Angle-based control of directed acyclic formations with three-leaders

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***Abstract*—We study angle-based control for directed acyclic formations. We consider a group of single-integrator modeled agents moving on the plane. The agent group consists of three leader agents and the remaining follower agents. The leader agents are able to measure their absolute position while the remaining agents measure the bearing angle of their three neighbors with respect to the x-axis. The bearing angle sensing topology is directed and acyclic. By adopting a direct position control law and an angle-based control law for the leader and follower agents, respectively, we drive the agents to their desired location. Based on input-to-state stability notion, we show that the desired locations are locally asymptotically stable. Simulation results are provided to validate the proposed control strategy.**

I. Introduction

Formation control is an interesting research topic within the realm of cooperative control. In an usual formation control problem, the objective of a group of agents is to achieve a prescribed geometric formation without a centralized sensing and processing unit. In order to accomplish the group task, the agents need to measure and control some variables related to the formation shape. Based on measured and controlled variables, one can classify various formation control problems. For instance, the authors of [1] have classified formation control into position-, displacement-, and distance-based schemes based on measured and controlled variables.

Though positions, displacements, and distances have been primarily used, bearing angles can be utilized for formation control [2], [3], [4], [5], [6]. The authors of [2] have proposed an formation control law for three-agents on the plane based on relative bearing angles. A similar result for four-agents on the plane is found in [3]. Considering that angle constraints specify formations up to scale, the authors of [4] have studied a three-agent formation control problem based on both angle and distance constraints. In [5], angle-based control has been applied to circular formations. The authors of [6] have proposed an angle-based control law for an agent to reach its desired position that is specified by angle-constraints with respect to three landmarks ion the plane. Note that angle-constraints by three landmarks determine an unique desired position for the agents.

The existing results on angle-based formation control have focused on a group of several agents. Further only formation shapes have been mainly considered. However, agent groups need to be scalable. Further the objective of formation control is often to drive agents to prescribed desired position. Based on these motivations, we consider a group of single-integrator modeled agents on the plane consisting of three leaders and the remaining followers. We assume that the leaders are able to measure their own position while the followers measure bearing angles of their exactly three neighbors. Under the assumption that the bearing angle measurement topology is modeled as a directed acyclic graph, we propose a position control law for the leader agents and an angle control law for the follower agents. Based on input-to-state and cascade system stability theory, we show that the desired formation for the agents is locally asymptotically stable.

The remainder of this paper is organized as follows. We summarize some preliminary results in Section II. In Section III, we formulate the formation control problem, propose a control strategy, and then analyze stability properties of the desired formation. A simulation result is presented in Section IV. We conclude the paper in Section V.

II. Preliminaries

A directed graph is defined as a pair $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of ordered pairs of the nodes, called edges. A directed edge $(i, j) \in \mathcal{G}$ exists if agent i measures (in its own local coordinate) agent j's bearing angle. We call agent j a neighbor of agent i and denote the neighbor set of i by $\mathcal{N}_i$. A directed path is defined as a sequence of edges $(i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k)$ in $\mathcal{G}$, where $i_1, i_2, ..., i_k \in \mathcal{V}$. i_1 and i_k are referred as the start vertex and the end vertex, respectively. A directed cycle is a directed path which has the same start vertex and end vertex. An acyclic directed graph is a directed graph which has no directed cycle. More results on graph theory can be found from [7].

Our stability analysis is based on input-to-state stability and cascade system theory. We summarize several relevant existing results in the following. Consider the following system

$$\dot{x} = f(x, u) \tag{1}$$

where $f : D_x \times D_u \mapsto \mathbb{R}^n$ is locally Lipschitz in x and u, and $D_x \subseteq \mathbb{R}^n$ and $D_u \subseteq \mathbb{R}^m$ are domains containing $x = 0$ and $u = 0$, respectively. Input-to-state stability (ISS) is defined as follows [8]:

Definition 1. *The system (1) is locally input-to-state stable if there exist a class $\mathcal{KL}$ function γ, and positive constants k_x*

and k_u such that for any initial state $x(0)$ with $\|x(0)\| < k_x$ and any input $u(t)$ with $\sup_{0\leqslant\tau\leqslant t} \|u(\tau)\| < k_x$, the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leqslant \beta(\|x(0)\|, t) + \gamma(\sup_{0\leqslant\tau\leqslant t} \|u(\tau)\|),\ t \geqslant 0. \quad (2)$$

A sufficient condition for local ISS [8] follows:

Theorem 1. *If there exists a neighborhood U of $(x = 0, u = 0)$ such that the function $f(x, u)$ in (1) is continuously differentiable and the unforced system $\dot{x} = f(x, 0)$ is asymptotically stable in U, then the system (1) is locally input-to-state stable.*

A condition for local asymptotic stability of cascade systems is provided in the following [8]:

Theorem 2. *For the cascade system*

$$\dot{x}_1 = f_{x_1}(x_1, x_2), \quad (3a)$$
$$\dot{x}_2 = f_{x_2}(x_2), \quad (3b)$$

where $f_{x_1} : D_{x_1} \times D_{x_2} \mapsto \mathbb{R}^{n_1}$ and $f_{x_2} : D_{x_2} \mapsto \mathbb{R}^{n_2}$ are locally Lipschitz in x_1 and x_2, if the system (2a), with x_2 as input, is locally input-to-state stable and the origin of the system (2b) is locally asymptotically stable, the origin of the cascade system (2) is locally asymptotically stable.

III. Formation Control Strategy

A. Problem Formulation

Consider the following N single-integrator modeled agents:

$$\dot{p_i} = u_i,\ i = 1, \ldots, N, \quad (4)$$

where $p_i \in \mathbb{R}^2$ and $u_i \in \mathbb{R}^2$ denote the position and control input of agent i, respectively. Without loss of generality, we assume that agents 1, 2, and 3 are leaders while the remaining agents are followers. We further assume that the leaders are able to measure their own position while the followers only measure the bearing angle of their exactly three neighbors. Let the bearing angle sensing topology be modeled by a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Let follower agent i have agents j, k, and l as its neighbors. We denote by β_{ij} the counter-clock wise bearing angle of agent j from the x-axis as depicted in Fig. 1. We define the bearing vector $\hat{u}_{ij}$ as

$$\hat{u}_{ij} := \frac{p_{ij}}{\|p_{ij}\|} = \mathbf{1}\angle\beta_{ij}, \quad (5)$$

where $\mathbf{1}$ is the unit vector.

Let $p_i^* \in \mathbb{R}^2$ be given for $i = 1, \ldots, N$. Then the objective of the agents is to achieve $p_i \to p_i^*$. Before stating the formation control problem, we summarize standing assumptions:

Assumption 1. *For the single-integrator modeled agents* (4), *we assume the following:*

- *Leader $i \in \{1, 2, 3\}$ measures p_i;*
- *Follower $i \in \{4, 5, ..., N\}$ has exactly three neighbors and it measures the bearing angle of its neighbors from the x-axis;*

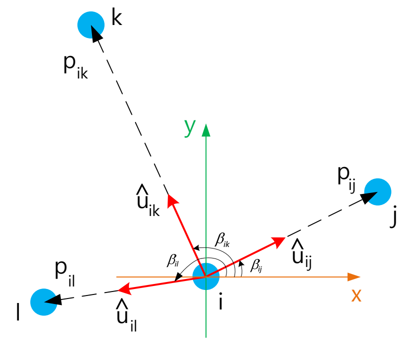

Fig. 1. Angle measurements of follower i.

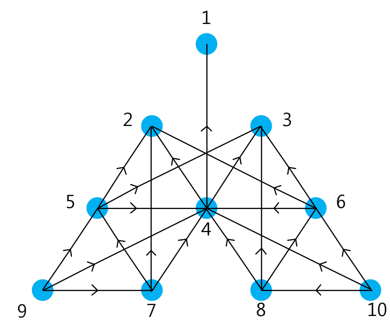

Fig. 2. Angle measurement topology for ten-agents.

- *The bearing angle measuring topology is modeled by a directed acyclic graph $\mathcal{G}$;*
- *Let agents j, k, and l be the neighbors of agent i. Then any three of p_i^*, p_j^*, p_k^*, and p_l^* are not collinear.*

We are ready to state the formation control problem for the single-integrator modeled agents (4):

Problem 1. *Consider the single-integrator modeled agents* (4). *Let p_i^* be given to leader i and $\hat{u}_{ij}^*$, $j \in \mathcal{N}_i$ be given to follower i. Under Assumption 1, design control laws for the leaders and followers such that $p^* = [p_1^{*T} \cdots p_N^{*T}]^T$ is asymptotically stable with respect to (4).*

B. Control strategy

We propose a formation control strategy that allows the followers to actively control bearing angles to their neighbors while allowing the leaders to directly control their position. For brevity, we introduce the following notations: For $1 \leq i \leq j \leq N$,

$$\mathcal{V}_{[i:j]} := \{i, ..., j\}, p_{[i:j]} = [p_i^T \ldots p_j^T]^T.$$

Consider the single-integrator modeled agents (4). Since the leaders measure their position and know their destination, it is natural to use the following control law:

$$u_i = k_L(p_i^* - p_i) = k_L\tilde{p}_i, \quad (6)$$

where $k_L > 0$.

For the follower $i \in \mathcal{V}_{[4:N]}$, we adopt the angle-based control law that has been proposed in [6]. To introduce the control law, we define the orthogonal vector of $\hat{u}_{ik}$ as

$$\hat{u}_{ik}^{\perp} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{u}_{ik}.$$

Then the control law for the followers can be designed as [6]:

$$u_i = -\sum_{j \in \mathcal{N}_i} (\hat{u}_{ij}^{*T} \hat{u}_{ij}^{\perp}) \hat{u}_{ij}^{\perp}. \tag{7}$$

Let $\tilde{p}_i := p_i - p_i^*$. Then the error dynamics of the agents can be described by

$$\dot{\tilde{p}}_i = -k_L \tilde{p}_i + \dot{p}_i^*, i \in \mathcal{V}_{[1:3]}, \tag{8a}$$

$$\dot{\tilde{p}}_i = f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}), i \in \mathcal{V}_{[4:N]}, \tag{8b}$$

where

$$f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}) := \sum_{j \in \mathcal{N}_i} (\hat{u}_{ij}^{*T} \hat{u}_{ij}^{\perp}) \hat{u}_{ij}^{\perp}. \tag{9}$$

C. Stability analysis

In below, we show that the origin $\tilde{p} = [\tilde{p}_1^T, \ldots, \tilde{p}_N^T]^T = 0$ is locally asymptotically stable under *Assumption 1* based on the approach adopted in [9]. Since p^* is constant, follower i can be described as

$$\dot{\tilde{p}}_i = f_i(\tilde{p}_i, \tilde{p}_{[1:i-1]}), \tag{10a}$$

$$\dot{\tilde{p}}_{[1:i-1]} = f_{[1:i-1]}(\tilde{p}_{[1:i-1]}), \tag{10b}$$

where

$$f_{[1:i-1]}(\tilde{p}_{[1:i-1]}) := \begin{bmatrix} -k_L \tilde{p}_{[1:3]} \\ f_4(\tilde{p}_4, \tilde{p}_{[1:3]}) \\ \vdots \\ f_{i-1}(\tilde{p}_{i-1}, \tilde{p}_{[1:i-2]}) \end{bmatrix}. \tag{11}$$

Our stability analysis consists of three steps as follows:

- *First Step*: We show that the origin $\tilde{p}_{[1:3]} = 0$ is exponentially stable with respect to $\dot{\tilde{p}}_{[1:3]} = -k_L \tilde{p}_{[1:3]}$.
- *Second Step*: We show that (10) is locally input-to-state stable (ISS) with $\tilde{p}_{[1:i-1]}$ as input. Based on *Theorem 2*, $\tilde{p}_{[1:i]}$ is locally asymptotically stable with respect to (10) if $\tilde{p}_{[1:i-1]} = 0$ is locally asymptotically stable with respect to (10b).
- *Third Step*: Finally we show that the origin is locally asymptotically stable with respect to $\dot{\tilde{p}}_{[1:N]} = f_{[1:N]}(\tilde{p}_i, \tilde{p}_{[1:N]})$ based on mathematical induction.

The first step is obvious. For the second step, we show local stability of the following unforced dynamics of (10a)

$$\dot{\tilde{p}}_i = f_i(\tilde{p}_i, 0), \; i \in \mathcal{V}_{[4:N]}. \tag{12}$$

We have the following lemma:

Lemma 1. *Let Assumption 1 hold. For $i \in \mathcal{V}_{[4:N]}$, the origin $\tilde{p}_i = 0$ is locally asymptotically stable with respect to (12).*

Proof. The unforced error dynamics (12) implies that all agents $1, \ldots, i-1$ are at their desired positions, which are non-collinear as a result of *Assumption 1*. It then follows from [2, *Theorem 6*] that $\tilde{p}_i = 0$ is locally asymptotically stable with respect to (12). □

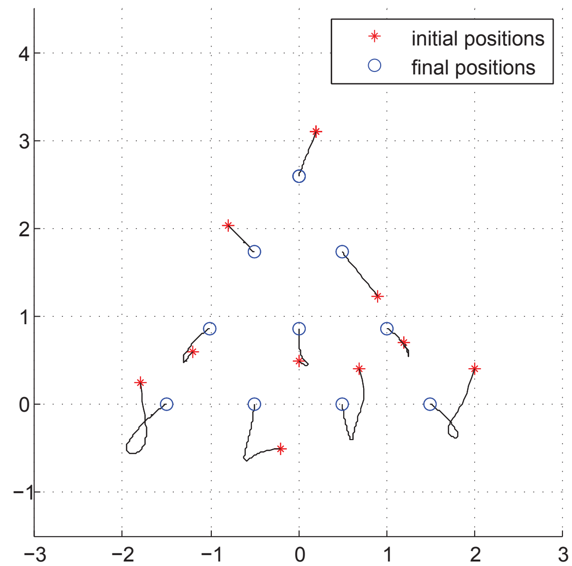

Fig. 3. Agent positions

The following lemma reveals local ISS of (10a):

Lemma 2. *Let Assumption 1 hold. For $i \in \mathcal{V}_{[4:N]}$, (10a) is locally input-to-state stable with $\tilde{p}_{[1:i-1]}$ as input.*

Proof. From *Lemma 1*, $\tilde{p}_i = 0$ is locally asymptotically stable with respect to (12). Further f_i defined in (9) is continuously differentiable in $\tilde{p}_i$ and $\tilde{p}_{[1:i-1]}$. Thus, it follows from *Theorem 1* that (10a) is locally ISS with $\tilde{p}_{[1:i-1]}$ as input. □

Finally, we have the following theorem:

Theorem 3. *Let Assumption 1 hold. The origin $\tilde{p} = 0$ is locally asymptotically stable with respect to (8).*

Proof. Consider the following cascade system

$$\dot{\tilde{p}}_4 = f_4(\tilde{p}_4, \tilde{p}_{[1:3]}), \tag{13a}$$

$$\dot{\tilde{p}}_{[1:3]} = -k_L \tilde{p}_{[1:3]}. \tag{13b}$$

It is obvious that the origin $\tilde{p}_{[1:3]} = 0$ is exponentially stable with respect to (13b). Based on *Lemma 2*, (13a) is locally stable with $\tilde{p}_{[1:3]}$ as input. It then follow from *Theorem 2* that $\tilde{p}_{[1:4]} = 0$ is locally asymptotically stable with respect to (13).

Next, suppose that, for any $i \in \mathcal{V}_{[L+1,N]}$, (10b) is locally asymptotically stable. *Lemma 2* guarantees the locally ISS stability of (10a) with $\tilde{p}_{[1:i-1]} = 0$ as input. Thus, from *Theorem 2*, we have $\tilde{p}_{[1:i]} = 0$ is also locally asymptotically stable with respect to (10b).

As a result of mathematical induction, for any $i \in \mathcal{V}_{[L+1,N]}$, $\tilde{p}_{[1:i]} = 0$ is locally asymptotically stable with respect to (10). Thus, we can conclude that $\tilde{p} = 0$ is locally asymptotically stable with respect to (8). □

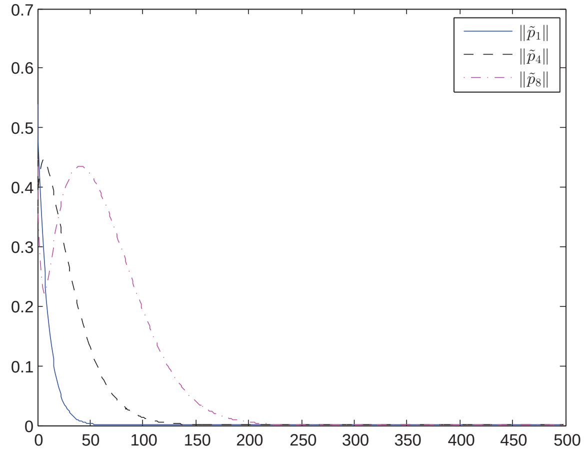

Fig. 4. Position error

IV. Simulation result

We consider ten agents whose sensing topology is depicted in Fig. 2. Let *Assumption 1* hold for the ten agents. The desired positions are given as $p_1^* = (0, -\frac{3\sqrt{3}}{2}), p_2^* = (-\frac{1}{2}, \sqrt{3}), p_3^* = (\frac{1}{2}, \sqrt{3}), p_4^* = (0, \frac{\sqrt{3}}{2}), p_5^* = (-1, \frac{\sqrt{3}}{2}), p_7^* = (-\frac{1}{2}, 0), p_8^* = (\frac{1}{2}, 0), p_9^* = (-\frac{3}{2}, 0), p_{10}^* = (\frac{3}{2}, 0)$. Note that the desired positions satisfy the forth condition in *Assumption 1* though some of the positions are collinear. Fig. 3 and Fig. 4 show the simulation result for the ten agents under the proposed control strategy. We can see from Fig. 3 that both of the leaders and the followers eventually reach their desired position. The position error magnitude of agents 1, 4, and 8 are shown in Fig. 4.

V. Conclusion

In this paper, we proposed a control strategy under which majority of agents can reach their desired position based on an angle-based control law. There are several further research directions. First, it is desirable to investigate global stability properties of the desired formation. Second, it would be interesting to consider the case that the leader agents are moving along their desired trajectory. Finally, it is desirable to extend this control strategy to more realistic agent models such as double-integrators and nonholonomic agents.

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