{2}), p_7^* = (-\frac{1}{2}, 0), p_8^* = (\frac{1}{2}, 0), p_9^* = (-\frac{3}{2}, 0), p_{10}^* = (\frac{3}{2}, 0)$. Note that the Assumption **??** is satisfied for this group of agents.

Simulation results for the group of ten agents under the proposed control strategy are shown in Fig. **??** and Fig. **??**. We can see from Fig. **??** that both leaders and followers eventually reach their desired positions. The position error of agents 1, 4, and 8 are shown in Fig. **??**. These errors are asymptotically reach to 0. Thus, the simulation results agree with our stability analysis.

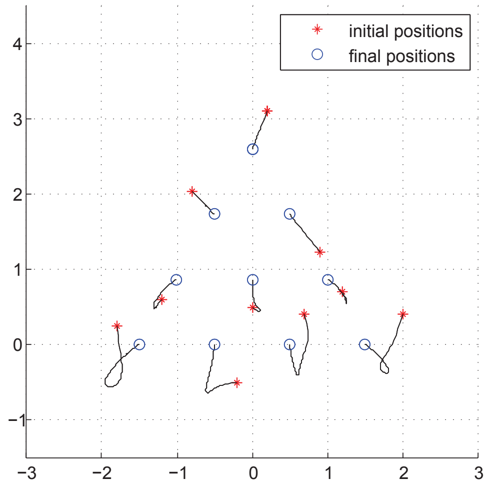

Figure 4.3. Trajectories of the agents.

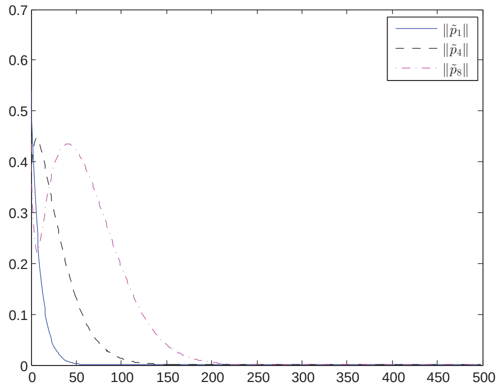

Figure 4.4. Magnitude of position errors.

4.4 Hardware experiments

Chapter 5

Control of a mobile agent using only bearing measurements with three stationary beacons

In this chapter, we present a new formation control framework. First, a bearing-only control law for a single integrator agent moving on the plane is developed. We prove that the control law is able to reach almost desired locations inside the triangular formed by three stationary beacons. Second, due to the symmetric of the problem, we introduce an algorithm to extend this control law to the entire plane. The novelty of this control law are the use of pure bearing vectors along with bearing angles in the control law and the invariant to any coordinate frame. Lastly, we formulate a formation control problem based on this control law.

5.1 Problem formulation

Consider a system of three non-colinear stationary beacons located at A_1, A_2, A_3 and an agent on the plane. The overall system is depicted in Fig. **??**. A single integrator model is used to describe the agent's motion. That is, the motion is governed by the equation

$$\dot{p} = v,$$

where $p, v \in \mathbb{R}^2$ are the position and velocity of the agent at time t correspondingly. In this chapter, the notation $\widehat{A_k}, k \in \{1, 2, 3\}$ is used to denote three interiors of the triangular $A_1A_2A_3$. We consider the following assumptions

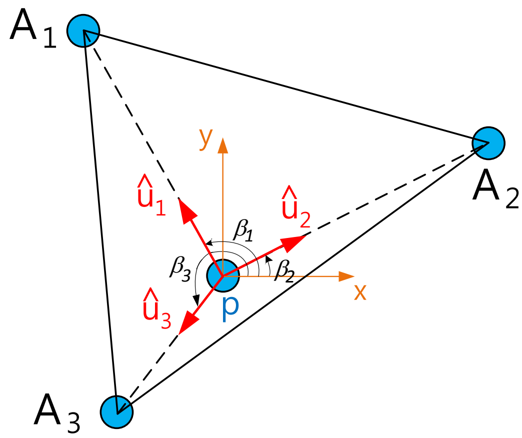

Figure 5.1. The agent measures the bearings $\beta_1, \beta_2, \beta_3$ with regard to the beacons A_1, A_2, A_3.

Assumption 5.1 *The agent's initial position is inside the triangular formed by the three beacons. Moreover, its initial position is not co-located with the three beacons' positions.*

Assumption 5.2 *The agent can measure the relative bearing angles to the three beacons $\beta_1, \beta_2, \beta_3$ in its local reference frame as depicted in Fig.* **??**. *Those bearing angles satisfy $0 \leq \beta_k < 2\pi$, for $k \in \{1, 2, 3\}$.*

Based on assumptions **??** - **??**, the agent can sense the directions toward the three beacons. In this chapter, we will use the following notation for the unit bearing vectors:

$$\hat{u}_k := \hat{u}(p_k - p) = \frac{p_k - p}{\|p_k - p\|} = \mathbf{1}\angle\beta_k, \tag{5.1}$$

for $k \in \{1, 2, 3\}$. Let $\vartheta_k = |\beta_{k-1} - \beta_{k+1}|$, then $0 \leq \vartheta_k < 2\pi$. We define the subtended angles can as

$$\alpha_k = \begin{cases} \vartheta_k & \text{if } 0 \leq \vartheta_k \leq \pi \\ 2\pi - \vartheta_k & \text{if } \pi \leq \vartheta_k \leq 2\pi \end{cases}, \tag{5.2}$$

for $k \in \{1,2,3\}$. Suppose that the agent wants to move to a desired position p^* inside the triangular A_1, A_2, A_3, where the three subtended angles are α_k^*, $k \in \{1,2,3\}$ as depicted in Fig. **??**,. Then, we have the following assumptions about those desired angles:

Assumption 5.3 *The desired subtended angles satisfy the following constraints*

$$\sum_{k=1}^{3} \alpha_k^* = 2\pi, \tag{5.3a}$$

$$\widehat{A_k} < \alpha_k^* \leq \pi. \tag{5.3b}$$

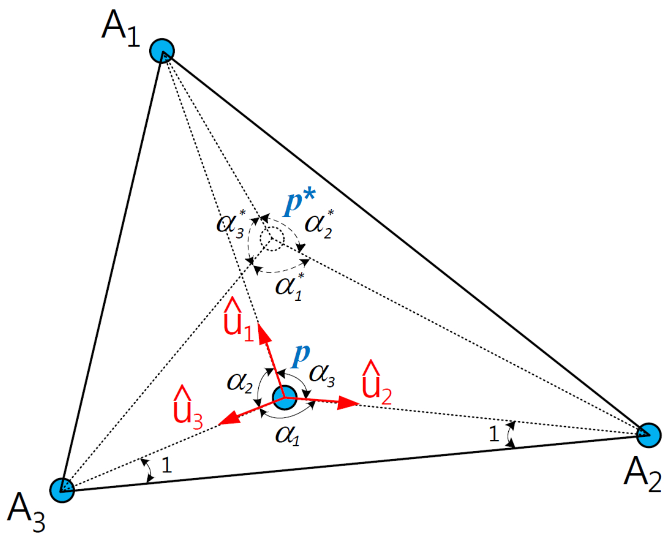

Figure 5.2. p^* is the desired position where three subtended angles are $\alpha_1^*, \alpha_2^*, \alpha_3^*$.

Now, we can formulate the problem as follows:

Problem 5.1 *Under the Assumptions* **??**-**??**, *design a control law for the agent to asymptotically reach its desired position.*

5.2 Control law and stability analysis

5.2.1 Proposed control law

Before proposing the navigation control law, we recall a simple result from planar geometry in the following lemma:

Lemma 5.1 *There is a unique point inside the triangular A_1, A_2, A_3 satisfying all three subtended angles α_k^*, $k \in \{1, 2, 3\}$ in the Assumption* **??**.

The proof of Lemma **??** is trivial. Based on Lemma **??**, it is natural to introduce a control law which continuously lets the agent move whenever three desired subtended angles are still not achieved. The following control law can be adopted

$$\dot{p} = u = u_1 + u_2 + u_3, \tag{5.4}$$

where

$$\begin{aligned} u_1 &= k_u(\alpha_1 - \alpha_1^*)\hat{u}_1 = k_u e_1 \hat{u}_1 \\ u_2 &= k_u(\alpha_2 - \alpha_2^*)\hat{u}_2 = k_u e_2 \hat{u}_2 \\ u_3 &= k_u(\alpha_3 - \alpha_3^*)\hat{u}_3 = k_u e_3 \hat{u}_3. \end{aligned}$$

In (**??**), $k_u > 0$ is a constant gain and e_k is the angle error defined by $e_k = \alpha_k - \alpha_k^*$, for $k \in \{1, 2, 3\}$. The control input consists of three components u_1, u_2, and u_3. The component u_k is heading toward or outward the $k-$th beacon depending on the agent's position. From hereby, we can set $k_u = 1$ to simply the analysis without loss of generality.

5.2.2 Stability Analysis

In below, we prove that the desired position is a unique asymptotically stable equilibrium point of system (**??**). The following lemma is essential for further analysis:

Lemma 5.2 *The agent under the control law (**??**) will never escape from the triangle $A_1A_2A_3$ if it is initially positioned inside that region.*

Proof: The agent may escape the triangle $A_1A_2A_3$ from one of three sides A_1A_2, A_2A_3 or A_3A_1. Without loss of generality, consider a case when the agent is about to escape the triangular from the side A_2A_3. Other cases can be treated similarly. As depicted in Figure **??**, at that time instance, A_2, the agent's position and A_3 are colinear. Thus, $\alpha_1 = \pi$, u_2 and u_3 lie in the same line A_2A_3. Due to Assumption **??**, $\alpha_1^* < \pi$. It follows from (**??**) that u_1 directs toward A_1 and the agent will move inside the triangular. ■

Lemma **??** guarantees that the agent will be the triangle $A_1A_2A_3$ for all time t. Therefore, the subtended angles must satisfy the following constraints

$$\sum_{k=1}^{3} \alpha_k = 2\pi \tag{5.5}$$

$$\widehat{A_k} < \alpha_k \leq \pi, \tag{5.6}$$

for $k \in \{1, 2, 3\}$. From equations (**??**) and (**??**), it follows that

$$\sum_{k=1}^{3} e_k = \sum_{k=1}^{3} (\alpha_k - \alpha_k^*) = \sum_{k=1}^{3} \alpha_k - \sum_{k=1}^{3} \alpha_k^* = 2\pi - 2\pi = 0 \tag{5.7}$$

Lemma 5.3 *There is an unique equilibrium point of (**??**) located inside the triangular $A_1A_2A_3$.*

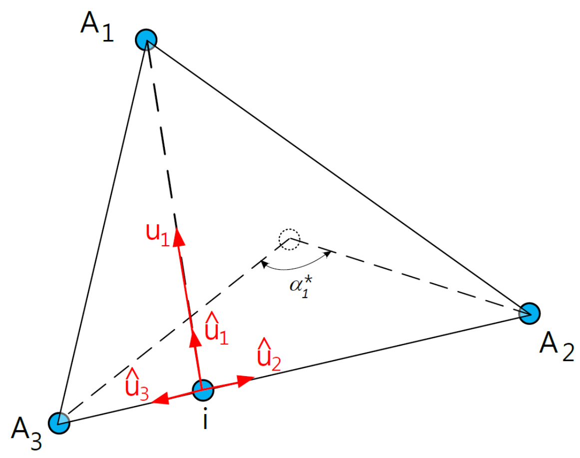

Figure 5.3. Illustration of Lemma **??**'s proof.

Proof: The equilibrium points satisfy the equation

$$\dot{p} = 0 \tag{5.8}$$

Obviously, equation (**??**) has a solution p^* where $\alpha_k = \alpha_k^*$, or $e_k = 0, \forall k \in \{1, 2, 3\}$. Based on Lemma **??**, p^* is the only point satisfying three desired subtended angles.

We will prove that p^* is also the unique solution of (**??**) by contradiction. Suppose that there exists p satisfying (**??**) but $\exists k \in \{1, 2, 3\}$ s.t. $e_k \neq 0$. Without loss of generality, we can assume that $k = 1$. It follows from Lemma **??** that all pairs of vectors $\{\hat{u}_1, \hat{u}_2\}$, $\{\hat{u}_2, \hat{u}_3\}$ and $\{\hat{u}_3, \hat{u}_1\}$ are linear independent. Therefore, from the linearly dependent of $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$, there exists unique $a, b \in \mathbb{R}$ such that $ab \neq 0$ and

$$\hat{u}_1 = a\hat{u}_2 + b\hat{u}_3 \tag{5.9a}$$

or,

$$\hat{u}_1 - a\hat{u}_2 - b\hat{u}_3 = 0 \tag{5.9b}$$

Equation (**??**) is equivalent to $e_1\hat{u}_1 + e_2\hat{u}_2 + e_3\hat{u}_3 = 0$. Since $e_1 \neq 0$, we have

$$\hat{u}_1 + \frac{e_2}{e_1}\hat{u}_2 + \frac{e_3}{e_1}\hat{u}_3 = 0 \tag{5.10}$$

Comparing (**??**) with (**??**), the uniqueness of a, b gives $e_2 = -ae_1$, $e_3 = -be_1$. Substituting into (**??**) yields $e_1 - ae_1 - be_1 = 0$, which is equivalent to $a + b = 1$. $(*)$

On the other hand, $\|\hat{u}_1\| = \|\hat{u}_2\| = \|\hat{u}_3\| = 1$. From (**??**), we have

$$\begin{aligned} 1 = \|\hat{u}_1\|^2 = \hat{u}_1^T\hat{u}_1 &= (a\hat{u}_2 + b\hat{u}_3)^T(a\hat{u}_2 + b\hat{u}_3) \\ &= a^2\|\hat{u}_2\|^2 + b^2\|\hat{u}_3\|^2 + 2ab\hat{u}_2^T\hat{u}_3 \\ &= a^2 + b^2 + 2ab\|\hat{u}_2\|\|\hat{u}_3\|\cos\alpha_1 \\ &= a^2 + b^2 + 2ab\cos\alpha_1 \end{aligned} \tag{5.11}$$

From (**??**), we have $\cos\alpha_1 < 1$. Thus, it follows from (**??**) that $1 < (a+b)^2$ or $|a+b| < 1$, which contradicts $(*)$. This contradiction implies that (**??**) has no other equilibrium point than p^*. ■

In the following analysis, we study the dynamics of the subtended angles with regard to each control input components u_1, u_2, u_3.

Firstly, we find the changes of α_1 caused by u_1. Define $\widehat{A}_{21}$, $\widehat{A}_{31}$ as the angles between A_2A_3 and line segments connecting the agent's position with A_2 and A_3 respectively (see Fig. **??**). The changes of two angles $\widehat{A}_{21}$ and $\widehat{A}_{31}$ under u_1, denoted by $\dot{\widehat{A}}_{21}^1$ and $\dot{\widehat{A}}_{31}^1$ respectively, are

$$\begin{aligned} \dot{\widehat{A}}_{21}^1 &= \frac{1}{r_2}(\alpha_1 - \alpha_1^*)\sin\alpha_3 = \frac{1}{r_2}\sin\alpha_3 e_1 \\ \dot{\widehat{A}}_{31}^1 &= \frac{1}{r_3}\sin\alpha_2 e_1, \end{aligned}$$

where $r_k = \|p_k - p\|$, $k \in \{1,2,3\}$ is the Euclidean distance from the agent to the $k-$th beacon. Let α_{11} the changes of α_1 under u_1. Since $\widehat{A}_{21} + \widehat{A}_{31} + \alpha_1 = \pi$, it follows that $\dot{\widehat{A}}^1_{21} + \dot{\widehat{A}}^1_{31} + \dot{\alpha}_{11} = 0$. Thus, the changes of α_1 caused by u_1 is given by

$$\dot{\alpha}_{11} = -(\frac{1}{r_3}\sin\alpha_2 + \frac{1}{r_2}\sin\alpha_3)e_1 = -g_{11}e_1. \tag{5.12}$$

Secondly, we find the changes of α_1 under u_2.

$$\dot{\widehat{A}}^2_{21} = 0$$

$$\dot{\widehat{A}}^2_{31} = -\frac{1}{r_3}\sin\alpha_1(\alpha_2 - \alpha_2^*) = -\frac{1}{r_3}\sin\alpha_1 e_2$$

Thus,

$$\dot{\alpha}_{12} = \frac{1}{r_3}\sin\alpha_1 e_2 = f_{12}e_2 \tag{5.13}$$

Thirdly, the changes of α_1 under u_3 is

$$\dot{\alpha}_{13} = \frac{1}{r_2}\sin\alpha_1 e_3 = f_{13}e_3 \tag{5.14}$$

Finally, from (**??**), (**??**) and (**??**), we can write

$$\begin{aligned}\dot{\alpha}_1 &= -(\frac{1}{r_3}\sin\alpha_2 + \frac{1}{r_2}\sin\alpha_3)e_1 + \frac{1}{r_3}\sin\alpha_1 e_2 \\ &\quad + \frac{1}{r_2}\sin\alpha_1 e_3 \\ &= -g_{11}e_1 + f_{12}e_2 + f_{13}e_3\end{aligned}$$

By a similar process, the changes of α_2 and α_3 due to u_1, u_2, u_3 can be written as follows

$$\begin{aligned}\dot{\alpha}_2 &= \frac{1}{r_3}\sin\alpha_2 e_1 - (\frac{1}{r_3}\sin\alpha_1 + \frac{1}{r_1}\sin\alpha_3)e_2 + \frac{1}{r_1}\sin\alpha_2 e_3 \\ &= f_{21}e_1 - g_{22}e_2 + f_{23}e_3\end{aligned}$$

$$\dot{\alpha}_3 = \frac{1}{r_2}\sin\alpha_3 e_1 + \frac{1}{r_1}\sin\alpha_3 e_2 - (\frac{1}{r_1}\sin\alpha_2 + \frac{1}{r_2}\sin\alpha_1)e_3$$
$$= f_{31}e_1 + f_{32}e_2 - g_{33}e_3$$

Let $\alpha = [\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array}]^T$ and $e = \left[\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}\right]^T$, then $\dot{\alpha} = \dot{e}$ and for $k \in \{1,2,3\}$,

$$\dot{e}_k = -g_{kk}e_k + f_{k(k+1)}e_{k+1} + f_{k(k-1)}e_{k-1}. \tag{5.15}$$

The angle error dynamics equations (**??**) can be combined as follows

$$\dot{e} = M(e)e \tag{5.16}$$

where

$$M(e) = \begin{bmatrix} -g_{11} & f_{12} & f_{13} \\ f_{21} & -g_{22} & f_{23} \\ f_{31} & f_{32} & -g_{33} \end{bmatrix}.$$

The nonlinear system (**??**) is defined in the manifold $\mathcal{M}_e = (\widehat{A}_1 - \alpha_1^*, \pi - \alpha_1^*] \times (\widehat{A}_2 - \alpha_2^*, \pi - \alpha_2^*] \times (\widehat{A}_3 - \alpha_3^*, \pi - \alpha_3^*]$. We can see that in $\mathcal{M}_e$, $g_{kk} \geq 0$, $f_{jk} \geq 0$ for $j, k \in \{1,2,3\}$ and $j \neq k$. Moreover, the column sums of M are zero

$$-g_{kk} + f_{(k+1)k} + f_{(k-1)k} = 0 \tag{5.17}$$

Let $\lambda(e) : \mathcal{M}_e \mapsto \mathbb{R} \cup \{+\infty\}$ be a convex, Lipschitz-continuous function with $\lambda > 0$ in $\mathcal{M}_e - \{0\}$. Suppose that $\lambda(e) = \Sigma_k \lambda_k(e_k)$ where each λ_k is a function of only e_k. Define the upper-right derivative [**?**], [**?**] of $\lambda(e)$ by

$$D^+\lambda(e) = [D^+\lambda_1(e_1), D^+\lambda_2(e_2), D^+\lambda_3(e_3)]^T$$

$$D^+\lambda_k(e_k) = \lim_{h\to 0^+}\sup\frac{\lambda_k(e_k+h) - \lambda_k(e_k)}{h}.$$

Note that the standard derivative coincides with the upper-right derivative at e_k when it exists. Now, we are at the point to state the main theorem.

Theorem 5.4 *Under the Assumptions* **??**–**??** *and the control law (***??***), the agent will asymptotically reach the desired position, or equivalently, the origin of the system (***??***) is asymptotically stable.*

Proof: Consider the Lyapunov candidate function $V(e) = \sum_{k=1}^{3} \lambda_k = \sum_{k=1}^{3} |e_k|$ which is positive definite in $\mathcal{M}_e$. Note that $\lambda_k = |e_k|$ is a convex, positive, Lipschitz continuous function in $\mathcal{M}_e - \{0\}$. Thus, λ_k is differentiable everywhere except at $e_k = 0$. Moreover, its upper-right derivative at $e_k = 0$ is $D^+\lambda_k(e_k) = 1$. Consider three cases:

- Case 1. ($e_k = 0$): $\dot{\lambda}_k = D^+\lambda_k(e_k)\dot{e}_k = \dot{e}_k$.

$$\begin{aligned}\dot{\lambda}_k = \dot{e}_k &= -g_{kk}e_k + f_{k(k+1)}e_{k+1} + f_{k(k-1)}e_{k-1} \\ &\leq -g_{kk}\lambda_k + f_{k(k+1)}\lambda_{k+1} + f_{k(k-1)}\lambda_{k-1}\end{aligned}$$

  where the inequality is an equality if and only if $e_{k+1} = 0, e_{k+1} = 0$.

- Case 2. ($e_k > 0$): $\lambda_k = e_k$ or $\dot{\lambda}_k = \dot{e}_k$.

$$\begin{aligned}\dot{\lambda}_k = \dot{e}_k &= -g_{kk}e_k + f_{k(k+1)}e_{k+1} + f_{k(k-1)}e_{k-1} \\ &\leq -g_{kk}\lambda_k + f_{k(k+1)}\lambda_{k+1} + f_{k(k-1)}\lambda_{k-1}\end{aligned}$$

  where the inequality is an equality if and only if $e_{k+1} \geq 0, e_{k+1} \geq 0$ which cannot happen because of (**??**).

- Case 3. ($e_k < 0$): $\lambda_k = -e_k$ or $\dot{\lambda}_k = -\dot{e}_k$.

$$\begin{aligned}\dot{\lambda}_k = \dot{e}_k &= g_{kk}e_k - f_{k(k+1)}e_{k+1} - f_{k(k-1)}e_{k-1} \\ &\leq -g_{kk}\lambda_k + f_{k(k+1)}\lambda_{k+1} + f_{k(k-1)}\lambda_{k-1}\end{aligned}$$

  where the inequality is an equality if and only if $e_{k+1} \leq 0, e_{k+1} \leq 0$ which also cannot happen because of (**??**).

Bringing the three cases together, for all $e \in \mathcal{M}_e$, the derivative of V can be written as

$$\dot{V} = \sum_k \dot{\lambda}_k \leq \sum_{k=1}^{3} (-g_{kk}\lambda_k + f_{k(k+1)}\lambda_{k+1} + f_{k(k-1)}\lambda_{k-1}) \tag{5.18a}$$

$$= \sum_{k=1}^{3} (-g_{kk} + f_{(k+1)k} + f_{(k-1)k})\lambda_k = 0, \tag{5.18b}$$

where, for the last equality, we have used (**??**). The equalities and inequalities in (**??**) are valid for all $e \in \mathcal{M}_e$ given $\dot{\lambda} = [D^+\lambda(e)]^T\dot{e}$. Moreover, the only possible case to have an equality is $e_k = 0$ for $k = 1, 2, 3$.

Since V is positive definite, $\dot{V}$ is negative semidefinite in $\mathcal{M}_e$ and $\dot{V} = 0$ if and only if $e = 0$, it follows from [**?**] that the origin $e = 0$ of (**??**) is asymptotically stable. ■

5.3 Extended navigation algorithm

5.3.1 Reaching the triangular region from outside

In this section, we study the Problem **??** when Assumption **??** is not available, i.e, the initial position of the agent is outside the triangular $A_1A_2A_3$. We propose the following bearing-only control law to move the agent into the triangular $A_1A_2A_3$:

$$\dot{p} = \hat{u}_1 + \hat{u}_2 + \hat{u}_3. \tag{5.19}$$

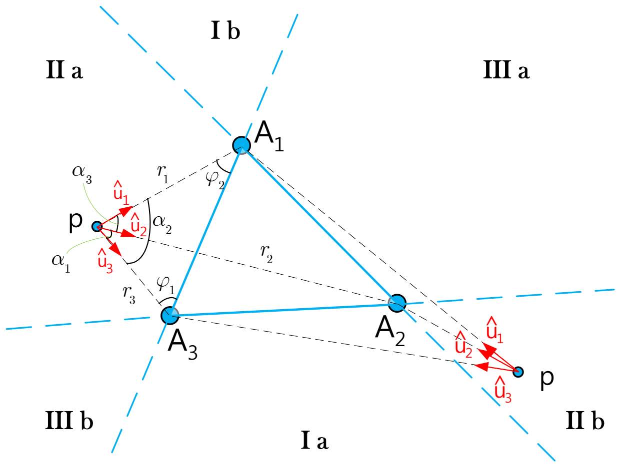

Figure 5.4. The control law when agent is outside the triangle$A_1A_2A_3$

Proposition 5.5 *Under the control law (**??**), the agent will move inside the triangle $A_1A_2A_3$.*

Proof: The proof consists of three steps:

- Step 1: Since the agent is outside the triangle, there exists an angle $\alpha_k, k \in \{1, 2, 3\}$ such that

$$\alpha_k = \alpha_{k-1} + \alpha_{k+1} \tag{5.20}$$

  Based on Eq. **??**, we divide the plane outside the triangle into 6 regions ($\mathbf{Ia}-\mathbf{IIIa}$, $\mathbf{Ib}-\mathbf{IIIb}$) as depicted in Fig. **??**. Note that the agent can check Eq. **??** to induce its current position. For instance, if $\alpha_2 = \alpha_1 + \alpha_3$, then the agent will be in region $\mathbf{IIa}$ or $\mathbf{IIb}$.

  Without loss of generality, we consider two cases: *(i)* the agent is inside region $\mathbf{IIa}$ and *(ii)* the agent is inside the region $\mathbf{IIb}$. The remaining regions can be treated similarly.

- Step 2: When the agent is inside region $\mathbf{IIa}$, we will prove that *(i)* the agent cannot escape to **Ib** or **IIIb** and *(ii)* inside **IIa**, the angle α_2 between $\hat{u}_1$ and $hatu_3$ is strictly increasing.

(i) When the agent is on the boundary of region **Ib** and **IIa** (the line A_2A1), the bearing vectors $\hat{u}_1$ and $\hat{u}_3$ lie along line A_1A2, while the bearing vector $\hat{u}_3$ directs into A_3. Thus, $\hat{u}_3$ lets the agent move into region **IIa**. When the agent is on the boundary of region **IIa** and **IIIb**, the proof is similar. Therefore, we can conclude that the agent will not move into **Ib** or **IIIb** from **IIa**.

(ii) Define the angles φ_1, φ_2 as in Fig. **??**. Consider the effect of each component $\hat{u}_1, \hat{u}_2, \hat{u}_3$ to α_2.

- Consider the change of α_2 under $\hat{u}_2$. Firstly, $\hat{u}_2$ affects φ_1 as follows: $\dot{\varphi}_{12} = -\frac{\sin\alpha_1}{r_3}$. Similarly, the change of φ_2 due to $\hat{u}_2$ is $\dot{\varphi}_{22} = -\frac{\sin\alpha_2}{r_1}$. Because $\alpha_2 + \varphi_1 + \varphi_2 = \pi$, we have $\dot{\alpha}_2 + \dot{\varphi}_1 + \dot{\varphi}_2 = 0$. Therefore, $\dot{\alpha}_{22} = \frac{\sin\alpha_1}{r_3} + \frac{\sin\alpha_2}{r_1}$.
- Similarly, the change of α_2 caused by $\hat{u}_1$ is $\dot{\alpha}_{21} = \frac{\sin\alpha_2}{r_3}$. And the change of α_2 caused by $\hat{u}_3$ is $\dot{\alpha}_{23} = \frac{\sin\alpha_2}{r_1}$.
- Finally, the total change of α_2 made by the control law is: $\dot{\alpha}_2 = \dot{\alpha}_{21} + \dot{\alpha}_{22} + \dot{\alpha}_{23} = \frac{\sin\alpha_1}{r_3} + \frac{\sin\alpha_2}{r_1} + (\frac{1}{r_1} + \frac{1}{r_3})\sin\alpha_2$. Because the agent is in the region **IIa**, the angles satisfy $0 < \alpha_2 \leq \pi$, $0 \leq \alpha_1 < \pi$. Consequently, $\dot{\alpha}_2 > 0$, or α_2 is strictly increasing whenever the agent is in the region **IIa**.

From *(i)* and *(ii)*, it follows that there must be a finite time t^* such that $\alpha_2 = 180^o$ or the agent is in A_1A_3.

- Step 3: When the agent is inside **IIb**, by similar arguments, the angle α_2 strictly increases. This implies that only two cases are possible: *(a)* the agent enters the triangle through beacon A_3, or *(b)* the agent enters region **Ia** or **IIa**. If *(a)* happens, then Lemma **??** is immediately proved. If *(b)* happens, then by using result from Step 2, we obtain the same result. Since the other cases can be treated similarly, we conclude that Lemma **??** is true.

■

Corollary 5.6 *If the agent is inside the triangular $A_1A_2A_3$, under the control law*

$$\dot{p} = -\hat{u}_1 + \hat{u}_2 + \hat{u}_3 \tag{5.21}$$

the agent moves outside the triangular through the segment A_2A_3.

The proof of Corollary **??** is similar to **??**. Symmetrically, the agent will get outside the triangular region under $\dot{p} = \hat{u}_1 - \hat{u}_2 + \hat{u}_3$ and $\dot{p} = \hat{u}_1 + \hat{u}_2 - \hat{u}_3$ through the segment A_1A_3 and A_1A_2 respectively.

Combining (**??**) and (**??**), we have a control law to reach almost points inside the triangular $A_1A_2A_3$, regardless of the agent's initial position.

$$\dot{p} = \begin{cases} \hat{u}_1 + \hat{u}_2 + \hat{u}_3, & \text{when the agent is outside triagular} A_1A_2A_3 \\ (\alpha_1 - \alpha_1^*)\hat{u}_1 + (\alpha_2 - \alpha_2^*)\hat{u}_2 + (\alpha_3 - \alpha_3^*)\hat{u}_3, & , \quad \text{otherwise.} \end{cases} \tag{5.22}$$

where $\hat{u}_i = \frac{p_i - p}{\|p_i - p\|}, i = 1, 2, 3.$

5.3.2 Reaching a point outside the triangular

In previous sections, we have developed a control law that can drive an agent to any desired position inside the triangular region $A_1A_2A_3$. In this section, our goal is to extend the control law to reach points outside this region. We have the following remarks:

Remark 5.7 *Using only pure bearing measurements, the agent cannot differentiate* $\mathbf{Ia}$ *from* $\mathbf{Ib}$*,* $\mathbf{IIa}$ *from* $\mathbf{IIb}$ *and* $\mathbf{IIIa}$ *from* $\mathbf{IIIb}$*.*

For example, when $\alpha_2 = \alpha_1 + \alpha_3$, the agent can induce that it is outside the triangular, in region $\mathbf{IIa}$ or $\mathbf{IIb}$. However, it does not know exactly whether its position is in $\mathbf{IIa}$ or $\mathbf{IIb}$.

Remark 5.8 *If the desired position is outside the triangular* $A_1A_2A_3$*, there may be two points satisfy* $\alpha_1^*, \alpha_2^*, \alpha_3^*$*. Moreover, the points are in different regions.*

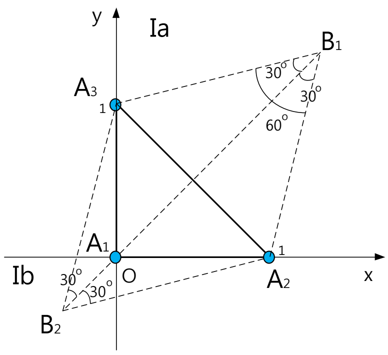

Figure 5.5. B_1 and B_2 satisfy $\alpha_1^* = 60^o, \alpha_2^* = 30^o, \alpha_3^* = 30^o$.

An example is given in Fig. **??**, the beacons are $A_1(0,0), A_2(1,0), A_3(0,1)$. We can see that both $B_1(\sqrt{2}\sin 75^o, \sqrt{2}\sin 75^o) \in \mathbf{Ia}$ and $B_2(-\sqrt{2}\sin 15^o, -\sqrt{2}\sin 15^o) \in \mathbf{Ib}$ satisfy $\alpha_1^* = 60^o, \alpha_2^* = 30^o, \alpha_3^* = 30^o$. From Remark **??**, we conclude that the agent cannot differentiate B_1 and B_2. This example shows that we cannot extend the control law to entire plane using only pure bearing measurements. Therefore, we will restrict the desired position to regions $\mathbf{Ia}, \mathbf{IIa}$ and $\mathbf{IIIa}$. Consider the following problem:

Problem 5.2 *If $p^* \in$ **Ia** (**Ib** or **Ic**) and initially the agent is inside the same region, design a control law to move the agent to the desired position.*

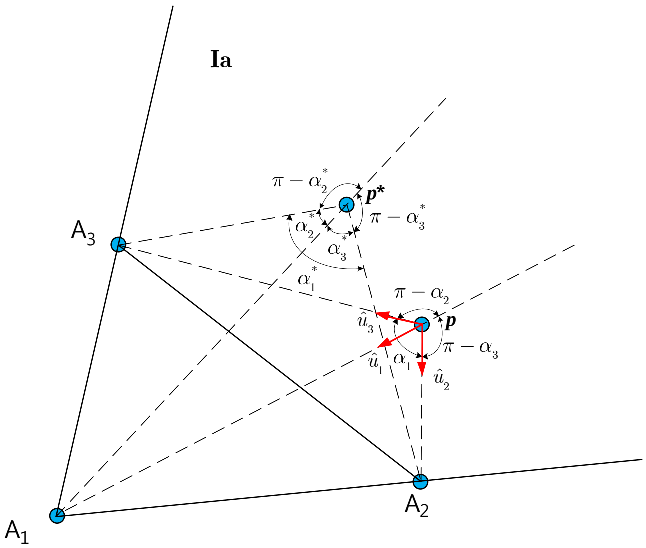

Figure 5.6. Illustration of Problem **??**: $p^* \in \mathbf{Ia}$.

Without loss of generality, consider $p^* \in \mathbf{Ia}$ and the initial position of the agent is in that region. The goal is equivalent to design a control law such that $\alpha_1 \to \alpha_1^*$, $(\pi - \alpha_2) \to (\pi - \alpha_2^*)$ and $(\pi - \alpha_3) \to (\pi - \alpha_3^*)$. Similar to Problem **??**, we propose the following control law:

$$u = (\alpha_1^* - \alpha_1)\hat{u}_1 + (\alpha_2^* - \alpha_2)\hat{u}_2 + (\alpha_3^* - \alpha_3)\hat{u}_3 \tag{5.23}$$

Theorem 5.9 *If $p^* \in$ **Ia** and initially the agent is inside the same region, under (**??**), the agent asymptotically reaches the desired position.*

Proof: The proof is similar to the proof of Theorem **??**. ■

All in all, the algorithm to reach almost desired location inside the triangular and regions $\mathbf{Ia} - \mathbf{IIIa}$ is given in the following flowchart

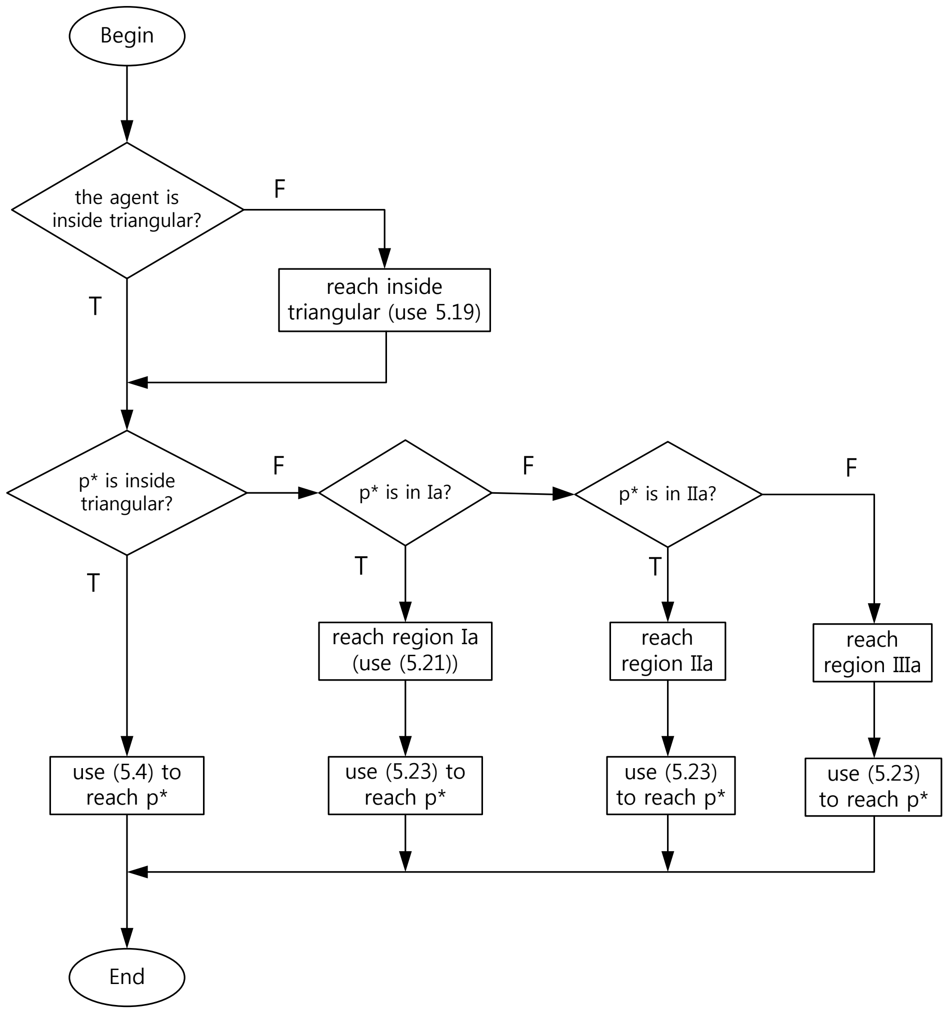

Figure 5.7. Algorithm to navigate all points inside triangular and regions $\mathbf{Ia} - \mathbf{IIIa}$.

5.4 Simulations

Simulation 1: Consider three beacons located at $A_1(-1;0)$, $A_2(4;0)$ and $A_3(0;5)$ and desired position are given by three desired angles $\alpha_1^* = \alpha_2^* = \alpha_3^* = 2\pi/3$. We simulate control law (**??**) for different initial positions of the agent inside the triangular.

The trajectories of the agent under the navigation law (**??**) corresponding to different initial positions are shown in Fig. **??**. Simulation results show that the agent will eventually

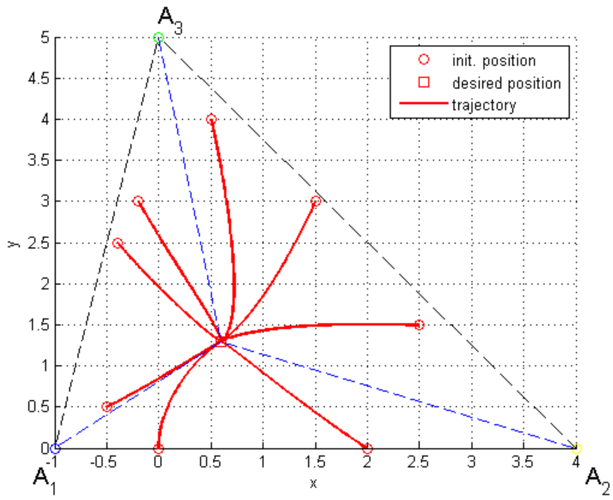

Figure 5.8. Trajectories of the agent with different initial positions under control law (**??**).

reach the desired destination. Consider the dynamics of the error angles e_k $(k = 1, 2, 3)$ when the agent's initial position is $(2.5; 1.5)$. As shown in Fig. **??**, $e_k \longrightarrow 0$ as $t \longrightarrow \infty$. This result agrees with our analysis.

Simulation 2 In this simulation, we keep the beacons' positions and desired position p^* unchanged. However, the initial position of the agent is outside the triangular region. The control law (**??**) is applied.

The simulation results are shown in Fig. **??**. The agent reaches desired position for different initial positions outside the triangular $A_1A_2A_3$.

Simulation 3

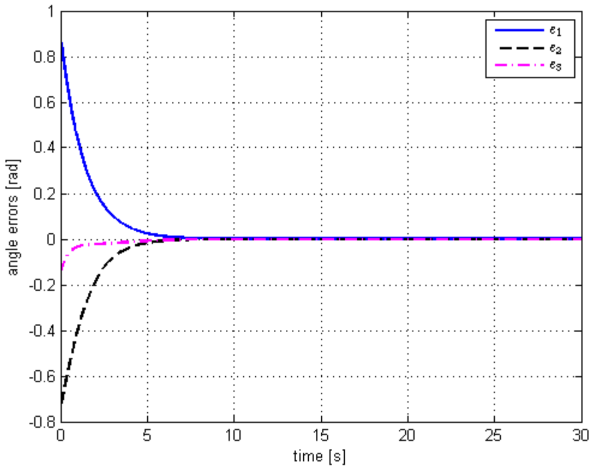

Figure 5.9. Angle errors when initial position of the agent is $(2.5; 1.5)$.

5.5 Hardware experiments

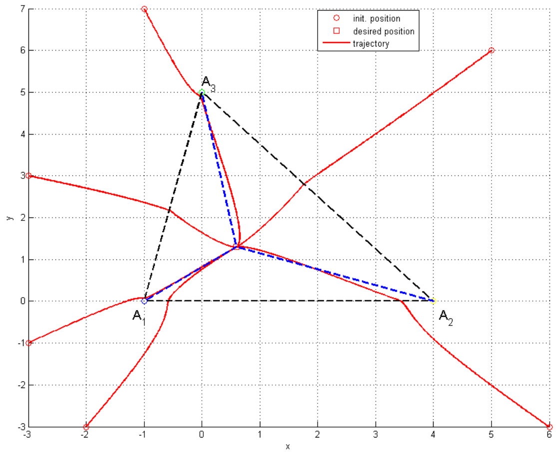

Figure 5.10. Trajectories of the agent with different initial positions under control law (**??**).

Chapter 6

The Fermat-Weber location problem in single integrator dynamics using only local bearing angles

6.1 Introduction

6.2 Convex Analysis

6.3 Algorithm to reach the solution when all beacons are stationary

6.4 Algorithm to reach the solution with moving beacons

6.5 Simulations

6.6 Remark

Chapter 7

Conclusion

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