Bearing-Based Formation Control of a Group of Agents With Leader-First Follower Structure

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Abstract—**This paper studies bearing-based formation control of a group of autonomous agents with the leader-first follower (LFF) structure in an arbitrary dimensional space. First, the bearing-based Henneberg construction and some properties of the LFF formation are introduced. Then, we propose and analyze bearing-only control laws that almost globally stabilize LFF formations to a desired formation. Further strategies to rotate and rescale the target formation are also discussed. Finally, simulation results are provided to support the analysis.**

Index Terms—**Bearing-only measurements, distributed control, Henneberg construction, leader-first follower, multi-agent systems.**

I. INTRODUCTION

FORMATION control is an ongoing research topic in the realm of multi-agent cooperative control [1], [2]. While distance-based formation control was extensively studied [3]–[7], bearing-based formation control has recently attracted much research interest due to the emergent technology of small

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UAVs equipped with vision sensors [8], [9]. In bearing-based formation control problems, a group of autonomous agents (mobile robots, UAVs) has to achieve a target formation specified by some bearing information (bearing vectors and/or subtended bearing angles) [10].

A focus of bearing-based formation control is designing decentralized control laws using only bearing information. Consider a small quadcopter, the relative bearing, which is the unit vector obtained from a relative position vector by normalizing its length, can be acquired from the onboard cameras, thanks to vision-based techniques [6], [11]. Since the camera is a passive sensor, in applications where exchanging signals is prohibited, bearing-only control is preferred [12]. Furthermore, the quadcopter system has a limited payload. To save the quadcopter's restricted payload, we can reduce the number of sensors in quadcopter systems by employing vision-based control laws [13].

Early works on bearing-based formation control focused on controlling the subtended bearing angle, which is invariant in each agent's local coordinate frame [14]–[18]. Another approach is based on bearing rigidity, in which the target formation is characterized by a set of desired bearing vectors, which are sufficient to specify the formation up to a scaling and a translation. In two-dimensional (2-D) space, the concept of bearing rigidity (or parallel rigidity) has been studied in [19], [20]. Based on parallel rigidity theory, the authors in [20] defined the bearing constrained rigidity matrix. Recently, the authors of [21] developed a theory of bearing rigidity and infinitesimal bearing rigidity in $\mathbb{R}^d$. A bearing-only stabilization control law for formations with undirected graphs in $\mathbb{R}^d$ has been proposed in [21]. Further applications of bearing rigidity theory in formation maneuvering and network localization have also been discussed in [22] and [23]. However, in these works, only undirected graphs were considered. That is, bearing-only control for directed graphs has been less investigated. Thus, differently from these existing works, we attempt to fill a gap in the literature on bearing-only formation control under directed graphs, based on the initial work [51]. Specifically, we focus on the *leader-first follower* (LFF) graphs that may be generated from a set of Henneberg extensions [31]–[33]. It is worth remarking that the analysis in the undirected case cannot be used in the directed case due to the asymmetry in the sensing graph [24]. The lack of symmetry raises difficulties in analysis, for example, the formation's centroid and scale are not invariant as in the undirected case.

There are several initial studies in bearing-based formation control of directed graphs [25], [27], [26], [24]. For instance, in

[25], by assuming the existence of three stationary beacons in the plane, it was proved that any n-agent system with an acyclic directed sensing graph is locally asymptotically stable. The local stability of planar formations with directed cycle graphs was studied in [24] and [26]. The authors in [27] introduced the bearing Laplacian from a set of desired bearing vectors and defined bearing persistence based on the null space of the bearing Laplacian. However, the proposed control law in [27] requires the relative positions between neighbors, which are not available from bearing measurements. The authors in [11], [28], [29] developed bearing-based rigidity theories in $SE(2)$, $\mathbb{R}^3 \times \mathcal{S}^1$, and $SE(3)$, in which the bearing vectors are defined in the body frame of each agent. Although a global reference frame is unnecessary in [11], [28], the proposed control law requires all neighbor agents to exchange their local information; thus, its applicability is limited.

The contributions of this work are as follows. First, we define the bearing-based Henneberg construction for LFF graphs and show some properties of the LFF graphs. Note that the bearing-based Henneberg construction, unlike the bearing rigidity theory given in [21], is a basic theoretical framework for bearing-based directed graphs. We extend the bearing-based Henneberg construction in [18] to generate all LFF graphs based on two graph operations, namely vertex addition and edge splitting. In practice, systems with LFF structure are easy to implement due to their cascade structure [30]. Moreover, the LFF formation is uniquely determined given the leader's position, the set of desired bearing vectors, and the formation scale. Second, we study the LFF formation [31]–[33] under the bearing-only control law in an arbitrary dimensional space. The analysis is based on the notion of almost global input-to-state stability of cascade systems [34], [35]. Third, we propose a modified bearing-only control law that guarantees the formation to escape from any undesired equilibrium, and globally asymptotically converge to the desired one. In practice, it may be unrealistic to assume the existence of a global reference frame. Even though all agents' local body frames are initially aligned, due to drift in inertial sensing, misalignment between local frames may still occur [36]. To address this issue, as the fourth contribution, a control strategy with orientation alignment is proposed. Under some assumptions, all local orientations are aligned with the leader's orientation; thus, the formation almost globally converges to the target formation under the proposed control strategy. Finally, we propose several extensions of the control law, including rotation and rescaling of the target formation. The ability to rotate and rescale the formation is an important feature for formation maneuvering [31].

The rest of this paper is organized as follows. In Section II, we introduce the bearing-based Henneberg construction and prove some properties of the LFF formation. In Sections III and IV, we analyze the LFF formation under the bearing-only control law for two cases: with and without a global reference frame. A bearing-only global stabilization control law is also studied in Section III. Strategies to rotate and rescale the target formation are discussed in Section V. Section VI provides numerical simulations to support the analysis. Finally, some concluding remarks and further research directions are reported in Section VII.

Notations: In this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. Bold font lower case letters denote vectors, while bold font upper case letters denote matrices. Let $\mathbf{x} = [x_1, \ldots, x_d]^\top$ denote a vector in $\mathbb{R}^d$. The orthogonal projection matrix of a nonzero vector $\mathbf{x}$ is defined as

$$\mathbf{P_x} := \mathbf{I}_d - \frac{\mathbf{x}}{\|\mathbf{x}\|} \frac{\mathbf{x}^\top}{\|\mathbf{x}\|} \in \mathbb{R}^{d \times d}. \tag{1}$$

Note that $\mathbf{P_x}$ is symmetric, positive semidefinite, and idempotent. Moreover, $\mathbf{P_x}$ has the nullspace $\mathcal{N}(\mathbf{P_x}) = \text{span}\{\mathbf{x}\}$, and the eigenvalue set $\{0, 1, \ldots, 1\}$ [21].

II. BEARING-BASED HENNEBERG CONSTRUCTION

A. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph with a vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$ of $|\mathcal{V}| = n$ vertices and an edge set $\mathcal{E} = \{e_{ij} = (v_i, v_j) | v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$ of $|\mathcal{E}| = m$ directed edges. A directed edge $e_{ij} = (v_i, v_j) \in \mathcal{E}$ is considered to be directed from v_i to v_j, and we refer to v_i and v_j as the start and the end vertex, respectively. If $e_{ij} \in \mathcal{E}$, we call vertex j a neighbor of vertex i and denote the neighbor set of vertex i by $\mathcal{N}_i := \{v_j \in \mathcal{V} | e_{ij} \in \mathcal{E}\}$. A directed path is a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots (v_{i_{k-1}}, v_{i_k})$ in $\mathcal{E}$. A directed cycle is a directed path having the same start and end node, i.e., $v_{i_1} \equiv v_{i_k}$. A graph $\mathcal{G}$ is called an *acyclic directed graph* if $\mathcal{G}$ has no directed cycle. If there exists a vertex $v_i \in \mathcal{V}$ such that for any vertex $v_j \neq v_i, v_j \in \mathcal{V}$, we can find at least one directed path connecting v_j to v_i, then $\mathcal{G}$ is called a rooted in-branching graph with a root vertex v_i. For an acyclic directed graph, we define *the parent set* of a vertex v_i as $\mathcal{P}_i = \{v_k | \exists \text{ a directed path from } v_k \text{ to } v_i\}$ [37].

For each vertex $v_i \in \mathcal{V}$, we associate each v_i with a point $\mathbf{p}_i \in \mathbb{R}^d$ in a global reference frame. Then, the stacked vector $\mathbf{p} = [\mathbf{p}_1^\top, \ldots, \mathbf{p}_n^\top]^\top \in \mathbb{R}^{dn}$ is referred to as a configuration of $\mathcal{G}$. The directed graph $\mathcal{G}$ and the configuration $\mathbf{p}$ together define a framework $\mathcal{G}(\mathbf{p})$ in the d-dimensional space [1].

Define $\mathbf{z}_{ij} := \mathbf{p}_j - \mathbf{p}_i$ as the displacement vector between $\mathbf{p}_i$ and $\mathbf{p}_j$. The distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ is $d_{ij} = \|\mathbf{z}_{ij}\|$. The relative bearing vector $\mathbf{g}_{ij} \in \mathbb{R}^d$ between two noncollocated points $\mathbf{p}_i$ and $\mathbf{p}_j$ is defined as the unit vector pointing from $\mathbf{p}_i$ to $\mathbf{p}_j$. In other words, $\mathbf{g}_{ij}$ is the vector obtained from $\mathbf{z}_{ij}$ by normalizing its length,

$$\mathbf{g}_{ij} := \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|} = \frac{\mathbf{z}_{ij}}{\|\mathbf{z}_{ij}\|}. \tag{2}$$

Consider the task of controlling a group of n autonomous agents in a d-dimensional space to take up a formation shape that is bearing congruent to a prescribed configuration $\mathbf{p}^* \in \mathbb{R}^{dn}$. Here, bearing congruency means that the formation and the target formation differ only by a translation and a dilation [21]. Let $\Gamma := \{\mathbf{g}_{ij}^* | i, j = 1, \ldots, n, i \neq j\}$ be the set of all bearing vectors in the target configuration $\mathbf{p}^*$. Supposing that all agents have access to a global reference frame, in order to guarantee bearing congruence between the formation with the configuration $\mathbf{p}^*$, it is unnecessary to control all bearing vectors. In fact, based on bearing rigidity theory [21], when a certain subset of
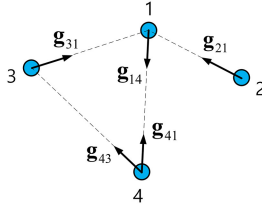
Fig. 1. An example of bearing constraint assignment: Agents 2 and 3 control their bearings toward agent 1, agents 1 and 4 both control the bearing between them, and agent 4 controls two bearings with regard to agents 1 and 3.
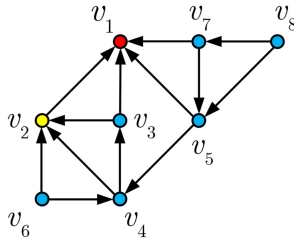

Fig. 2. An LFF graph of eight vertices: vertex 1 (the leader) has no neighbor, vertex 2 (the first follower) has one neighbor, and each vertex i ($i = 3, \ldots, 8$) has two neighbors.
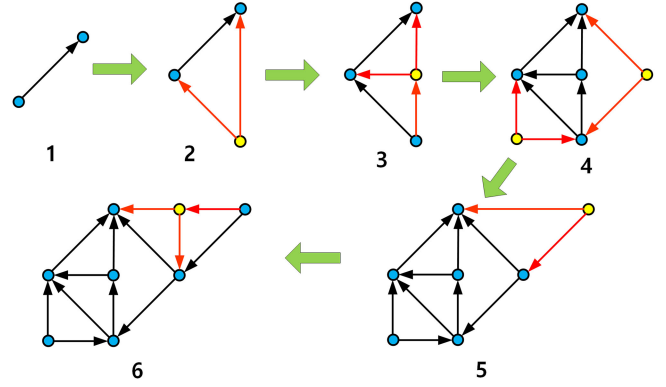

Fig. 3. An example of a Henneberg construction. In each step, the added vertex and added edges are in yellow and red, respectively. Vertex addition is used in steps 2, 4, and 5 while edge splitting is used in steps 3 and 6.

the desired bearing vectors in Γ is achieved, the target formation shape will be attained.

Therefore, the formation control task is distributed to every agent in the group, and each agent must only maintain one or more local bearing vectors with regard to other agents in the system. The directed graph $\mathcal{G}$ is used to describe this task assignment. We use a slight abuse of terminology here to refer to $\mathcal{G}(\mathbf{p})$ also as a formation. A directed edge e_{ij} in $\mathcal{E}$ is understood to imply that the task of controlling $\mathbf{g}_{ij}$ is assigned to agent i while a double-edge e_{ij} and e_{ji} means that both agents i and j are assigned to control $\mathbf{g}_{ij}$ and $\mathbf{g}_{ji}$, respectively. An example of task allocation on a group of four agents is illustrated in Fig. 1.

Besides achieving the group's task, in formation control, it is desirable that the control scheme has a scalability property and should be cost effective. A possible design strategy is minimizing the number of bearing vectors that have to be controlled. Let all agents have access to a common global reference frame, then it holds $\mathbf{g}_{ij} = -\mathbf{g}_{ji}$. The role of controlling a bearing vector between two agents i and j can be assigned to only one of the two agents, for example, to agent i, and then agent j moves without awareness of this task. The rest of this paper will focus on a task distribution strategy in a special structure termed "leader-first follower" or "two-leader formation" [18], [31].

B. Bearing-Based Henneberg Construction

The underlying graph of an LFF formation is constructed from a bearing-based Henneberg construction. For example, an LFF graph of eight vertices is given in Fig. 2. The Henneberg construction starts from a directed edge followed by a sequence of operations namely *vertex addition* and *edge splitting* and is defined as follows:

Definition 1 (Henneberg construction): Start from a pair of vertices v_1 and v_2 and a directed edge (v_2, v_1) joining them. Define the *degree of cascade* of a vertex as the length of its longest directed path from this vertex to the vertex v_1. Then, vertex 1 has degree 0, vertex 2 has degree 1, and we denote $\mathrm{doc}(v_1) = 0$, $\mathrm{doc}(v_2) = 1$. In each step, we perform one of the following two operations:

1) *Vertex Addition:* Add a new vertex v_i to the graph, together with two directed edges to two existing vertices v_j, v_k in the graph. The degree of cascade of the new vertex is defined by $\mathrm{doc}(v_i) = \max[\mathrm{doc}(v_j), \mathrm{doc}(v_k)] + 1$.
2) *Edge Splitting:* Consider a vertex v_i having precisely two neighbors v_j and v_m in the graph. Remove an edge (v_i, v_j) from the graph and add a new vertex v_k together with three directed edges (v_i, v_k), (v_k, v_j), and (v_k, v_l), where $\mathrm{doc}(v_l) \leq \mathrm{doc}(v_i)$. Then, update the degrees of cascade of v_k and all its parent vertices, $\mathcal{P}_i$, in the new graph: $\mathrm{doc}(v_k) = \max[\mathrm{doc}(v_j), \mathrm{doc}(v_l)] + 1$, $\mathrm{doc}(v_i) = \max[\mathrm{doc}(v_k), \mathrm{doc}(v_m)] + 1, \ldots$.

Fig. 3 depicts an example on constructing the eight-vertex graph in Fig. 2. Any graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of n vertices obtained from a Henneberg construction is acyclic and rooted in-branching. Furthermore, $\mathcal{G}$ has exactly $2n - 3$ directed edges [1], [38], and except for vertex 1 and vertex 2, each vertex in $\mathcal{G}$ has precisely two neighbors. It is not difficult to see that in each step the degree of the new vertex is two, and the degree of existing vertices in the graph before and after the step is unaltered. An induction argument then shows that all vertices other than v_1 and v_2 have degree two.

Let each vertex in $\mathcal{V}$ represent an agent in the group and each edge in $\mathcal{E}$ represent a bearing vector assignment. There is an agent 1 (the leader) with no neighbor. Agent 2 is the first follower, which is supposed to control one bearing vector to the leader. Agent 3 (the second follower) has to control exactly two bearing vectors to the leader and the first follower. Similarly, each agent i ($3 \leq i \leq n$) (a follower) has to control two bearing vectors to two agents $j, k \in \{1, \ldots, i-1\}$. The Henneberg construction together with a bearing assignment is referred to as a *bearing-based Henneberg construction*.
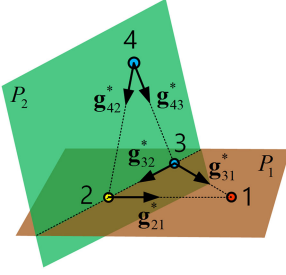
Fig. 4. Since agents 1, 2, and 3 are in the same plane $\mathcal{P}_1$, the two desired bearing vectors $\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$ of agent 3 must be in the same plane with $\mathbf{g}_{21}^*$. The position $\mathbf{p}_3^*$ in $\mathbb{R}^d$ can be uniquely determined from $\mathbf{p}_1^*$, $\mathbf{p}_2^*$ and $\mathbf{g}_{31}^*$, $\mathbf{g}_{32}^*$.

Consider an n-agent formation in $\mathbb{R}^d$, we examine the degrees of freedom specifying the formation shape. With n agents, there are dn coordinates, from which $d + 1$ (d accounting for position of the centroid and 1 for the scale) should be subtracted. Thus, $dn - d - 1$ scalar values specify the formation shape.

Now, consider an LFF formation obtained from a bearing-based Henneberg construction. Consider agent 2 which is assigned only one unit bearing vector $\mathbf{g}_{21}^* \in \mathbb{R}^d$. Note that any vector in $\mathbb{R}^d$ contains d pieces of data. Since one constraint $\|\mathbf{g}_{21}^*\| = 1$ was used, there are $d - 1$ independent pieces of bearing data[1] in $\mathbf{g}_{21}^*$. Next, consider agent 3, which is assigned two bearing vectors $\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$ as depicted in Fig. 4. The two vectors $\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$ cannot be chosen arbitrarily. If we choose $\mathbf{g}_{31}^*$ first, since $\mathbf{g}_{31}^* \neq \pm\mathbf{g}_{21}^*$ and $\|\mathbf{g}_{31}^*\| = 1$, we have $d - 1$ independent pieces of bearing data in $\mathbf{g}_{31}^*$. Now, because the positions of three agents 1, 2, and 3 define a plane, it follows that $\mathbf{g}_{31}^*$ and $\mathbf{g}_{21}^*$ define the same plane, call it, $\mathcal{P}_1$. Next, we choose $\mathbf{g}_{32}^*$. Besides the constraints $\|\mathbf{g}_{32}^*\| = 1$, $\mathbf{g}_{32}^* \neq \pm\mathbf{g}_{31}^*$, and $\mathbf{g}_{32}^* \neq \pm\mathbf{g}_{21}^*$, $\mathbf{g}_{32}^*$ must be additionally restricted to the plane $\mathcal{P}_1$, which is called *coplanarity restriction*. By choosing a direction in a plane, we have only one degree of freedom. This implies that only one piece of bearing data in $\mathbf{g}_{32}^*$ can be freely chosen. Thus, there are total d [i.e., $(d - 1) + 1$] independent pieces of bearing data chosen by agent 3. From a similar argument, for each agent $i > 3$ with two neighbors $1 \leq j < k < i$, there are d independent pieces of bearing data from $\mathbf{g}_{ij}^*$ and $\mathbf{g}_{ik}^*$. Hence, for the overall LFF formation with $2n - 3$ bearing vectors, there are exactly $(d - 1) + d(n - 2) = dn - d - 1$ pieces of independent bearing data that can be chosen.

Remark 1: For planar LFF formations ($d = 2$), each bearing vector contains exactly one independent bearing data. Thus, the number of independent bearing data specifying the formation ($dn - d - 1 = 2n - 3$) matches the number of edges in the graph ($m = 2n - 3$). Hence, $2n - 3$ is the minimal number of bearing vectors to specify a formation in the plane. This observation is consistent with [18].

[1]When we measure a relative bearing vector in $\mathbb{R}^d$, we obtain $d - 1$ scalar pieces of information, which we term bearing data. As far as a single measurement is concerned, they are all independent, in the sense that no relation is implied among them. When a collection of such measurements is obtained for a number of agents in a general formation, relations may exist, and then the data would not be independent.

For LFF formations in $\mathbb{R}^d$, there are totally $2n - 3$ bearing vectors that specify the formation. For $d \geq 3$, the $2n - 3$ bearing vectors give rise to more pieces of data than degrees of freedom due to the coplanarity restriction, i.e., there is redundant data in the collection of bearing vectors. Determining whether $2n - 3$ is also the minimal number of bearing vectors to specify a formation with general directed graph in $\mathbb{R}^d, d \geq 3$ is beyond the scope of this paper. We refer readers to [39] for a further discussion on this topic. Henceforth, we shall assume that all specifications of bearings are consistent with the coplanarity restriction described before.

C. Properties of LFF Formations

This section studies some properties of LFF formations generated from a bearing-based Henneberg construction.

Lemma 1 (Uniqueness of the target formation): Consider an LFF formation with the position of the leader $\mathbf{p}_1^*$, the distance $d_{21}^* = \|\mathbf{p}_2^* - \mathbf{p}_1^*\|$ and the bearings $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}}$. If each agent i ($i \geq 3$) has two neighbors $1 \leq j \neq k < i$ with $\mathbf{g}_{ij}^* \neq \mathbf{g}_{ik}^*$, the location $\mathbf{p}_i^*$ is uniquely determined from its neighbors' positions and the desired bearing vectors. More specifically, $\mathbf{p}_i^*$ is calculated iteratively by

$$\mathbf{p}_i^* = \left(\sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}^*}\right)^{-1} \left(\sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}^*}\mathbf{p}_j^*\right). \tag{3}$$

Proof: For agent 2, since $\mathbf{g}_{21}^* = (\mathbf{p}_1^* - \mathbf{p}_2^*)/d_{21}^*$, we have $\mathbf{p}_2^* = \mathbf{p}_1^* - d_{21}^*\mathbf{g}_{21}^*$.

Consider agent 3. The position $\mathbf{p}_3^*$ of agent 3 satisfies two bearing vectors $\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$ as depicted in Fig. 4. Thus,

$$\mathbf{P}_{\mathbf{g}_{31}^*}(\mathbf{p}_1^* - \mathbf{p}_3^*) = \mathbf{0} \text{ and } \mathbf{P}_{\mathbf{g}_{32}^*}(\mathbf{p}_2^* - \mathbf{p}_3^*) = \mathbf{0}. \tag{4}$$

From (4), it follows that

$$(\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})\mathbf{p}_3^* = \mathbf{P}_{\mathbf{g}_{31}^*}\mathbf{p}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*}\mathbf{p}_2^*. \tag{5}$$

Consider the matrix $(\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})$. We have $\mathcal{N}(\mathbf{P}_{\mathbf{g}_{31}^*}) = \text{span}(\mathbf{g}_{31}^*)$ and $\mathcal{N}(\mathbf{P}_{\mathbf{g}_{32}^*}) = \text{span}(\mathbf{g}_{32}^*)$. Since $\mathbf{g}_{31}^* \neq \pm\mathbf{g}_{32}^*$, and recalling that $\mathbf{P}_{\mathbf{g}_{31}^*}$ and $\mathbf{P}_{\mathbf{g}_{32}^*}$ are positive semidefinite matrices, the nullspaces of $\mathbf{P}_{\mathbf{g}_{31}^*}$ and $\mathbf{P}_{\mathbf{g}_{32}^*}$ intersect at only $\{\mathbf{0}\}$. As a result, $(\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})$ is invertible, and $\mathbf{p}_3^*$ can be calculated from (5) as

$$\mathbf{p}_3^* = \left(\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*}\right)^{-1} \left(\mathbf{P}_{\mathbf{g}_{31}^*}\mathbf{p}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*}\mathbf{p}_2^*\right) \tag{6}$$

which can be written in a compact form as (3). For $i = 4, \ldots, n$, the position can be calculated in a similar way. ∎

Lemma 2 (Translation of the target formation): For an LFF formation, given d_{21}^* and $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}}$, the translation of the leader's position determines the translation of the entire formation.

Proof: We only need to prove that if $\mathbf{p}_1^*$ is changed to $\mathbf{q}_1^* = \mathbf{p}_1^* + \boldsymbol{\delta}$, then $\mathbf{p}_i^*$ for all i will be changed to $\mathbf{q}_i^* = \mathbf{p}_i^* + \boldsymbol{\delta}$. For agent 2, it is obvious that $\mathbf{q}_2^* = \mathbf{q}_1^* - d_{21}^*\mathbf{g}_{21}^* = \mathbf{p}_1^* + \boldsymbol{\delta} -$

$d_{21}^* \mathbf{g}_{21}^* = \mathbf{p}_2^* + \boldsymbol{\delta}$. For agent 3, we have

$$
\begin{aligned}
\mathbf{q}_3^* &= \left(\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*}\right)^{-1} \left(\mathbf{P}_{\mathbf{g}_{31}^*} \mathbf{q}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*} \mathbf{q}_2^*\right) \\
&= (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1} (\mathbf{P}_{\mathbf{g}_{31}^*} \mathbf{p}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*} \mathbf{p}_2^* \\
&\quad + (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})\boldsymbol{\delta}) \\
&= \mathbf{p}_3^* + \boldsymbol{\delta}.
\end{aligned}
$$

For agent i $(i = 4, \ldots, n)$, the proof is similar. ∎

Although the main goal is achieving a formation shape defined by some desired bearing vectors, it is important to have a measure to compare the size between two LFF formations. To this end, we have the following definition.

Definition 2 (Formation scale): Consider an LFF formation $\mathcal{G}(\mathbf{p})$, the *formation scale* is defined as the average of all the interagent distances defined by the edge set, $\mathcal{E}$,

$$
\zeta(\mathcal{G}(\mathbf{p})) := \frac{1}{|\mathcal{E}|} \sum_{(i,j) \in \mathcal{E}} \|\mathbf{p}_i - \mathbf{p}_j\| = \frac{1}{|\mathcal{E}|} \sum_{(i,j) \in \mathcal{E}} d_{ij}.
$$

Lemma 3 (Scale of the target formation): For an LFF formation, if d_{21}^* is scaled by α, all interagent distances will be scaled by α, i.e., the formation scale is determined by d_{21}^*.

Proof: Suppose that $\mathbf{p}_2^* - \mathbf{p}_1^*$ is changed to $\alpha(\mathbf{p}_2^* - \mathbf{p}_1^*)$ for $\alpha \neq 0$, then for any $(i,j) \in \mathcal{E}$, $\mathbf{p}_i^* - \mathbf{p}_j^*$ will be changed to $\alpha(\mathbf{p}_i^* - \mathbf{p}_j^*)$. We only consider $n = 3$ without loss of generality. Since $\mathbf{p}_3^* - \mathbf{p}_1^* = (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1}(\mathbf{P}_{\mathbf{g}_{31}^*} \mathbf{p}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*} \mathbf{p}_2^*) - \mathbf{p}_1^* = (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1}(\mathbf{P}_{\mathbf{g}_{31}^*} \mathbf{p}_1^* + \mathbf{P}_{\mathbf{g}_{32}^*} \mathbf{p}_2^* - (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*}) \mathbf{p}_1^*) = (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1} \mathbf{P}_{\mathbf{g}_{32}^*} (\mathbf{p}_2^* - \mathbf{p}_1^*)$, which shows that $\mathbf{p}_3^* - \mathbf{p}_1^*$ is a linear mapping of $\mathbf{p}_2^* - \mathbf{p}_1^*$. Thus, when $\mathbf{p}_2^* - \mathbf{p}_1^*$ is changed to $\alpha(\mathbf{p}_2^* - \mathbf{p}_1^*)$, $\mathbf{p}_3^* - \mathbf{p}_1^*$ will be changed to $\alpha(\mathbf{p}_3^* - \mathbf{p}_1^*)$. ∎

III. BEARING-ONLY CONTROL OF LFF FORMATIONS

A. Problem Formulation

Consider a group of n agents modeled by a single integrator model

$$
\dot{\mathbf{p}}_i = \mathbf{u}_i, \quad i = 1, \ldots, n. \tag{7}
$$

where $\mathbf{p}_i \in \mathbb{R}^d$ and $\mathbf{u}_i \in \mathbb{R}^d$ are the position and the control input of agent i at time instance t, respectively. All agents in the group have access to a common global reference frame and each agent can sense the relative bearing vectors to its neighbor agents. We assume that the n-agent system satisfies the following assumptions.

Assumption 1: The sensing graph of the group is characterized by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ generated from a Henneberg construction. Each agent can measure the bearing vectors with regard to its neighbor agents.

Assumption 2: The information of a desired formation is given as a set of feasible desired bearing constraints $\mathcal{B} = \{\mathbf{g}_{ij}^* \in \mathbb{R}^d \,|\, e_{ij} \in \mathcal{E}\}$. The feasibility conditions are i) there exists a configuration $\bar{\mathbf{p}} \in \mathbb{R}^{dn}$ such that $\mathbf{g}_{ij}^* = \frac{\bar{\mathbf{p}}_j - \bar{\mathbf{p}}_i}{\|\bar{\mathbf{p}}_j - \bar{\mathbf{p}}_i\|}, \forall \mathbf{g}_{ij}^* \in \mathcal{B}$; ii) Agent i's $(3 \leq i \leq n)$ desired position is not collinear with its two neighbor agents j, k $(1 \leq j \neq k < i)$, i.e., $\mathbf{g}_{ij}^* \neq \pm \mathbf{g}_{ik}^*$.
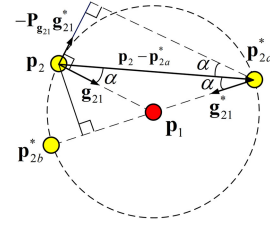

Fig. 5.   Agent 2 adopts the control law (9). There are two isolated equilibria $\mathbf{p}_{2a}^*$ and $\mathbf{p}_{2b}^*$ corresponding to $\mathbf{g}_{21} = \mathbf{g}_{21}^*$ and $\mathbf{g}_{21} = -\mathbf{g}_{21}^*$, respectively.

Note that Assumption 2 implies that the desired bearing vectors have been chosen to guarantee the coplanarity condition as discussed in Section II-B.

Assumption 3: Initially, the positions of the agents are not collocated, i.e., $\mathbf{p}_i(0) \neq \mathbf{p}_j(0), \forall 1 \leq i \neq j \leq n$.

This section aims to solve the following problem.

Problem 1: Under the Assumptions 1–3, design control laws for the agents using only local bearing information such that all desired bearing vectors in $\mathcal{B}$ are asymptotically achieved as $t \to \infty$.

B. Almost Global Stabilization of LFF Formations

The following bearing-only control law is proposed for each agent i $(i = 1, \ldots, n)$:

$$
\dot{\mathbf{p}}_i = \mathbf{u}_i = -\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}} \mathbf{g}_{ij}^*. \tag{8}
$$

We will prove that the control law (8) almost globally stabilizes the n-agent system to the target formation, satisfying all bearing vectors in $\mathcal{B}$. Note that almost global stability is understood in the sense that every trajectory starting in $\mathbb{R}^{nd} \setminus \mathcal{A}$ asymptotically converges to the target formation, where $\mathcal{A}$ is a set of measure zero in $\mathbb{R}^{nd}$ [34], [35]. The analysis starts from the leader and the first follower to other followers. Due to the cascade structure of LFF formations, mathematical induction will be invoked to establish almost global stability of the n-agent LFF formation.

1) The Leader and the First Follower: Since the leader (agent 1) has no neighbor, from (8), $\dot{\mathbf{p}}_1 = \mathbf{u}_1 = \mathbf{0}$ and the leader's position is fixed at $\mathbf{p}_1 = \mathbf{p}_1^*$ for all $t \geq 0$.

The first follower (agent 2) can measure one bearing vector $\mathbf{g}_{21}$ and has to asymptotically reach to $\mathbf{p}_{2a}^* = \mathbf{p}_1^* - d_{21} \mathbf{g}_{21}^*$ corresponding to $\mathbf{g}_{21} = \mathbf{g}_{21}^*$ (see Fig. 5). The control law for agent 2 is proposed as

$$
\dot{\mathbf{p}}_2 = \mathbf{u}_2 = -\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*. \tag{9}
$$

We have the following lemma on the equilibria of the first follower.

Lemma 4: Under Assumptions 1–3 and control law (9): i) d_{21} is invariant; ii) there are two equilibria of (9): $\mathbf{p}_{2a}^* = \mathbf{p}_1^* - d_{21} \mathbf{g}_{21}^*$, where $\mathbf{g}_{21} = \mathbf{g}_{21}^*$, and $\mathbf{p}_{2b}^* = \mathbf{p}_1^* + d_{21} \mathbf{g}_{21}^*$, where $\mathbf{g}_{21} = -\mathbf{g}_{21}^*$. The equilibrium $\mathbf{p}_{2a}^*$ is almost globally exponentially stable, whereas the equilibrium $\mathbf{p}_{2b}^*$ is unstable.

Proof: i) We have

$$\frac{d}{dt}d_{21}^2 = \frac{d}{dt}(\mathbf{z}_{21}^\top \mathbf{z}_{21}) = 2\mathbf{z}_{21}^\top \dot{\mathbf{z}}_{21}$$

$$= 2\mathbf{z}_{21}^\top (\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_2) = 2\mathbf{z}_{21}^\top \mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^* = 0 \qquad (10)$$

where the last equality follows from $\mathbf{z}_{21} = d_{21}\mathbf{g}_{21}$ and $\mathbf{g}_{21}^\top \mathbf{P}_{\mathbf{g}_{21}} = (\mathbf{P}_{\mathbf{g}_{21}}\mathbf{g}_{21})^\top = \mathbf{0}^\top$. Consequently, d_{21} is invariant under the control law (9).

(ii) It follows from (9) and the property of the projection matrix that $\dot{\mathbf{p}}_2 = \mathbf{0}$ if and only if $\mathbf{g}_{21} = \mathbf{g}_{21}^*$ or $\mathbf{g}_{21} = -\mathbf{g}_{21}^*$. Since d_{21} is invariant, in $\mathbb{R}^d$ there are only two equilibrium points: $\mathbf{p}_{2a}^*$ corresponding to $\mathbf{g}_{21} = \mathbf{g}_{21}^*$ and $\mathbf{p}_{2b}^*$ corresponding to $\mathbf{g}_{21} = -\mathbf{g}_{21}^*$, as depicted in Fig. 5.

Consider the Lyapunov function $V_b = \frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_{2b}^*\|^2$, which is continuously differentiable everywhere since $d_{21} = d_{21}(0) \neq 0$ for all $t \geq 0$. Moreover, V_b is positive definite and $V_b = 0$ if and only if $\mathbf{p}_2 = \mathbf{p}_{2b}^*$. The derivative of V_b along a trajectory of system (9) is

$$\dot{V}_b = (\mathbf{p}_2 - \mathbf{p}_{2b}^*)^\top \dot{\mathbf{p}}_2 = -(\mathbf{p}_2 - \mathbf{p}_{2b}^*)^\top \mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*$$

$$= (\mathbf{p}_2 - \mathbf{p}_{2b}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}}(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_2 - \mathbf{p}_{2b}^*)$$

$$= (\mathbf{p}_2 - \mathbf{p}_{2b}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}}(\mathbf{p}_2 - \mathbf{p}_{2b}^*) \geq 0 \qquad (11)$$

since $\mathbf{P}_{\mathbf{g}_{21}}\mathbf{z}_{21} = \mathbf{0}$ and $\mathbf{P}_{\mathbf{g}_{21}}$ is positive semidefinite. Therefore, $\mathbf{p}_2 = \mathbf{p}_{2b}^*$ is unstable by Chetaev's instability theorem [40, Th. 4.3].

Similarly, consider the Lyapunov function $V_a = \frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|^2$, which is continuously differentiable, radially unbounded. Moreover, V_a is positive definite, $V_a = 0$ if and only if $\mathbf{p}_2 = \mathbf{p}_{2a}^*$. Along a trajectory of system (9),

$$\dot{V}_a = (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \dot{\mathbf{p}}_2 = -(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*$$

$$= -(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}}(\mathbf{p}_2 - \mathbf{p}_{2a}^*) \leq 0. \qquad (12)$$

Note $\dot{V}_a = 0$ if and only if $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ or $\mathbf{p}_2 = \mathbf{p}_{2b}^*$ (see Fig. 5). Since $\mathbf{p}_{2b}^*$ is unstable, $\mathbf{p}_{2a}^*$ is almost globally asymptotically stable due to LaSalle's invariance principle.

Moreover, consider $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$, we can write

$$\dot{V}_a = -\frac{2\sin^2 \alpha}{d_{21}}V_a \leq -\frac{2\sin^2 \alpha(0)}{d_{21}}V_a = -\kappa V_a$$

where α is the angle as depicted in Fig. 5, $\alpha(0) \in (0, \frac{\pi}{2}]$ for $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$, and $\kappa = 2d_{21}^{-1}\sin^2 \alpha(0) > 0$. It follows that $\mathbf{p}_2 \to \mathbf{p}_{2a}^*$ exponentially fast if $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$. ∎

2) The Second Follower: We will analyze the dynamics of agent 3 (the second follower), whose neighbors are agents 1 and 2. The other agent's dynamics can be treated later in a similar way. The dynamics of agent 3 is

$$\dot{\mathbf{p}}_3 = \mathbf{u}_3(\mathbf{p}_3, \mathbf{p}_2) = -\mathbf{P}_{\mathbf{g}_{31}}\mathbf{g}_{31}^* - \mathbf{P}_{\mathbf{g}_{32}}\mathbf{g}_{32}^*. \qquad (13)$$

We consider (13) as a cascade system with $\mathbf{p}_2$ being an input to the unforced system

$$\dot{\mathbf{p}}_3 = \mathbf{u}_3(\mathbf{p}_3, \mathbf{p}_{2a}^*) = -\mathbf{P}_{\mathbf{g}_{31}}\mathbf{g}_{31}^* - \mathbf{P}_{\mathbf{g}_{32}}\mathbf{g}_{32}^*. \qquad (14)$$

The unforced system (14) characterizes the motion of agent 3 when agent 2 is located at its desired position $\mathbf{p}_{2a}^*$. However, if agent 2 is initially located at the undesired equilibrium $\mathbf{p}_2(0) = \mathbf{p}_{2b}^*$, then $\dot{\mathbf{p}}_2(t) = \mathbf{0}$ and the dynamics of agent 3 changes to

$$\dot{\mathbf{p}}_3 = \mathbf{u}_3(\mathbf{p}_3, \mathbf{p}_{2b}^*). \qquad (15)$$

The following lemma characterizes the equilibrium set of (14) and (15).

Lemma 5: i) The system (14) has a unique equilibrium point $\mathbf{p}_{3a}^* = (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1}(\mathbf{P}_{\mathbf{g}_{31}^*}\mathbf{p}_1 + \mathbf{P}_{\mathbf{g}_{32}^*}\mathbf{p}_{2a}^*)$ corresponding to $\mathbf{g}_{31} = \mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = \mathbf{g}_{32}^*$. ii) The system (15) has a unique equilibrium point $\mathbf{p}_{3b}^* = (\mathbf{P}_{\mathbf{g}_{31}^*} + \mathbf{P}_{\mathbf{g}_{32}^*})^{-1}(\mathbf{P}_{\mathbf{g}_{31}^*}\mathbf{p}_1 + \mathbf{P}_{\mathbf{g}_{32}^*}\mathbf{p}_{2b}^*)$ corresponding to $\mathbf{g}_{31} = -\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = -\mathbf{g}_{32}^*$.

Proof: (i) The equilibria of (14) satisfy

$$\dot{\mathbf{p}}_3 = -(\mathbf{P}_{\mathbf{g}_{31}}\mathbf{g}_{31}^* + \mathbf{P}_{\mathbf{g}_{32}}\mathbf{g}_{32}^*) = \mathbf{0}. \qquad (16)$$

Premultiplying $\mathbf{g}_{31}^\top$ on both sides of (16), we have

$$\mathbf{g}_{31}^\top(\mathbf{P}_{\mathbf{g}_{31}}\mathbf{g}_{31}^* + \mathbf{P}_{\mathbf{g}_{32}}\mathbf{g}_{32}^*) = 0$$

$$\text{or} \qquad \mathbf{g}_{31}^\top \mathbf{P}_{\mathbf{g}_{32}}\mathbf{g}_{32}^* = 0. \qquad (17)$$

Equation (17) is satisfied if and only if $\mathbf{g}_{31} = \pm\mathbf{g}_{32}$ or $\mathbf{g}_{32}^* = \pm\mathbf{g}_{32}$. The condition $\mathbf{g}_{32} = \pm\mathbf{g}_{31}$ happens if and only if agent 3 is collinear with agents 1 and 2. In this case, $\mathbf{P}_{\mathbf{g}_{31}} = \mathbf{P}_{\mathbf{g}_{32}} = \mathbf{P}_{\mathbf{g}_{12}^*}$. Substituting them into (16) gives $\mathbf{P}_{\mathbf{g}_{12}^*}(\mathbf{g}_{31}^* + \mathbf{g}_{32}^*) = \mathbf{0}$, or equivalently

$$\mathbf{g}_{32}^* + \mathbf{g}_{31}^* = k\mathbf{g}_{12}^* \qquad (18)$$

where k is a nonzero constant.

On the other hand, from the assumption on feasibility of the target formation, the desired position of agent 3 and two leaders must be coplanar. Thus, there exist positive scalars d_{12}^*, d_{31}^*, and d_{32}^* such that

$$d_{12}^*\mathbf{g}_{12}^* - d_{32}^*\mathbf{g}_{32}^* + d_{31}^*\mathbf{g}_{31}^* = \mathbf{0}. \qquad (19)$$

Substitute (18) into (19), it follows that

$$\frac{d_{12}^*}{k}(\mathbf{g}_{31}^* + \mathbf{g}_{32}^*) - d_{32}^*\mathbf{g}_{32}^* + d_{31}^*\mathbf{g}_{31}^* = \mathbf{0}$$

$$\text{or} \quad (d_{12}^* + kd_{31}^*)\mathbf{g}_{31}^* + (-kd_{32}^* + d_{12}^*)\mathbf{g}_{32}^* = \mathbf{0}$$

which implies $\mathbf{g}_{31}^*$ is parallel with $\mathbf{g}_{32}^*$. This contradicts Assumption 2 that $\mathbf{g}_{31}^* \neq \pm\mathbf{g}_{32}^*$. Thus, i) cannot happen and (17) holds if and only if $\mathbf{g}_{32} = \pm\mathbf{g}_{32}^*$. Substituting $\mathbf{g}_{32} = \pm\mathbf{g}_{32}^*$ into (14), it follows that $\mathbf{g}_{31} = \pm\mathbf{g}_{31}^*$.

The feasibility of $\mathcal{B}$ guarantees the existence of $\mathbf{p}_{3a}^*$, where $\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$ are both achieved. The equilibrium $\mathbf{p}_{3a}^*$ is uniquely determined as in Lemma 1. Note that when $\mathbf{p}_2 = \mathbf{p}_{2a}^*$, other combinations $-\mathbf{g}_{31}^*$ and $\mathbf{g}_{32}^*$, or $\mathbf{g}_{31}^*$ and $-\mathbf{g}_{32}^*$, or $-\mathbf{g}_{31}^*$ and $-\mathbf{g}_{32}^*$ are unrealizable in $\mathbb{R}^d$.

ii) Following the same process as the above-mentioned, the equilibrium has to satisfy $\mathbf{g}_{31} = \pm\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = \pm\mathbf{g}_{32}^*$. The existence of $\mathbf{p}_{3a}^*$ in case (i) implies the existence of $\mathbf{p}_{3b}^*$, which is symmetric with $\mathbf{p}_{3a}^*$ about $\mathbf{p}_1$ as depicted in Fig. 6. At $\mathbf{p}_{3b}^*$, the bearing vectors with regard to $\mathbf{p}_1$ and $\mathbf{p}_{2b}^*$ are $\mathbf{g}_{31} = -\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = -\mathbf{g}_{32}^*$, respectively. ∎
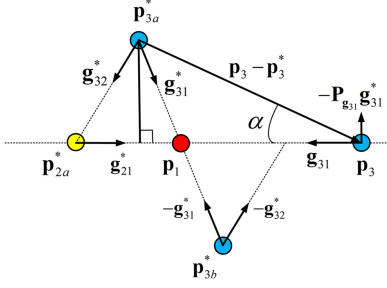
Fig. 6.   Illustration when the position of agent 3, $\mathbf{p}_3$, is collinear with $\mathbf{p}_1$ and $\mathbf{p}_{2a}^*$.

We discuss on stability of the equilibria of two systems (14) and (15) in the following lemma.

Lemma 6: i) The equilibrium $\mathbf{p}_{3a}^*$ corresponding to $\mathbf{g}_{31} = \mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = \mathbf{g}_{32}^*$ of the unforced system (14) is globally asymptotically stable. ii) The equilibrium $\mathbf{p}_{3b}^*$ corresponding to $\mathbf{g}_{31} = -\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = -\mathbf{g}_{32}^*$ of the unforced system (15) is unstable.

Proof: i) Consider the Lyapunov candidate function $V = \frac{1}{2}\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2$, which is positive definite, radially unbounded, and continuously differentiable. We have

$$\dot{V} = -(\mathbf{p}_3 - \mathbf{p}_{3a}^*)^\top (\mathbf{P}_{\mathbf{g}_{31}} \mathbf{g}_{31}^* + \mathbf{P}_{\mathbf{g}_{32}} \mathbf{g}_{32}^*)$$

$$= -(\mathbf{p}_3 - \mathbf{p}_{3a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|}(\mathbf{p}_1 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3a}^*)$$

$$- (\mathbf{p}_3 - \mathbf{p}_{3a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|}(\mathbf{p}_2 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3a}^*)$$

$$= -(\mathbf{p}_3 - \mathbf{p}_{3a}^*)^\top \underbrace{\left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|}\right)}_{:=\mathbf{M}}(\mathbf{p}_3 - \mathbf{p}_{3a}^*). \quad (20)$$

Since $\mathbf{P}_{\mathbf{g}_{31}}$ and $\mathbf{P}_{\mathbf{g}_{32}}$ are positive semidefinite matrices, $\mathbf{M}$ is also positive semidefinite. Thus, $\dot{V} \le 0$. Moreover, $\dot{V} = 0$ if and only if $(\mathbf{p}_3 - \mathbf{p}_{3a}^*) \in \mathcal{N}(\mathbf{M})$. We consider two cases:

1) If $\mathbf{g}_{31} = \pm\mathbf{g}_{32}$ or three agents are in collinear positions, then, $\mathcal{N}(\mathbf{M}) = \text{span}\{\mathbf{g}_{31}\}$. Due to Assumption 2 on the feasibility of $\mathcal{B}$, $\mathbf{p}_{3a}^*$ is not collinear with $\mathbf{p}_1$ and $\mathbf{p}_{2a}^*$, i.e., $(\mathbf{p}_3 - \mathbf{p}_{3a}^*) \notin \mathcal{N}(\mathbf{M})$. Therefore

$$\dot{V} \le -\gamma\sin^2\alpha\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2 = -\gamma\sin^2\alpha V \le 0$$

where α is the angle between the line connecting $\mathbf{p}_3$ and $\mathbf{p}_{3a}^*$ and the line connecting $\mathbf{p}_1$ and $\mathbf{p}_{2a}^*$, as depicted in Fig. 6, and $\gamma = \|\mathbf{z}_{31}^*\|^{-1} + \|\mathbf{z}_{32}^*\|^{-1}$. It is easy to see that $\alpha \in (0, \pi)$.

2) If $\mathbf{g}_{31} \ne \pm\mathbf{g}_{32}$, $\mathbf{M}$ is positive definite (see the proof of Lemma 1). As a result

$$\dot{V} \le -\lambda_{\min}(\mathbf{M}(t))\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2 = -\lambda_{\min}(\mathbf{M}(t))V \le 0$$

where $\lambda_{\min}(\mathbf{M}(t)) > 0$ is the smallest eigenvalue of $\mathbf{M}$ at time t and $\dot{V} = 0$ if and only if $\mathbf{p}_3 = \mathbf{p}_3^*$.

Choosing $\kappa = \min\{\inf_t\{\lambda_{\min}(\mathbf{M})\}, \gamma\sin^2\alpha\} > 0$, it follows that $\dot{V} \le -\kappa V \le 0$. As a result, V is negative definite and $\dot{V} = 0$ if and only if $\mathbf{p}_3 = \mathbf{p}_{3a}^*$. Thus, $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ of (14) is globally asymptotically stable [40].

ii) Consider the function $V = \frac{1}{2}\|\mathbf{p}_3 - \mathbf{p}_{3b}^*\|^2$. Similar to i), along a trajectory of system (15), we have

$$\dot{V} = (\mathbf{p}_3 - \mathbf{p}_{3b}^*)^\top (\mathbf{P}_{\mathbf{g}_{31}}(-\mathbf{g}_{31}^*) + \mathbf{P}_{\mathbf{g}_{32}}(-\mathbf{g}_{32}^*))$$

$$= (\mathbf{p}_3 - \mathbf{p}_{3b}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|}(\mathbf{p}_1 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3b}^*)$$

$$+ (\mathbf{p}_3 - \mathbf{p}_{3b}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|}(\mathbf{p}_2 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3b}^*)$$

$$= (\mathbf{p}_3 - \mathbf{p}_{3b}^*)^\top \mathbf{M}(\mathbf{p}_3 - \mathbf{p}_{3b}^*) \ge 0.$$

Thus, $\dot{V} > 0$ if $\mathbf{p}_3 \ne \mathbf{p}_{3b}^*$. The equilibrium $\mathbf{p}_3 = \mathbf{p}_{3b}^*$ is unstable and $\|\mathbf{p}_3 - \mathbf{p}_3^*\|$ grows unbounded in this case. Thus, $\mathbf{p}_2(0) \ne \mathbf{p}_{2b}^*$ is required to avoid the divergence of $\mathbf{p}_3$. ∎

Since the bearing vectors are undefined when the neighbor agents are collocated, the analysis is valid when collision avoidance is guaranteed. In practice, when each agent is equipped with vision sensors, collision avoidance can be treated independently by vision-based techniques, see [41], [42] and the references therein for examples. Next, we give a sufficient condition for collision-free between agent 3 and its leaders under the dynamics (14).

Lemma 7: In the system (14), agent 3 never collides with agents 1 and 2 if

$$\|\mathbf{p}_3(0) - \mathbf{p}_3^*\| < \min\{\|\mathbf{p}_3^* - \mathbf{p}_1^*\|, \|\mathbf{p}_3^* - \mathbf{p}_{2a}^*\|\}. \quad (21)$$

Proof: Agent 3 never collides with agent 1 if $\|\mathbf{p}_3 - \mathbf{p}_1\| = \|\mathbf{p}_3 - \mathbf{p}_1^*\| > 0, \forall t \ge 0$. Since

$$\|\mathbf{p}_3 - \mathbf{p}_1^*\| = \|(\mathbf{p}_3 - \mathbf{p}_3^*) + (\mathbf{p}_3^* - \mathbf{p}_1^*)\|$$

$$\ge \|\mathbf{p}_3^* - \mathbf{p}_1^*\| - \|\mathbf{p}_3 - \mathbf{p}_3^*\|$$

and $\mathbf{p}_3 \to \mathbf{p}_3^*$ asymptotically [Lemma 6(i)], the following condition is sufficient to avoid collision between agents 1 and 3:

$$\|\mathbf{p}_3(0) - \mathbf{p}_3^*\| < \|\mathbf{p}_3^* - \mathbf{p}_1^*\|.$$

Similarly, a sufficient condition for collision-free between agents 2 and 3 is given as

$$\|\mathbf{p}_3(0) - \mathbf{p}_3^*\| < \|\mathbf{p}_3^* - \mathbf{p}_{2a}^*\|.$$

Thus, condition (21) guarantees collision-free between agent 3 and its leaders. ∎

Remark 2: In [43], a bearing-only navigation problem in a 2-D space with three stationary landmarks was studied. The authors in [43] proposed a 2D version of the control law (14) to guide an agent to any desired position in $\mathbb{R}^2$. Lemma 6(i) improved the result in [43, Proposition 1] by showing that it is sufficient to use only two stationary beacons to reach any position in $\mathbb{R}^d$ that is not collinear with the two landmarks.

At this stage, we can prove the following result on the stability of the system (13).

Proposition 1: The system (13) has an almost globally asymptotically stable equilibrium $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ corresponding to $\mathbf{g}_{31} = \mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = \mathbf{g}_{32}^*$.

Proof: We will show that the system (13) satisfies the ultimate boundedness property. Consider the Lyapunov function

$V = \frac{1}{2}\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2$, which is positive definite, radially unbounded, and continuously differentiable. If $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$, the derivative of V along a trajectory of system (13) is

$$
\begin{aligned}
\dot{V} =& -2(\mathbf{p}_3 - \mathbf{p}_3^*)^\top (\mathbf{P}_{\mathbf{g}_{31}} \mathbf{g}_{31}^* + \mathbf{P}_{\mathbf{g}_{32}} \mathbf{g}_{32}^*) \\
=& -(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} (\mathbf{p}_1 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3a}^*) \right. \\
& \left. + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} (\mathbf{p}_{2a}^* - \mathbf{p}_2 + \mathbf{p}_2 - \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_{3a}^*) \right) \\
=& -(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} \right) (\mathbf{p}_3 - \mathbf{p}_3^*) \\
& + (\mathbf{p}_3 - \mathbf{p}_3^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} (\mathbf{p}_2 - \mathbf{p}_{2a}^*) \\
\leq& -(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} \right) (\mathbf{p}_3 - \mathbf{p}_3^*) \\
& + \|\mathbf{p}_3 - \mathbf{p}_3^*\| \frac{\|\mathbf{P}_{\mathbf{g}_{32}}\|}{\|\mathbf{z}_{32}^*\|} \|\mathbf{p}_2 - \mathbf{p}_{2a}^*\| \\
\leq& -(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} \right) (\mathbf{p}_3 - \mathbf{p}_3^*) \\
& + \frac{2d_{21}}{\|\mathbf{z}_{32}^*\|} \|\mathbf{p}_3 - \mathbf{p}_3^*\|.
\end{aligned}
\tag{22}
$$

When $\|\mathbf{p}_3\|$ is large, the second term in (22) is $\mathcal{O}(\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|)$ while the first term is $-\mathcal{O}(\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2)$. This implies $\dot{V} < 0$ when $\|\mathbf{p}_3\|$ is large. Equivalently, $\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|$ is ultimately bounded and so is $\|\mathbf{p}_3\|$. Since the unforced system (14) has a globally asymptotically stable equilibrium $\mathbf{p}_{3a}^*$ as shown in Lemma 6 and satisfies the ultimate boundedness property, the system (13) is input-to-state stable (ISS) with regard to the input $\mathbf{p}_2$. On the other hand, according to Lemma 4, the input $\mathbf{p}_2$ exponentially converges to $\mathbf{p}_{2a}^*$ if it is not initially located at $\mathbf{p}_{2b}^*$. Thus, the equilibrium $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ is almost globally asymptotically stable [34, Th. 2].

By Proposition 1, we have proved that the desired equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$, $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ of the cascade connection

$$
\begin{aligned}
\dot{\mathbf{p}}_2 &= \mathbf{u}_2(\mathbf{p}_2) \\
\dot{\mathbf{p}}_3 &= \mathbf{u}_3(\mathbf{p}_3, \mathbf{p}_2)
\end{aligned}
\tag{23}
$$

is almost globally asymptotically stable. All trajectories of (23) converge to the desired positions except for those starting at $\mathbf{p}_2(0) = \mathbf{p}_{2b}^*$. Moreover, the undesired equilibrium $\mathbf{p}_2 = \mathbf{p}_{2b}^*$, $\mathbf{p}_3 = \mathbf{p}_{3b}^*$ is unstable.

3) The n-Agent System: Consider the LFF formation of n-agents ($n \geq 3$) satisfying all assumptions in Problem 1. From the assumption on the graph $\mathcal{G}$, each agent i ($3 \leq i \leq n$) has two neighbors $1 \leq j \neq k \leq i-1$ and must control two bearing vectors $\mathbf{g}_{ij}$, $\mathbf{g}_{ik}$. The control law for agent i is explicitly given as

$$
\dot{\mathbf{p}}_i = \mathbf{u}_i(\mathbf{p}_i, \mathbf{p}_{i-1}, \dots, \mathbf{p}_2) = -\mathbf{P}_{\mathbf{g}_{ij}} \mathbf{g}_{ij}^* - \mathbf{P}_{\mathbf{g}_{ik}} \mathbf{g}_{ik}^*.
\tag{24}
$$

The dynamics of n agents can be expressed in the form of a cascade system:

$$
\dot{\mathbf{p}} = \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \\ \vdots \\ \dot{\mathbf{p}}_i \\ \vdots \\ \dot{\mathbf{p}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_2(\mathbf{p}_2) \\ \mathbf{u}_3(\mathbf{p}_3, \mathbf{p}_2) \\ \vdots \\ \mathbf{u}_i(\mathbf{p}_i, \mathbf{p}_{i-1}, \dots, \mathbf{p}_2) \\ \vdots \\ \mathbf{u}_n(\mathbf{p}_n, \mathbf{p}_{n-1}, \dots, \mathbf{p}_2) \end{bmatrix}
\tag{25}
$$

where $(\mathbf{p}_{i-1}, \dots, \mathbf{p}_2)$ is considered as an input to the dynamics of an agent i ($i = 3, \dots, n$).

From Lemmas 5 and 6, for all $i = 3, \dots, n$, it follows that $\mathbf{p}_i = \mathbf{p}_{ia}^* = (\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}^*})^{-1} (\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}^*} \mathbf{p}_{ja}^*)$ is a globally asymptotically stable equilibrium of the unforced subsystem

$$
\dot{\mathbf{p}}_i = \mathbf{u}_i(\mathbf{p}_i, \mathbf{p}_{(i-1)a}^*, \dots, \mathbf{p}_{2a}^*).
\tag{26}
$$

Based on the stability of cascade interconnected systems [34], we can prove almost global stability of the system (25) in the following theorem.

Theorem 1: Under the Assumptions 1–3 and the proposed control laws, the system (25) has two equilibiria. The equilibrium $\mathbf{p}_a^* = [\mathbf{p}_1^{*\top}, \mathbf{p}_{2a}^{*\top}, \dots, \mathbf{p}_{na}^{*\top}]^\top$ satisfying all desired bearings constraints in $\mathcal{B}$ is almost globally asymptotically stable. The equilibrium $\mathbf{p}_b^* = [\mathbf{p}_1^\top, \mathbf{p}_{2b}^{*\top}, \dots, \mathbf{p}_{nb}^{*\top}]^\top$ is unstable. All trajectories starting with $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$ asymptotically converge to $\mathbf{p}_a^*$.

Proof: We will prove this theorem by mathematical induction. Consider $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$. First, for $l = 2$, we have $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ almost globally asymptotically stable and $\mathbf{p}_2 = \mathbf{p}_{2b}^*$ unstable based on Lemma 4. Thus, Theorem 1 is true for $l = 2$. Second, Theorem 1 is also true for $l = 3$ based on Proposition 1.

Second, suppose that the claim of Theorem 1 is true for $3 \leq l \leq i-1$. That is, $\mathbf{p}_i = \mathbf{p}_{ia}^*$ is globally asymptotically stable for all $3 \leq l \leq i-1$. We have to prove that the theorem is also true for $l = i$. By following a similar process as in the proof of Lemma 6, we can show that $\mathbf{p}_{ia}^*$ is a globally asymptotically stable equilibrium of the unforced system (26).

We will next show that $\mathbf{p}_i(t)$ is bounded. To this end, suppose i has two neighbor agents j and k, $1 \leq j \neq k < i$. Consider the Lyapunov function $V = \frac{1}{2}\|\mathbf{p}_i - \mathbf{p}_{ia}^*\|^2$, which is positive definite, radially unbounded, and continuously differentiable. The derivative of V along a trajectory of the system (26) is given by

$$
\begin{aligned}
\dot{V} =& -(\mathbf{p}_i - \mathbf{p}_{ia}^*)^\top (\mathbf{P}_{\mathbf{g}_{ij}} \mathbf{g}_{ij}^* + \mathbf{P}_{\mathbf{g}_{ik}} \mathbf{g}_{ik}^*) \\
=& -(\mathbf{p}_i - \mathbf{p}_{ia}^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}^*\|} (\mathbf{p}_{ja}^* - \mathbf{p}_j + \mathbf{p}_j - \mathbf{p}_i + \mathbf{p}_i - \mathbf{p}_{ia}^*) \right. \\
& \left. + \frac{\mathbf{P}_{\mathbf{g}_{ik}}}{\|\mathbf{z}_{ik}^*\|} (\mathbf{p}_{ka}^* - \mathbf{p}_k + \mathbf{p}_k - \mathbf{p}_i + \mathbf{p}_i - \mathbf{p}_{ia}^*) \right) \\
=& -(\mathbf{p}_i - \mathbf{p}_{ia}^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{ik}}}{\|\mathbf{z}_{ik}^*\|} \right) (\mathbf{p}_i - \mathbf{p}_{ia}^*)
\end{aligned}
$$

$$- (\mathbf{p}_i - \mathbf{p}_{ia}^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}^*\|} (\mathbf{p}_{ja}^* - \mathbf{p}_j) + \frac{\mathbf{P}_{\mathbf{g}_{ik}}}{\|\mathbf{z}_{ik}^*\|} (\mathbf{p}_{ka}^* - \mathbf{p}_k) \right)$$

$$\leq - (\mathbf{p}_i - \mathbf{p}_{ia}^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{ij}}}{\|\mathbf{z}_{ij}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{ik}}}{\|\mathbf{z}_{ik}^*\|} \right) (\mathbf{p}_i - \mathbf{p}_{ia}^*)$$

$$+ \|\mathbf{p}_i - \mathbf{p}_{ia}^*\| \left(\frac{\|\mathbf{P}_{\mathbf{g}_{ij}}\|}{\|\mathbf{z}_{ij}^*\|} \|\mathbf{p}_{ja}^* - \mathbf{p}_j\| + \frac{\|\mathbf{P}_{\mathbf{g}_{ik}}\|}{\|\mathbf{z}_{ik}^*\|} \|\mathbf{p}_{ka}^* - \mathbf{p}_k\| \right).$$

Because Theorem 1 is true for $l \leq i - 1$, $\|\mathbf{p}_j - \mathbf{p}_{ja}^*\|$ and $\|\mathbf{p}_k - \mathbf{p}_{ka}^*\|$ are bounded and converge to zero as $t \to +\infty$. It follows that $\|\mathbf{p}_i - \mathbf{p}_{ia}^*\|$ is bounded, which implies that $\|\mathbf{p}_i\|$ is also bounded. Thus, the equilibrium $\mathbf{p}_{ia}^*$ is asymptotically stable and all trajectories with $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$ converge to $\mathbf{p}_{ia}^*$ [34, Th. 2]. Furthermore, if $\mathbf{p}_2(0) = \mathbf{p}_{2b}^*$, the system has an unstable equilibrium $\mathbf{p}_{ib}^*$ due to Lemma 6. Therefore, $\mathbf{p}_i = \mathbf{p}_{ia}^*$ is almost globally asymptotically stable and Theorem 1 is also true for $l = i$.

Finally, from mathematical induction, the claim holds for all $l \geq 3$. Thus, the n-agent system (25) is almost globally asymptotically stable. All trajectories satisfying $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$ converge to a formation satisfying all desired bearing vectors in $\mathcal{B}$. If $\mathbf{p}_2(0) = \mathbf{p}_{2b}^*$, the system has an undesired equilibrium, where $\mathbf{g}_{ij} = -\mathbf{g}_{ij}^*$ for all $\mathbf{g}_{ij}^* \in \mathcal{B}$. This undesired equilibrium is unstable. ∎

C. Global Stabilization of LFF Formations

In the previous subsection, the fact that instead of a global stabilization we have an almost global stabilization of the overall formation is due to the possibility that $\mathbf{p}_2(0) = \mathbf{p}_{2b}^*$, which is an unstable equilibrium. Of course, in practice, noise may displace the system from $\mathbf{p}_{2b}^*$ if it is initialized there. However, instead of relying on noise, we can propose the following modified bearing-only control law for agent 2:

$$\mathbf{u}_2 = -\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^* - k \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| \mathbf{P}_{\mathbf{g}_{21}} (\mathbf{sgn}(\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) + \mathbf{n}).$$
(27)

In this control law, $k > 0$ is a control gain, sgn denotes the signum function, $\mathbf{sgn}(\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) := [\mathrm{sgn}([\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*]_1), \dots,$ $\mathrm{sgn}([\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*]_d)]^\top$; $\mathbf{n} = \mathbf{n}(t) = [n_1(t), \dots, n_d(t)]$, where $n_1(t), \dots, n_d(t)$ are time-varying continuous functions satisfying $\sum_{k=1}^d n_k^2(t) = c$, and c is a constant satisfying $0 < c < 1$.[2]

In the control law (27), the first term is the same as the control law (9) while the last term is added to guarantee global convergence of $\mathbf{g}_{21}$ to $\mathbf{g}_{21}^*$. Note that the adjustment term in (27) was originally introduced in another form in [4]. Observe that under the control law (27), we have

$$\frac{d}{dt} d_{21}^2 = \frac{d}{dt} (\mathbf{z}_{21}^\top \mathbf{z}_{21}) = 2\mathbf{z}_{21}^\top (\dot{\mathbf{p}}_1 - \dot{\mathbf{p}}_2)$$

$$= -2\mathbf{z}_{21}^\top \mathbf{P}_{\mathbf{g}_{21}} (\mathbf{g}_{21}^* + k \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\|$$

$$(\mathbf{sgn}(\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) + \mathbf{n}))$$

$$= 0.$$
(28)

[2]When $d = 2$, we may choose $\mathbf{n} = \sqrt{c}[\cos t, \sin t]^\top$.

Thus, d_{21} is invariant under the control law (27). Furthermore, it can be checked that $\mathbf{p}_{2b}^*$ is not an equilibrium of (27) due to the adjustment term. We prove the following result on stability of the agent 2.

Proposition 2: Under the control law (27), the equilibrium $\mathbf{p}_{2a}^* = \mathbf{p}_1^* - d_{21} \mathbf{g}_{21}^*$ corresponding to $\mathbf{g}_{21} = \mathbf{g}_{21}^*$ is globally asymptotically stable and almost globally exponentially stable.

Proof: We consider the solution $\mathbf{p}_2$ of the nonsmooth system (27) in the Filippov sense [44], [45]. For almost all time,

$$\dot{\mathbf{p}}_2 \in -\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^* - k \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| \mathbf{P}_{\mathbf{g}_{21}} (K[\mathbf{sgn}](\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) + \mathbf{n})$$
(29)

where $K[\mathbf{f}](\mathbf{x})$ denotes the Fillipov set-valued mapping of $\mathbf{f}(\mathbf{x})$ [44]. Consider the Lyapunov function $V = \frac{1}{2} \|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|^2$, which is continuously differentiable, radially unbounded, and positive definite. Then, at each point, $\mathbf{p}_2 \in \mathbb{R}^d$, $\partial V = (\mathbf{p}_2 - \mathbf{p}_{2a}^*)$. Based on [44, Th. 2.2], $\dot{V}$ exists almost everywhere (a.e.) and $\dot{V} \in^{a.e.} \tilde{\dot{V}}$, where

$$\tilde{\dot{V}} = \bigcap_{\boldsymbol{\xi} \in \partial V} \boldsymbol{\xi}^\top \dot{\mathbf{p}}_2$$

$$= - (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*$$

$$\quad - k \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{P}_{\mathbf{g}_{21}}$$

$$\quad (K[\mathbf{sgn}](\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) + \mathbf{n})$$

$$= - (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*)$$

$$\quad - k d_{21} \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| (\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*)^\top (K[\mathbf{sgn}](\mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^*) + \mathbf{n}).$$

Define $\boldsymbol{\eta} := \mathbf{P}_{\mathbf{g}_{21}} \mathbf{g}_{21}^* = [\eta_1, \dots, \eta_d]^\top$, we have

$$\tilde{\dot{V}} \leq - (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*)$$

$$\quad - k d_{21} \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| (\boldsymbol{\eta}^\top K[\mathbf{sgn}](\boldsymbol{\eta}) - |\boldsymbol{\eta}^\top \mathbf{n}|). \quad (30)$$

From the property of **sgn** function, we can write

$$\boldsymbol{\eta}^\top K[\mathbf{sgn}](\boldsymbol{\eta}) = \sum_{i=1}^d \eta_k K[\mathrm{sgn}](\eta_k).$$

Recall from [44] that

$$K[\mathrm{sgn}](\eta_k) = \begin{cases} 1 & \eta_k > 0 \\ [-1, 1] & \eta_k = 0 \\ -1 & \eta_k < 0 \end{cases}.$$

Thus, $\eta_k K[\mathrm{sgn}](\eta_k) = |\eta_k|$, and

$$\boldsymbol{\eta}^\top K[\mathbf{sgn}](\boldsymbol{\eta}) = \sum_{k=1}^d |\eta_k|. \quad (31)$$

Moreover,

$$\left| \sum_{k=1}^d \eta_k n_k \right| \leq \sum_{k=1}^d |\eta_k n_k| \leq \sum_{k=1}^d |\eta_k| |n_k| \leq \sqrt{c} \sum_{k=1}^d |\eta_k|$$

where the last inequality follows from the fact that $|n_k| \leq \sqrt{\sum_{k=1}^{d} n_k^2} = \sqrt{c} < 1$. Therefore

$$\dot{V} \leq -(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*)$$

$$- k(1 - \sqrt{c}) d_{21} \|\mathbf{g}_{21} - \mathbf{g}_{21}^*\| \sum_{k=1}^{d} |\eta_k|$$

$$\leq -(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*) \leq 0. \quad (32)$$

It follows that $\dot{V} = 0$ if and only if $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ or $\mathbf{p}_2 = \mathbf{p}_{2b}^*$. Since $\dot{\mathbf{p}}_2|_{\mathbf{p}_2 = \mathbf{p}_{2b}^*} \neq \mathbf{0}$, based on LaSalle's invariance principle for nonsmooth system [44, Th. 3.2], every trajectory of (29) asymptotically converges to $\mathbf{p}_{2a}^*$.

Next, let α be the angle between $\mathbf{p}_{2a}^* - \mathbf{p}_2$ and $\mathbf{g}_{21}^*$ as depicted in Fig. 6, we have $\alpha \in [0, \pi/2]$. Furthermore, we can write $\|\mathbf{P}_{\mathbf{g}_{21}}(\mathbf{p}_2 - \mathbf{p}_{2a}^*)\| = \sin\alpha \|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|$. For all $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$, we have $\alpha(0) > 0$. Since $\mathbf{p}_2 \to \mathbf{p}_{2a}^*$ asymptotically, we have $\alpha(t) \geq \alpha(0) > 0, \forall t > 0$. It follows from (32) that

$$\dot{V} \leq -(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{12}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*)$$

$$= -\frac{\sin^2\alpha}{d_{21}} \|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|^2$$

$$\leq -\frac{2\sin^2\alpha(0)}{d_{21}} V = -\kappa V \leq 0$$

where $\kappa = 2d_{21}^{-1} \sin^2\alpha(0) > 0$. Therefore, the equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ is globally asymptotically stable and almost globally exponentially stable. ∎

Theorem 2: Under Assumptions 1–3, if agent 2 employs the control law (27) and agent i $(3 \leq i \leq n)$ employs the control law (24), the formation globally asymptotically reaches the desired formation satisfying all bearing vectors in $\mathcal{B}$. ∎

Proof: The proof involves the same steps as in Section III-B. The only difference is agent 2 always reaches $\mathbf{p}_{2a}^*$ from any initial condition. Thus, $\mathbf{p}_i \to \mathbf{p}_{ia}^*, \forall 3 \leq i \leq n$, or, i.e., the LFF formation globally asymptotically converges to the desired formation satisfying all bearing vectors in $\mathcal{B}$. ∎

IV. BEARING-BASED CONTROL OF LFF FORMATIONS WITHOUT A GLOBAL ORIENTATION

In this section, we extend the result in the previous section to a more general setup. The model of each agent in this section is given in $\mathbb{R}^3 \times SO(3)$, thus including both position and orientation of the agent.

A. Problem Formulation

Consider a group of n autonomous agents in the 3-D space $\mathbb{R}^3$. The position, linear velocity, and angular velocity of agent i given in a global reference frame are denoted as $\mathbf{p}_i$, $\mathbf{u}_i$, and $\mathbf{w}_i \in \mathbb{R}^3$, respectively. Each agent i maintains a local reference frame ${}^i\Sigma$; the linear and the angular velocity of agent i expressed in ${}^i\Sigma$ are given by $\mathbf{u}_i^i = [u_x^i, u_y^i, u_z^i]^\top$ and $\mathbf{w}_i^i = [w_x^i, w_y^i, w_z^i]^\top$,

respectively. Let $\mathbf{R}_i \in SO(3)$ be the rotation from ${}^i\Sigma$ to a global reference frame ${}^g\Sigma$, we have $\det(\mathbf{R}_i) = 1$ and $\mathbf{R}_i \mathbf{R}_i^\top = \mathbf{I}_3$. The position and orientation dynamics of agent i written in the global reference frame are

$$\dot{\mathbf{p}}_i = \mathbf{R}_i \mathbf{u}_i^i \quad (33)$$

$$\dot{\mathbf{R}}_i = \mathbf{R}_i \mathbf{S}_i \quad (34)$$

where

$$\mathbf{S}_i = \begin{bmatrix} 0 & -w_z^i & w_y^i \\ w_z^i & 0 & -w_x^i \\ -w_y^i & w_x^i & 0 \end{bmatrix}$$

is a skew-symmetric matrix. Note that from (33)–(34), the dynamics of agent i are now defined in $\mathbb{R}^3 \times SO(3)$. We follow the Assumptions 1–3 of Problem 1 on the sensing graph and the initial position of the agents. Furthermore, we assume that in addition to the local bearing vectors $\mathbf{g}_{ij}^i = \mathbf{R}_i^\top \mathbf{g}_{ij}$, agent i can also obtain the relative orientation $\mathbf{R}_i^\top \mathbf{R}_j$ with regard to each neighboring agent j. Finally, we adopt the following assumption on the initial orientations of the agents.

Assumption 4: The initial orientations of all agents are contained within a closed ball $\bar{B}_r(\mathbf{R}_1)$ of radius r less than $\pi/2$. Equivalently, the symmetric part of $\mathbf{R}_1^\top \mathbf{R}_i(0)$ is positive definite, $\forall i = 2, \ldots, n$ [46].

At this point, we can formulate the following problem.

Problem 2: Given an n-agent system with initial position $\mathbf{p}(0)$ and orientations $\{\mathbf{R}_i(0)\}_{i \in \mathcal{V}}$, satisfying Assumptions 1–4, design $\mathbf{u}_i^i$ and $\mathbf{w}_i^i$ for agent $i \in \mathcal{V}$ based on local bearing measurements and relative orientation measurements, such that $\{\mathbf{R}_i(t)\}_{i \in \mathcal{V}}$, converges to $\mathbf{R}_1(0)$ and $\mathbf{g}_{ij}^i \to \mathbf{g}_{ij}^*$ for all $\mathbf{g}_{ij}^* \in \mathcal{B}$.

B. Proposed Control Strategy

To solve Problem 2, we propose a two-layer control strategy for the n-agent system. The two layers will be referred to as the orientation alignment layer and the formation control layer. On the orientation alignment layer, we use a consensus algorithm to synchronize all agents' orientations. Simultaneously, on the formation control layer, we implement the bearing-only control law proposed earlier in Problem 1 in each agent's local frame to achieve the desired formation. Note that this two-layer control strategy was also used in distance-based formation control problems with different setups [5], [6], [21], [47].

1) Orientation Alignment Layer: The following orientation alignment control law for each agent i $(1 \leq i \leq n)$ is adopted in this paper:

$$\mathbf{S}_i = -\sum_{j \in \mathcal{N}_i} (\mathbf{R}_j^\top \mathbf{R}_i - \mathbf{R}_i^\top \mathbf{R}_j). \quad (35)$$

The control law (35) is adopted from the attitude synchronization control law in [46], [48], [49]. Since $\mathbf{R}_j^\top \mathbf{R}_i = (\mathbf{R}_i^\top \mathbf{R}_j)^\top$, the control law (35) requires only the *local relative orientations* of agent i with regard to its neighbors, i.e., communication between agents is not needed [49].

Because the leader has no neighbor, we let $\dot{\mathbf{R}}_1 = \mathbf{0}$. Thus, the orientation of the leader is time invariant, i.e., $\mathbf{R}_1(t) = \mathbf{R}_1(0)$, $\forall t > 0$.

From (34), the angular velocity in the global reference frame can be rewritten as follows:

$$\dot{\mathbf{R}}_i = -\sum_{j \in \mathcal{N}_i} \mathbf{R}_i (\mathbf{R}_j^\top \mathbf{R}_i - \mathbf{R}_i^\top \mathbf{R}_j). \tag{36}$$

Unlike [5], [6], [21], [47], where the interaction graphs are assumed to be undirected, the alignment (35) is performed in a directed graph $\mathcal{G}$ built up via a Henneberg construction, i.e., a rooted directed graph with a root at vertex v_1. This setup leads to a different result. When the interaction graph is bidirectional, the final orientation is determined by all agents' initial orientations [49]. However, when the graph is directed and has a rooted spanning tree, the aligned orientation is determined by the orientations of the agent locating at the root of the graph, as stated in the following lemma.

Lemma 8: [49], [50, Th. 3.2] Assume that $\mathcal{G}$ has a rooted spanning tree. If there is $\mathbf{R} \in SO(3)$, such that the orientations of all agents are initially contained within a closed ball $\bar{B}_r(\mathbf{R})$ of radius r less than $\pi/2$ centered around $\mathbf{R}$, then the controller (35) is a synchronization controller, i.e., $\mathbf{R}_i^\top \mathbf{R}_j \to \mathbf{I}_3$ asymptotically for all $i, j \in \mathcal{V}$.

The following result is implied from Lemma 8 and Corollary 2 in [46].

Lemma 9: Under Assumption 4 and the orientation alignment control law (34), if the directed graph $\mathcal{G}$ is built up by a Henneberg construction, all agents' orientations will asymptotically converge to the leader's orientation, i.e., for $i = 2, \ldots, n$, $\mathbf{R}_i(t)^\top \mathbf{R}_1 \to \mathbf{I}_3$ asymptotically, as $t \to \infty$.

Proof: Since the graph $\mathcal{G}$ is built up by a Henneberg construction, it has a rooted spanning tree. Thus, all conditions of Lemma 8 are satisfied and orientations of all agents will converge to a common aligned orientation. Under the control law (35), $\mathbf{R}_1(t) = \mathbf{R}_1(0)$, for all time $t > 0$, and thus $\mathbf{R}_i(t) \to \mathbf{R}_1$, as $t \to \infty$. ∎

2) Formation Control Layer: In this layer, we use a locally implemented version of the control laws in Section III. The leader is stationary, i.e., $\mathbf{u}_1^1 = \mathbf{0}$. The first follower's position control law written in its local reference frame $^2\Sigma$ is designed as[3]

$$\mathbf{u}_2^2 = -\mathbf{P}_{\mathbf{g}_{21}^2}(\mathbf{I}_3 + \mathbf{R}_2^\top \mathbf{R}_1)\mathbf{g}_{21}^*. \tag{37}$$

For each follower agent i $(3 \le i \le n)$, the position control law written in $^i\Sigma$ is

$$\mathbf{u}_i^i = -\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}^i}(\mathbf{I}_3 + \mathbf{R}_i^\top \mathbf{R}_j)\mathbf{g}_{ij}^* \tag{38}$$

where $\mathbf{P}_{\mathbf{g}_{ij}^i} = \mathbf{I}_3 - \mathbf{g}_{ij}^i(\mathbf{g}_{ij}^i)^\top$ is the orthogonal projection matrix. Using the following derivation:

$$\mathbf{R}_i \mathbf{P}_{\mathbf{g}_{ij}^i}(\mathbf{I}_3 + \mathbf{R}_i^\top \mathbf{R}_j)\mathbf{g}_{ij}^*$$
$$= \mathbf{R}_i(\mathbf{I}_3 - \mathbf{R}_i^\top \mathbf{g}_{ij}\mathbf{g}_{ij}^\top \mathbf{R}_i)(\mathbf{I}_3 + \mathbf{R}_i^\top \mathbf{R}_j)\mathbf{g}_{ij}^*$$
$$= \mathbf{R}_i \mathbf{R}_i^\top(\mathbf{I}_3 - \mathbf{g}_{ij}\mathbf{g}_{ij}^\top)\mathbf{R}_i(\mathbf{I}_3 + \mathbf{R}_i^\top \mathbf{R}_j)\mathbf{g}_{ij}^*$$
$$= \mathbf{P}_{\mathbf{g}_{ij}}(\mathbf{R}_i + \mathbf{R}_j)\mathbf{g}_{ij}^*$$

---

[3] The global stabilization control law (27) cannot be used here since it uses global information.

and (33) and (38), we can express the dynamics of agent 2 in the global frame as follows:

$$\dot{\mathbf{p}}_2 = \underbrace{-2\mathbf{P}_{\mathbf{g}_{21}}\mathbf{R}_1\mathbf{g}_{21}^*}_{:= \mathbf{f}_2(\mathbf{p}, t)} + \underbrace{\mathbf{P}_{\mathbf{g}_{21}}(\mathbf{R}_1 - \mathbf{R}_2)\mathbf{g}_{21}^*}_{:= \mathbf{h}_2(\mathbf{p}, t)}. \tag{39}$$

Similarly, the dynamics of an agent i $(i = 3, \ldots, n)$ can be expressed in $^g\Sigma$ as

$$\dot{\mathbf{p}}_i = -\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}}(\mathbf{R}_i + \mathbf{R}_j)\mathbf{g}_{ij}^*$$

$$= \underbrace{-2\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}}\mathbf{R}_1\mathbf{g}_{ij}^*}_{:= \mathbf{f}_i(\mathbf{p}, t)} + \underbrace{\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}}(2\mathbf{R}_1 - \mathbf{R}_i - \mathbf{R}_j)\mathbf{g}_{ij}^*}_{:= \mathbf{h}_i(\mathbf{p}, t)}.$$

Then, the position dynamics of the n-agent system can be expressed in the following compact form:

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}) + \mathbf{h}(\mathbf{p}, t) \tag{40}$$

where $\mathbf{f}(\mathbf{p}) = [\mathbf{f}_1^\top, \ldots, \mathbf{f}_n^\top]^\top$, $\mathbf{h}(\mathbf{p}, t) = [\mathbf{h}_1^\top, \ldots, \mathbf{h}_n^\top]^\top$, and $\mathbf{f}_1 = \mathbf{0}$, $\mathbf{h}_1 = \mathbf{0}$. We will analyze the system (40) in the next section using the results on almost global ISS [35]. Note that the approach in the next section is similar to [21].

C. Stability Analysis

1) Input to the Nominal System: Observe that in the compact form (40), $\mathbf{h}(t)$ can be considered as an input to the nominal system

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}). \tag{41}$$

We have the following lemma on $\mathbf{h}(t)$.

Lemma 10: Under Assumptions 1–4, the input $\mathbf{h}(t)$ from the orientation alignment layer to the formation control layer is bounded. Moreover, $\mathbf{h}(t)$ asymptotically converges to $\mathbf{0}$ as $t \to \infty$.

Proof: This proof is similar to the proof of [21, Lemma 12] and will be omitted. ∎

2) The First Follower: The dynamics of agent 2 (the first follower) is given by

$$\dot{\mathbf{p}}_2 = \mathbf{f}_2(\mathbf{p}_2) + \mathbf{h}_2(t)$$
$$= -2\mathbf{P}_{\mathbf{g}_{21}}\mathbf{R}_1\mathbf{g}_{21}^* + \mathbf{P}_{\mathbf{g}_{21}}(\mathbf{R}_1 - \mathbf{R}_2)\mathbf{g}_{21}^* \tag{42}$$
$$\dot{\mathbf{R}}_2 = -\mathbf{R}_2(\mathbf{R}_1^\top \mathbf{R}_2 - \mathbf{R}_2^\top \mathbf{R}_1). \tag{43}$$

We have the following lemma on the unforced system $\dot{\mathbf{p}}_2 = \mathbf{f}_2(\mathbf{p}_2)$, whose proof is similar to the proof of Lemma 4.

Lemma 11: The unforced system $\dot{\mathbf{p}}_2 = \mathbf{f}_2(\mathbf{p}_2)$ has two equilibria. The first equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ corresponding to $\mathbf{g}_{21} = \mathbf{R}_1\mathbf{g}_{21}^*$ is almost globally asymptotically stable. The second equilibrium $\mathbf{p}_2 = \mathbf{p}_{2b}^*$ corresponding to $\mathbf{g}_{21} = -\mathbf{R}_1\mathbf{g}_{21}^*$ is (exponentially) unstable.

In fact, every initial condition, other than the unstable equilibirium point, is in the region of attraction of the equilibirium point $\mathbf{p}_{2a}^*$.

Lemma 12: The system (42) has two equilibria. The equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ is almost globally asymptotically stable. The equilibrium $\mathbf{p}_2 = \mathbf{p}_{2b}^*$ is exponentially unstable. All trajectories

with $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$, $\mathbf{R}_2(0) \neq \mathbf{R}_1$ asymptotically converge to the stable equilibrium.

Proof: We first prove the system (42) satisfies the ultimate boundedness property. Consider the potential function $V = \frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|^2$, which is positive definite, radially unbounded, and $V = 0$ if and only if $\mathbf{p}_2 = \mathbf{p}_{2a}^*$. Then

$$\dot{V} = (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \dot{\mathbf{p}}_2$$
$$= -2(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{P}_{\mathbf{g}_{21}} \mathbf{R}_1 \mathbf{g}_{21}^* + (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{h}_2$$
$$= -2(\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{21}} (\mathbf{p}_2 - \mathbf{p}_{2a}^*) + (\mathbf{p}_2 - \mathbf{p}_{2a}^*)^\top \mathbf{h}_2$$
$$\leq -\frac{2\sin^2\alpha(0)}{d_{21}}\|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|^2 + \|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|\|\mathbf{h}_2\|$$
$$\leq -\kappa V + 2d_{21}\|\mathbf{h}_2\| \qquad (44)$$

where $\kappa = 4d_{21}^{-1}\sin^2\alpha(0)$, and α is the angle between $\mathbf{p}_{2a}^* - \mathbf{p}_2$ and $\mathbf{g}_{21}^*$. Since $\|\mathbf{p}_2 - \mathbf{p}_{2a}^*\|$ is bounded, it follows from (44) that the system (42) satisfies the ultimate boundedness property [35, Proposition 3].

When $\mathbf{h}_2(t) = \mathbf{0}$, the unforced system has two isolated equilibria with properties given in Lemma 11. Since the system (42) satisfies Assumptions A0–A2 in [35] and the ultimate boundedness property, (42) is almost globally ISS with respect to the equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ based on [35, Proposition 2].

Since $\mathbf{h}_2(t) \to \mathbf{0}$ as proved in Lemma 10, the equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$ of (42) is almost globally asymptotically stable [34, Th. 2]. ∎

3) The Second Follower: The second follower's dynamics is given by

$$\dot{\mathbf{p}}_3 = \mathbf{f}_3(\mathbf{p}_3, \mathbf{p}_2) + \mathbf{h}_3(t)$$
$$= -2\mathbf{P}_{\mathbf{g}_{31}}\mathbf{R}_1\mathbf{g}_{31}^* - 2\mathbf{P}_{\mathbf{g}_{32}}\mathbf{R}_1\mathbf{g}_{32}^* + \mathbf{h}_3(t). \qquad (45)$$

Lemma 13 is about the unforced systems

$$\dot{\mathbf{p}}_3 = \mathbf{f}_3(\mathbf{p}_3, \mathbf{p}_{2a}^*) \qquad (46)$$
$$\dot{\mathbf{p}}_3 = \mathbf{f}_3(\mathbf{p}_3, \mathbf{p}_{2b}^*). \qquad (47)$$

Lemma 13: The system (46) has a globally asymptotically stable equilibrium $\mathbf{p}_{3a}^*$, where $\mathbf{g}_{31} = \mathbf{R}_1\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = \mathbf{R}_1\mathbf{g}_{32}^*$. The system (47) has an unstable equilibrium $\mathbf{p}_{3b}^*$, where $\mathbf{g}_{31} = -\mathbf{R}_1\mathbf{g}_{31}^*$ and $\mathbf{g}_{32} = -\mathbf{R}_1\mathbf{g}_{32}^*$.

Proof: The result follows from Lemma 6. ∎

Lemma 14: The cascade system (42), (45) has two equilibria. The equilibrium $\mathbf{p}_2 = \mathbf{p}_{2a}^*$, $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ is almost globally asymptotically stable. The equilibrium $\mathbf{p}_2 = \mathbf{p}_{2b}^*$, $\mathbf{p}_3 = \mathbf{p}_{3b}^*$ is unstable. All trajectories starting out of the undesired equilibrium asymptotically converge to the stable equilibrium.

Proof: As in Lemma 12, we first prove that the system (45) satisfies the ultimate boundedness property if $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$. Consider the Lyapunov function $V = \frac{1}{2}\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|$, the

derivative of V is given by

$$\dot{V} = -2(\mathbf{p}_3 - \mathbf{p}_3^*)^\top (\mathbf{P}_{\mathbf{g}_{31}}\mathbf{R}_1\mathbf{g}_{31}^* + \mathbf{P}_{\mathbf{g}_{32}}\mathbf{R}_1\mathbf{g}_{32}^* - \mathbf{h}_3)$$
$$= -2(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \left(\frac{\mathbf{P}_{\mathbf{g}_{31}}}{\|\mathbf{z}_{31}^*\|} + \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|} \right) (\mathbf{p}_3 - \mathbf{p}_3^*)$$
$$\quad + (\mathbf{p}_3 - \mathbf{p}_3^*)^\top \frac{\mathbf{P}_{\mathbf{g}_{32}}}{\|\mathbf{z}_{32}^*\|}(\mathbf{p}_2 - \mathbf{p}_{2a}^*) + (\mathbf{p}_3 - \mathbf{p}_3^*)^\top \mathbf{h}_3$$
$$\leq -2(\mathbf{p}_3 - \mathbf{p}_3^*)^\top \mathbf{M}(\mathbf{p}_3 - \mathbf{p}_3^*)$$
$$\quad + \|\mathbf{p}_3 - \mathbf{p}_3^*\| \left(\frac{\|\mathbf{P}_{\mathbf{g}_{32}}\|}{\|\mathbf{z}_{32}^*\|}\|\mathbf{p}_2 - \mathbf{p}_{2a}^*\| + \|\mathbf{h}_3\| \right)$$
$$\leq -2\kappa\|\mathbf{p}_3 - \mathbf{p}_3^*\|^2 + \|\mathbf{p}_3 - \mathbf{p}_3^*\| \left(\frac{2d_{21}}{\|\mathbf{z}_{32}^*\|} + \|\mathbf{h}_3\| \right) \quad (48)$$

where $\kappa > 0$ as defined in Lemma 6. Furthermore, in (48), $\|\mathbf{h}_3\|$ is bounded from Lemma 10. Thus, if $\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|$ is large enough, the second term is $\mathcal{O}(\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|)$ while the first term is $\mathcal{O}(-\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|^2)$, and $\dot{V} < 0$. Consequently, $\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|$ is bounded, and we can choose m_3 such that $m_3 > \max g g_{t \geq 0}\|\mathbf{p}_3 - \mathbf{p}_{3a}^*\|$. It follows that

$$\dot{V} \leq -2\kappa V + 2d_{21}\|\mathbf{z}_{32}^*\|^{-1}m_3 + m_3\|\mathbf{h}_3\|$$

or the system (45) satisfies the ultimate boundedness property. When $\mathbf{h}_3(t) = \mathbf{0}$, the unforced system has two isolated equilibria with properties given in Lemma 13. Since the system (45) satisfies Assumptions–A2 and the ultimate boundedness property, (45) is almost globally ISS with respect to the equilibrium $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ [35, Proposition 3].

Because $\mathbf{h}_3(t) \to \mathbf{0}$ as proved in Lemma 10, the equilibrium $\mathbf{p}_3 = \mathbf{p}_{3a}^*$ of (42) is almost globally asymptotically stable [34, Th. 2]. ∎

4) The Overall System: Consider the n-agent system (40)

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}) + \mathbf{h}(\mathbf{p}, t).$$

We have the following lemma whose proof follows from Lemma 11 and repetitively applying Lemma 12.

Lemma 15: The unforced system $\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p})$ has two equilibria. The first equilibrium $\mathbf{p} = \mathbf{p}_a^*$ corresponding to $\mathbf{g}_{ij} = \mathbf{R}_1\mathbf{g}_{ij}^*$, $\forall \mathbf{g}_{ij}^* \in \mathcal{B}$ is almost globally asymptotically stable. The second equilibrium $\mathbf{p} = \mathbf{p}_b^*$ corresponding to $\mathbf{g}_{ij} = -\mathbf{R}_1\mathbf{g}_{ij}^*$, $\forall \mathbf{g}_{ij}^* \in \mathcal{B}$ is unstable.

Finally, the main result of this section is given in the following Theorem.

Theorem 3: Consider the system (33)–(34). Under Assumptions 1–4 and the proposed control laws (35), (37), and (38), $\mathbf{R}_i \to \mathbf{R}_1$ ($i = 1, \ldots, n$) and $\mathbf{p} \to \mathbf{p}_a^*$ asymptotically if initially $\mathbf{R}_2(0) \neq \mathbf{R}_1$, $\mathbf{p}_2(0) \neq \mathbf{p}_{2b}^*$.

Proof: We have $\mathbf{R}_i \to \mathbf{R}_1$, $\forall i = 1, \ldots, n$ according to Lemma 9. The convergence of $\mathbf{p}$ to the target formation follows from Lemma 12, Lemma 14, and by invoking mathematical induction as in Theorem 1. ∎

V. REGULATING THE TARGET FORMATION

In this section, we study two strategies to regulate the LFF formations given that the n-agent system starts from a formation that is bearing congruent to the desired formation. First,

we propose a strategy to control the formation's orientation by switching the leader's orientation. Second, we show that by controlling the distance between the leader and the first follower, we can control the formation scale.

A. Controlling the Formation's Orientation

As proved in Section IV, under the two-layer control strategy, the n-agent system (33)–(34) almost globally asymptotically converges to the desired formation $\mathbf{p}_a^*$ corresponding to $\mathbf{g}_{ij} = \mathbf{R}_1 \mathbf{g}_{ij}^*$, $\forall (i,j) \in \mathcal{E}$. The desired formation's orientation with regard to the global reference frame is thus determined by the leader's orientation. When the actual formation is identical with the desired formation $\mathbf{g}_{ij} = \mathbf{g}_{ij}^*$, $\mathbf{R}_i = \mathbf{R}_1$ for all $1 \leq i \leq n$, the leader can control the overall formation's orientation with regard to the global reference frame by switching its orientation $\mathbf{R}_1$ to a new orientation $\mathbf{R}_1'$. The new orientation must satisfy the following assumption.

Assumption 5: The new orientation $\mathbf{R}_1'$ is contained within a closed ball $\bar{B}_r(\mathbf{R}_1)$ of radius r less than $\pi/2$, or equivalently, the symmetric part of $(\mathbf{R}_1')^\top \mathbf{R}_1$ is positive definite.

Corollary 1: Under Assumptions 1–3 and control laws (36)–(38), if initially the formation is at a desired equilibrium satisfying $\mathbf{g}_{ij}(0) = \mathbf{R}_1 \mathbf{g}_{ij}^*, \forall \mathbf{g}_{ij}^* \in \mathcal{B}$, $\mathbf{R}_i(0) = \mathbf{R}_1, \forall i = 1, \ldots, n$, and agent 1 switches its orientation to $\mathbf{R}_1'$ satisfying Assumption 5, the formation asymptotically converges to a formation with the same formation scale satisfying $\mathbf{g}_{ij} = \mathbf{R}_1' \mathbf{g}_{ij}^*$, $\forall \mathbf{g}_{ij}^* \in \mathcal{B}$.

Proof: Since the new orientation $\mathbf{R}_1'$ satisfies Assumption 5, after the leader switches its orientation, the convergence of all other agents' orientations to $\mathbf{R}_1'$ is guaranteed and thus $\mathbf{R}_i \to \mathbf{R}_1', 2 \leq i \leq n$.

The new desired formation has to satisfy $\mathbf{g}_{ij} = \mathbf{R}_1' \mathbf{g}_{ij}^*$, $\forall \mathbf{g}_{ij}^* \in \mathcal{B}$. Because $\mathbf{g}_{21}(0) = \mathbf{R}_1 \mathbf{g}_{21}^*$, $\mathbf{R}_2(0) = \mathbf{R}_1$, after the leader switches its orientation, agent 2 cannot be at the new undesired equilibrium, i.e., $\mathbf{g}_{21}(0) \neq -\mathbf{R}_1' \mathbf{g}_{21}^*$, $\mathbf{R}_2(0) \neq \mathbf{R}_1'$. Therefore, the convergence of the formation to the new desired formation follows immediately from Theorem 3. ∎

B. Rescaling the Formation

In practice, it may be desired to control the scale of the formation. If only the bearing information is measured, there is apparently no basis to control the size of the overall formation. Suppose the formation is in its desired shape. Furthermore, assume that one distance, d_{12}, between the leader and first follower, for which there is an associated desired distance constraint d^*, can be measured by the leader. It turns out that by controlling d_{12}, the whole other distances in the LFF formation will be controlled. The scale adjustment control law is proposed as

$$\dot{\mathbf{p}}_1 = \alpha_1 (d_{12}^2 - (d^*)^2)(\mathbf{p}_2 - \mathbf{p}_1) \qquad (49)$$

where $\alpha_1 > 0$ is a control gain.

Proposition 3: Under Assumptions 1–3, if the LFF formation is initially in a desired formation, agent 1 moves under the control law (49) and other agents move under the control law

(37)–(38), then the LFF formation asymptotically converges to a new desired formation with formation scale specified by d^*.

Proof: First, since the formation is assumed to be initially at a desired formation, all local orientations are aligned and will not be changed with time.

Second, the first follower is initially in its desired position, that is $\mathbf{g}_{21}(0) = \mathbf{R}_1 \mathbf{g}_{21}^*$, and it will not move ($\dot{\mathbf{p}}_2 = \mathbf{0}$) because the motion of the leader preserves $\mathbf{g}_{21}$. This fact follows from

$$\frac{\partial \mathbf{g}_{21}}{\partial \mathbf{p}_1} \dot{\mathbf{p}}_1 = \frac{\mathbf{P}_{\mathbf{g}_{21}}}{d_{12}} \alpha_1 (d_{12}^2 - (d^*)^2)(\mathbf{p}_2 - \mathbf{p}_1)$$

$$= \alpha_1 \frac{d_{12}^2 - (d^*)^2}{d_{12}} \mathbf{P}_{\mathbf{g}_{21}} \mathbf{z}_{12} = \mathbf{0}. \qquad (50)$$

We prove that d_{21} converges to d^* exponentially fast. To this end, consider the distance dynamics

$$\frac{d}{dt}(d_{12}^2) = 2\alpha_1 (d_{12}^2 - (d^*)^2)\mathbf{z}_{21}^\top (\mathbf{p}_2 - \mathbf{p}_1)$$

$$= -2\alpha_1 d_{12}^2 (d_{12}^2 - (d^*)^2). \qquad (51)$$

Consider also the Lyapunov function $V = \frac{1}{4}(d_{12}^2 - (d^*)^2)^2$, which is positive definite, continuously differentiable, and radially unbounded. Furthermore, $V = 0$ if and only if $d_{12} = d^*$. We have

$$\dot{V} = -\alpha_1 d_{12}^2 (d_{12}^2 - (d^*)^2)^2 < 0, \ \forall d_{12} \neq d^*$$

and $\dot{V} = 0$ if and only if $d_{21}^2 = (d^*)^2$. Since $\dot{V} \leq 0$, it follows that d_{12}^2 is bounded and the variable d_{12} increases or decreases monotonically to the desired distance d_{12}^*. Thus, there exists $\kappa = \alpha_1 \min_{t \geq 0} d_{12}^2(t) > 0$ and

$$\dot{V} = -\alpha_1 d_{12}^2 (d_{12}^2 - (d^*)^2)^2 \leq -\kappa V \leq 0.$$

It follows that d_{12} converges to d^* exponentially fast [40].

The remaining proof for convergence of other followers is similar to the proof of Theorem 1.

Consequently, the formation scale asymptotically converges to the desired one, which is fully determined by the distance between the leader and the first follower as discussed in Lemma 3. ∎

VI. SIMULATION RESULTS

In this section, we consider an eight-agent system with an LFF graph as depicted in Fig. 2. The desired bearing vectors were chosen satisfying Assumption 3 and such that the desired formation is a cube in $\mathbb{R}^3$.

A. Simulation 1: Achieving the Desired Formation

In this simulation, the leader's initial conditions are $\mathbf{p}_1(0) = [0, 0, 0]^\top$, $\mathbf{R}_1(0) = \mathbf{I}_3$. Other agents' orientations were randomly chosen such that Assumption 4 is satisfied. Agent 1 is placed at the origin. Agent 2's initial position is chosen at $\mathbf{p}_2(0) = [1, \sqrt{3}, 0]^\top$, which is not an undesired equilibrium.

Fig. 7 depicts trajectories and orientations of eight agents. The initial orientations and the final orientations are colored black and red, respectively. Observe that agent 1 does not move
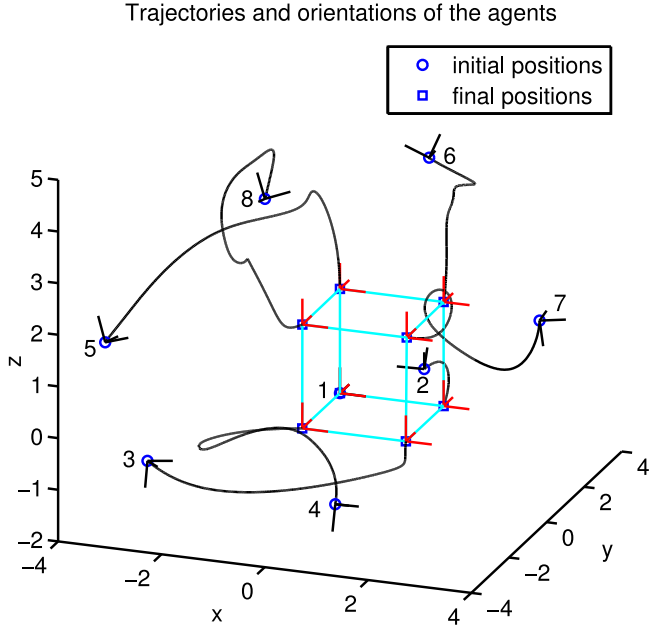
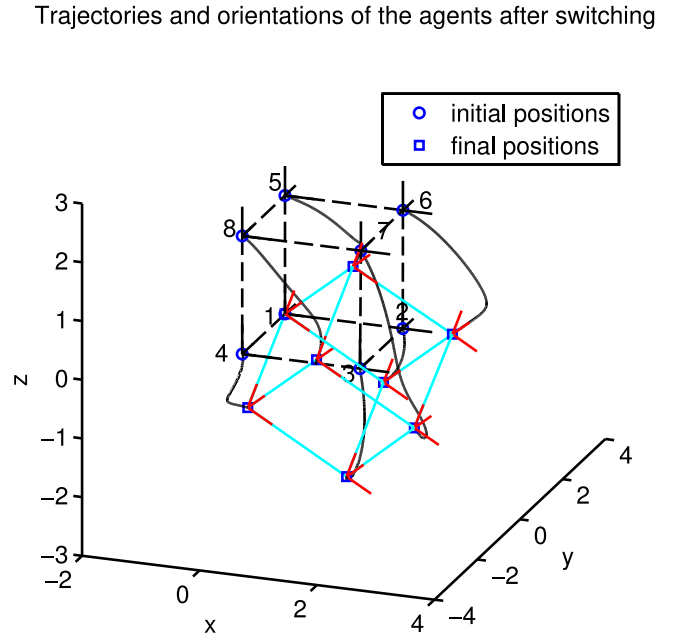Fig. 7. Simulation 1: Achieving the desired formation with orientation alignment.

Fig. 8. Simulation 2: Rotating the target formation by switching the leader's orientation.

in this simulation, and $d_{21}(t) = d_{21}(0) = 2$, $\forall t \geq 0$. In the final formation, all orientations are aligned and all desired bearing vectors are satisfied. Thus, the simulation result is consistent with Theorem 3.

B. Simulation 2: Rotating Formation by Switching Leader's Orientation

This simulation continues from the end of Simulation 1, i.e., eight agents have taken up the desired formation shape described in the previous simulation. Agent 1 switches its orientation from $\mathbf{I}_3$ to

$$\mathbf{R}_1' = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0.3536 & 0.8660 & -0.3536 \\ -0.6124 & 0.5000 & 0.6124 \end{bmatrix}$$

which satisfies Assumption 5. Fig. 8 depicts trajectories and orientations of eight agents after agent 1 switched its orientation. The final formation is rotated by $\mathbf{R}_1'$ from the initial formation and all agent's local orientations converge to $\mathbf{R}_1'$. Agent 1 does not move in this simulation. Also, the formation's scale does not change during the system's evolution and $d_{21}(t) = 2$, $\forall t \geq 0$.

C. Simulation 3: Rescaling the Formation

This simulation continues from the end of Simulation 2. The leader starts to control the scale. It is shown in Fig. 9 that the formation is rescaled to the desired formation scale, and $d_{21}(t) \to d^* = 1$. Agent 1 moves along a straight line toward agent 2 while agent 2 does not move since its bearing constraint $\mathbf{g}_{21}^*$ is always satisfied. Thus, the simulation result is consistent with Proposition 3.
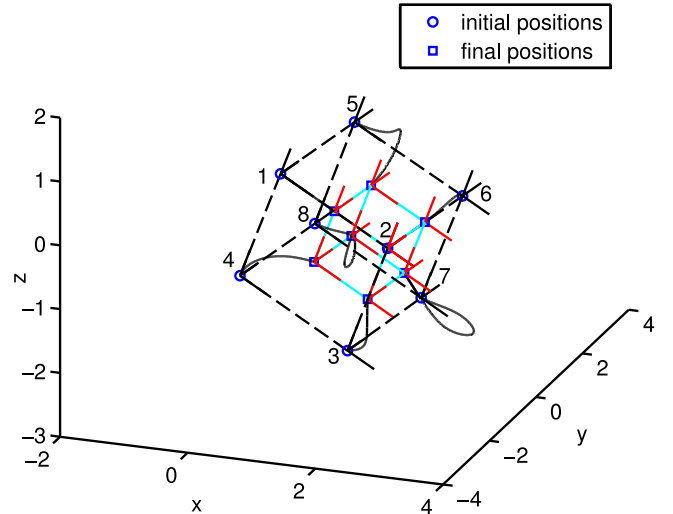

Fig. 9. Simulation 3: Rescaling the target formation by controlling the distance d_{12}.

VII. CONCLUSION

This paper studied bearing-based LFF formation control in an arbitrary dimensional space. The stability of LFF formations under the proposed bearing-only control law was extensively examined. As far as we know, this is the first paper fully dedicated to the stability analysis of a directed bearing-constrained formation in an arbitrary dimensional space. Additionally, strategies to achieve the desired formation without a common reference

frame, to rotate and to rescale the formation, were also addressed.

Several problems in bearing-only based formation control are still open. For example, a bearing-based persistence theory on directed formations has not yet been developed. Furthermore, studies on formations containing directed cycles may lead to some ideas for solving this problem. We are also planning to implement the control law in quadcopter systems with vision sensors. Hardware implementation may raise many practical issues in bearing-based formation control including agent's nonlinear dynamics, bearing measurement errors, and vision sensor's range.

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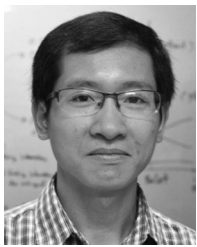
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